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Exercises "VU Finanzstatistik" (SS 2014)

15. Let X, Y and Z be continuous local martingales and $a, b \in \mathbb{R}$. Show the following properties of the covariation only with the help of its definition:
    a) $[X, Y] = [Y, X]$
    b) $[aX, bY] = ab[X, Y]$
    c) $[X + Y, Z] = [X, Z] + [Y, Z]$
    d) $[X - X_0, Z] = [X, Z]$
16. Let $M \in \mathcal{M}^2$ and $H \in \mathcal{P}_2(M)$. Show with the help of Theorem 2.18 that $\|H \cdot M\| = \|H\|_M$ (Ito-Isometry).
17. Let $h : [0, \infty) \to \mathbb{R}$ be a continuous mapping and $(B_t)_{t \geq 0}$ a Brownian motion. Show that $(h \cdot B)_t = \int_0^t h(s) dB_s$ defines a centered Gaussian process with covariance function $\Gamma(s, t) = \int_0^{\min\{s,t\}} h^2(u) du$.
18. For two continuous semimartingales X and Y, the Stratonovich integral is defined as $(X \circ Y)_t := (X \cdot Y)_t + \frac{1}{2}[X, Y]_t$, $t \geq 0$. Now, let X be a continuous semimartingale in $\mathbb{R}^d$ and $f \in C^3(\mathbb{R}^d)$, $d \in \mathbb{N}$. Show that

    $$f(X) = f(X_0) + \sum_{i=1}^{d} f^{(i)}(X) \circ X^i,$$

    where $f^{(i)}$ is the partial derivative of f with respect to its ith component and X^i is the ith component of X.
19. Let $(B_t)_{t \geq 0}$. Show that
    a) $\int_0^t s dB_s = tB_t - \int_0^t B_s ds$,
    b) $\int_0^t B_s^2 dB_s = \frac{1}{3} B_t^3 - \int_0^t B_s ds$.
20. Show that $X_t = \sin(B_t)$ solves the stochastic differential equation

    $$dX_t = -\frac{1}{2} X_t dt + \sqrt{1 - X_t^2} dB_t$$

    for all $t < \inf_s \{s > 0 : B_s \notin [-\pi/2, \pi/2]\}$.