

Modelling Ferroelectric and Ferroelastic Hysteresis: Thermodynamical Consistency by Hysteresis Potentials

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joint work with

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Overview

- motivation
- Preisach operators
- a phenomenological model using Preisach operators
- thermodynamic consistency and hysteresis potentials
- a thermodynamically consistent material law for ferroelectricity and ferroelasticity
- well-posedness

Piezoelectric Transducers

Direct effect: apply mechanical force → measure electric voltage

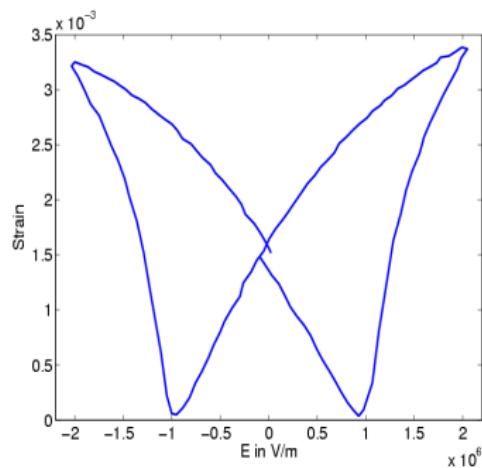
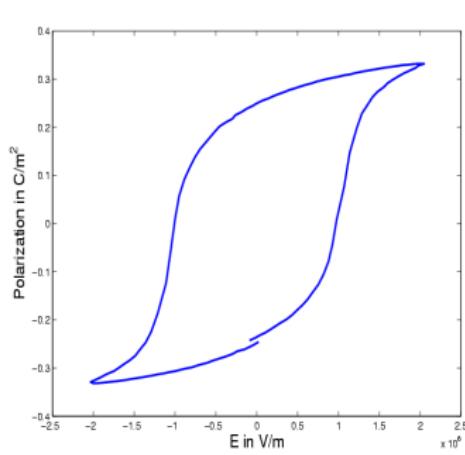
Indirect effect: impress electric voltage → observe mechanical displacement

Application Areas:

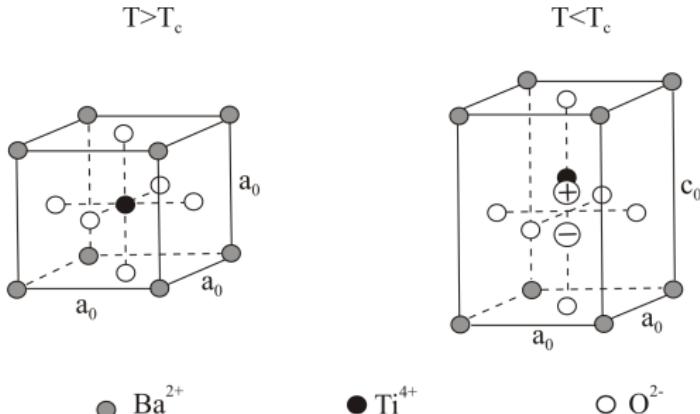
- Ultrasound (medical imaging & therapy)
- Force- and acceleration Sensors
- Actor injection valves (common-rail Diesel engines)
- SAW (surface-acoustic-wave) sensors
- ...

Hysteresis in Piezoelectricity

e.g. ferroelectric hysteresis:
dielectric displacement and mechanical strain
at high electric field intensities ($E \sim 2\text{MV/m}$):



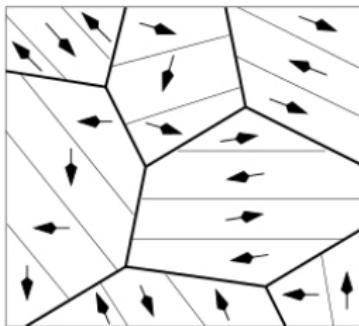
Piezoelectricity and Ferroelectricity on Unit Cell Level



Unit cell of BaTiO_3 above (left) and below (right) Curie temperature T_c ,
the latter exhibiting spontaneous polarization and strain

courtesy to M.Kamlah

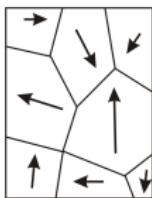
Grain and Domain Structure



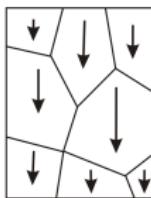
Grains with same unit cell orientation
domains with same polarization direction

courtesy to M.Kamlah

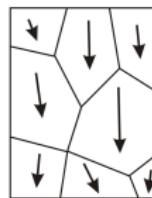
Poling Process



Initial state



$E > E_c$

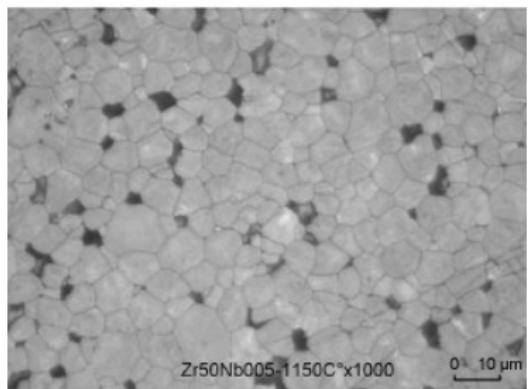


$E=0$

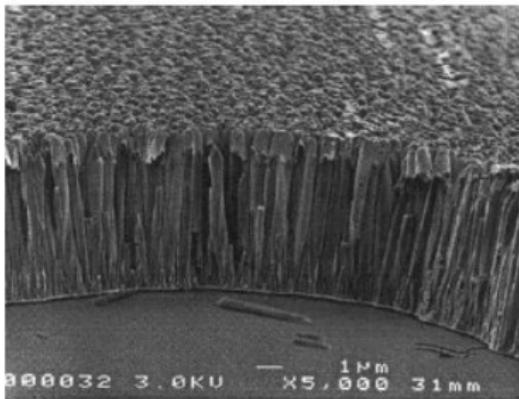
Orientation of the total polarization of the grains at initial state (left),
due to a strong external electric field (middle)
and after switching it off, leading to a remanent polarization and strain
(right)

courtesy to M.Kamlah

Grains



Zr50Nb005-1150C°x1000 0 10 μm

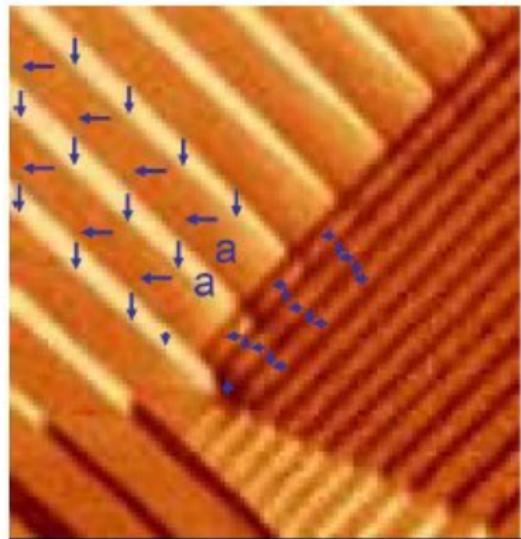


left: Nb doped PZT bulk grain structure (courtesy to CeramTech)

right: Sputter deposited ZnO thin films with columnar grains

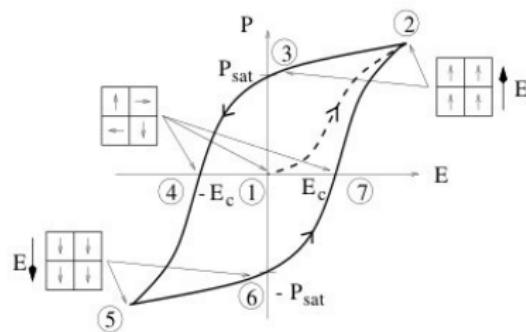
[Damjanovic 1998]

Domains

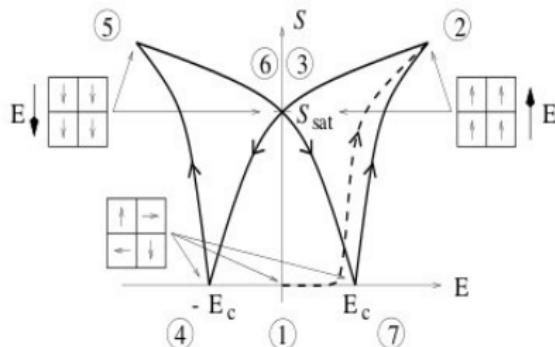


Domain structure of single PZT grain with 90 degree and 180 degree domain walls (courtesy to G.Schneider, TU Hamburg)

Ferroelectricity



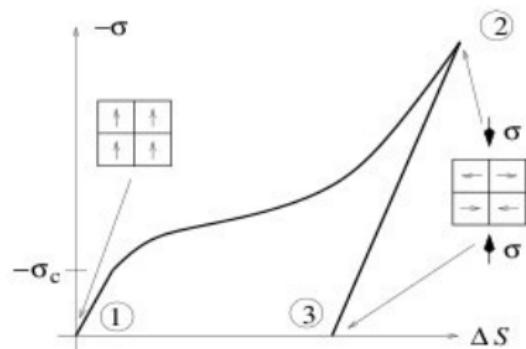
polarization hysteresis



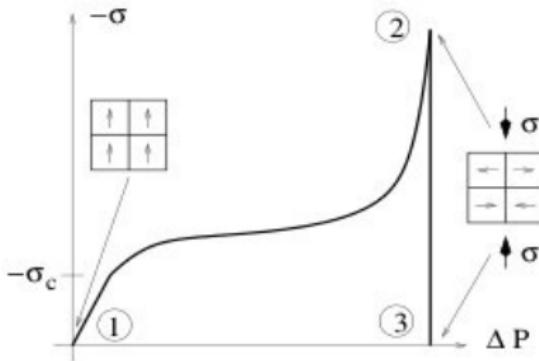
butterfly hysteresis

courtesy to M.Kamlah

Ferroelasticity



stress-strain relation



mechanical depolarization

courtesy to M.Kamlah

Piezoelectricity – Ferroelectricity – Ferroelasticity

Piezoelectricity

... linear coupling between electric and mechanical fields
(reversible)

Ferroelectricity

... external electric field influences polarization
(irreversible, hysteretic)

Ferroelasticity

... external mechanical field influences polarization
(irreversible, hysteretic)

Models of Ferroelectricity/Ferroelasticity

① Thermodynamically consistent models

macroscopic view, 2nd law of thermodynamics

Bassiouny&Ghaleb'89, Kamlah&Böhle'01, Landis'04,

Schröder&Romanowski'05, Su&Landis'07,

Linnemann&Klinkel&Wagner'09, Mielke&Timofte'06,

Alber&Kraynyukova'12

② Micromechanical models

consider material on level of single grains

Huber&Fleck'01, Fröhlich'01, Delibas&Arockiarajan&Seemann'05,

Belov&Kreher'06, Huber'06, McMeeking&Landis&Jimeneza'07

③ Phase field models

transition between phases (domain wall motion)

Wang&Kamlah&Zhang'10,

Xu&Schrade&Müller&Gross&Granzow&Rödel'10,

④ Multiscale models

Schröder&Keip'10, Zäh&Liefer&Rosato&Miehe'10

⑤ Phenomenological models using hysteresis operators

from input-output description for control purposes

Hughes&Wen'95, Kuhnhen'01, Cimaa&Laboure&Muralt'02,

Smith&Seelecke&Ounaies&Smith'03, Pasco&Berry'04,

Kuhnhen&Krejčí'07, Ball&Smith&Kim&Seelecke'07

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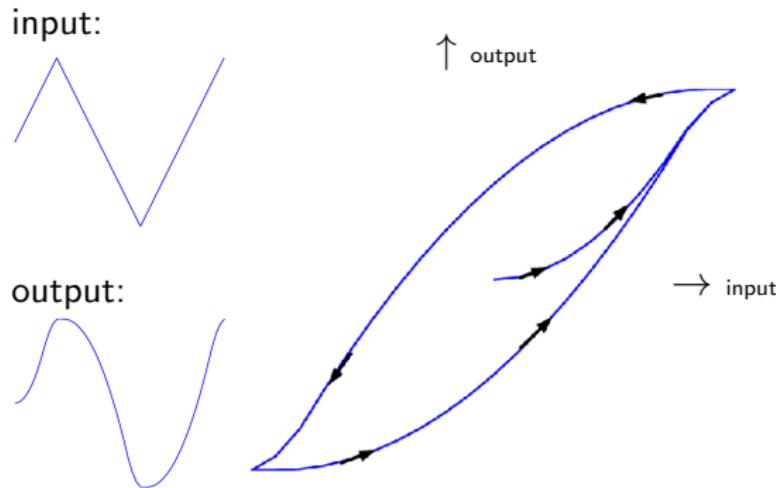
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Hegewald&BK&MK&Lerch'08,'09

Preisach Operators

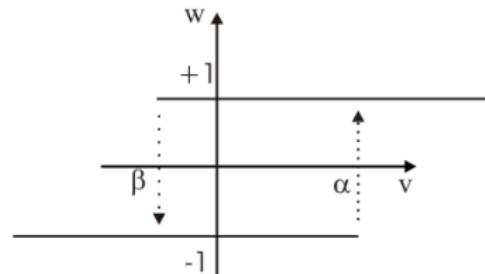
Hysteresis and Hysteresis Operators



- magnetics
- piezoelectricity
- plasticity
- ...
- * memory
- * Volterra property
- * rate independence

Krasnoselksii-Pokrovskii (1983), Mayergoyz (1991), Visintin (1994),
Krejčí (1996), Brokate-Sprekels (1996)

A Simple Example I: The Relay

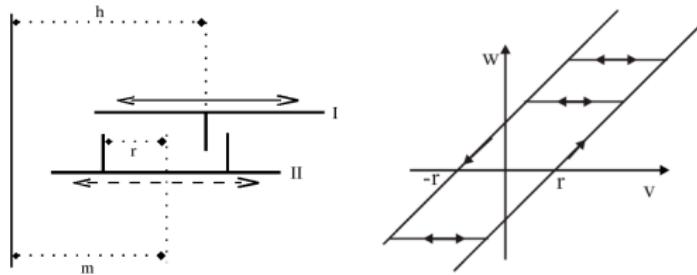


$$\mathcal{R}_{\beta,\alpha}[v](t) = w(t)$$

$$= \begin{cases} +1 & \text{if } v(t) > \alpha \text{ or } (w(t_i) = +1 \wedge v(t) > \beta) \\ -1 & \text{if } v(t) < \beta \text{ or } (w(t_i) = -1 \wedge v(t) < \alpha) \end{cases} \quad t \in [t_i, t_{i+1}]$$

t_0, t_1, t_2, \dots sequence of local extrema of v ,
i.e., v monotone on $[t_i, t_{i+1}]$.

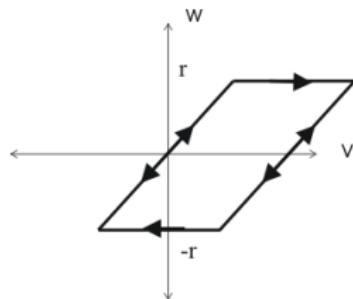
A Simple Example II: The Mechanical Play



$$\mathcal{F}_r[v](t) = w(t) = \max\{v(t)-r, \min\{v(t)+r, w(t_i)\}\} \quad t \in [t_i, t_{i+1}]$$

Relation to Relay operator: $\mathcal{F}_r[v](t) = \frac{1}{2} \int_{-\infty}^{\infty} \mathcal{R}_{s-r, s+r}[v](t) ds$

A Simple Example III: The Elastic-Plastic Element



$v \sim$ strain

$w \sim$ stress

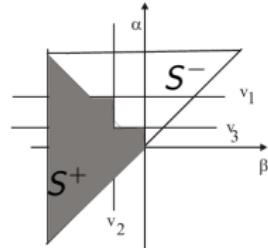
$r \sim$ yield stress

$$\mathcal{S}_r[v](t) = w(t) = \min\{r, \max\{-r, v(t)\}\} \quad t \in [t_i, t_{i+1}]$$

Relation to mechanical Play: $\mathcal{S}_r[v](t) = v(t) - \mathcal{F}_r[v](t)$

A General Hysteresis Model: the Preisach Operator

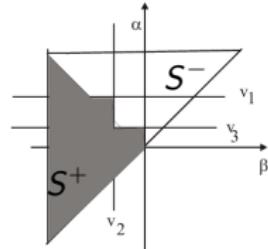
weighted superposition of elementary relays
with Preisach weight function ω defined on
Preisach plane $S = S^+ \cup S^-$:



$$\begin{aligned}\mathcal{P}^\omega[v](t) &= \iint_{\alpha, \beta \in S} \omega(\beta, \alpha) \mathcal{R}_{\beta, \alpha}[v](t) d(\alpha, \beta) \\ &= \iint_{\alpha, \beta \in S^+(t)} \omega(\beta, \alpha) d(\alpha, \beta) - \iint_{\alpha, \beta \in S^-(t)} \omega(\beta, \alpha) d(\alpha, \beta)\end{aligned}$$

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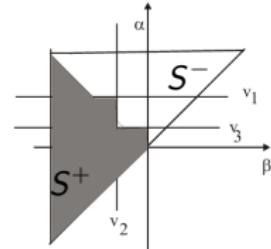


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- ± high dimensionality
- + can model minor loops
- + can model saturation
- + highly efficient evaluation

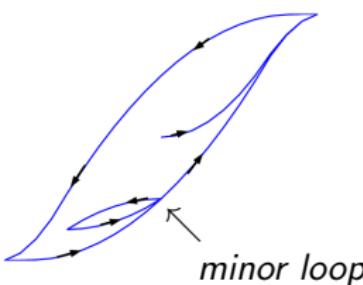
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Regularity and Monotonicity Properties of Preisach Operators

Lipschitz continuity on $C[0, T]$

$$\text{if } \int_0^\infty \sup_{s \in \mathbb{R}} |\omega(s-r, s+r)| dr < \infty$$

Coercivity: $\mathcal{P}[v]_t(t)v_t(t) \geq \underline{\mu}|v_t(t)|^2$

$$\text{if } 0 < \underline{\mu} \leq \partial_2 \mathcal{E}$$

Convexity: $\frac{\partial}{\partial t} (v_t \mathcal{P}[v]_t)(t) \leq 2v_{tt}(t) \mathcal{P}[v]_t(t)$ if $\text{sign}(v^* - v_*) \partial_2^2 \mathcal{E}(v_*, v^*) \leq 0$

$$\mathcal{E}(v_*, v^*) = \begin{cases} 2 \iint_{v_* \leq \beta \leq \alpha \leq v^*} \omega(\beta, \alpha) d(\alpha, \beta) & \text{if } v_* \leq v^* \\ \mathcal{E}(-v^*, -v_*) = \mathcal{E}(v_*, v^*) & \text{if } v_* > v^* \end{cases}$$

However...

- \mathcal{P} is not differentiable in a classical sense¹
- \mathcal{P} is not a monotone operator

¹[Brokate&Krejčí'14] generalized differentiability

A Phenomenological Model for Hysteresis in Piezoelectricity

$$\begin{aligned}\underline{\underline{S}} &= \underline{\underline{S}}^r + \underline{\underline{S}}^i & \underline{\underline{S}}^r &= \mathbf{s}^E \underline{\underline{\sigma}} + \mathbf{d}_{\vec{P}}^T \vec{E} & \vec{D}^i &= \vec{P} = \mathcal{P}^\omega [E] e_{\vec{P}} \\ \vec{D} &= \vec{D}^r + \vec{D}^i & \vec{D}^r &= \mathbf{d}_{\vec{P}} \underline{\underline{\sigma}} + \varepsilon^\sigma \vec{E} & \underline{\underline{S}}^i &= f_S(\vec{P}) (\frac{3}{2} e_{\vec{P}} e_{\vec{P}}^T - \frac{1}{2} I) \\ &&&& \mathbf{d}_{\vec{P}} &= f_d(P) = \frac{P}{P_{sat}} \mathbf{d}\end{aligned}$$

$\underline{\underline{S}}^r$... reversible strain

$\underline{\underline{\sigma}}$... mech. stress

\mathbf{s}^E ... elast. coeff.

\vec{D}^r ... reversible polarization

\vec{E} ... electr. field

ε^σ ... dielectr. coeff.

$\underline{\underline{S}}^i$... irreversible strain

\vec{P} ... polarization

\mathbf{d} ... coupling coeff.

\vec{D}^i ... irreversible polarization

hysteresis identification: [Hegewald&B.K.&M.K.&Lerch, J.Int.Mat.Sys.Struct.'08]

finite element formulation: [M.K&B.K.&Hegewald&Lerch J.Int.Mat.Sys.Struct.'09]

A Phenomenological Model for Hysteresis in Piezoelectricity

$$\frac{\underline{S}}{\vec{D}} = \frac{\underline{S}^r + \underline{S}^i}{\vec{D}^r + \vec{D}^i}$$

$$\begin{aligned}\underline{S}^r &= \mathbf{s}^E \underline{\sigma} + \mathbf{d}_{\vec{P}}^T \vec{E} \\ \vec{D}^r &= \mathbf{d}_{\vec{P}} \underline{\sigma} + \varepsilon^\sigma \vec{E}\end{aligned}$$

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good qualitative and quantitative reproduction of measurements
highly efficient in FEM simulations

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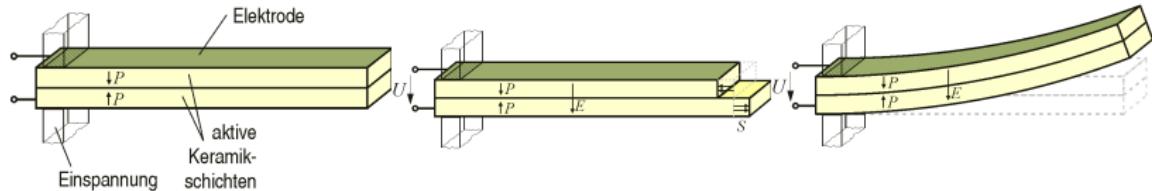
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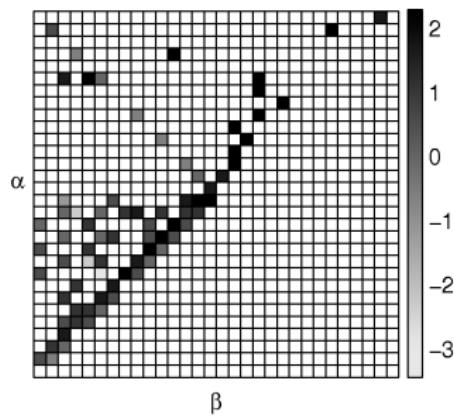
... however, thermodynamic consistency
cannot be shown for this model

Simulation Results: Bending Actuator



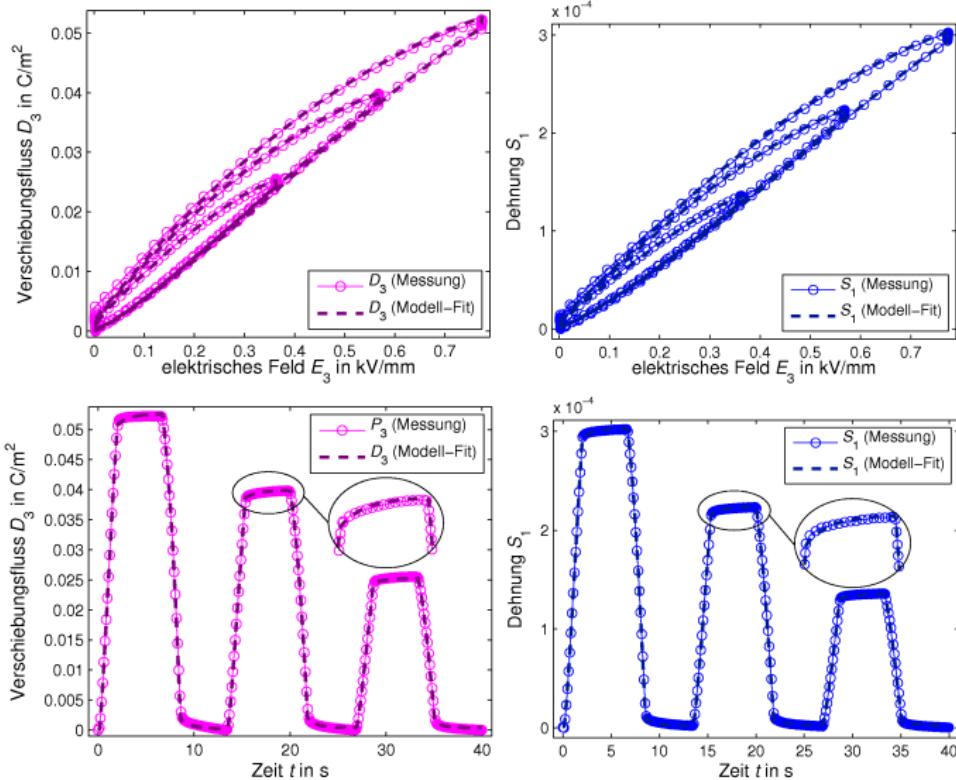
Simulation Results: Bending Actuator

Identified Preisach weight function:



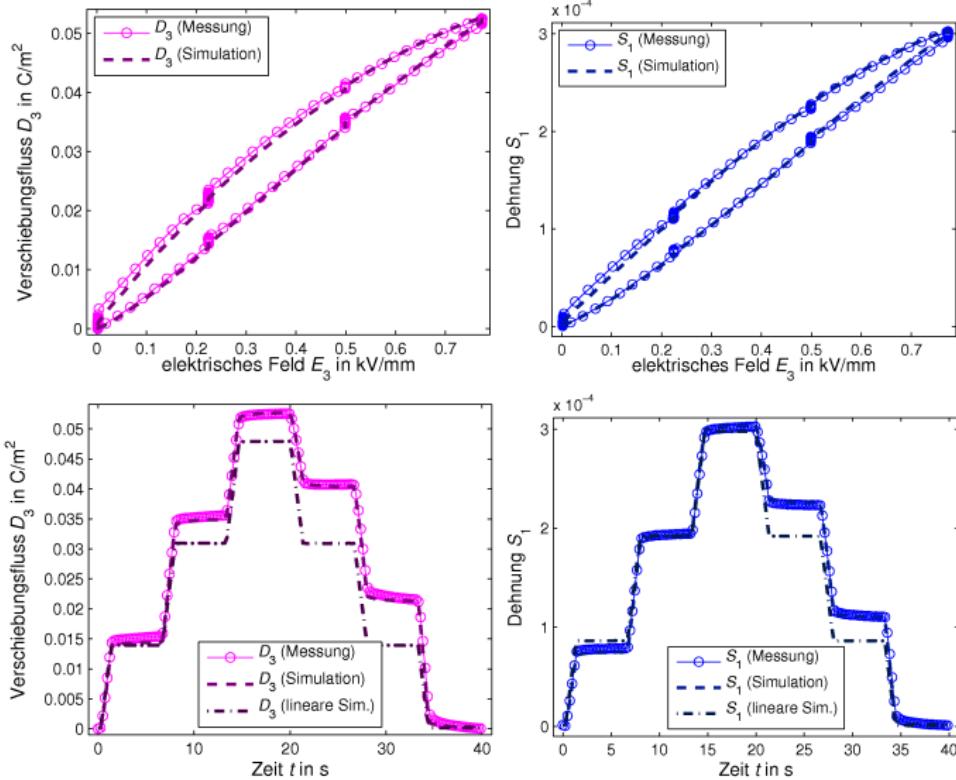
Simulation Results: Bending Actuator

comparison measurement – simulation with fitted Preisach operators:

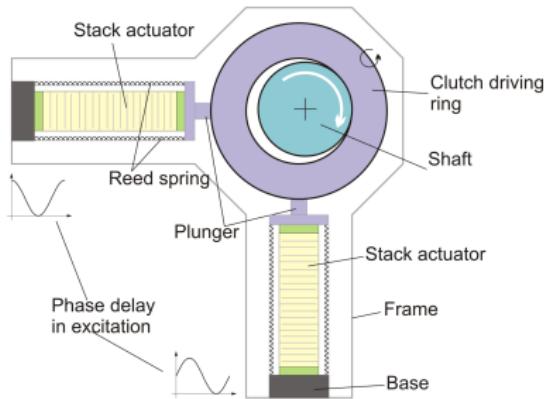


Simulation Results: Bending Actuator

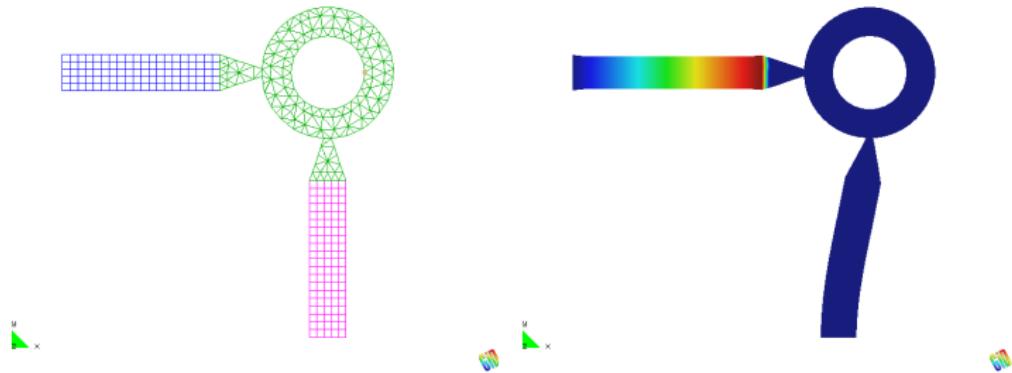
comparison measurement – simulation for alternative input signal:



Simulation Results: Revolving Drive



Simulation Results: Revolving Drive



Thermodynamic Consistency and Hysteresis Potentials

Thermodynamic Consistency

$\underline{\sigma}$... mechanical stress

$\underline{\varepsilon}$... mechanical strain

\vec{E} ... electric field

\vec{D} ... dielectric displacement

W ... work done by electric and mechanical forces

F ... free energy

D ... energy dissipation

$$W(t_1, t_2) = \int_{t_1}^{t_2} \dot{\underline{\varepsilon}}(t) : \underline{\sigma}(t) + \dot{\vec{D}}(t) \cdot \vec{E}(t) dt$$

2nd law of thermodynamics:

$$D(t_1, t_2) = W(t_1, t_2) - (F(t_2) - F(t_1)) \geq 0$$

differential form:

$$\dot{\underline{\varepsilon}} : \underline{\sigma} + \dot{\vec{D}}(t) \cdot \vec{E} - \dot{F} \geq 0$$

Hysteresis Potentials

clockwise:

$$\mathcal{P}[q](t)\dot{q}(t) - \mathcal{U}[q](t) \geq 0 \text{ a.e.} \quad \forall q \in W^{1,1}(0, T)$$

cOUNTERCLOCKWISE:

$$q(t)\mathcal{P}[q](t) - \mathcal{U}[q](t) \geq 0 \text{ a.e.} \quad \forall q \in W^{1,1}(0, T)$$

q ... internal state

\mathcal{P} ... hysteresis operator

\mathcal{U} ... hysteresis potential

Hysteresis Potentials for Preisach Operators

Theorem (e.g., [Brokate&Sprekels, 1996], [Krejčí, 1996])

$$\begin{aligned}\tilde{\mathcal{P}}[v](t) = \mathcal{P}^\omega[v](t) &= \iint_{\alpha, \beta \in S} \omega(\beta, \alpha) \mathcal{R}_{\beta, \alpha}[v](t) d(\alpha, \beta) \\ \tilde{\mathcal{U}}[v](t) &= 2 \iint_{\alpha, \beta \in S} (\alpha + \beta) \omega(\beta, \alpha) \mathcal{R}_{\beta, \alpha}[v](t) d(\alpha, \beta)\end{aligned}$$

with $\omega \geq 0$.

Then $\tilde{\mathcal{P}}$ is a counterclockwise piecewise convex hysteresis operator and $\tilde{\mathcal{U}}$ is a counterclockwise hysteresis potential for $\tilde{\mathcal{P}}$.

Hysteresis Potentials for Preisach Operators

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Then $\tilde{\mathcal{P}}$ is a counterclockwise piecewise convex hysteresis operator and $\tilde{\mathcal{U}}$ is a counterclockwise hysteresis potential for $\tilde{\mathcal{P}}$.

. . . clockwise case via inverse of $\tilde{\mathcal{P}}$.

A Thermodynamically Consistent Material Law for Ferroelectricity and Ferroelasticity

A Thermodynamically Consistent Material Law

see [Davino&Krejčí&Visone'13] for magnetostriction

Ansatz:
$$\begin{cases} \sigma = c\varepsilon - eE + a\mathcal{P}[q] + b\mathcal{U}[q] \\ D = e\varepsilon + \kappa E + c\mathcal{P}[q] + d\mathcal{U}[q] \\ F = \frac{c}{2}\varepsilon^2 + \frac{\kappa}{2}E^2 + \xi\mathcal{P}[q] + \eta\mathcal{U}[q] \end{cases}$$

$q = q(\varepsilon, E)$... internal variable

$a = a(\varepsilon, E)$, $b = b(\varepsilon, E)$, $c = c(\varepsilon, E)$, $d = d(\varepsilon, E)$, $\xi = \xi(\varepsilon, E)$,

$\eta = \eta(\varepsilon, E)$... coefficient functions

\mathcal{U} counterclockwise hysteresis potential for \mathcal{P}

thermodynamically consistent choice (X , Y ... Gibbs potentials):

$$a = \frac{\partial X}{\partial \varepsilon} \quad b = \frac{\partial Y}{\partial \varepsilon} \quad c = -\frac{\partial X}{\partial E} \quad d = -\frac{\partial Y}{\partial E}$$

$$\xi = X + cE \quad \eta = Y + dE$$

$$q = -\frac{X}{Y} \quad 0 < Y$$

A Thermodynamically Consistent Material Law

see [Davino&Krejčí&Visone'13] for magnetostriction

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$\eta = \eta(\varepsilon, E)$... coefficient functions

\mathcal{U} counterclockwise hysteresis potential for \mathcal{P}

linear part: piezoelectric coupling

nonlinear hysteretic part: ferroelectricity and ferroelasticity

Simulations for some Simple Test Case

Ansatz:

$$\begin{cases} \sigma = \underline{c}\varepsilon - eE + a\mathcal{P}[q] + b\mathcal{U}[q] \\ D = e\varepsilon + \kappa E + c\mathcal{P}[q] + d\mathcal{U}[q] \\ F = \frac{\underline{c}}{2}\varepsilon^2 + \frac{\kappa}{2}E^2 + \xi\mathcal{P}[q] + \eta\mathcal{U}[q] \end{cases}$$

q ... internal variable

\mathcal{U} counterclockwise hysteresis potential for \mathcal{P}

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\mathcal{U} counterclockwise hysteresis potential for \mathcal{P}

thermodynamically consistent choice ($X = -EY(\varepsilon)$, $Y = Y(\varepsilon)$):

$$a = -EY'(\varepsilon) \quad b = Y'(\varepsilon) \quad c = Y(\varepsilon) \quad d = 0$$

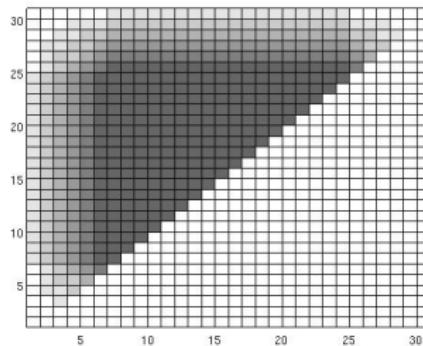
$$\xi = 0 \quad \eta = Y(\varepsilon)$$

$$q = E \quad 0 < Y$$

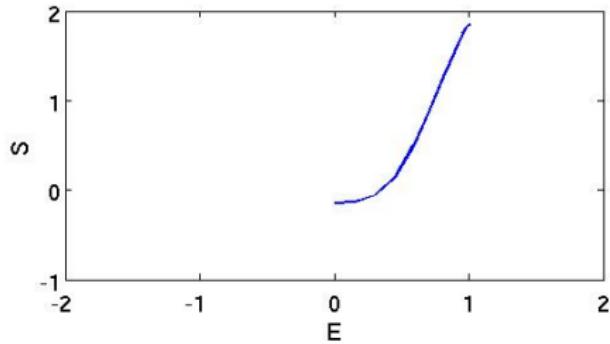
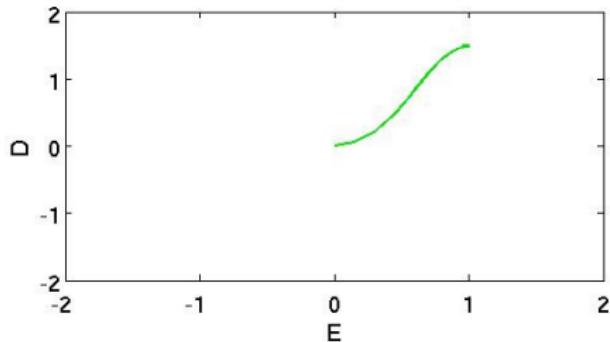
$$\begin{cases} \sigma = \underline{c}\varepsilon - eE - Y'(\varepsilon)(E\mathcal{P}[E] - \mathcal{U}[E]) \\ D = e\varepsilon + \kappa E + Y(\varepsilon)\mathcal{P}[E] \\ F = \frac{c}{2}\varepsilon^2 + \frac{\kappa}{2}E^2 + Y(\varepsilon)\mathcal{U}[E] \end{cases}$$

A Simple Test (I)

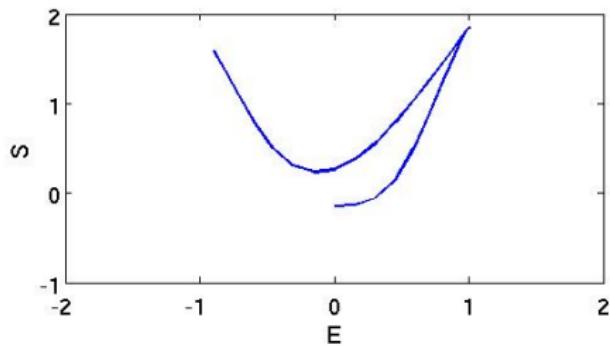
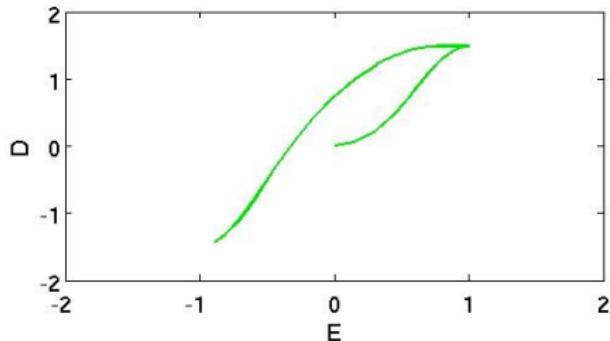
Preisach weight function:



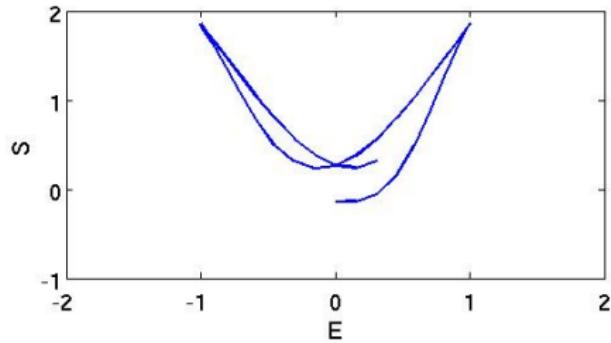
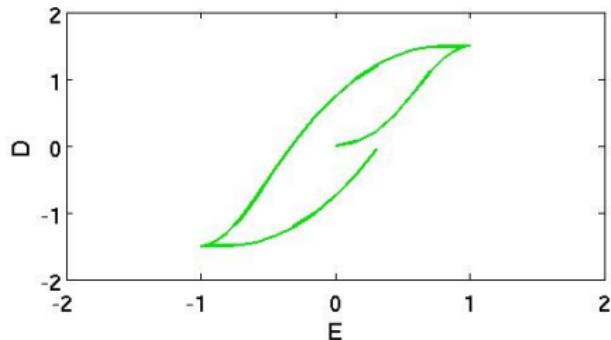
A Simple Test (I): Ferroelectricity



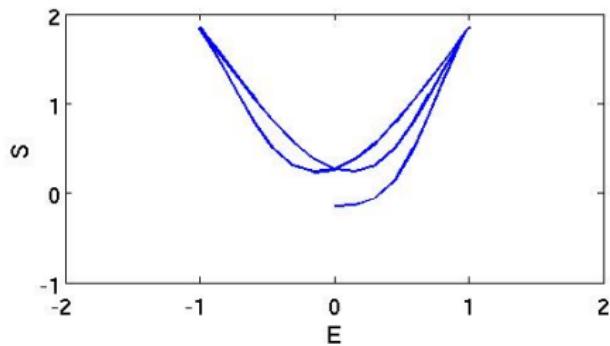
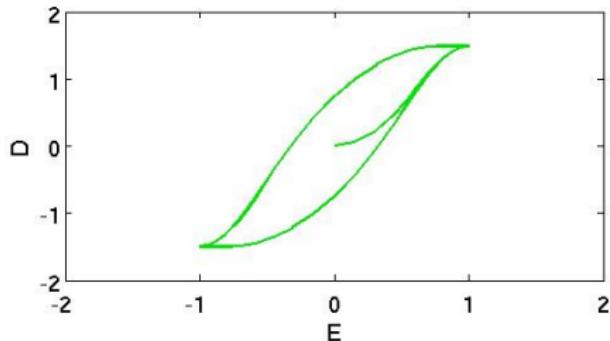
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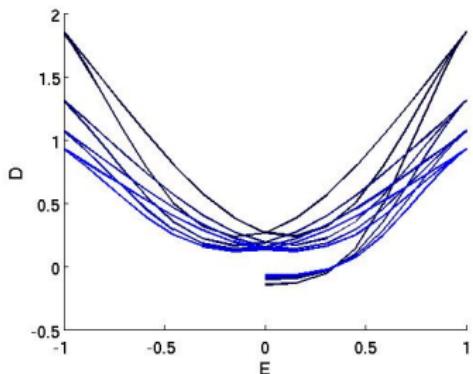
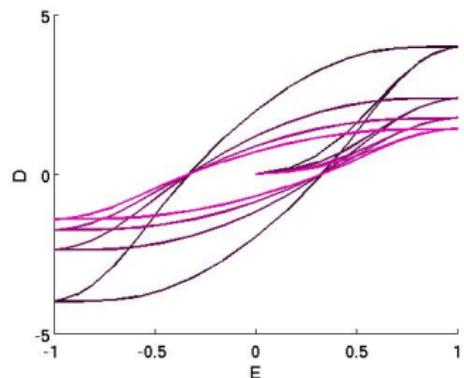
A Simple Test (I): Ferroelectricity



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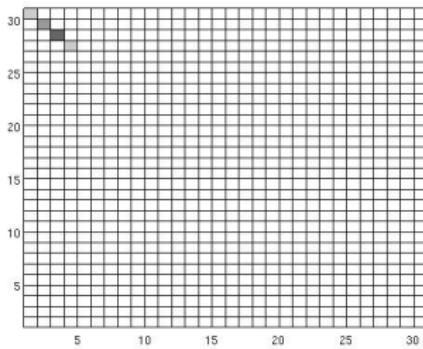


A Simple Test (I): Ferroelectricity and Ferroelasticity

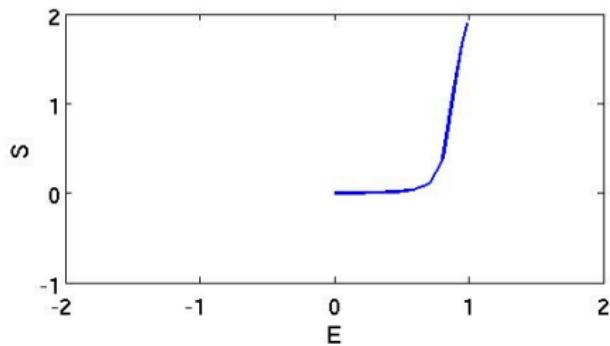
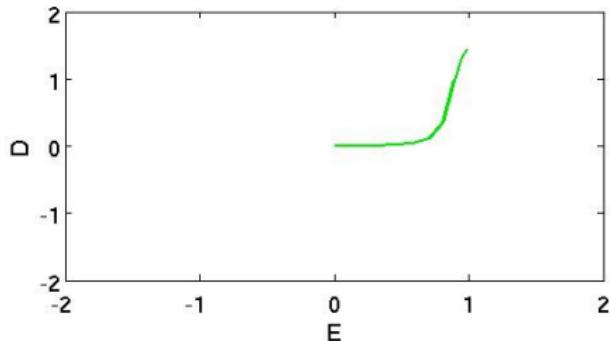


A Simple Test (II)

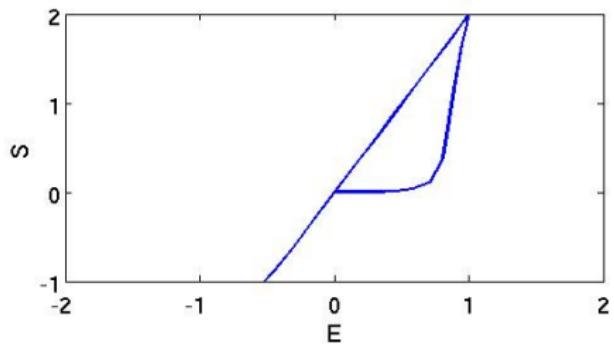
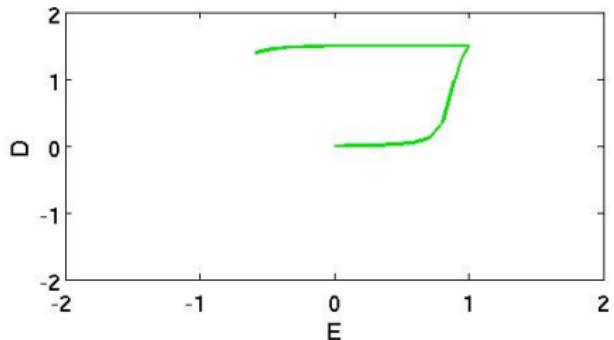
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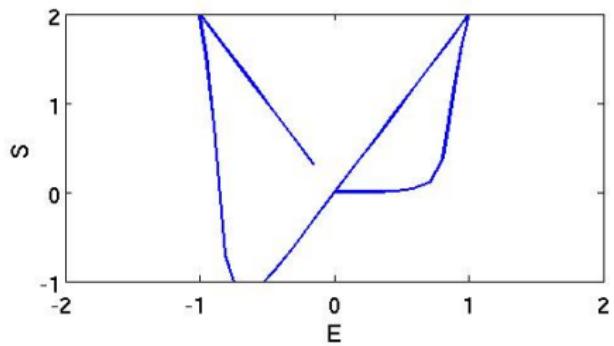
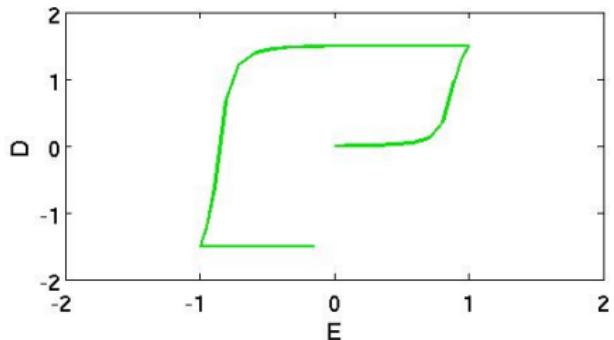
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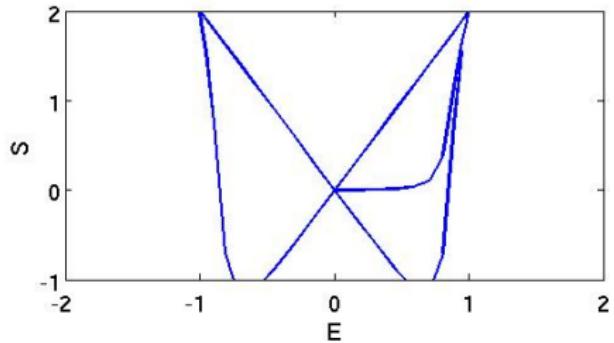
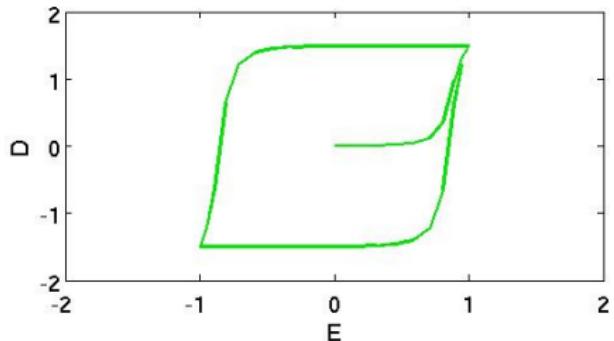
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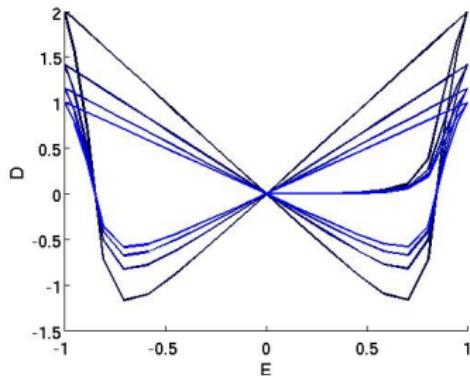
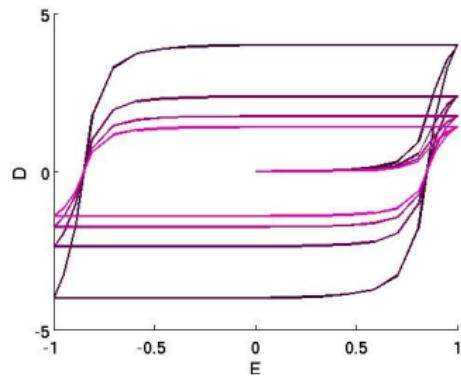
A Simple Test (II): Ferroelectricity



A Simple Test (II): Ferroelectricity



A Simple Test (II): Ferroelectricity and Ferroelasticity



The Full PDE Model

$$\rho \ddot{d} - \nabla_s^T \sigma = 0 \quad \text{Newton's law}$$
$$-\nabla D = 0 \quad \text{Gauß's law}$$

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$\varepsilon = \nabla_s d$ $d \dots$ mechanical displacement

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$\varepsilon = \nabla_s d$... mechanical displacement

$E = \nabla \phi$... electric potential,

boundary conditions:

$$n \cdot \sigma = f_m \text{ on } \partial\Omega$$

$$\phi = \phi_e \text{ on } \Gamma$$

$$n \cdot D = f_e \text{ on } \partial\Omega \setminus \Gamma.$$

Well-posedness

Well-posedness in the Context of Hysteresis Operators

challenges:

- only very weak regularity properties
- no monotonicity
- no compactness

Well-posedness via Thermodynamic Consistency?

Well-posedness via Thermodynamic Consistency?

$$\rho \ddot{d} - \nabla_s^T \sigma = 0 \quad (\text{Newton}) \quad (1)$$

$$-\nabla \dot{D} = 0 \quad (\frac{d}{dt} \text{Gauss}) \quad (2)$$

Test (1) with \dot{d} and (2) with ϕ :

$$\int_{\Omega} \left\{ \frac{1}{2} \frac{d}{dt} \left(\rho \dot{d}^2 \right) + \sigma : \dot{\varepsilon} + \dot{D} \cdot E \right\} dx = \int_{\partial\Omega} f_m \dot{d} ds + \int_{\Gamma} n \cdot D\phi_0 ds + \int_{\partial\Omega \setminus \Gamma} f_e \phi ds \\ =: \langle f, (\dot{d}, \phi) \rangle_{V^*, V}$$

From thermodynamic consistency

$$\dot{\varepsilon} : \sigma + \dot{D}(t) \cdot E - \dot{F} \geq 0$$

and our ansatz for the free energy

$$F = \frac{c}{2} \varepsilon^2 + \frac{\kappa}{2} E^2 + \xi \mathcal{P}[q] + \eta \mathcal{U}[q]$$

we get:

$$d \int \left[1 - \frac{1}{2} \varepsilon^2 - \frac{1}{2} E^2 - \frac{1}{2} \xi \mathcal{P}[q] - \frac{1}{2} \eta \mathcal{U}[q] \right] dx$$

$$\int_{\Omega} \left\{ \frac{1}{2} \frac{d}{dt} (\rho \dot{d}^2) + \sigma : \dot{\varepsilon} + \dot{D} \cdot E \right\} dx = \int_{\partial\Omega} f_m \dot{d} ds + \int_{\Gamma} n \cdot D\phi_0 ds + \int_{\partial\Omega \setminus \Gamma} f_e \phi ds \\ =: \langle f, (\dot{d}, \phi) \rangle_{V^*, V}$$

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$$F = \frac{\underline{c}}{2} \varepsilon^2 + \frac{\kappa}{2} E^2 + \xi \mathcal{P}[q] + \eta \mathcal{U}[q]$$

we get:

$$\frac{d}{dt} \left\{ \underbrace{\frac{1}{2} |\sqrt{\rho} \dot{d}^2|_H^2 + \frac{1}{2} |\sqrt{\underline{c}} \varepsilon|_H^2 + \frac{1}{2} |\sqrt{\kappa} E|_H^2}_{=: \mathcal{E}(t)} + \xi \mathcal{P}[q] + \eta \mathcal{U}[q] \right\} \leq \langle f, (\dot{d}, \phi) \rangle_{V^*, V}$$

i.e., setting $\xi = 0$ and $\eta \geq 0$ and $\mathcal{U} \geq 0$

\rightsquigarrow estimate of energy $\mathcal{E}(t)$ (decreasing for $f = 0$)

$$\frac{d}{dt} \left\{ \underbrace{\frac{1}{2} |\sqrt{\rho} \dot{d}^2|_H^2 + \frac{1}{2} |\sqrt{c} \varepsilon|_H^2 + \frac{1}{2} |\sqrt{\kappa} E|_H^2}_{=: \mathcal{E}(t)} + \xi \mathcal{P}[q] + \eta \mathcal{U}[q] \right\} \leq \langle f, (\dot{d}, \phi) \rangle_{V^*, V}$$

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\rightsquigarrow estimate of energy $\mathcal{E}(t)$ (decreasing for $f = 0$)

However, time discretization and passage to the limit not possible:
 Hysteresis operators need $C[0, T]$ convergence of input $q = q(\varepsilon, E)$,
 but energy estimate only yields $C[0, T]$ boundedness of input q
 . . . lack of compactness

Well-posedness via Mechanical Damping

$$\rho \ddot{d} - \nabla_s^T (k\dot{\varepsilon} + \sigma) = 0$$

$$-\nabla D = 0$$

$$\sigma = c\varepsilon - eE + (a\mathcal{P}[q] + b\mathcal{U}[q])M$$

$$D = e^T \varepsilon + \kappa E + (c\mathcal{P}[q] + d\mathcal{U}[q])p$$

$$\varepsilon = \nabla_s d$$

$$E = \nabla \phi,$$

weak form:

$$\langle \Lambda \ddot{u}, w \rangle_{V^*, V} + (AJ\dot{u} + \Pi Ju + \Phi[Ju], Jw)_H = \langle f, w \rangle_{V^*, V} \quad \forall w \in V$$

Well-posedness via Mechanical Damping

$$\rho \ddot{d} - \nabla_s^T (\textcolor{blue}{k} \dot{\varepsilon} + \sigma) = 0$$

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$$D = e^T \varepsilon + \kappa E + (c \mathcal{P}[q] + d \mathcal{U}[q]) p$$

$$\varepsilon = \nabla_s d$$

$$E = \nabla \phi,$$

weak form:

$$\langle \Lambda \ddot{u}, w \rangle_{V^*, V} + (\textcolor{blue}{A} J \dot{u} + \Pi J u + \Phi[J u], J w)_H = \langle f, w \rangle_{V^*, V} \quad \forall w \in V$$

strong damping

nonlinearity

with

$$u = (d, \phi), \quad w = (v, \psi),$$

$$V = H^1(\Omega)^{d+1}, \quad H = L^2(\Omega)^{\frac{d(d+1)}{2} + d},$$

$$\Lambda : V^* \rightarrow V^*, \quad \Lambda(d, \phi) = (\rho d, 0),$$

$$A : H \rightarrow H, \quad A(\varepsilon, E) = (k\varepsilon, 0)$$

$$J : V \rightarrow H, \quad J(d, \phi) = (\nabla_s d, \nabla \phi),$$

$$\Pi : H \rightarrow H, \quad \Pi(\varepsilon, E) = (\underline{c}\varepsilon - eE, e^T \varepsilon + \kappa E),$$

$$\Phi : C(0, T; H) \rightarrow C(0, T; H),$$

$$\Phi[(\varepsilon, E)] = ((a\mathcal{P}[q] + b\mathcal{U}[q])M, (c\mathcal{P}[q] + d\mathcal{U}[q])p)$$

$$\langle f(t), w \rangle_{V^*, V} = \int_{\partial\Omega} (f_m(t)v + f_e(t)\psi) \, ds.$$

An Abstract Well-posedness Result

$$\langle \Lambda \ddot{u}, w \rangle_{V^*, V} + (AJ\dot{u} + \Pi Ju + \Phi[Ju], Jw)_H = \langle f, w \rangle_{V^*, V} \quad \forall w \in V$$

assumptions:

$$(A1) \left\{ \begin{array}{l} V \hookrightarrow H \hookrightarrow V^* \\ \Lambda \in \mathcal{L}(V^*, V^*) , \quad \langle \Lambda \dot{v}, v \rangle_{V^*, V} = \frac{1}{2} \frac{d}{dt} \lambda(v) \text{ for some continuous function} \\ \lambda : \mathcal{D}(\lambda) \subseteq V^* \rightarrow \mathbb{R}_0^+, \quad \lambda(0) = 0 \\ A \in \mathcal{L}(H, H), \quad (Ah, h)_H \geq \alpha |h|_H^2 \text{ for some } \alpha > 0 \\ \Pi \in \mathcal{L}(H, H), \quad (\Pi h, \dot{h})_H = \frac{d}{dt} \frac{1}{2} \pi(h) + \theta(h, \dot{h}) \text{ for some continuous fctns} \\ \pi \in H \rightarrow \mathbb{R}_0^+, \quad \theta \in H \rightarrow \mathbb{R}, \quad \theta(h, \dot{h}) \leq L_\theta \sqrt{\pi(h)} \|\dot{h}\|_H, \quad \pi(0) = 0 \\ |\Phi[h](t) - \Phi[\tilde{h}](t)|_H \leq L_\Phi \left(\int_0^t |\dot{h}(\tau) - \dot{\tilde{h}}(\tau)|_H^2 d\tau + |h(0) - \tilde{h}(0)|_H^2 \right)^{\frac{1}{2}} \\ J \in \mathcal{L}(V, H) \end{array} \right.$$

An Abstract Well-posedness Result

$$\langle \Lambda \ddot{u}, w \rangle_{V^*, V} + (AJ\dot{u} + \Pi Ju + \Phi[Ju], Jw)_H = \langle f, w \rangle_{V^*, V} \quad \forall w \in V \quad (3)$$

Theorem

Under conditions (A1), for any $T > 0$, initial conditions $Ju(0) \in H$, $\dot{u}(0) \in \mathcal{D}(\lambda)$, and right hand side $f \in L^2(0, T; V^*)$ such that the linearization of (3) at 0 has a solution $u_{lin} \in W = \{v \in L^2(0, T; V^*) : \lambda(\dot{v}), \pi(Jv) \in C[0, T], J\dot{v} \in L^2(0, T; H)\}$, also the nonlinear equation (3) has a solution $u \in W$, and the solution is unique in a sufficiently small neighborhood of u_{lin} .

Idea of proof: Banach's Fixed Point Theorem

Application to the Piezoelectric Model

$$\begin{aligned}\ddot{\rho d} - \nabla_s^T (\textcolor{blue}{k}\dot{\varepsilon} + \sigma) &= 0 \\ -\nabla D &= 0 \\ \sigma &= \underline{c}\varepsilon - eE + (a\mathcal{P}[q] + b\mathcal{U}[q])M \\ D &= e^T\varepsilon + \kappa E + (c\mathcal{P}[q] + d\mathcal{U}[q])p \\ \varepsilon &= \nabla_s d \\ E &= \nabla\phi,\end{aligned}\tag{4}$$

Assume Lipschitz continuity pf Φ

$$|\Phi[h](t) - \Phi[\tilde{h}](t)|_H \leq L_\Phi \left(\int_0^t |\dot{h}(\tau) - \dot{\tilde{h}}(\tau)|_H^2 d\tau + |h(0) - \tilde{h}(0)|_H^2 \right)^{\frac{1}{2}} \tag{5}$$

in

$$H = L^2(\Omega)^{\frac{d(d+1)}{2} + d}$$

typically satisfied for Preisach hysteresis operators and potentials cf. [Brokate&Sprekels'96, Krejčí'96].

Application to the Piezoelectric Model

Corollary

Let $\rho, k, l > 0$, \underline{c}, κ be positive definite tensors, and let

$$\begin{aligned}\Phi : C(0, T; L^2(\Omega)^{\frac{d(d+1)}{2}+d}) &\rightarrow C(0, T; L^2(\Omega)^{\frac{d(d+1)}{2}+d}) \\ \Phi[(\varepsilon, E)] &= ((a\mathcal{P}[q] + b\mathcal{U}[q])M, (c\mathcal{P}[q] + d\mathcal{U}[q])p)\end{aligned}$$

satisfy the Lipschitz condition (5).

Then for any $T > 0$, initial conditions

$(d_0, \phi_0) = (d(0), \phi(0)) \in H^1(\Omega)^{d+1}$, $d_1 = \dot{d}(0) \in L^2(\Omega)^d$, and boundary conditions $(f_m, f_e) \in L^2(0, T; H^{-1/2}(\partial\Omega)^{d+1})$, (4) has a weak solution $(d, \phi) \in H^1([0, T]; H^1(\Omega)^{d+1})$ with $\dot{d} \in C([0, T]; L^2(\Omega)^d)$, and the solution is unique in a sufficiently small neighborhood of (d_{lin}, ϕ_{lin}) solving the linearization of (4) at 0 with initial conditions d_0, ϕ_0, d_1 .

Well-posedness via Mechanical Viscosity

$$\begin{aligned}\rho \ddot{\mathbf{d}} - \nabla_s^T (\mathbf{k} \dot{\varepsilon} + \boldsymbol{\sigma}) &= \mathbf{0} \\ -\nabla D &= \mathbf{0} \\ \boldsymbol{\sigma} &= \underline{c} \varepsilon - \mathbf{e} E + (a \mathcal{P}[q] + b \mathcal{U}[q]) M \\ \mathbf{D} &= \mathbf{e}^T \varepsilon + \kappa E + (c \mathcal{P}[q] + d \mathcal{U}[q]) p \\ \varepsilon &= \nabla_s \mathbf{d} \\ \mathbf{E} &= \nabla \phi\end{aligned}\tag{6}$$

Corollary

Existence, uniqueness and stability of $(\mathbf{d}, \phi) \in C([0, T]; H^1(\Omega)^{d+1})$ with $\dot{\mathbf{d}} \in C([0, T]; L^2(\Omega)^d) \cap L^2(0, T; H^1(\Omega)^d)$

Idea of proof:

elimination of ϕ via 2nd, 4th, and 6th equation in (6).

Well-posedness via Mechanical Viscosity

$$\rho \ddot{\mathbf{d}} - \nabla_s^T (\mathbf{k} \dot{\varepsilon} + \boldsymbol{\sigma}) = \mathbf{0}$$

$$-\nabla D = \mathbf{0}$$

$$\boldsymbol{\sigma} = \underline{c} \varepsilon - e \mathbf{E} + (a \mathcal{P}[q] + b \mathcal{U}[q]) \mathbf{M}$$

$$\mathbf{D} = \mathbf{e}^T \varepsilon + \kappa \mathbf{E} + (c \mathcal{P}[q] + d \mathcal{U}[q]) \mathbf{p}$$

$$\varepsilon = \nabla_s \mathbf{d}$$

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Existence, uniqueness and stability of $(\mathbf{d}, \phi) \in C([0, T]; H^1(\Omega)^{d+1})$ with $\dot{\mathbf{d}} \in C([0, T]; L^2(\Omega)^d) \cap L^2(0, T; H^1(\Omega)^d)$

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10th International Piezo Workshop

Sept. 21-24, 2014

Vienna University of Technology

- theory
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- advanced material characterization
- fatigue, damage, cracks and reliability
- manufacturing processes
- performance of piezoelectric transducers

http://www.mec.tuwien.ac.at/piezo2014/piezoworkshop_2014/

Thank you for your attention!