

Mathematics of Nonlinear Acoustics: Modeling, Numerics, Optimization

Barbara Kaltenbacher

Alpen-Adria-Universität Klagenfurt

19th European Conference on Mathematics for Industry (ECMI 2016), June 16, 2016

Mathematics of Nonlinear Acoustics: Modeling, Numerics, Optimization

Barbara Kaltenbacher

Alpen-Adria-Universität Klagenfurt

19th European Conference on Mathematics for Industry (ECMI 2016), June 16, 2016

joint work with:

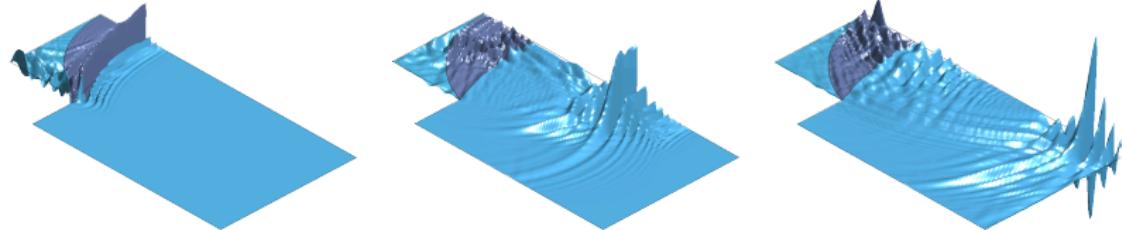
Rainer Brunnhuber, AAU, Vanja Nikolić, AAU,
Christian Clason, U Duisburg-Essen, Manfred Kaltenbacher, TU Vienna,
Irena Lasiecka, U Memphis, Richard Marchand, Slippery Rock U,
Gunther Peichl, U Graz, Maria K. Pospieszalska, La Jolla Institute,
Petronela Radu, U Nebraska at Lincoln, Igor Shevchenko, UCL,
Mechthild Thalhammer, U Innsbruck

Harald Eizenhöfer, Dornier MedTech

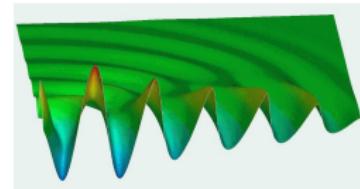
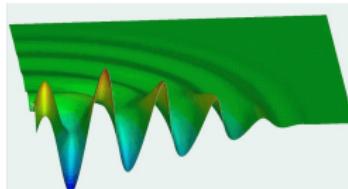
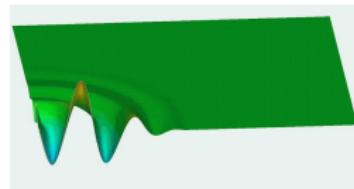
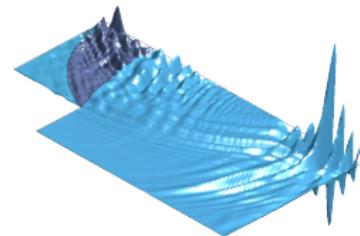
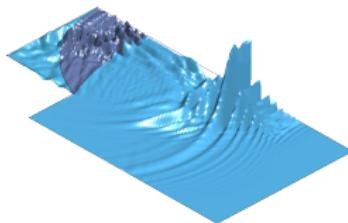
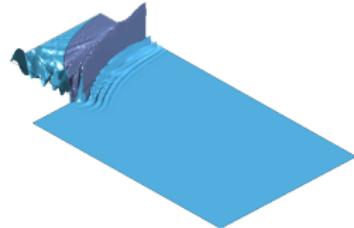


Mathematics of Nonlinear Acoustics

Nonlinear Acoustic Wave Propagation



Nonlinear Acoustic Wave Propagation



Applications of High Intensity Focused Ultrasound HIFU



lithotripsy



thermotherapy

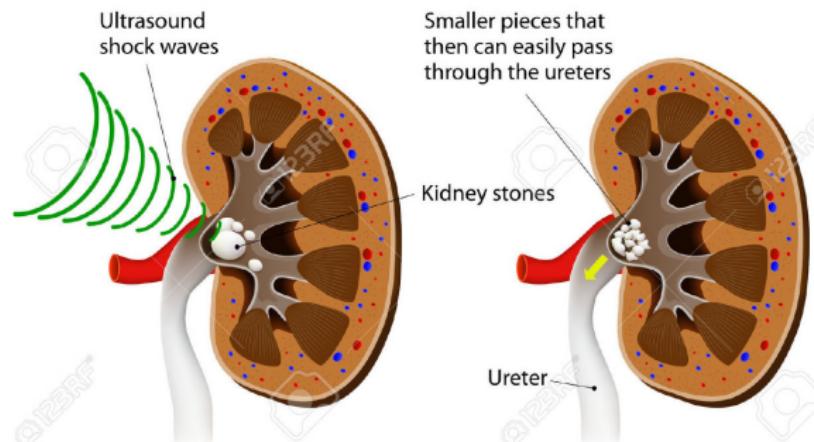


cleaning



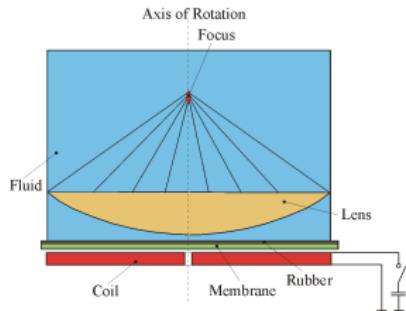
welding

Lithotripsy

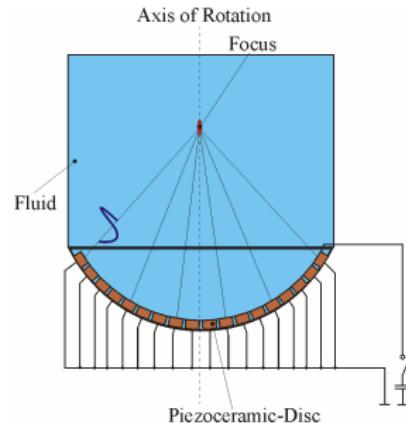


extracorporeal shock wave lithotripsy (ESWL)

Lithotripsy: Focusing Principles



electromagnetic shock
wave emitter (EMSE)
focusing by acoustic
(silicone) lens
Dornier MedTech



piezoelectric array
(self)focusing by array shaping
Wolf PiezoLith

Outline

- modeling
- degeneracy
- coupling
- time integration
- nonreflecting boundary conditions
- optimization

modeling

Physical Principles

main physical quantities:

- acoustic particle velocity \vec{v} ;
- acoustic pressure p ;
- mass density ϱ ;

decomposition into mean and fluctuating part:

$$\vec{v} = \vec{v}_0 + \vec{v}_{\sim} = \vec{v}, \quad p = p_0 + p_{\sim}, \quad \varrho = \varrho_0 + \varrho_{\sim}$$

Physical Principles

main physical quantities:

- acoustic particle velocity \vec{v} ;
- acoustic pressure p ;
- mass density ϱ ;

decomposition into mean and fluctuating part:

$$\vec{v} = \vec{v}_0 + \vec{v}_{\sim} = \vec{v}, \quad p = p_0 + p_{\sim}, \quad \varrho = \varrho_0 + \varrho_{\sim}$$

governing equations:

- Navier Stokes equation (under the assumption $\nabla \times \vec{v} = 0$)

$$\varrho \left(\vec{v}_t + \nabla(\vec{v} \cdot \vec{v}) \right) + \nabla p = \left(\frac{4\mu v}{3} + \zeta v \right) \Delta \vec{v}$$

- equation of continuity

$$\nabla \cdot (\varrho \vec{v}) = -\varrho_t$$

- state equation

$$\varrho_{\sim} = \frac{1}{c^2} p_{\sim} - \frac{1}{\varrho_0 c^4} \frac{B}{2A} p_{\sim}^2 - \frac{\kappa}{\varrho_0 c^4} \left(\frac{1}{c_V} - \frac{1}{c_p} \right) p_{\sim t}$$

Derivation of Wave Equation

governing equations:

- Navier Stokes equation (under the assumption $\nabla \times \vec{v} = 0$)

$$\varrho \left(\vec{v}_t + \nabla(\vec{v} \cdot \vec{v}) \right) + \nabla p = \left(\frac{4\mu\nu}{3} + \zeta_V \right) \Delta \vec{v}$$

- equation of continuity

$$\nabla \cdot (\varrho \vec{v}) = -\varrho_t$$

- state equation

$$\varrho_{\sim} = \frac{p_{\sim}}{c^2} - \frac{1}{\varrho_0 c^4} \frac{B}{2A} p_{\sim}^2 - \frac{\kappa}{\varrho_0 c^4} \left(\frac{1}{c_V} - \frac{1}{c_p} \right) p_{\sim t}$$

Derivation of Wave Equation

governing equations:

- Navier Stokes equation (under the assumption $\nabla \times \vec{v} = 0$)

$$\varrho_0 \left(\vec{v}_t + \nabla(\vec{v} \cdot \vec{v}) \right) + \nabla p = \left(\frac{4\mu\nu}{3} + \zeta_V \right) \Delta \vec{v}$$

- equation of continuity

$$\nabla \cdot (\varrho_0 \vec{v}) = -\varrho_t$$

- state equation

$$\varrho_{\sim} = \frac{p_{\sim}}{c^2} - \frac{1}{\varrho_0 c^4} \frac{B}{2A} p_{\sim}^2 - \frac{\kappa}{\varrho_0 c^4} \left(\frac{1}{c_V} - \frac{1}{c_p} \right) p_{\sim t}$$

Derivation of Wave Equation

governing equations:

- Navier Stokes equation (under the assumption $\nabla \times \vec{v} = 0$)

$$\varrho_0 \left(\vec{v}_t + \nabla(\vec{v} \cdot \vec{v}) \right) + \nabla p = \left(\frac{4\mu\nu}{3} + \zeta_V \right) \Delta \vec{v}$$

- equation of continuity

$$\nabla \cdot (\varrho_0 \vec{v}) = -\varrho_t$$

- state equation

$$\varrho_{\sim} = \frac{p_{\sim}}{c^2} - \frac{1}{\varrho_0 c^4} \frac{B}{2A} p_{\sim}^2 - \frac{\kappa}{\varrho_0 c^4} \left(\frac{1}{c_V} - \frac{1}{c_p} \right) p_{\sim t}$$

- fluctuating quantities:

$$\nabla p = \nabla p_{\sim}, \quad \varrho_t = \varrho_{\sim t}$$

Derivation of Wave Equation

$$\varrho_0 \vec{v}_t + \nabla p_{\sim} = 0$$

$$\varrho_0 \nabla \cdot \vec{v} = -\varrho_{\sim t} = -\frac{1}{c^2} p_{\sim t}$$

Derivation of Wave Equation

$$\varrho_0 \vec{v}_t + \nabla p_{\sim} = 0$$

$$\varrho_0 \nabla \cdot \vec{v} + \frac{1}{c^2} p_{\sim t} = 0$$

Derivation of Wave Equation

$$-\nabla \cdot \quad \varrho_0 \vec{v}_t + \nabla p_{\sim} = 0$$

$$\frac{\partial}{\partial t} \quad \varrho_0 \nabla \cdot \vec{v} + \frac{1}{c^2} p_{\sim t} = 0$$

$$\frac{1}{c^2} p_{\sim tt} - \Delta p_{\sim} = 0$$

Classical Models of Nonlinear Acoustics I

- **Kuznetsov's equation** [Lesser & Seebass 1968, Kuznetsov 1971]

$$p_{\sim tt} - c^2 \Delta p_{\sim} - b \Delta p_{\sim t} = - \left(\frac{B}{2A \varrho_0 c^2} p_{\sim}^2 + \varrho_0 |\vec{v}|^2 \right)_{tt}$$

where $\varrho_0 \vec{v}_t = -\nabla p$

for the **particle velocity** \vec{v} and the **pressure** p , i.e.,

$$\psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t = - \left(\frac{B}{2A c^2} (\psi_t)^2 + |\nabla \psi|^2 \right)_t$$

since $\nabla \times \vec{v} = 0$ hence $\vec{v} = -\nabla \psi$ for a **velocity potential** ψ

- **Westervelt equation** [Westervelt 1963]

$$p_{\sim tt} - c^2 \Delta p_{\sim} - b \Delta p_{\sim t} = - \frac{1}{\varrho_0 c^2} \left(1 + \frac{B}{2A} \right) p_{\sim tt}^2$$

via $\varrho_0 |\vec{v}|^2 \approx \frac{1}{c^2} (p_{\sim t})^2$

Classical Models of Nonlinear Acoustics II

- **Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation**
[Zabolotskaya & Khokhlov 1969]

$$2cp_{\sim xt} - c^2 \Delta_{yz} p_{\sim} - \frac{b}{c^2} p_{\sim ttt} = \frac{\beta_a}{\rho_0 c^2} p_{\sim tt}^2$$

$x\dots$ direction of sound propagation

- **Burgers' equation** [Burgers 1974]

$$p_{\sim t} - \frac{b}{2c^2} p_{\sim \tau \tau} = \frac{\beta_a}{\rho_0 c^3} p_{\sim} p_{\sim \tau}$$

$\tau = t - \frac{x}{c}\dots$ retarded time

Advanced Models of Nonlinear Acoustics (Examples)

- **Blackstock-Crighton** equation [Brunnhuber & Jordan 2016], [Blackstock 1963], [Crighton 1979]

$$(\partial_t - a\Delta) (\psi_{tt} - c^2 \Delta \psi - b\Delta \psi_t) - r\Delta \psi_t = - \left(\frac{B}{2Ac^2} (\psi_t^2) + |\nabla \psi|^2 \right)_{tt}$$

$$a = \frac{\nu}{Pr} \dots \text{thermal conductivity}$$

- **Jordan-Moore-Gibson-Thompson** equation [Jordan 2009, 2014], [Christov 2009], [Straughan 2010]

$$\tau \psi_{ttt} + \psi_{tt} - c^2 \Delta \psi - b\Delta \psi_t = - \left(\frac{B}{2Ac^2} (\psi_t)^2 + |\nabla \psi|^2 \right)_t$$

$\tau \dots$ relaxation time

- cf. Kuznetsov:

$$\psi_{tt} - c^2 \Delta \psi - b\Delta \psi_t = - \left(\frac{B}{2Ac^2} (\psi_t^2) + |\nabla \psi|^2 \right)_t$$

Models and their Analysis

- **further models:** [Angel & Aristegui 2014], [Christov & Christov & Jordan 2007], [Kudryashov & Sinelshchikov 2010], [Ockendon & Tayler 1983], [Makarov & Ochmann 1996], [Rendón & Ezeta & Pérez-López 2013], [Rasmussen & Sørensen & Christiansen 2008], [Soderholm 2006], ...
- **resonances, shock waves:** [Ockendon & Ockendon & Peake & Chester 1993], [Ockendon & Ockendon 2001, 2004, 2016], ...
- **traveling waves solutions:** [Jordan 2004], [Chen & Torres & Walsh 2009], [Keiffer & McNorton & Jordan & Christov, 2014], [Gaididei & Rasmussen & Christiansen & Sørensen, 2016], ...
- **well-posedness and asymptotic behaviour:**
for KZK: [Rozanova-Pierrat 2007, 2008, 2009, 2010]
for Westervelt, Kuznetsov, Blackstock-Crighton, JMGT:
based on semigroup theory and energy estimates: [BK & Lasiecka 2009, 2012], [BK & Lasiecka & Veljović 2011], [BK & Lasiecka & Marchand 2012], [BK & Lasiecka & Pospiezska 2012], [Lasiecka & Wang 2015], [Liu & Triggiani 2013], [Marchand & McDevitt & Triggiani 2012], [Nikolić 2015], [Nikolić & BK 2016]
based on maximal L_p regularity: [Meyer & Wilke 2011, 2013], [Meyer & Simonett 2016], [Brunnhuber & Meyer 2016], [BK 2016]

Setting

consider:

Westervelt / Kuznetsov / Jordan-Moore-Gibson-Thompson /
Blackstock-Crighton on some domain $\Omega \subseteq \mathbb{R}^d$
+boundary conditions on $\partial\Omega$
+initial conditions at $t = 0$

e.g.,

$$u_{tt} - c^2 \Delta u - b \Delta u_t = -\frac{k}{2} (u^2)_{tt} \quad \text{in } \Omega$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega$$

$$u(t=0) = u_0, \quad u_t(t=0) = u_1 \quad \text{in } \Omega$$

where u ... pressure

degeneracy

Degeneracy

e.g., for Westervelt (u . . . pressure)

$$u_{tt} - c^2 \Delta u - b \Delta u_t = -\frac{k}{2} (u^2)_{tt}$$

Degeneracy

e.g., for Westervelt (u . . . pressure)

$$u_{tt} - c^2 \Delta u - b \Delta u_t = -\frac{k}{2} (u^2)_{tt} = -k u u_{tt} - k (u_t)^2$$

Degeneracy

e.g., for Westervelt (u . . . pressure)

$$u_{tt} - c^2 \Delta u - b \Delta u_t = -\frac{k}{2} (u^2)_{tt} = -k u u_{tt} - k (u_t)^2$$

$$(1 + ku) u_{tt} - c^2 \Delta u - b \Delta u_t = -k (u_t)^2$$

Degeneracy

e.g., for Westervelt (u ... pressure)

$$u_{tt} - c^2 \Delta u - b \Delta u_t = -\frac{k}{2} (u^2)_{tt} = -k u u_{tt} - k (u_t)^2$$

$$(1 + ku) u_{tt} - c^2 \Delta u - b \Delta u_t = -k (u_t)^2$$

⇒ degeneracy for $u \leq -\frac{1}{k}$

Degeneracy

e.g., for Westervelt (u . . . pressure)

$$u_{tt} - c^2 \Delta u - b \Delta u_t = -\frac{k}{2} (u^2)_{tt} = -k u u_{tt} - k (u_t)^2$$

$$(1 + ku) u_{tt} - c^2 \Delta u - b \Delta u_t = -k (u_t)^2$$

\Rightarrow degeneracy for $u \leq -\frac{1}{k}$

similarly for Kuznetsov, Jordan-Moore-Gibson-Thompson,
Blackstock-Crighton.

Degeneracy

e.g., for Westervelt (u ... pressure)

$$u_{tt} - c^2 \Delta u - b \Delta u_t = -\frac{k}{2} (u^2)_{tt} = -k u u_{tt} - k (u_t)^2$$

$$(1 + ku) u_{tt} - c^2 \Delta u - b \Delta u_t = -k (u_t)^2$$

⇒ degeneracy for $u \leq -\frac{1}{k}$

similarly for Kuznetsov, Jordan-Moore-Gibson-Thompson,
Blackstock-Crighton.

- ~~> employ energy estimates to obtain bound on u in $C(0, T; H^2(\Omega))$
- ~~> use smallness of u in $C(0, T; H^2(\Omega))$ and $H^2(\Omega) \rightarrow L_\infty(\Omega)$ embedding to guarantee $1 + ku \geq \underline{\alpha} > 0$

Degeneracy

e.g., for Westervelt (u ... pressure)

$$u_{tt} - c^2 \Delta u - b \Delta u_t = -\frac{k}{2} (u^2)_{tt} = -k u u_{tt} - k (u_t)^2$$

$$(1 + ku) u_{tt} - c^2 \Delta u - b \Delta u_t = -k (u_t)^2$$

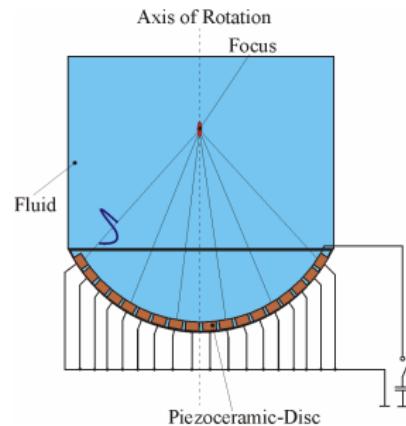
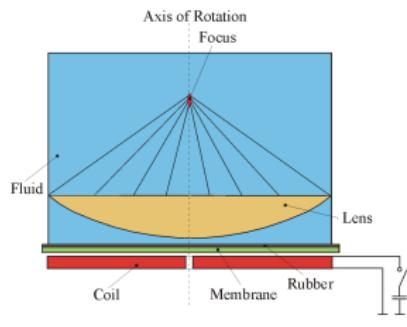
⇒ degeneracy for $u \leq -\frac{1}{k}$

similarly for Kuznetsov, Jordan-Moore-Gibson-Thompson,
Blackstock-Crighton.

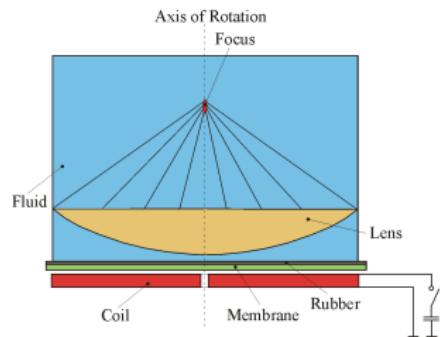
- ~~> employ energy estimates to obtain bound on u in $C(0, T; H^2(\Omega))$
- ~~> use smallness of u in $C(0, T; H^2(\Omega))$ and $H^2(\Omega) \rightarrow L_\infty(\Omega)$ embedding to guarantee $1 + ku \geq \underline{\alpha} > 0$
- ~~> fixed point argument

coupling

Lithotripsy: Focusing Principles

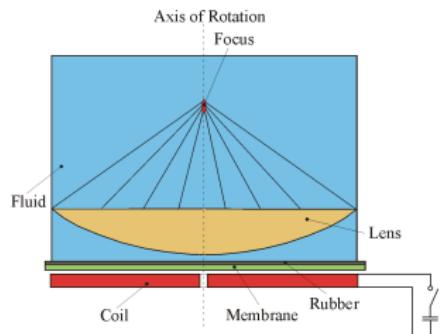


Lithotripsy: Focusing Principles



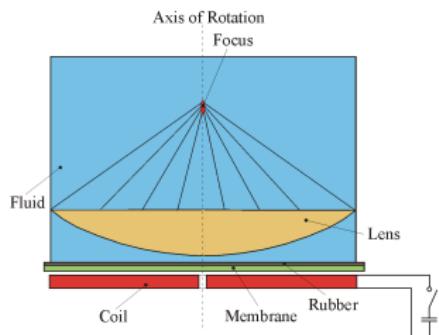
Lithotripsy: Focusing Principles

- interface coupling between fluid Ω_- (nonlinearly acoustic) and lens Ω_+ (linearly acoustic or elastic)
 $\overline{\Omega} = \overline{\Omega}_- \cup \overline{\Omega}_+, \quad \Omega_- \cap \Omega_+ = \emptyset$

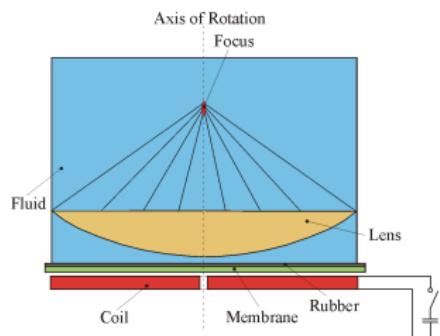


Lithotripsy: Focusing Principles

- interface coupling between fluid Ω_- (nonlinearly acoustic) and lens Ω_+ (linearly acoustic or elastic)
 $\overline{\Omega} = \overline{\Omega}_- \cup \overline{\Omega}_+$, $\Omega_- \cap \Omega_+ = \emptyset$
- No global $H^2(\Omega)$ regularity!

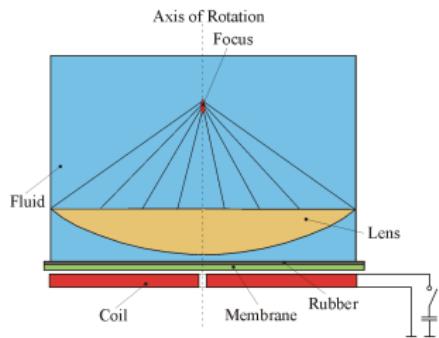


Lithotripsy: Focusing Principles



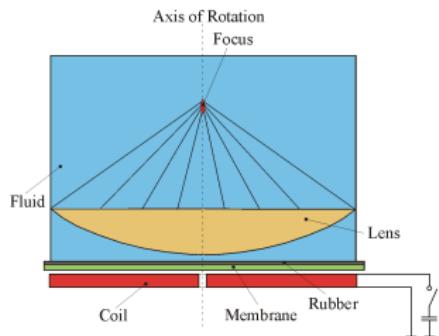
- interface coupling between fluid Ω_- (nonlinearly acoustic) and lens Ω_+ (linearly acoustic or elastic)
 $\overline{\Omega} = \overline{\Omega}_- \cup \overline{\Omega}_+, \quad \Omega_- \cap \Omega_+ = \emptyset$
- No global $H^2(\Omega)$ regularity!
- How to control $L^\infty(\Omega)$ norm of u ?

Lithotripsy: Focusing Principles



- interface coupling between fluid Ω_- (nonlinearly acoustic) and lens Ω_+ (linearly acoustic or elastic)
 $\overline{\Omega} = \overline{\Omega}_- \cup \overline{\Omega}_+, \quad \Omega_- \cap \Omega_+ = \emptyset$
- No global $H^2(\Omega)$ regularity!
- How to control $L^\infty(\Omega)$ norm of u ?
- Introduce p -Laplace damping (power law fluids / regularization) and use $W^{p+1}(\Omega) \rightarrow L_\infty(\Omega)$ embedding

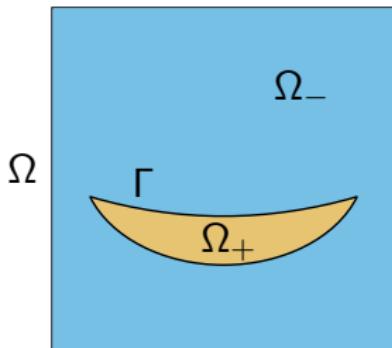
Lithotripsy: Focusing Principles



- interface coupling between fluid Ω_- (nonlinearly acoustic) and lens Ω_+ (linearly acoustic or elastic)
 $\overline{\Omega} = \overline{\Omega}_- \cup \overline{\Omega}_+, \quad \Omega_- \cap \Omega_+ = \emptyset$
- No global $H^2(\Omega)$ regularity!
- How to control $L^\infty(\Omega)$ norm of u ?
- Introduce p -Laplace damping (power law fluids / regularization) and use $W^{p+1}(\Omega) \rightarrow L_\infty(\Omega)$ embedding

$$(1 + ku)u_{tt} - c^2 \Delta u - b \operatorname{div} \left(\nabla u_t + \delta |\nabla u_t|^{p-1} \nabla u_t \right) = -k(u_t)^2,$$

Monodomain Formulations for Coupling: Acoustic-Acoustic

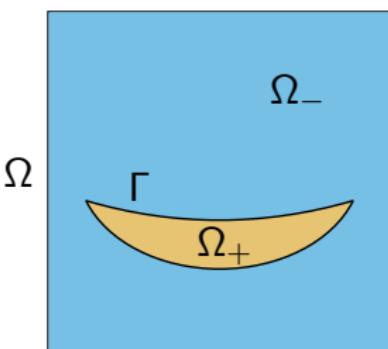


$$\begin{aligned} & \frac{1}{\lambda}(1 + ku)u_{tt} - \operatorname{div}\left(\frac{1}{\varrho}\nabla u\right) \\ & - \operatorname{div}\left(b(1 + \delta|\nabla u_t|^{p-1})\nabla u_t\right) = -\frac{k}{\lambda}(u_t)^2, \end{aligned}$$

where

- u ... pressure
- $\lambda = \varrho c^2$... bulk modulus
- $\lambda, \varrho, k, b, \delta$ piecewise constant
- $b, k \equiv 0$ on Ω_+

Monodomain Formulations for Coupling: Acoustic-Acoustic



$$\begin{aligned} \frac{1}{\lambda}(1 + ku)u_{tt} - \operatorname{div}\left(\frac{1}{\varrho}\nabla u\right) \\ - \operatorname{div}\left(b(1 + \delta|\nabla u_t|^{p-1})\nabla u_t\right) = -\frac{k}{\lambda}(u_t)^2, \end{aligned}$$

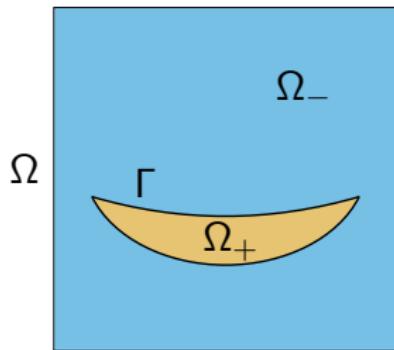
where

- u ... pressure
- $\lambda = \varrho c^2$... bulk modulus
- $\lambda, \varrho, k, b, \delta$ piecewise constant
- $b, k \equiv 0$ on Ω_+

interface conditions:

- $\operatorname{tr}_\Gamma(u|_{\Omega_-}) = \operatorname{tr}_\Gamma(u|_{\Omega_+})$
i.e., continuity of pressure in a trace sense
- $\int_\Gamma \left\{ \vec{v}_u|_{\Omega_-} - \vec{v}_u|_{\Omega_+} \right\} \nu \cdot \nu \, ds = 0 \quad \forall \nu \in H^1(\Omega)$
where $\vec{v}_u := \frac{1}{\varrho}\nabla u + b(1 + \delta|\nabla u_t|^{p-1})\nabla u_t$
i.e., continuity of (modified) normal velocity in a variational sense.

Monodomain Formulations for Coupling: Acoustic-Elastic

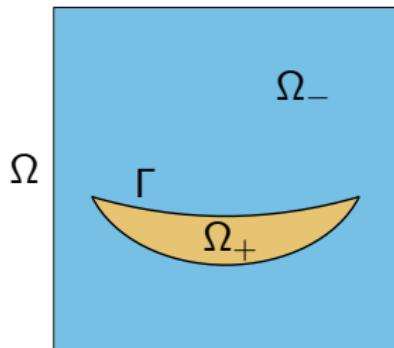


$$\begin{aligned} \varrho \vec{v}_{tt} - \operatorname{div} \frac{1}{1 + \tilde{k} \psi_t} [\lambda] \nabla_s \vec{v} \\ - \operatorname{div} \left((1 + \delta |\nabla_s \vec{v}_t|^{p-1}) [b] \nabla_s \vec{v}_t \right) = 0 \end{aligned}$$

where

- $\vec{v} = \nabla \psi + \nabla \times \vec{A}$... velocity
- $\varrho, [\lambda], k, [b], \delta$ piecewise constant
- $[b], k \equiv 0$ on Ω_+

Monodomain Formulations for Coupling: Acoustic-Elastic



$$\varrho \vec{v}_{tt} - \operatorname{div} \frac{1}{1 + \tilde{k} \psi_t} [\lambda] \nabla_s \vec{v} \\ - \operatorname{div} \left((1 + \delta |\nabla_s \vec{v}_t|^{p-1}) [b] \nabla_s \vec{v}_t \right) = 0$$

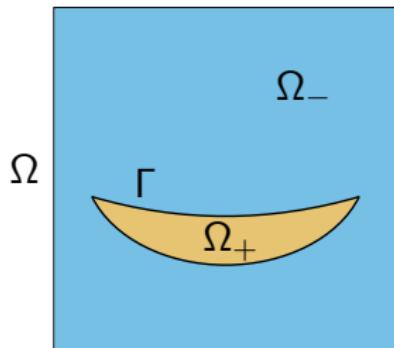
where

- $\vec{v} = \nabla \psi + \nabla \times \vec{A}$... velocity
- $\varrho, [\lambda], k, [b], \delta$ piecewise constant
- $[b], k \equiv 0$ on Ω_+

interface conditions:

- $\operatorname{tr}_\Gamma(\vec{v}|_{\Omega_-}) = \operatorname{tr}_\Gamma(\vec{v}|_{\Omega_+})$
i.e., continuity of velocity in a trace sense
- $\int_\Gamma \left\{ \sigma_v|_{\Omega_-} - \sigma_v|_{\Omega_+} \right\} \vec{v} \cdot \vec{w} \, ds = 0 \quad \forall \vec{w} \in (H^1(\Omega))^d$
where $\sigma_v := [\lambda] \nabla_s \vec{v} + (1 + \delta |\nabla_s \vec{v}_t|^{p-1}) [b] \nabla_s \vec{v}_t$
i.e., continuity of (modified) normal stresses in a variational sense.

Monodomain Formulations for Coupling: Acoustic-Elastic



$$\begin{aligned} \varrho \vec{v}_{tt} - \operatorname{div} \frac{1}{1 + \tilde{k} \psi_t} [\lambda] \nabla_s \vec{v} \\ - \operatorname{div} \left((1 + \delta |\nabla_s \vec{v}_t|^{p-1}) [b] \nabla_s \vec{v}_t \right) = 0 \end{aligned}$$

where

- $\vec{v} = \nabla \psi + \nabla \times \vec{A}$... velocity
- $\varrho, [\lambda], k, [b], \delta$ piecewise constant
- $[b], k \equiv 0$ on Ω_+

interface conditions:

- $\operatorname{tr}_\Gamma(\vec{v}|_{\Omega_-}) = \operatorname{tr}_\Gamma(\vec{v}|_{\Omega_+})$
i.e., continuity of velocity in a trace sense
- $\int_\Gamma \left\{ \sigma_v|_{\Omega_-} - \sigma_v|_{\Omega_+} \right\} \vec{v} \cdot \vec{w} \, ds = 0 \quad \forall \vec{w} \in (H^1(\Omega))^d$
where $\sigma_v := [\lambda] \nabla_s \vec{v} + (1 + \delta |\nabla_s \vec{v}_t|^{p-1}) [b] \nabla_s \vec{v}_t$
i.e., continuity of (modified) normal stresses in a variational sense.

[Brunnhuber & BK & Radu 2014, Nikolić 2015, Nikolić & BK 2015, 16]

time integration

Westervelt Equation: Motivation for Operator Splitting

$$\psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t + \frac{k}{2} (\psi_t)^2 = 0$$

$$\frac{1}{2} \psi_{tt} + \frac{1}{2} \psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t + \frac{k}{2} (\psi_t)^2 = 0$$

Westervelt Equation: Motivation for Operator Splitting

$$\psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t + \frac{k}{2} (\psi_t)^2 = 0$$

$$\frac{1}{2} \psi_{tt} + \frac{1}{2} \psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t + \frac{k}{2} (\psi_t)^2 = 0$$

wave equation:

$$\frac{1}{2} \psi_{tt} - c^2 \Delta \psi = 0$$

nonlinear heat equation ($p = \psi_t$):

$$\frac{1}{2} p_t - b \Delta p + \frac{k}{2} (p^2)_t = 0$$

additive splitting [Espedal Karlsen, 2000]

Westervelt Equation as a Nonlinear Evolutionary System

$$\psi_{tt} - \frac{b}{1+k\psi_t} \Delta \psi_t - \frac{c^2}{1+k\psi_t} \Delta \psi = 0$$

- + homogeneous Dirichlet boundary conditions
- + initial conditions

Westervelt Equation as a Nonlinear Evolutionary System

$$\psi_{tt} - \frac{b}{1+k\psi_t} \Delta \psi_t - \frac{c^2}{1+k\psi_t} \Delta \psi = 0$$

- + homogeneous Dirichlet boundary conditions
- + initial conditions

abstract Cauchy problem for $u : [0, T] \rightarrow X : t \mapsto u(t) = \begin{pmatrix} \psi(\cdot, t) \\ \psi_t(\cdot, t) \end{pmatrix}$:

$$\begin{cases} u_t(t) = F(u(t)) , & t \in (0, T] \\ u(0) = u_0 , \end{cases}$$

where $F : D(F) \rightarrow X$ (quasilinear) $\tilde{b}(v_2) = \frac{b}{1+k v_2}$, $\tilde{c}^2(v_2) = \frac{c^2}{1+k v_2}$

$$F(v) = \begin{pmatrix} v_2 \\ \tilde{b}(v_2) \Delta v_2 + \tilde{c}^2(v_2) \Delta v_1 \end{pmatrix} = \begin{pmatrix} 0 & I \\ \tilde{c}^2(v_2) \Delta & \tilde{b}(v_2) \Delta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Time-Splitting Methods

$$u_t(t) = F(u(t)), \quad t \in (0, T], \quad (1)$$

$u : [0, T] \rightarrow X$ Banach space $(X, \|X\| \cdot)$.

splitting:

$$F = A + B$$

efficient numerical solvers available for subproblems

$$v_t(t) = A(v(t)), \quad w_t(t) = B(w(t)), \quad t \in (0, T]. \quad (2)$$

e.g., first-order Lie–Trotter splitting method:

$$u_1 = \mathcal{S}_F(h, u_0) = \mathcal{E}_B\left(h, \mathcal{E}_A(h, u_0)\right) \approx u(h) = \mathcal{E}_F(h, u(0)),$$

$\mathcal{E}_F, \mathcal{E}_A, \mathcal{E}_B \dots$ evolution operators for subproblems (1) and (2).

Time-Splitting Methods

$$u_t(t) = F(u(t)), \quad t \in (0, T], \quad (1)$$

$u : [0, T] \rightarrow X$ Banach space $(X, \|X\| \cdot)$.

splitting:

$$F = A + B$$

efficient numerical solvers available for subproblems

$$v_t(t) = A(v(t)), \quad w_t(t) = B(w(t)), \quad t \in (0, T]. \quad (2)$$

e.g., first-order Lie–Trotter splitting method:

$$u_1 = \mathcal{S}_F(h, u_0) = \mathcal{E}_B(h, \mathcal{E}_A(h, u_0)) \approx u(h) = \mathcal{E}_F(h, u(0)),$$

$\mathcal{E}_F, \mathcal{E}_A, \mathcal{E}_B \dots$ evolution operators for subproblems (1) and (2).

higher order splitting methods, e.g., Strang

Decompositions for Westervelt

$$F(v) = \begin{pmatrix} v_2 \\ \tilde{b}(v_2) \Delta v_2 + \tilde{c}^2(v_2) \Delta v_1 \end{pmatrix} \quad F = A + B$$

Decomposition I:

$$A(v) = \begin{pmatrix} v_2 \\ \tilde{b}(v_2) \Delta v_2 \end{pmatrix}$$

$$B(v) = \begin{pmatrix} 0 \\ \tilde{c}^2(v_2) \Delta v_1 \end{pmatrix}$$

Decomposition II:

$$A(v) = \begin{pmatrix} \frac{1}{2} v_2 \\ \tilde{b}(v_2) \Delta v_2 \end{pmatrix}$$

$$B(v) = \begin{pmatrix} \frac{1}{2} v_2 \\ \tilde{c}^2(v_2) \Delta v_1 \end{pmatrix}$$

Decomposition III:

$$A(v) = \begin{pmatrix} 0 \\ \tilde{b}(v_2) \Delta v_2 \end{pmatrix}$$

$$B(v) = \begin{pmatrix} v_2 \\ \tilde{c}^2(v_2) \Delta v_1 \end{pmatrix}$$

Decomposition VI:

$$A(v) = \begin{pmatrix} 0 \\ \tilde{b}(v_2) \Delta v_2 - k v_2 \tilde{c}^2(v_2) \Delta v_1 \end{pmatrix}$$

$$B(v) = \begin{pmatrix} v_2 \\ c^2 \Delta v_1 \end{pmatrix}$$

Decompositions for Westervelt

$$F(v) = \begin{pmatrix} v_2 \\ \tilde{b}(v_2) \Delta v_2 + \tilde{c}^2(v_2) \Delta v_1 \end{pmatrix} \quad F = A + B$$

Decomposition I:

$$A(v) = \begin{pmatrix} v_2 \\ \tilde{b}(v_2) \Delta v_2 \end{pmatrix}$$

$$B(v) = \begin{pmatrix} 0 \\ \tilde{c}^2(v_2) \Delta v_1 \end{pmatrix}$$

Decomposition II:

$$A(v) = \begin{pmatrix} \frac{1}{2} v_2 \\ \tilde{b}(v_2) \Delta v_2 \end{pmatrix}$$

$$B(v) = \begin{pmatrix} \frac{1}{2} v_2 \\ \tilde{c}^2(v_2) \Delta v_1 \end{pmatrix}$$

Decomposition III:

$$A(v) = \begin{pmatrix} 0 \\ \tilde{b}(v_2) \Delta v_2 \end{pmatrix}$$

$$B(v) = \begin{pmatrix} v_2 \\ \tilde{c}^2(v_2) \Delta v_1 \end{pmatrix}$$

Decomposition VI:

$$A(v) = \begin{pmatrix} 0 \\ \tilde{b}(v_2) \Delta v_2 - k v_2 \tilde{c}^2(v_2) \Delta v_1 \end{pmatrix}$$

$$B(v) = \begin{pmatrix} v_2 \\ c^2 \Delta v_1 \end{pmatrix}$$

[BK & Nikolić & Thalhammer 2014]

Decomposition I:

subproblems to be solved:

$$A : \begin{cases} \Psi_{1t}(x, t) = \Psi_2(x, t), \\ \Psi_{2t}(x, t) = \frac{b}{1+k\Psi_2(x, t)} \Delta \Psi_2(x, t), \end{cases}$$

nonlinear diffusion equation for second component Ψ_2 ;
plain time integration to obtain first component Ψ_1 from Ψ_2

$$B : \begin{cases} \Psi_{1t}(x, t) = 0, \\ \Psi_{2t}(x, t) = \frac{c^2}{1+k\Psi_2(x, t)} \Delta \Psi_1(x, t), \end{cases}$$

first component Ψ_1 remains constant;
explicit representation for second component:

$$\Psi_2(x, t) = \frac{1}{k} \left(1 - \sqrt{\left(1 + k \Psi_2(x, 0) \right)^2 + 2 c^2 k t \Delta \Psi_1(x, 0)} \right)$$

Numerical Experiments

1-d Test problem:
unit parameters:

$$b = 1, \quad c^2 = 1, \quad k = -1,$$

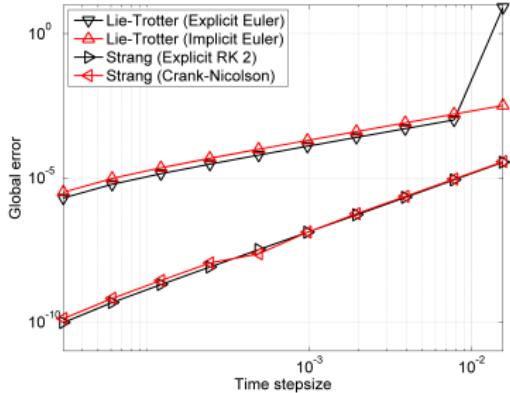
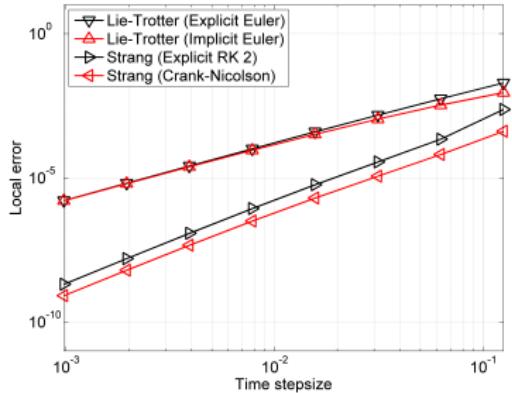
homogeneous Dirichlet conditions :

$$\psi(-a, t) = 0 = \psi(a, t), \quad t \in [0, T],$$

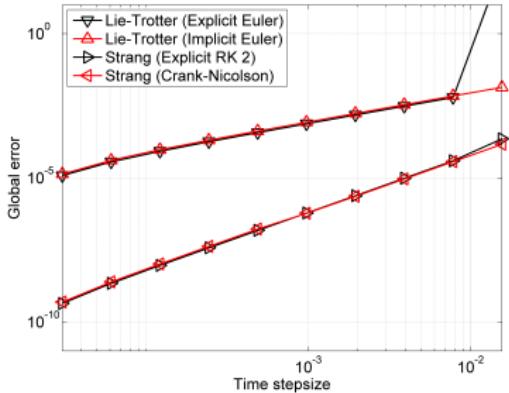
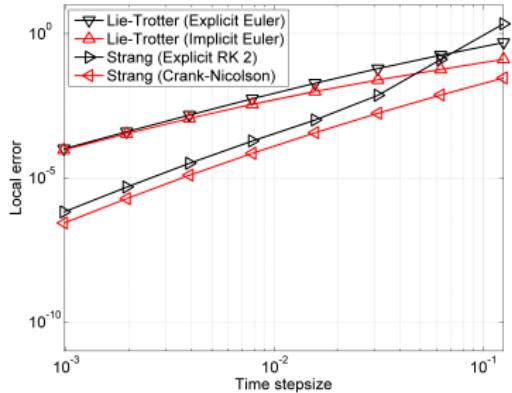
regular initial conditions:

$$\psi(x, 0) = \exp(-x^2), \quad \psi_t(x, 0) = -x \exp(-x^2), \quad x \in [-a, a].$$

$$a = 8, \quad T = 1$$



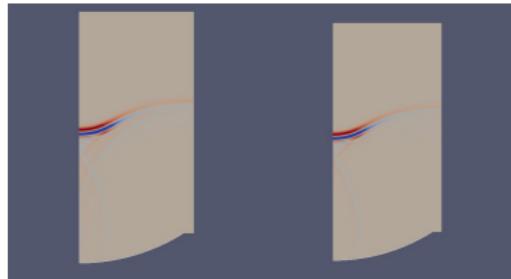
Local (left) and global (right) errors for the Lie–Trotter and Strang splitting methods with respect to the $L^2 \times L^2$ -norm obtained for Decomposition I. Comparison of different time integration methods for the numerical solution of the subproblems.



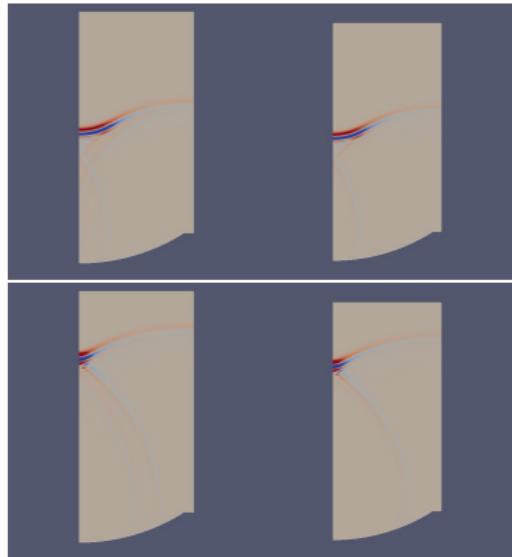
Local (left) and global (right) errors for the Lie–Trotter and Strang splitting methods with respect to the $H^3 \times H^1$ -norm obtained for Decomposition I. Comparison of different time integration methods for the numerical solution of the subproblems.

nonreflecting boundary conditions

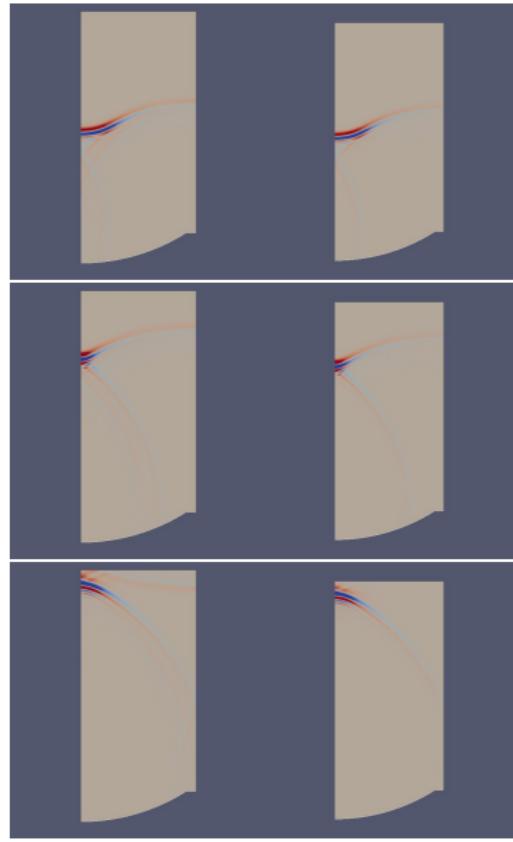
Sound-Hard versus Absorbing Boundary Conditions



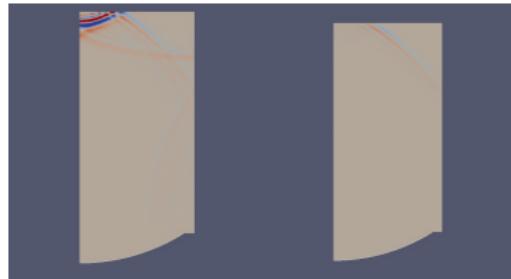
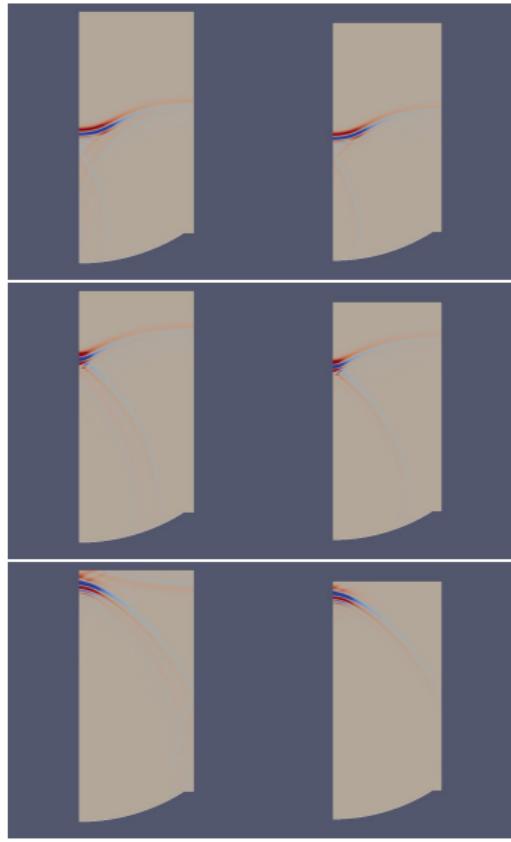
Sound-Hard versus Absorbing Boundary Conditions



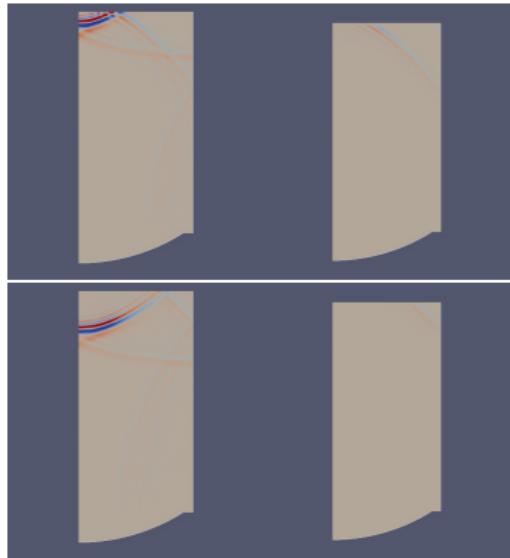
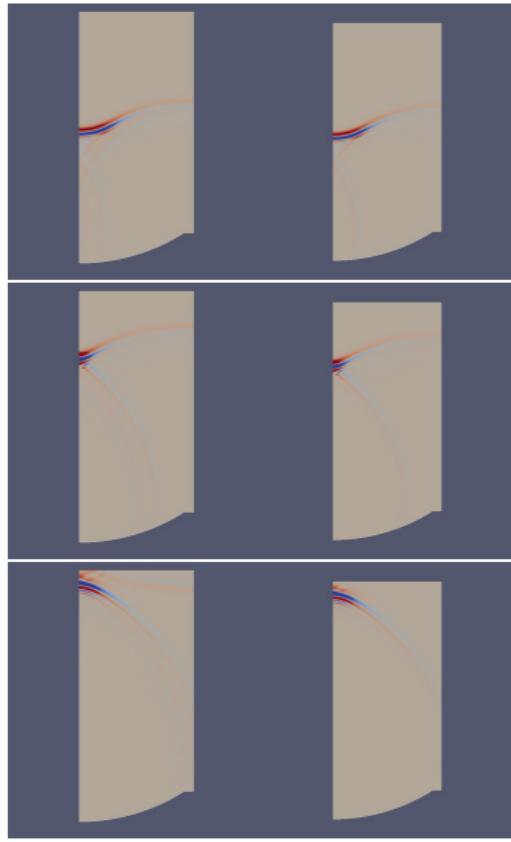
Sound-Hard versus Absorbing Boundary Conditions



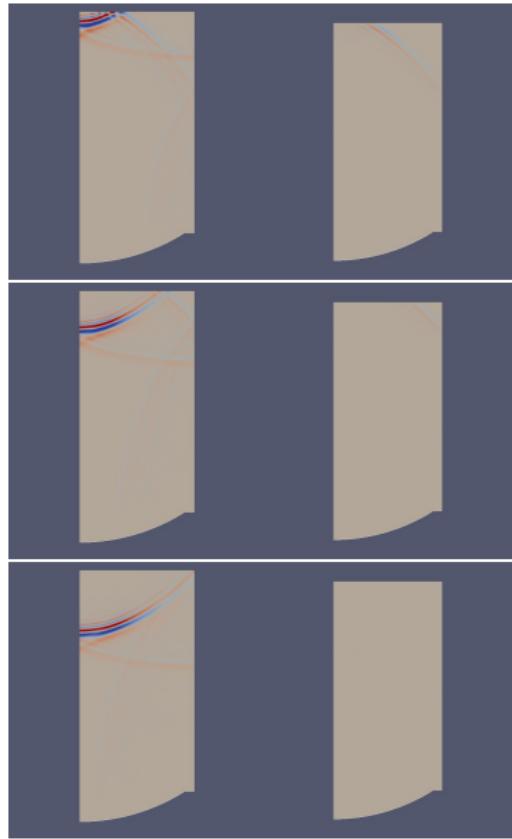
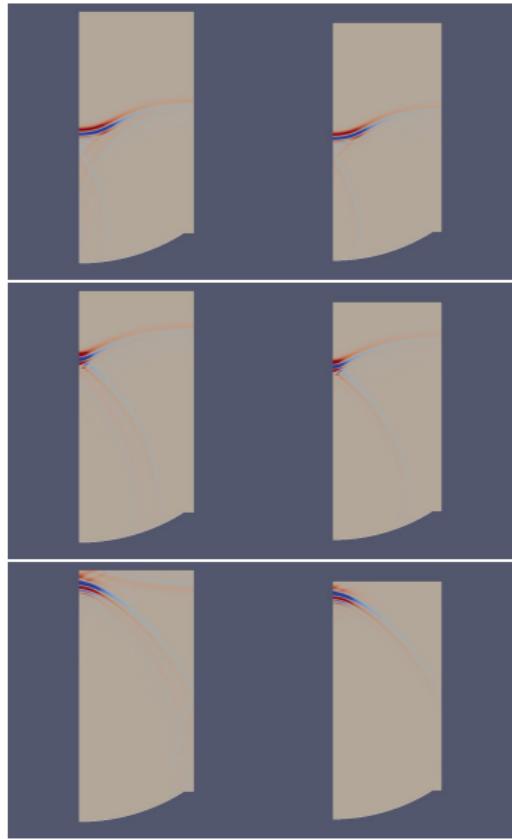
Sound-Hard versus Absorbing Boundary Conditions



Sound-Hard versus Absorbing Boundary Conditions



Sound-Hard versus Absorbing Boundary Conditions



Absorbing Boundary Conditions for the Wave Equation

- 1-dim. linear case: $u_{tt} - c^2 u_{xx} = 0$ on $[-1, 1]$

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = (\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0$$

Absorbing Boundary Conditions for the Wave Equation

- 1-dim. linear case: $u_{tt} - c^2 u_{xx} = 0$ on $[-1, 1]$

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = (\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0$$

annihilate reflected waves:

$$(\partial_t - c\partial_x)u = 0 \text{ at } x = -1 \quad (\partial_t + c\partial_x)u = 0 \text{ at } x = 1$$

Absorbing Boundary Conditions for the Wave Equation

- 1-dim. linear case: $u_{tt} - c^2 u_{xx} = 0$ on $[-1, 1]$

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = (\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0$$

annihilate reflected waves:

$$(\partial_t - c\partial_x)u = 0 \text{ at } x = -1 \quad (\partial_t + c\partial_x)u = 0 \text{ at } x = 1$$

- d-dim. linear case: $u_{tt} - c^2 \Delta u = 0$ in $\Omega \subseteq \mathbb{R}^d$

$$u_n = -\frac{1}{c}u_t \text{ on } \partial\Omega$$

Absorbing Boundary Conditions for the Wave Equation

- 1-dim. linear case: $u_{tt} - c^2 u_{xx} = 0$ on $[-1, 1]$

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = (\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0$$

annihilate reflected waves:

$$(\partial_t - c\partial_x)u = 0 \text{ at } x = -1 \quad (\partial_t + c\partial_x)u = 0 \text{ at } x = 1$$

- d-dim. linear case: $u_{tt} - c^2 \Delta u = 0$ in $\Omega \subseteq \mathbb{R}^d$

$$u_n = -\frac{1}{c}u_t \text{ on } \partial\Omega \quad \rightsquigarrow \text{decreases energy}$$

Absorbing Boundary Conditions for the Wave Equation

- 1-dim. linear case: $u_{tt} - c^2 u_{xx} = 0$ on $[-1, 1]$

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = (\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0$$

annihilate reflected waves:

$$(\partial_t - c\partial_x)u = 0 \text{ at } x = -1 \quad (\partial_t + c\partial_x)u = 0 \text{ at } x = 1$$

- d-dim. linear case: $u_{tt} - c^2 \Delta u = 0$ in $\Omega \subseteq \mathbb{R}^d$

$$u_n = -\frac{1}{c}u_t \text{ on } \partial\Omega \quad \rightsquigarrow \text{decreases energy}$$

- 2-dim. nonlinear case:

$$(1 + ku)u_{tt} - c^2 \Delta u - b\Delta u_t = -k(u_t)^2 \text{ in } \Omega \subseteq \mathbb{R}^2$$

$$\text{zero order: } u_n = -\frac{1}{c}\sqrt{1 + ku} u_t \text{ on } \partial\Omega$$

$$\text{first order: } u_{n tt} = -\frac{1}{c}\sqrt{1 + ku} u_{ttt} + \frac{1}{2c}\sqrt{1 + ku} u_{\vartheta\vartheta t} + \dots \text{ on } \partial\Omega$$

Absorbing Boundary Conditions for the Wave Equation

- 1-dim. linear case: $u_{tt} - c^2 u_{xx} = 0$ on $[-1, 1]$

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = (\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0$$

annihilate reflected waves:

$$(\partial_t - c\partial_x)u = 0 \text{ at } x = -1 \quad (\partial_t + c\partial_x)u = 0 \text{ at } x = 1$$

- d-dim. linear case: $u_{tt} - c^2 \Delta u = 0$ in $\Omega \subseteq \mathbb{R}^d$

$$u_n = -\frac{1}{c}u_t \text{ on } \partial\Omega \quad \rightsquigarrow \text{decreases energy}$$

- 2-dim. nonlinear case:

$$(1 + ku)u_{tt} - c^2 \Delta u - b\Delta u_t = -k(u_t)^2 \text{ in } \Omega \subseteq \mathbb{R}^2$$

$$\text{zero order: } u_n = -\frac{1}{c}\sqrt{1+ku} u_t \text{ on } \partial\Omega$$

$$\text{first order: } u_{n tt} = -\frac{1}{c}\sqrt{1+ku} u_{ttt} + \frac{1}{2c}\sqrt{1+ku} u_{\vartheta\vartheta t} + \dots \text{ on } \partial\Omega$$

(formal) pseudodifferential calculus to decompose differential operator
+ energy estimates

Absorbing Boundary Conditions for the Wave Equation

- 1-dim. linear case: $u_{tt} - c^2 u_{xx} = 0$ on $[-1, 1]$

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = (\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0$$

annihilate reflected waves:

$$(\partial_t - c\partial_x)u = 0 \text{ at } x = -1 \quad (\partial_t + c\partial_x)u = 0 \text{ at } x = 1$$

- d-dim. linear case: $u_{tt} - c^2 \Delta u = 0$ in $\Omega \subseteq \mathbb{R}^d$

$$u_n = -\frac{1}{c}u_t \text{ on } \partial\Omega \quad \rightsquigarrow \text{decreases energy}$$

- 2-dim. nonlinear case:

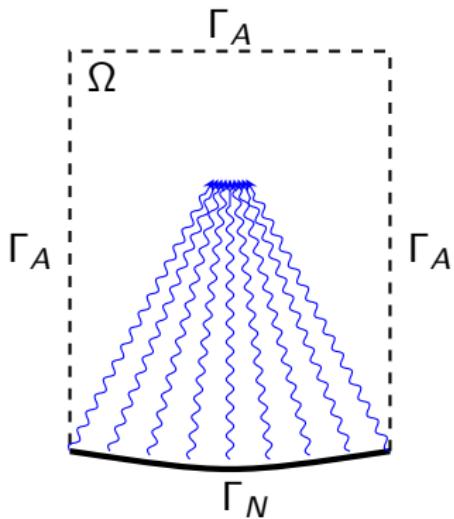
$$(1 + ku)u_{tt} - c^2 \Delta u - b\Delta u_t = -k(u_t)^2 \text{ in } \Omega \subseteq \mathbb{R}^2$$

$$\text{zero order: } u_n = -\frac{1}{c}\sqrt{1 + ku} u_t \text{ on } \partial\Omega$$

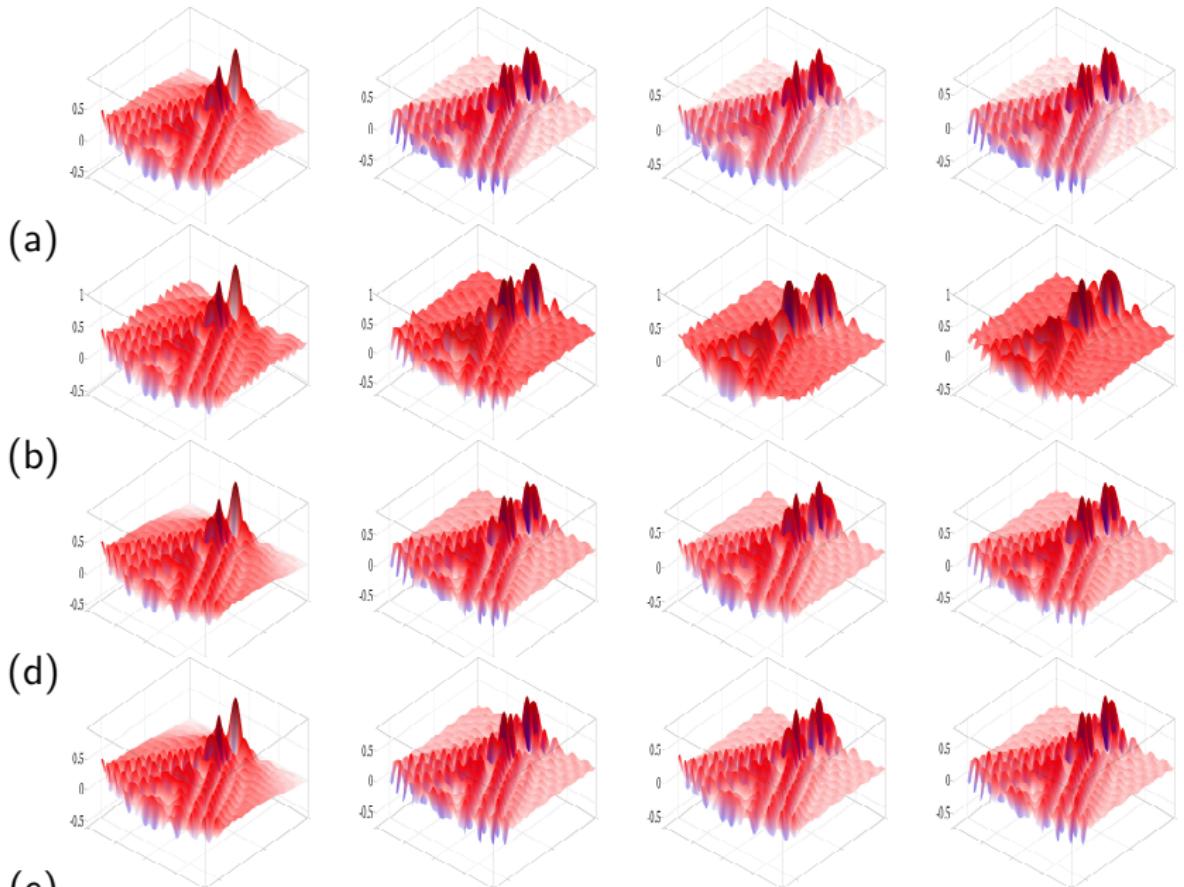
$$\text{first order: } u_{n tt} = -\frac{1}{c}\sqrt{1 + ku} u_{ttt} + \frac{1}{2c}\sqrt{1 + ku} u_{\vartheta\vartheta t} + \dots \text{ on } \partial\Omega$$

(formal) pseudodifferential calculus to decompose differential operator
+ energy estimates [BK&Shevchenko 2015, 2016]

Numerical Experiments



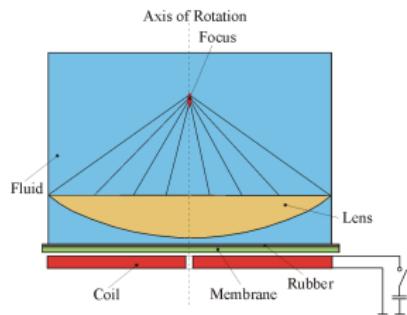
geometric setup for the high-intensity focused ultrasound problem.



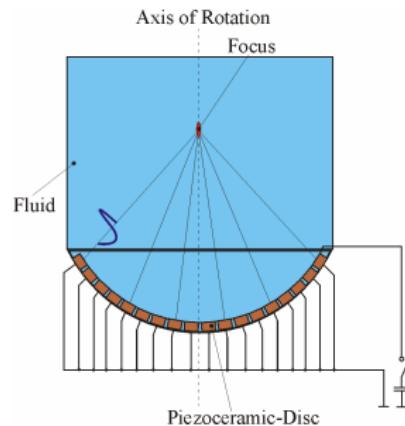
(a) reference solution (b) $\text{ABC}_{\text{lin}}^{2,1}$, (c) $\text{ABC}_{\text{nl}}^{2,0}$, (d) $\text{ABC}_{\text{nl}}^{2,1}$.

optimization

Lithotripsy: Focusing Principles

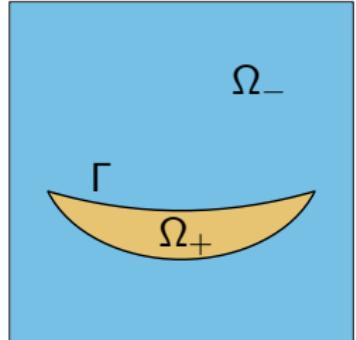


~~ shape optimization of the acoustic lens



~~ shape optimization of the piezo mosaic/
optimal control of the excitation signal

Shape Optimization of Acoustic Lens



$$\min_{\substack{\Omega_+ \in \mathcal{O}_{ad} \\ u \in L^2(\Omega \times [0, T])}} \int_0^T \int_{\Omega} (u - u_d)^2 dx ds$$

subject to

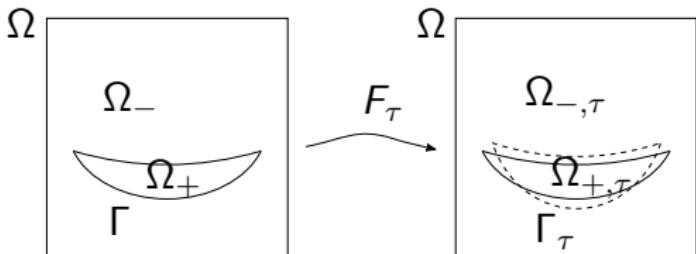
$$\left\{ \begin{array}{l} \frac{1+ku}{\lambda} u_{tt} - \operatorname{div}\left(\frac{1}{\varrho} \nabla u\right) - \operatorname{div}(b(1 + \delta |\nabla u_t|^{q-1}) \nabla u_t) = -\frac{k}{\lambda} (u_t)^2 \\ \quad \text{in } \Omega_+ \cup \Omega_- \\ \llbracket u \rrbracket = 0 \quad \text{on } \Gamma = \partial\Omega_+ \\ \left[\frac{1}{\varrho} \frac{\partial u}{\partial n_+} + b \frac{\partial u_t}{\partial n_+} + b\delta |\nabla u_t|^{q-1} \frac{\partial u_t}{\partial n_+} \right] = 0 \quad \text{on } \Gamma = \partial\Omega_+ \\ u = 0 \quad \text{on } \partial\Omega \\ (u, u_t)|_{t=0} = (u_0, u_1) \end{array} \right.$$

$$\lambda = \begin{cases} \lambda_+ & \text{in } \Omega_+ \\ \lambda_- & \text{in } \Omega_- \end{cases}, \quad \varrho = \begin{cases} \varrho_+ & \text{in } \Omega_+ \\ \varrho_- & \text{in } \Omega_- \end{cases}, \quad b = \begin{cases} b_+ & \text{in } \Omega_+ \\ b_- & \text{in } \Omega_- \end{cases}, \quad k = \begin{cases} k_+ & \text{in } \Omega_+ \\ k_- & \text{in } \Omega_- \end{cases}$$

Shape Optimization of Acoustic Lens

method of mappings:

$$F_\tau = id + \tau h, \text{ for } \tau \in \mathbb{R}.$$



Eulerian derivative of J at Ω_+ in the direction of the vector field h

$$dJ(u, \Omega_+)h = \lim_{\tau \rightarrow 0} \frac{1}{\tau} (J(u_\tau, \Omega_{+,\tau}) - J(u, \Omega_+)),$$

Strong shape derivative (Delfour-Hadamard-Zolésio Structure Thm)

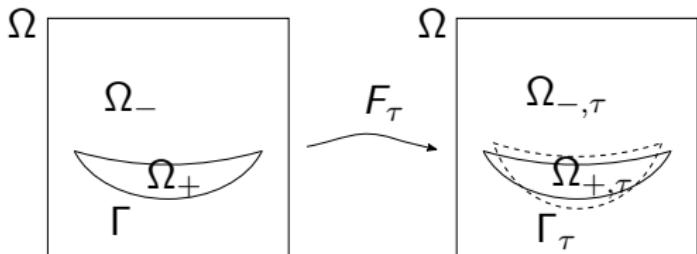
$$\begin{aligned} dJ(u, \Omega_+)h &= \int_0^T \int_{\Gamma} \left[-\frac{1}{\lambda} (1 + ku) u_{tt} p - \frac{1}{\varrho} \nabla u \cdot \nabla p - b(1 + \delta |\nabla u_t|^{q-1}) \nabla u_t \cdot \nabla p \right. \\ &\quad - \frac{k}{\lambda} (u_t)^2 p + \frac{2}{\varrho} \frac{\partial u}{\partial n_+} \frac{\partial p}{\partial n_+} + 2b(1 + \delta |\nabla u_t|^{q-1}) \frac{\partial u_t}{\partial n_+} \frac{\partial p}{\partial n_+} \\ &\quad \left. + b\delta(q-1) |\nabla u_t|^{q-3} (\nabla u_t \cdot \nabla p) \left| \frac{\partial u_t}{\partial n_+} \right|^2 \right] h^T n_+ dx ds. \end{aligned}$$

with p ... adjoint state

Shape Optimization of Acoustic Lens

method of mappings:

$$F_\tau = id + \tau h, \text{ for } \tau \in \mathbb{R}.$$



Eulerian derivative of J at Ω_+ in the direction of the vector field h

$$dJ(u, \Omega_+)h = \lim_{\tau \rightarrow 0} \frac{1}{\tau} (J(u_\tau, \Omega_{+,\tau}) - J(u, \Omega_+)),$$

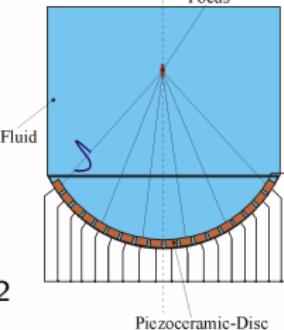
Strong shape derivative (Delfour-Hadamard-Zolésio Structure Thm)

$$\begin{aligned} dJ(u, \Omega_+)h &= \int_0^T \int_{\Gamma} \left[-\frac{1}{\lambda} (1 + ku) u_{tt} p - \frac{1}{\varrho} \nabla u \cdot \nabla p - b(1 + \delta |\nabla u_t|^{q-1}) \nabla u_t \cdot \nabla p \right. \\ &\quad - \frac{k}{\lambda} (u_t)^2 p + \frac{2}{\varrho} \frac{\partial u}{\partial n_+} \frac{\partial p}{\partial n_+} + 2b(1 + \delta |\nabla u_t|^{q-1}) \frac{\partial u_t}{\partial n_+} \frac{\partial p}{\partial n_+} \\ &\quad \left. + b\delta(q-1) |\nabla u_t|^{q-3} (\nabla u_t \cdot \nabla p) \left| \frac{\partial u_t}{\partial n_+} \right|^2 \right] h^T n_+ dx ds. \end{aligned}$$

with p ... adjoint state [Nikolić & BK 2016, Peichl & BK 2016]

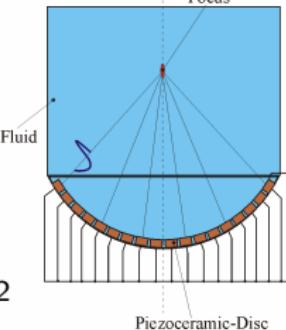
Optimal Boundary Control of Piezoarray

$$\left\{ \begin{array}{ll} \min_{\substack{g \in \mathcal{G}_{ad} \\ u \in L^2(\Omega \times [0, T])}} & \int_0^T \int_{\Omega} (u - u_d)^2 \, dx \, ds \quad \text{s.t.} \\ (P_g) & \begin{cases} (1 + ku)u_{tt} - c^2 \Delta u - b \Delta u_t = -\frac{k}{\lambda} (u_t)^2 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = g \text{ on } \Gamma \\ \frac{\partial u}{\partial n} = -u_t \text{ on } \partial\Omega \setminus \Gamma \\ (u, u_t)|_{t=0} = (u_0, u_1) \end{cases} \end{array} \right.$$



Optimal Boundary Control of Piezoarray

$$(P_g) \quad \left\{ \begin{array}{l} \min_{\substack{g \in \mathcal{G}_{ad} \\ u \in L^2(\Omega \times [0, T])}} \int_0^T \int_{\Omega} (u - u_d)^2 dx ds \quad \text{s.t.} \\ \left\{ \begin{array}{l} (1 + ku)u_{tt} - c^2 \Delta u - b \Delta u_t = -\frac{k}{\lambda} (u_t)^2 \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} = g \text{ on } \Gamma \\ \frac{\partial u}{\partial n} = -u_t \text{ on } \partial\Omega \setminus \Gamma \\ (u, u_t)|_{t=0} = (u_0, u_1) \end{array} \right. \end{array} \right.$$



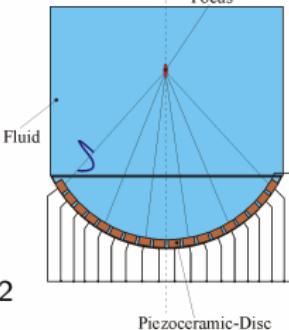
conventional approach:

define control-to-state map $S : g \mapsto u$ solution to (P_g) , consider

$$\min_{g \in \mathcal{G}_{ad}} \int_0^T \int_{\Omega} (S(g) - u_d)^2 dx ds$$

avoid degeneracy $\Leftarrow u = S(g)$ small $\Leftarrow g$ small

Optimal Boundary Control of Piezoarray



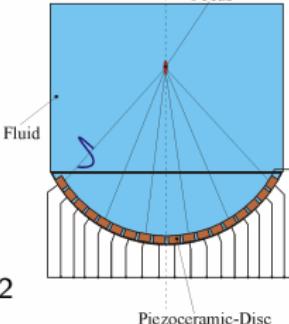
$$(P_g) \quad \left\{ \begin{array}{l} \min_{\substack{g \in \mathcal{G}_{ad} \\ u \in L^2(\Omega \times [0, T])}} \int_0^T \int_{\Omega} (u - u_d)^2 dx ds \quad \text{s.t.} \\ \left\{ \begin{array}{l} (1 + ku)u_{tt} - c^2 \Delta u - b \Delta u_t = -\frac{k}{\lambda} (u_t)^2 \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} = g \text{ on } \Gamma \\ \frac{\partial u}{\partial n} = -u_t \text{ on } \partial\Omega \setminus \Gamma \\ (u, u_t)|_{t=0} = (u_0, u_1) \end{array} \right. \end{array} \right.$$

conventional approach:

define control-to-state map $S : g \mapsto u$ solution to (P_g) , consider

$$\min_{g \in \mathcal{G}_{ad}} \int_0^T \int_{\Omega} (S(g) - u_d)^2 dx ds$$

avoid degeneracy $\Leftarrow u = S(g)$ small $\Leftarrow g$ small \rightsquigarrow **too restrictive!**

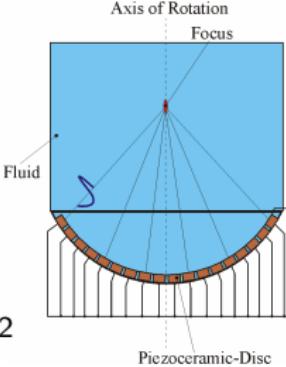


Optimal Boundary Control of Piezoarray

$$\left\{ \begin{array}{ll} \min_{\substack{g \in \mathcal{G}_{ad} \\ u \in L^2(\Omega \times [0, T])}} & \int_0^T \int_{\Omega} (u - u_d)^2 \, dx \, ds \quad \text{s.t.} \\ (P_g) & \begin{cases} (1 + ku)u_{tt} - c^2 \Delta u - b \Delta u_t = -\frac{k}{\lambda} (u_t)^2 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = g \text{ on } \Gamma \\ \frac{\partial u}{\partial n} = -u_t \text{ on } \partial\Omega \setminus \Gamma \\ (u, u_t)|_{t=0} = (u_0, u_1) \end{cases} \end{array} \right.$$

Optimal Boundary Control of Piezoarray

$$\left\{ \begin{array}{ll} \min_{\substack{g \in \mathcal{G}_{ad} \\ u \in L^2(\Omega \times [0, T])}} & \int_0^T \int_{\Omega} (u - u_d)^2 dx ds \\ & \text{s.t.} \\ (P_g) & \begin{cases} (1 + ku)u_{tt} - c^2 \Delta u - b \Delta u_t = -\frac{k}{\lambda} (u_t)^2 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = g \text{ on } \Gamma \\ \frac{\partial u}{\partial n} = -u_t \text{ on } \partial\Omega \setminus \Gamma \\ (u, u_t)|_{t=0} = (u_0, u_1) \end{cases} \end{array} \right.$$



alternative approach:

add state constraint $u(x) \geq -0.99/k \quad \forall x \in \Omega$

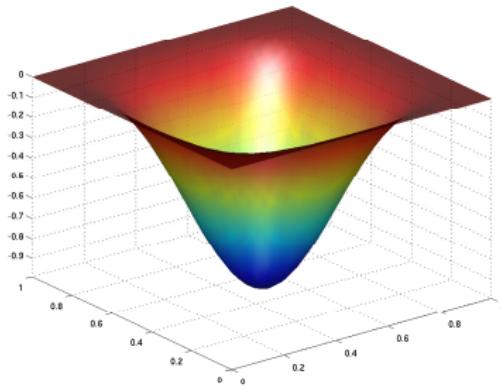
[Clason & BK 2015]

Numerical Experiments

$$J(a, u) = \frac{1}{2} \|u - u_d\|_{L^2}^2 + \frac{\alpha}{2} \|g\|_{L^2}^2$$

MEMS model, static PDE $-\Delta u + \frac{g(x)}{(1+u)^2} = 0$

$\Omega = (0, 1)^2$, $\alpha = 10^{-6}$, 64×64 grid.

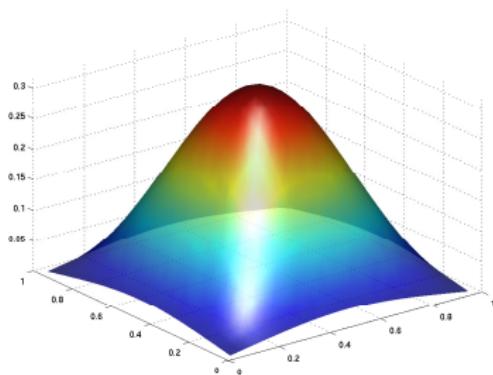


Target u_d

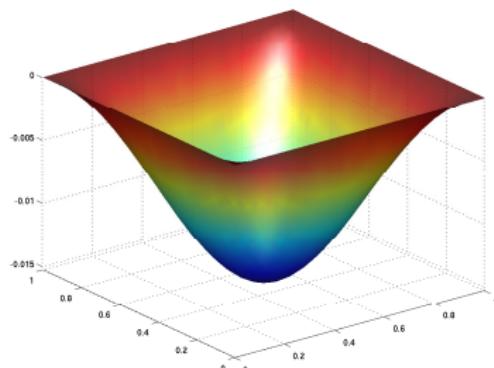
Numerical Experiments: conventional approach

impose control constraints: $\|g\|_{L^2} \leq 0.14815$
to guarantee well-definedness of control-to-state map

optimal control g



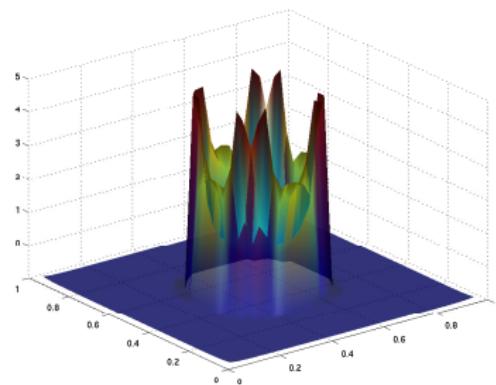
optimal state u



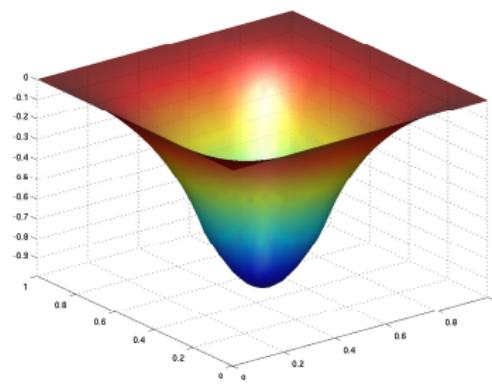
Numerical Experiments: alternative approach

impose pointwise state constraints: $u(x) \geq -0.99$
to avoid singularity

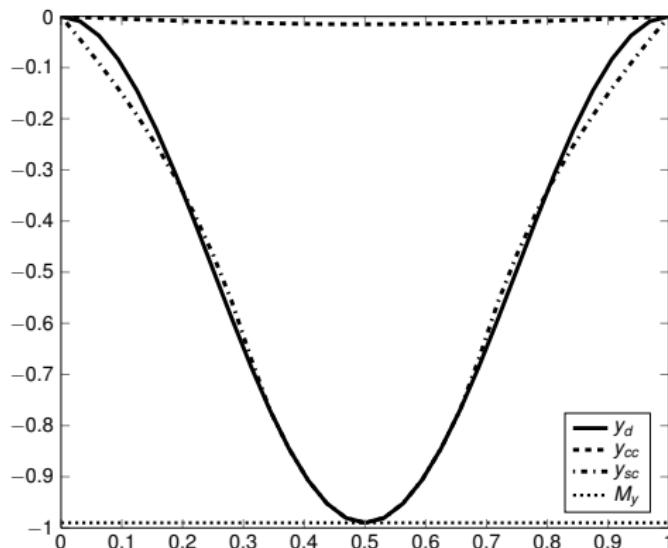
optimal control g



optimal state u



Comparison: with vs without control-to-state map



Cross sections of states for conventional (dashed)
and alternative (dash-dotted) approach,
as well as target y_d (solid) and bound -0.99 (dotted)

Outlook

- implementation of shape optimization using isogeometric FE
(Vanja Nikolić, Linus Wunderlich, TU München)
- modeling and analysis: temperature coupling, cavitation, fractional order damping
- consider first order system of conservation laws;
space-time adaptive FE

Thank you for your attention!



ifip tc7.2+4 workshop

UNIVERSITÄT
DUISBURG
ESSEN

Offen im Denken

Optimal Control Meets Inverse Problems

University of Duisburg-Essen

September 5-9, 2016

Organizers:

Christian Clason

Arnd Rösch

Irwin Yousept

<https://www.uni-due.de/mathematik/agclason/ifip2016>