Methods for Inverse Problems: Introduction

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Mathematical modeling:
Formulate the underlying physical/biological/economic... laws in a mathematical language (usually partial differential equations PDEs)

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forward problem:

$$\textbf{cause} \quad \Longrightarrow \quad \textbf{effect}$$

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inverse problem:

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PDE coefficients,
initial conditions,
boundary conditions,
source terms,
shapes,
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data:

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boundary traces
Dirichlet-to-Neumann map
far field
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inverse problem:

cause $\overset{\textbf{foward}}{\underset{\textbf{operator}}{\Longrightarrow}}$ **effect**

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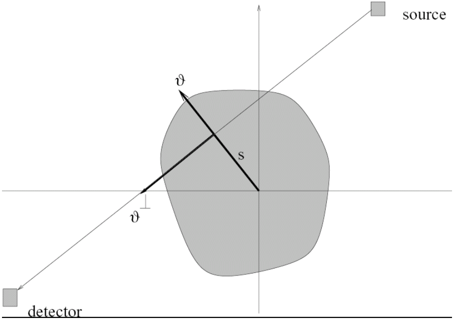
examples

A classical example: Computerized tomography

$$\int_{t_S}^{t_D} \rho(s\vec{\vartheta} + t\vec{\vartheta}^{\perp})dt = -\log\left(\frac{I_D(s,\vec{\vartheta})}{I_E(s,\vec{\vartheta})}\right)$$



Johann Radon, 1887–1956

$$\rho \mapsto \int_{t_S}^{t_D} \rho(s\vec{\vartheta} + t\vec{\vartheta}^{\perp})dt \ldots \text{ Radon transform}$$

An elementary example: Numerical differentiation

Given perturbed values $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \ldots, \tilde{f}_n$ of a differentiable function f
with $|\tilde{f}_i - f(x_i)| \leq \delta, \quad x_i = i\,h,\ i = 1, 2, \ldots, n$

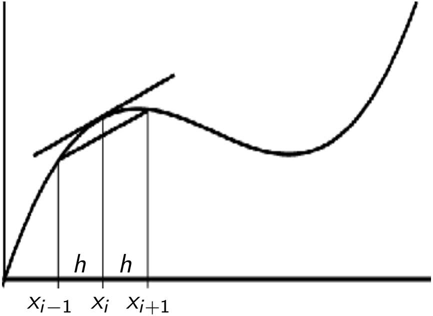
Find f'

secant approximation:

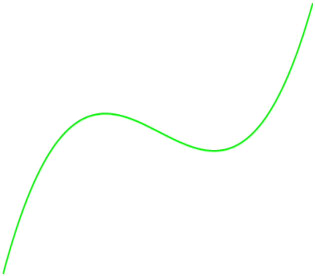
$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_{i-1})}{2h} =: f_h'(x_i)$$

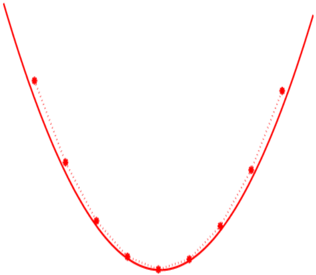
insert given measured data:

$$f'(x_i) \approx \frac{\tilde{f}_{i+1} - \tilde{f}_{i-1}}{2h} = \tilde{f}_h'(x_i)$$



exact data: secant approximation with $h = \frac{1}{10}$:





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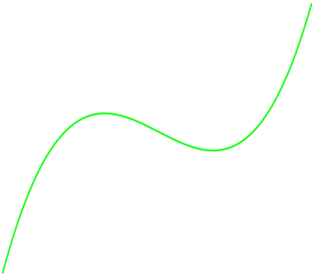

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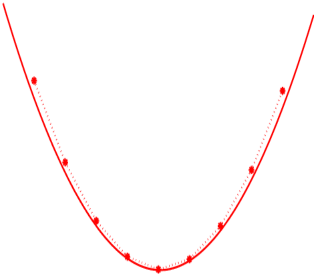
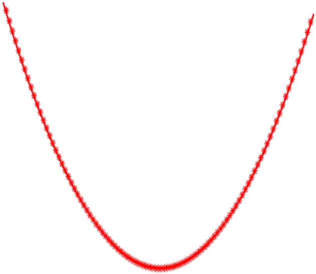
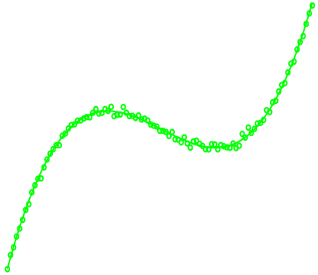


secant approximation with $h = \frac{1}{100}$:



perturbed data (1% noise): secant approximation with $h = \frac{1}{100}$:



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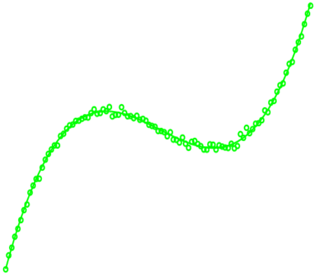

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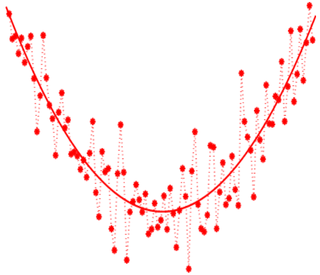

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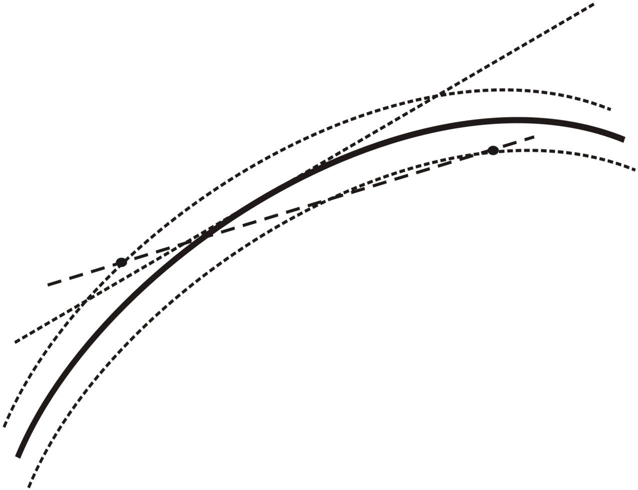

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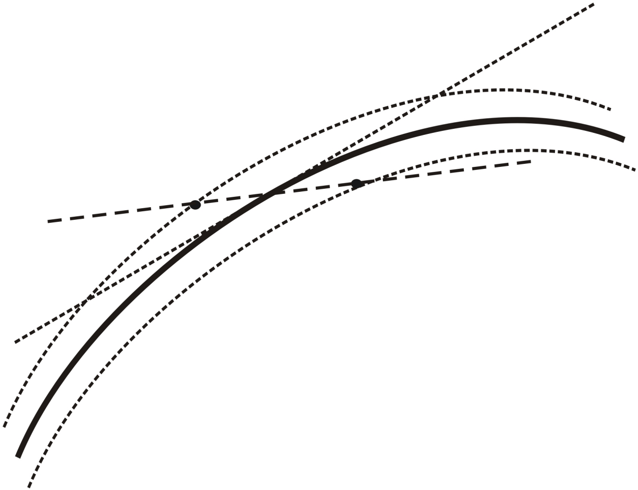

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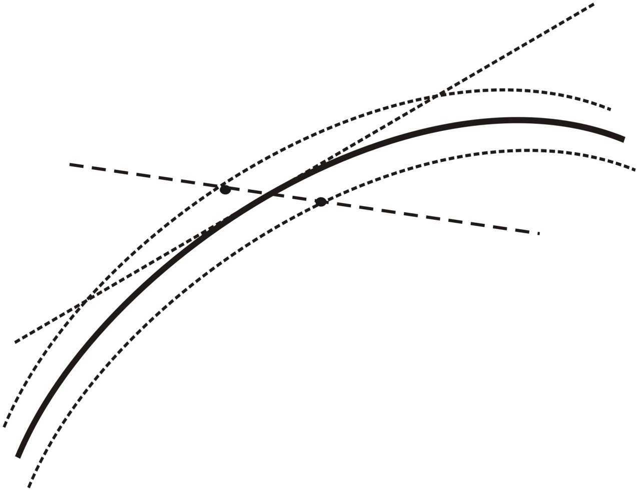

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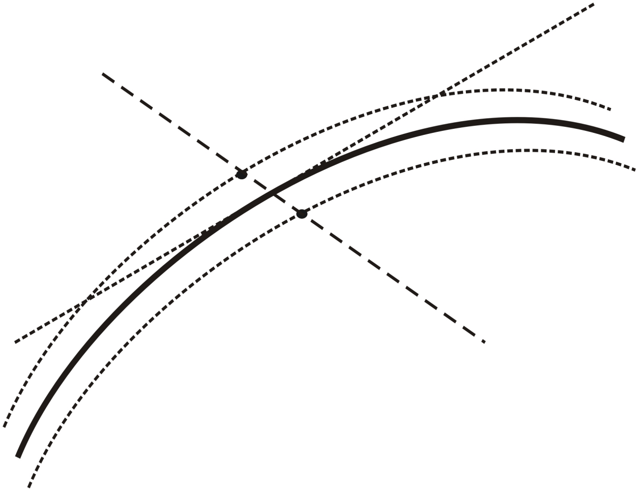

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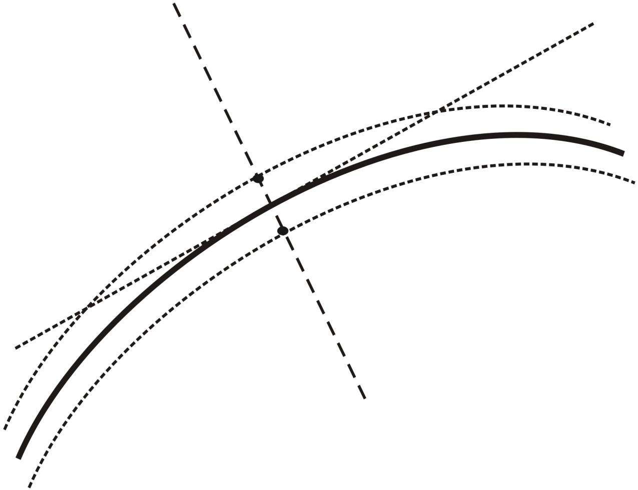

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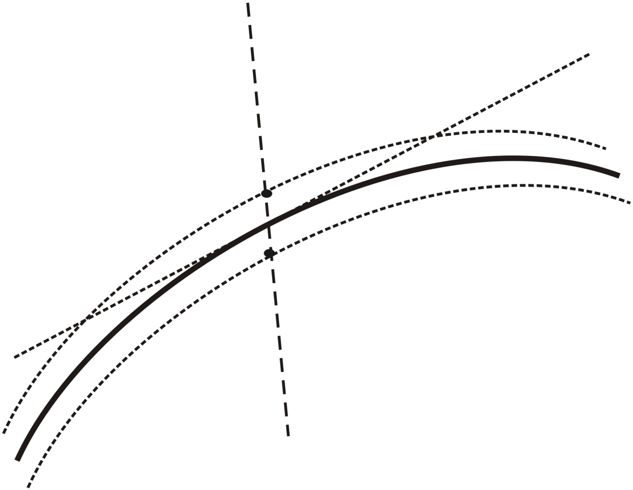

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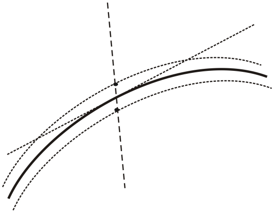
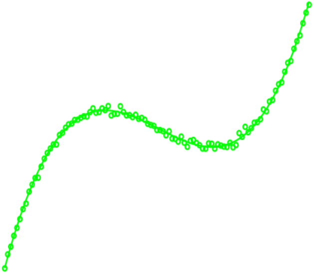


$$|\tilde{f}_h'(x_i) - f_h'(x_i)| = |\frac{\tilde{f}_{i+1} - \tilde{f}_{i-1}}{2h} - \frac{f_{i+1} - f_{i-1}}{2h}|$$
$$= \frac{1}{2h}|\underbrace{\tilde{f}_{i-1} - f_{i-1}}_{\leq\delta} - \underbrace{\tilde{f}_{i+1} - f_{i+1}}_{\geq-\delta}| \leq \frac{2\delta}{2h} = \frac{\delta}{h}$$

perturbed data (1% noise):



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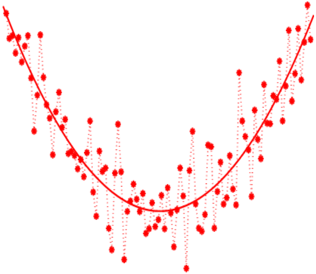
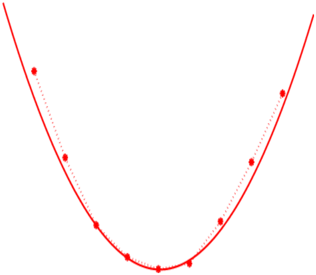


secant approximation with h_{opt}:



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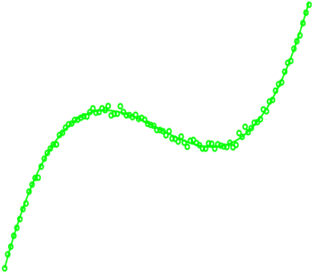

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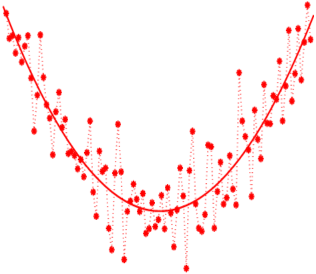

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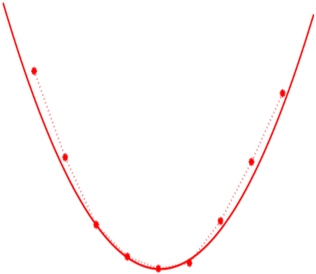

$\rightsquigarrow$ Regularization!

How large is the deviation from the exact derivative?

$$|\tilde{f}_h'(x_i) - f'(x_i)| \le \underbrace{|f_h'(x_i) - f'(x_i)|}_{\text{approximation error}} + \underbrace{|\tilde{f}_h'(x_i) - f_h'(x_i)|}_{\text{propagated noise}}$$

$$\xrightarrow{h\to 0} 0 \qquad\qquad \le \delta/h \xrightarrow{h\to 0} \infty \quad \to \text{instability}$$

Regularization:

choice of $h = h(\delta)$ such that $h(\delta) \xrightarrow{\delta\to 0} 0$ and $\delta/h(\delta) \xrightarrow{\delta\to 0} 0$

$\Rightarrow |\tilde{f}_h'(x_i) - f'(x_i)| \xrightarrow{\delta\to 0} 0$ d.h.: smaller noise $\Rightarrow$ better result

Parameter Identification in Differential Equations: Some Examples

- Identify spatially varying coefficients/source a, b, c in linear elliptic boundary value problem on $\Omega \subseteq \mathbb{R}^d$, $d \in \{1, 2, 3\}$

$$-\nabla(a\nabla u) + cu = b \text{ in } \Omega\,, \qquad \frac{\partial u}{\partial n} = j \text{ on } \partial\Omega\,,$$

from boundary or (restricted) interior observations of u.

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$$\dot{u}(t) = f(t, u(t), \vartheta) \; t \in (0, T)\,, \quad u(0) = u_0$$

  from discrete of continuous observations of u.
  $y_i = g_i(u(t_i))$, $i \in \{1, \dots, m\}$ or $y(t) = g(t, y(t))$, $t \in (0, T)$

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  applications in population dynamics, epidemology, combustion (parabolic); imaging with waves (hyperbolic),...