CORRIGENDUM ON: A NOTE ON THE DICHOTOMY SPECTRUM

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My paper "A note on the dichotomy spectrum" (J. Difference Equ. Appl. 15, no. 10, 1021–1025 (2009)) contains a serious error. In fact, [Pötz09, Lemma 7] is wrong with the consequence that also the final [Pötz09, Thm. 8] on the ℓ₀-roughness of exponential dichotomies (EDs for short) for

\[(L) \quad x_{k+1} = A_k x_k \quad \text{for all } k \in \mathbb{Z}\]

does not hold.

Throughout this corrigendum, suppose \(A_k, B_k \in L(X), \ k \in \mathbb{Z}\) are bounded sequences of bounded linear operators on a Banach space \(X\) and we borrow our further notation and terminology from [Pötz09]. The faulty [Pötz09, Thm. 8] states that the dichotomy spectra \(\Sigma(A)\) for the linear difference equation \((L)\) and \(\Sigma(A+B)\) for the linear-homogeneously perturbed difference equation

\[(P) \quad x_{k+1} = [A_k + B_k] x_k \quad \text{for all } k \in \mathbb{Z}\]

are the same, provided the respective transition operators \(\Phi\) of both \((L)\) and \((P)\) satisfy the injectivity assumption \(N(\Phi(k,k-n)) = \{0\}\) for all \(k \in \mathbb{Z}, n \in \mathbb{N}\), and

\[\lim_{k \to \pm\infty} \|B_k\| = 0.\]

The dichotomy spectrum yielding the appropriate hyperbolicity concept for non-autonomous problems dates back to the pioneering work of Sacker & Sell [SS78]. Using a flexible perturbation result for linear skew-product flows, [SS78, Sects. 5–6] shows that \(\Sigma(A)\) depends upper-semicontinuously on perturbations of the right-hand side in \((L)\). Furthermore, in a finite-dimensional situation, the claimed invariance of \(\Sigma(A)\) under perturbations \(B_k \in \mathbb{R}^{d \times d}\) decaying to 0 is known for difference equations defined on semi-lines \(\mathbb{Z}^\pm_\kappa := \{-\infty, \kappa, \kappa + 1, \ldots\}\) or \(\mathbb{Z}^-_\kappa := \{-\kappa, -\kappa + 1, \ldots\}\) (we refer to [BG93, Thm. 2.3] for invertible coefficient matrices \(A_k \in \mathbb{R}^{d \times d}\)). In this sense, the dichotomy spectrum on semi-lines is essential spectrum.

When dealing with problems \((L)\) on the full line \(\mathbb{Z}\), nevertheless, this statement is not necessarily true. Indeed, the author realized the faultiness of [Pötz09, Thm. 8] while becoming aware of [Hen81, p. 235, Thm. 7.6.9]; the latter result precisely indicates that when passing over from \((L)\) to the perturbed equation \((P)\), point spectrum might occur. To explicitly falsify [Pötz09, Thm. 8] we need the subsequent characterization for EDs on \(\mathbb{Z}\):

**Lemma 1.** Let \(\kappa \in \mathbb{Z}\). Equation \((L)\) has an ED on \(\mathbb{Z}\) if and only if it admits EDs on both semi-lines \(\mathbb{Z}^\pm_\kappa\) with corresponding projectors \(P^\pm_k\) satisfying

\[
R(P^+_k) \oplus N(P^-_k) = X.
\]

**Proof.** Referring to [Bas00, Cor. 2.1] and [Hen81, p. 230, Thm. 7.6.5], EDs on both semi-lines \(\mathbb{Z}^+_\kappa\) and \(\mathbb{Z}^-_\kappa\) extend to the whole line \(\mathbb{Z}\) under (1). Conversely, for an ED on \(\mathbb{Z}\) the projector \(P^+_k\) is uniquely determined and clearly fulfills (1). \(\square\)
From this we obtain the following counterexample to [Pöt09, Thm. 8]:

**Example 1.** In $\mathbb{R}^2$ we consider a piecewise constant difference equation $(L)$ with

$$A_k := \begin{pmatrix} a_k & 0 \\ 0 & a_k^{-1} \end{pmatrix}, \quad a_k := \begin{cases} 2, & k \geq 0, \\ \frac{1}{2}, & k < 0. \end{cases}$$

Given $\gamma > 0$, its scaled counterpart

$$(L_\gamma) \quad x_{k+1} = \gamma^{-1} A_k x_k \quad \text{for all } k \in \mathbb{Z}$$

has the following dichotomy properties: If $\gamma \not\in \left(\frac{1}{2}, 2\right)$ then we have EDs on both semi-lines $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$ with constant projectors $P_+$ resp. $P_-$. They are given by

- $\gamma < \frac{1}{2}$: $P_+ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $P_- = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
- $\frac{1}{2} < \gamma < 2$: $P_+ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $P_- = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $2 < \gamma$: $P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $P_- = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

which yields the relations

$$R(P_+) \cap N(P_-) = \begin{cases} \{0\}, & \gamma < \frac{1}{2} \text{ or } 2 < \gamma, \\ \mathbb{R} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \frac{1}{2} < \gamma < 2, \end{cases}$$

$$R(P_+) + N(P_-) = \begin{cases} \mathbb{R}^2, & \gamma < \frac{1}{2} \text{ or } 2 < \gamma, \\ \mathbb{R} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \frac{1}{2} < \gamma < 2. \end{cases}$$

With Lemma 1 this shows that $(L_\gamma)$ admits an ED on $\mathbb{Z}$, if and only if $\gamma \not\in \left(\frac{1}{2}, 2\right]$, i.e. $(L)$ has the dichotomy spectrum $\Sigma(A) = \left[\frac{1}{2}, 2\right]$. We perturb $(L)$ with the matrix

$$B_k := \begin{pmatrix} 0 & b_k \\ 0 & 0 \end{pmatrix}, \quad b_k := \begin{cases} \left(\frac{1}{2}\right)^k, & k \geq 0, \\ 0, & k < 0. \end{cases}$$

satisfying $\lim_{k \to \pm \infty} \|B_k\| = 0$ even exponentially. Due to [BG93, Thm. 2.3] the scaled perturbed difference equation

$$(P_\gamma) \quad x_{k+1} = \gamma^{-1} [A_k + B_k] x_k \quad \text{for all } k \in \mathbb{Z}$$

admits an ED on the semi-line $\mathbb{Z}_0^+$ if and only if $(L_\gamma)$ has the same property. Using the general forward solution

$$\varphi_\gamma(k; 0, \xi, \eta) = \begin{pmatrix} \left(\frac{2}{\gamma}\right)^k \left(\xi + \frac{\xi}{\eta}\right) - \frac{4}{\gamma} \eta \left(\frac{1}{\gamma}\right)^k \\ \left(\frac{1}{\gamma}\right)^k \eta \end{pmatrix}$$

for all $k \in \mathbb{Z}_0^+$, $\xi, \eta \in \mathbb{R}$, of equation $(P_\gamma)$, the corresponding projector $\bar{P}_k^+$ for the ED of $(P_\gamma)$ on $\mathbb{Z}_0^+$ satisfies the relation (cf. [Pal88, Prop. 2.3(1)])

$$R(\bar{P}_k^+) = \left\{(\xi, \eta) \in \mathbb{R}^2 : \sup_{k \geq 0} \|\varphi_\gamma(k; 0, \xi, \eta)\| < \infty \right\} = \begin{cases} \{0\}, & \gamma < \frac{1}{2}, \\ \mathbb{R} \left(\frac{1}{\gamma}\right), & \frac{1}{2} < \gamma < 2, \\ \mathbb{R}^2, & 2 < \gamma. \end{cases}$$

Both difference equations $(L_\gamma)$ and $(P_\gamma)$ coincide on $\mathbb{Z}_{-1}$ and therefore the perturbed projector $\bar{P}_k^-$ for the ED of $(P_\gamma)$ on $\mathbb{Z}_{-1}$ satisfies $N(\bar{P}_k^-) = N(P_k^-)$. This
ED extends to the semi-line \( \mathbb{Z}^+ \) and the invariance \( N(\hat{P}_0^+) = A_{-1}N(P_-) \) implies \( R(\hat{P}_0^+) \oplus N(\hat{P}_0^+) = \mathbb{R}^2 \) for all \( \gamma \notin \{\frac{1}{2}, 2\} \). Using Lemma 1 we arrive at

\[
\Sigma(A + B) = \{\frac{1}{2}, 2\} \neq \Sigma(A).
\]

We point out that our \( \ell_0 \)-robustness result \([\text{Pö}t09, \text{Thm. 8}]\) fails due to the preparatory but erroneous \([\text{Pö}t09, \text{Lemma 7}]\). Its proof relies on the abstract Lemma 2.

For every \( A_k \in L(X), k \in \mathbb{Z} \), is invertible with \( \sup_{k \in \mathbb{Z}} \|A_k^{-1}\| < \infty \), then the essential spectrum \( \sigma_{\text{ess}}(T_A) \) of \( T_A \) satisfies \( \partial\sigma(T_A) \subseteq \sigma_{\text{ess}}(T_A) \subseteq \sigma(T_A) \).

Proof. Since \( \sigma(T_A) \) is rotationally symmetric w.r.t. \( 0 \in \mathbb{C} \), its only possible isolated spectral point is 0. Yet, our assumptions guarantee that \( T_A \in L(\ell^\infty) \) is invertible with bounded inverse \( (T_A^{-1}\psi)_k = A_k^{-1}\psi_{k+1} \) and in particular \( 0 \notin \sigma(T_A) \). This yields \( \sigma(T_A) = \emptyset \) and using \([\text{Har}88, \text{p. 371, Thm. 9.8.4}]\) it follows \( \partial\sigma(T_A) \setminus \sigma_{\text{ess}}(T_A) = \emptyset \), i.e. one has \( \partial\sigma(T_A) \subseteq \sigma_{\text{ess}}(T_A) \). The inclusion \( \sigma_{\text{ess}}(T_A) \subseteq \sigma(T_A) \) holds trivially. \( \square \)

Lemma 3. If \( B_k \in L(X), k \in \mathbb{Z} \), is a sequence of compact operators satisfying \( \lim_{k \to \pm \infty} \|B_k\| = 0 \), then also \( T_B \in L(\ell^\infty) \) is compact with \( \sigma(T_B) = \{0\} \).

Proof. For every \( n \in \mathbb{N} \) we define compact operators \( T_B^n \in L(\ell^\infty) \),

\[
(T_B^n \phi)_k := \begin{cases} B_{k-1}\phi_{k-1}, & |k-1| \leq n, \\ 0, & |k-1| > n. \end{cases}
\]

Thus, thanks to \( \|T_B - T_B^n\|_{L(\ell^\infty)} \leq \sup_{|k| > n} \|B_k\| \to 0 \) also the uniform limit \( T_B \) is compact (cf. \([\text{Yos}80, \text{p. 278, Thm. (iii)}]\)). Since the spectrum of compact operators consists of isolated points with zero as the only possible accumulation point (see \([\text{Yos}80, \text{p. 284, Thm. 2}]\)), the rotational invariance of \( \sigma(T_B) \) implies \( \sigma(T_B) = \{0\} \). \( \square \)

Using Lemma 2 and 3 we can establish an accurate counterpart to \([\text{Pö}t09, \text{Thm. 8}]\) under essentially two additional assumptions: First, \( (L) \) is supposed to have discrete dichotomy spectrum, which e.g. occurs for autonomous or periodic equations. Second, the coefficient operator of \( (L) \) and the perturbation sequence \( B_k \) need to commute. Precisely, we have

**Theorem 1.** (\( \ell_0 \)-roughness) Under the assumptions

(1) every \( A_k \in L(X), k \in \mathbb{Z} \), is invertible with \( \sup_{k \in \mathbb{Z}} \|A_k^{-1}\| < \infty \),

(2) every \( B_k \in L(X), k \in \mathbb{Z} \), is compact with \( \lim_{k \to \pm \infty} \|B_k\| = 0 \)

and \( \partial\Sigma(A) = \Sigma(A) \) the following holds:

(a) \( \Sigma(A) \subseteq \Sigma(A + B) \),

(b) if \( B_{k+1}A_k = A_{k+1}B_k, k \in \mathbb{Z} \), then \( \Sigma(A) = \Sigma(A + B) \).

Proof. (a) Referring to \([\text{Pö}t09, \text{Thm. 1}]\) we have \( \sigma(T_A) = \partial\sigma(T_A) \). Consequently, the above Lemma 2 guarantees

\[
\sigma(T_A) = \sigma_{\text{ess}}(T_A) = \sigma_{\text{ess}}(T_A + T_B) \subseteq \sigma(T_A + T_B) = \sigma(T_{A+B}),
\]

since \( T_B \) is compact due to Lemma 3 and compact perturbations leave the essential spectrum invariant (cf. \([\text{Kat}80, \text{p. 244, Thm. 5.35}]\)). Then again our \([\text{Pö}t09, \text{Thm. 1}]\) implies the claimed inclusion.
(b) Our assumption ensures that $T_A$ and $T_B$ commute. Hence, we obtain the inclusion $\sigma(T_{A+B}) = \sigma(T_A + T_B) \subseteq \sigma(T_A) + \sigma(T_B)$ (cf. [ARS94, Thm. 7.2]) and by means of Lemma 3 this in turn yields $\sigma(T_{A+B}) \subseteq \sigma(T_A)$. With [Pöt09, Thm. 1] we conclude $\Sigma(A + B) \subseteq \Sigma(A)$ and a combination with assertion (a) implies our claim. □

References


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