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Nonautonomous Dynamical Systems*

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*Draft: Corrections are welcome!

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Preface

The theory of dynamical systems is a well-developed and successful mathematical framework to describe time-varying phenomena. Its wide area of applications ranges from simple predator-prey models to complicated signal transduction pathways in biological cells, from the motion of a pendulum to complex climate models in physics, and beyond that to further fields as diverse as chemistry (reaction kinetics), economics, engineering, sociology or demography. In particular, this broad scope of applications has provided a significant impact on the theory of dynamical systems itself, and is one of the main reasons for its popularity over the last decades.

As a general principle, before abstract mathematical tools can be applied to real-world phenomena from the above areas, one needs corresponding models. From a conceptional level, in developing such models one distinguishes an actual dynamical *system* from its surrounding *environment*. The system is given in terms of physical or internal feedback laws yielding an evolutionary equation. The parameters in this equation describe the current state of the environment. The latter may or may not be variable, but is assumed to be unaffected by the system.

For classical *autonomous dynamical systems* the basic law of evolution is static in the sense that the environment does not change with time. However, in many applications such a static approach is too restrictive and a temporally fluctuating environment favorable:

- Parameters in real-world situations are rarely constant over time. This has various reasons, like absence of lab conditions, adaption processes, seasonal effects, changes in nutrient supply, or an intrinsic "background noise".
- On the other hand, sometimes it is desirable to include regulation or control strategies into a model (e.g. harvesting, dosing of drugs, stimulating chemicals or catalytic submissions) and to study their influence.

Consequently, in reasonable models adapted to and well-suited for problems in temporally fluctuating environments, the evolutionary equations have to depend explicitly on time. But also within the autonomous theory, time-dependent problems occur naturally: The investigation of a nonconstant solution ϕ to the autonomous ODE $\dot{x} = f(x)$ leads to the *equation of perturbed motion*

$$\dot{x} = f(x + \phi(t)) - f(\phi(t)),$$

which is clearly nonautonomous.

In order to study such realistic problems from applications as well as from mathematics, the classical theory of dynamical systems has to be extended. Indeed, one has to dismiss certain classical concepts known from the autonomous theory. This should be illustrated by the following examples:

Example 0.0.1. *The scalar autonomous ODE*

$$\dot{x} = f_1(x) := -x \tag{0.0a}$$

has the general solution $\varphi(t,\tau,\xi) = e^{\tau-t}\xi$ *for all* $\tau,\xi \in \mathbb{R}$ *and induces the flow* $\phi(t,\xi) = e^{-t}\xi$. *Its unique equilibrium* $x^* = 0$ *can be obtained from* $f_1(x^*) = 0$ *and* $Df_1(x^*) = -1 < 0$ *guarantees its asymptotic stability. Moreover, for every* $\xi \in \mathbb{R}$ *one has the invariant limit set* $\omega(\xi) = \{0\}$.

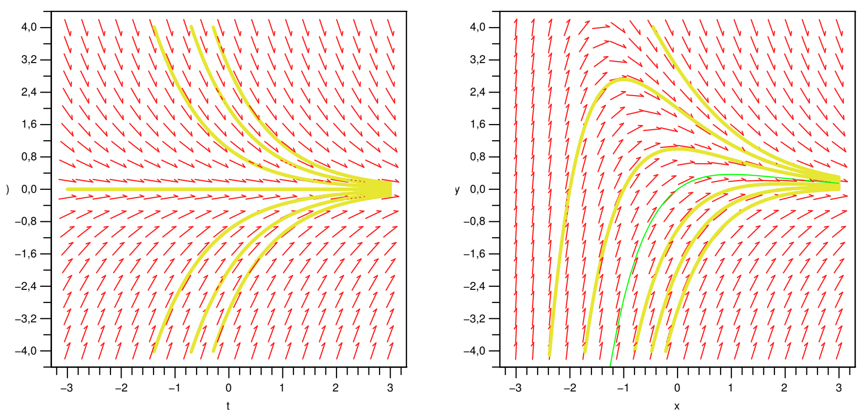

Fig. 0.1 Solution portraits of the equations (0.0a) (left) and (0.0b) (right)

Now we perturb (0.0a) *with an inhomogeneity decaying to* 0 *exponentially in time and obtain*

$$\dot{x} = f_2(t,x) := -x + e^{-t}. \tag{0.0b}$$

As opposed to (0.0a), *this equation is nonautonomous and has the general solution* $\varphi(t,\tau,\xi) = te^{-t} - e^{\tau-t}(\tau e^{-\tau} - \xi)$ *for all* $\tau,\xi \in \mathbb{R}$. *There are no equilibria (constant solutions) and the function* $x^*(t) = e^{-t}$ *obtained from* $f_2(t,x^*(t)) = 0$ *is obviously not a solution to* (0.0b). *Yet, every solution to* (0.0b) *is asymptotically stable. Due to the limit relation*

$$\lim_{t\to\infty} \varphi(t,\tau,\xi) = 0 \quad \textit{for all } \tau,\xi \in \mathbb{R}$$

a naive definition of (forward) limit sets yields $\omega(\tau,\xi)=\{0\}$. *This set, however, is by no means invariant w.r.t.* (0.0b).

The above example motivates two questions: (1) What is a nonautonomous counterpart of an equilibrium? (2) How can one define limit sets, which are invariant?

Furthermore, Ex. 0.0.1 allowed to determine stability properties using eigenvalues. In general, this is not possible, as the following periodic example illustrates:

Example 0.0.2. *We consider the linear ODE*

$$\dot{x}=A(t)x, \qquad A(t):=\begin{pmatrix}-1-2\cos(4t) & 2+2\sin(4t)\\ -2+2\sin(4t) & -1+2\cos(4t)\end{pmatrix}, \tag{0.0c}$$

whose coefficient matrix satisfies $\sigma(A(t))\equiv\{-1\}$ *for all* $t\in\mathbb{R}$. *This property, however, does not guarantee the (asymptotic) stability. In fact,* (0.0c) *is unstable, since it possesses the unbounded solution*

$$\phi(t)=e^t\begin{pmatrix}\sin(2t)\\ \cos(2t)\end{pmatrix}$$

The goal of this class is to tackle the above problems and to present adequate solutions.

München, July 29, 2011 Christian Pötzsche

Contents

Chapter 1
Basics

Young man, in mathematics you don't understand things. You just get used to them.
John von Neumann

We try to present the theory of discrete and continuous dynamical systems in a parallel fashion. Thereto, it is useful to introduce the notation $\mathbb{T}$ for the *time axis*, which is one of the sets $\mathbb{Z}$, $\mathbb{R}$ and we define its nonnegative part

$$\mathbb{T}_+ := \{t \in \mathbb{T} : t \geq 0\}.$$

Note that $\mathbb{T}$ and $\mathbb{T}_+$ are additive ordered semigroups.

1.1 Autonomous dynamical systems

In this section, we suppose X is a metric (or topological) space.

The solutions of autonomous ordinary differential equations

$$\dot{x} = f(x) \tag{1.1a}$$

and difference equations

$$x_{t+1} = f(x_t) \tag{1.1b}$$

are translational invariant in time. This means, given a solution ϕ, also the shifted function $\phi(\tau + \cdot)$ resp. sequence $\phi_{\tau+\cdot}$, $\tau \in \mathbb{T}$, is again a solution. Thus, it suffices to restrict to the initial time 0. An abstraction of this solution concept led to

Definition 1.1.1 (dynamical system)**.** A *semidynamical system* is a family of mappings $\phi_t : X \to X$, $t \in \mathbb{T}_+$, satisfying

(i) $\phi_0 = \mathrm{id}_X$ (initial value condition)

(ii) $\phi_{t+s} = \phi_t \phi_s$ for all $s, t \in \mathbb{T}_+$ (semigroup property)
(iii) $(t, x) \mapsto \phi_t(x)$ is continuous.

In case the above conditions hold with the semiaxis $\mathbb{T}_+$ replaced by $\mathbb{T}$ one speaks of a *dynamical system*, for a *continuous* system it is $\mathbb{T} = \mathbb{R}$ and for a *discrete* system one has $\mathbb{T} = \mathbb{Z}$.

Remark 1.1.2. *(1) We frequently use the convenient notation $\phi_t\phi_s = \phi_t \circ \phi_s$, although ϕ_t is not assumed to be linear. Similarly, we sometimes write $\phi(t, x) = \phi_t x$.*

(2) For dynamical systems, the mapping $\phi_t : X \to X$, $t \in \mathbb{T}$, is a homeomorphism with inverse $\phi_t^{-1} = \phi_{-t}$.

Example 1.1.3 (time shift)**.** *If $X = \mathbb{T}$, then $\phi_s t := t + s$ is a dynamical system on $\mathbb{T}$.*

Example 1.1.4 (continuous dynamical system)**.** *Let $X = \mathbb{R}^d$, suppose that $f : \mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz continuous and that all solutions to the autonomous ODE* (1.1a) *exist on $\mathbb{R}$; the latter assumptions hold for linearly bounded right hand sides f, i.e. there exist $a, b \geq 0$ with $\left\| f(x) \right\| \leq a + b\|x\|$ for all $x \in \mathbb{R}^d$ (cf. [Aul04]). If ϕ denotes the general (time-independent) solution to* (1.1a)*, which means that $\phi(\cdot, \xi)$ satisfies the initial condition $x(0) = \xi$, $\xi \in \mathbb{R}^d$, then $\phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a continuous dynamical system on $\mathbb{R}^d$.*

Example 1.1.5 (discrete dynamical system)**.** *Suppose $f : X \to X$ is a continuous mapping. The forward iterates*[1] *$\phi(t, \xi) = f^t(\xi)$, $t \geq 0$, i.e. the forward solutions to* (1.1b) *define a discrete semidynamical system $\phi : \mathbb{Z}_+ \times X \to X$. If f is a homeomorphism, then the iterates yield a discrete dynamical system $\phi : \mathbb{Z} \times X \to X$.*

Example 1.1.6 (cocycles)**.** *We consider autonomous ordinary differential resp. difference equations*

$$\begin{cases} \dot{p} = f(p), \\ \dot{x} = g(p, x) \end{cases} \qquad \begin{cases} p_{t+1} = f(p_t), \\ x_{t+1} = g(p_t, x_t) \end{cases} \tag{1.1c}$$

with variables $p \in \mathbb{R}^n$, $x \in \mathbb{R}^d$ and mappings f, g satisfying global existence and uniqueness of forward solutions. Then (1.1c) *generates a semidynamical system ϕ on $\mathbb{R}^n \times \mathbb{R}^d$ which can be written in component form as*

$$\phi(t, p_0, x_0) = \begin{pmatrix} \theta(t, p_0) \\ \lambda(t, p_0, x_0) \end{pmatrix}$$

with initial condition $\phi(0, p_0, x_0) = (p_0, x_0)$. We point out two important aspects in this formulation: First, the p-component in (1.1c) *generates an independent semi-*

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[1] the *iterates* of a mapping $f : X \to X$ are defined recursively via

$$f^0 := \mathrm{id}_X, \qquad f^{t+1} := f \circ f^t \quad \text{for all } t \in \mathbb{N}_0;$$

in case f is bijective, one sets $f^{-t} := (f^{-1})^t$ for $t \in \mathbb{N}_0$.

dynamical system θ on $\mathbb{R}^n$ of its own right; in particular it fulfills the semigroup property

$$\theta(t+s,p_0)=\theta(t,\theta(s,p_0)) \quad \text{for all } t,s\geq 0,\ p_0\in\mathbb{R}^n. \tag{1.1d}$$

Second, the semigroup property of ϕ can be represented as

$$\begin{aligned}\begin{pmatrix}\theta(s,\theta(t,p_0))\\ \lambda(t+s,p_0,x_0)\end{pmatrix} &\overset{(1.1d)}{=} \begin{pmatrix}\theta(t+s,p_0)\\ \lambda(t+s,p_0,x_0)\end{pmatrix}=\phi(t+s,p_0,x_0)=\phi(s,\phi(t,p_0,x_0))\\ &= \begin{pmatrix}\theta(s,\theta(t,p_0))\\ \lambda(s,\theta(t,p_0),\lambda(t,p_0,x_0))\end{pmatrix}\end{aligned}$$

yielding the cocycle property

$$\lambda(t+s,p_0,x_0)=\lambda(s,\theta(t,p_0),\lambda(t,p_0,x_0)) \quad \text{for all } s,t\geq 0,\ p_0\in\mathbb{R}^n,\ x_0\in\mathbb{R}^d.$$

Remark 1.1.7 (Warning!)**.** *One often encounters the remark that every nonautonomous equation $\dot{x}=g(t,x)$ resp. $x_{t+1}=g(t,x_t)$ can be written as an autonomous equation by considering t as a state space variable; this means one chooses $\dot{p}=1$ resp. $p_{t+1}=p_t+1$ in* (1.1c)*. From a dynamical systems point of view this approach is useless, as well as pointless for the following reasons:*

- *The resulting equation* (1.1c) *has no equilibria*
- *every solution to* (1.1c) *is unbounded*
- *thus, all the limit sets (and attractors) are empty.*

Example 1.1.8. *Let $X=BC(\mathbb{T},\mathbb{R}^d)$ denote the Banach space of all bounded continuous functions $x:\mathbb{T}\to\mathbb{R}^d$ equipped with the natural norm*

$$\|x\|:=\sup_{t\in\mathbb{T}}|x(t)|.$$

On the infinite-dimensional space X we define the shift operator

$$\phi_t x:=x(t+\cdot) \quad \text{for all } t\in\mathbb{T}$$

and obtain that $\phi_t:X\to X$ defines a dynamical system on X.

Further examples of continuous semidynamical systems on infinite-dimensional Banach spaces are the solution operators to functional and delay differential equations (it is $X=C[-1,0]$, cf. [HVL93]), or of semilinear parabolic equations (where X is an ambient Sobolev space, cf. [Hen81]).

1.2 Nonautonomous dynamical systems and examples

Let P and X be nonempty sets. We consistently use the notation

$$\mathscr{X}:=P\times X$$

and a subset $\mathscr{S} \subseteq \mathscr{X}$ is called *nonautonomous set* with *p-fiber* (see Fig. 1.1)

$$\mathscr{S}(p) := \{x \in X : (p,x) \in \mathscr{S}\} \quad \text{for all } p \in P.$$

Throughout these notes, calligraphic letters ($\mathscr{S}, \mathscr{X}$ and so on) denote nonautonomous sets. The *cartesian product* of two nonautonomous sets $\mathscr{S}_1, \mathscr{S}_2 \subseteq \mathscr{X}$ is defined to be the following set of triples

$$\mathscr{S}_1 \times \mathscr{S}_2 := \{(p, x_1, x_2) \in P \times X \times X : x_1 \in \mathscr{S}_1(p), x_2 \in \mathscr{S}_2(p)\}.$$

Accordingly, inclusions, intersections or unions of nonautonomous sets are defined fiber-wise. Sometimes it is convenient to identify a function $\mu : P \to X$ with its graph $\{(p, \mu(p)) : p \in P\}$.

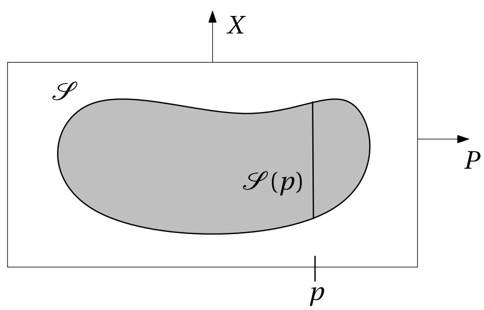

Fig. 1.1 A nonautonomous set $\mathscr{S}$ and the p-fiber $\mathscr{S}(p)$

If (X,d) is a metric space, then $\mathscr{S}$ is said to be *open, closed, compact* (or another topological property), if every fiber $\mathscr{S}(p)$, $p \in P$, is open, closed or compact etc., w.r.t. the topology given on X. We write $\mathrm{cl}\,\mathscr{S} := \{(p,x) \in \mathscr{X} : x \in \mathrm{cl}\,\mathscr{S}(p)\}$ for the *closure* of $\mathscr{S}$ and proceed analogously with the *interior* $\mathrm{int}\,\mathscr{S}$ or the *boundary* $\mathrm{bd}\,\mathscr{S}$. A nonautonomous set $\mathscr{S}$ in metric spaces X is called *bounded*, provided each fiber $\mathscr{S}(p)$, $p \in P$, is bounded. In metric linear spaces (X,d) we denote $\mathscr{X}$ as *uniformly bounded*, if there exists a $R > 0$ so that one has

$$\|\mathscr{S}(p)\| := \sup_{x \in \mathscr{S}(p)} d(x,0) \le R \quad \text{for all } p \in P.$$

For finite sets P the notions of a bounded and a uniformly bounded nonautonomous set coincide. A *neighborhood* of $\mathscr{S}$ is a nonautonomous set containing a so-called ε*-neighborhood*

$$\mathscr{B}_\varepsilon(\mathscr{S}) := \{(p,\xi) \in \mathscr{X} : \mathrm{dist}(\xi, \mathscr{S}(p)) < \varepsilon\}^2$$

with some given $\varepsilon > 0$; in metric linear spaces we abbreviate $\mathscr{B}_\rho := \mathscr{B}_\rho(0)$.

[2] the *distance* of a point $\xi \in X$ to a set $S \subseteq X$ is defined as $\mathrm{dist}(\xi, S) := \inf_{x \in S} d(\xi, x)$.

A *vector bundle*[3] is a nonautonomous set $\mathscr{S}$ with every fiber $\mathscr{S}(p)$, $p \in P$, being a linear space. For vector bundles $\mathscr{X}_1, \mathscr{X}_2 \subseteq \mathscr{X}$ we define

$$\mathscr{X}_1 + \mathscr{X}_2 := \left\{(p,x) \in \mathscr{X} : x \in \mathscr{X}_1(p) + \mathscr{X}_2(p)\right\},$$
$$\mathscr{X}_1 \oplus \mathscr{X}_2 := \left\{(p,x) \in \mathscr{X} : x \in \mathscr{X}_1(p) \oplus \mathscr{X}_2(p)\right\}$$

and the latter expression is denoted as *Whitney sum* of $\mathscr{X}_1$ and $\mathscr{X}_2$. Trivial examples of vector bundles are the *zero bundle* resp. the extended state space

$$\mathscr{O} := P \times \{0\}, \qquad \mathscr{X} = P \times X.$$

An abstract nonautonomous dynamical system consists of two components, a driving system modeling the nonautonomy and a so-called *cocycle* (see Fig. 1.2):

Definition 1.2.1 (nonautonomous dynamical system)**.** A *nonautonomous dynamical system* (NDS for short) is a pair of mappings (θ, λ) with the following properties:

(i) The *base flow* $\theta_t : P \to P$, $t \in \mathbb{T}$, satisfies the *group property*

$$\theta_0 = \mathrm{id}_P, \qquad \theta_{t+s} = \theta_t \theta_s \quad \text{for all } t, s \in \mathbb{T}, \tag{1.2a}$$

(ii) $\lambda : \mathbb{T}_+ \times P \times X \to X$ satisfies the *cocycle property*

$$\lambda(0,p) = \mathrm{id}_X, \qquad \lambda(t+s,p) = \lambda(t, \theta_s p)\lambda(s,p) \tag{1.2b}$$

for all $t, s \in \mathbb{T}_+$, $p \in P$. We denote the set X as *state space*, P as *base space* and $\mathscr{X} = P \times X$ as *extended state space*. If the semiaxis $\mathbb{T}_+$ can be replaced by $\mathbb{T}$ in (ii), one speaks of an *invertible* NDS.

Remark 1.2.2. *(1) If P is a singleton, X a topological space and λ is continuous in the last variable, then Def. 1.2.1 reduces to the usual definition of a semidynamical system (cf. Def. 1.1.1).*

(2) Given topological spaces P, X one speaks of a continuous NDS*, if the mapping $(t,x) \mapsto \lambda(t,p,x)$, $p \in P$, is continuous. Given a linear space X over $\mathbb{K}$, a* linear NDS *fulfills*

$$\lambda(t,p,\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \lambda(t,p,x_1) + \alpha_2 \lambda(t,p,x_2) \quad \textit{for all } \alpha_1, \alpha_2 \in \mathbb{K},\, x_1, x_2 \in X$$

and $t \in \mathbb{T}_+$, $p \in P$.

(3) For an invertible NDS one has $\lambda(t,p)^{-1} = \lambda(-t, \theta_t p)$ for all $t \in \mathbb{T}$, $p \in P$.

[3] We refer to [AMR88, p. 166, Definition 3.4.1] for the general notion of a *vector bundle* in differential topology.

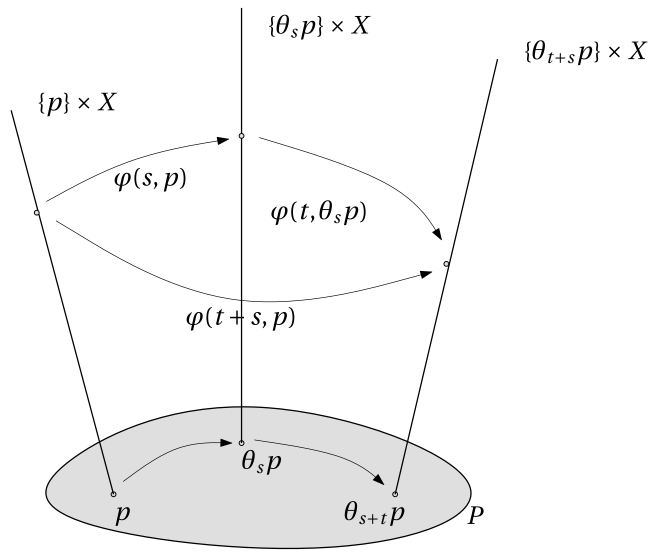

Fig. 1.2 A nonautonomous dynamical system (θ, λ)

Corollary 1.2.3 (skew-product flow)**.** *Let P, X be topological spaces. If the mappings θ and λ are continuous, then $\phi = (\theta, \lambda) : \mathbb{T}_+ \times P \times X \to P \times X$ is a semidynamical system on $P \times X$, a so-called* skew-product semiflow. *Accordingly, invertible NDSs generate* skew-product flows.

Proof. Let $p \in P$ and $x \in X$. Obviously, the mapping ϕ is continuous and satisfies

$$\phi(0, p, x) = \begin{pmatrix} \theta_0 p \\ \lambda(0, \theta_0 p, x) \end{pmatrix} \overset{(1.2a)}{=} \begin{pmatrix} p \\ \lambda(0, p, x) \end{pmatrix} \overset{(1.2b)}{=} \begin{pmatrix} p \\ x \end{pmatrix}.$$

On the other hand, one obtains

$$\begin{aligned} \phi(t, \phi(s, p, x)) &= \phi(t, \theta_s p, \lambda(s, p, x)) = \begin{pmatrix} \theta_t \theta_s p \\ \lambda(t, \theta_s p, \lambda(s, p, x)) \end{pmatrix} \\ &= \begin{pmatrix} \theta_{t+s} p \\ \lambda(t+s, p, x) \end{pmatrix} \overset{(1.2a)}{=} \phi(t+s, p, x) \quad \text{for all } t, s \in \mathbb{T}_+ \end{aligned}$$

and therefore the claim. □

In Ex. 1.1.6 we have seen one possibility to construct NDSs. Next we illustrate that the concept is in fact significantly broader:

1.2.1 Processes

Let $\mathbb{I}$ denote a $\mathbb{T}$-interval, i.e. the intersection of $\mathbb{T}$ with a real interval.

Among the several approaches to describe the dynamics of nonautonomous evolutionary equations, probably the most straight-forward one is given as

Definition 1.2.4 (process)**.** A family of mappings $\varphi(t,s) : X \to X$, $s,t \in \mathbb{I}$, $s \le t$ is called a *process* or a 2*-parameter semigroup* on X, if it satisfies

$$\varphi(\tau,\tau) = \mathrm{id}_X, \qquad \varphi(t,s)\varphi(s,\tau) = \varphi(t,\tau) \quad \text{for all } \tau \le s \le t. \tag{1.2c}$$

A 2*-parameter group* satisfies (1.2c) for all $\tau, s, t \in \mathbb{T}$.

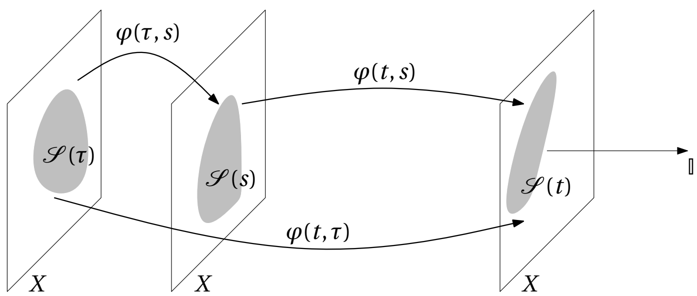

Fig. 1.3 2-parameter semigroup property of $\varphi(t,\tau) : X \to X$, $\tau \le t$

Remark 1.2.5. *For a* 2*-parameter group the mapping* $\varphi(t,\tau) : X \to X$ *is invertible with inverse* $\varphi(t,\tau)^{-1} := \varphi(\tau,t)$.

Example 1.2.6 (nonautonomous ODEs)**.** *Consider a nonautonomous ODE*

$$\dot{x} = f(t,x) \tag{1.2d}$$

with a right hand side $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ *guaranteeing global existence and uniqueness of solutions to initial value problems. Let* $\varphi(t,s,\xi)$ *denote the general solution to* (1.2d) *which starts at time* $s \in \mathbb{R}$ *in* $\xi \in \mathbb{R}^d$*, i.e.* $\varphi(s,s,\xi) = \xi$*. Then the mapping* $\varphi(t,s,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ *defines a process on* $\mathbb{R}^d$ *with* $\mathbb{T} = \mathbb{R}$.

From this, an invertible NDS can be constructed as follows: We choose the time axis $\mathbb{T} = \mathbb{R}$*, the base space* $P = \mathbb{R}$ *and the time shift* $\theta_t : \mathbb{R} \to \mathbb{R}$, $\theta_t p := t + p$ *as base flow. If* $\lambda(t,p,x) := \varphi(t+p,p,x)$*, then the pair* $\phi = (\theta,\lambda)$ *is a NDS.*

It has the additional properties that (θ,λ) *is continuous (even differentiable, if* f *is differentiable); however, the base space* $P = \mathbb{R}$ *is not compact. In case* (1.2d) *is* *periodic, i.e.* $f(t+2\pi,\cdot) = f(t,\cdot)$ *for all* $t \in \mathbb{R}$*, one can choose* $P = \mathbb{S}^1$ *and obtains a compact base space.*

Example 1.2.7 (nonautonomous difference equations)**.** *Let* X *be a set. An analogous construction is possible for nonautonomous difference equations*

$$x_{t+1} = f_t(x_t) \tag{1.2e}$$

with a right hand side $f_t : X \to X$, $t \in \mathbb{Z}$*. Here, the general solution can be constructed explicitly by means of the composition*

$$\varphi(t,s) := \begin{cases} \mathrm{id}_X, & t = s, \\ f_{t-1} \circ \ldots \circ f_s, & s < t, \end{cases}$$

which defines a process on X*. It induces a NDS with cocycle* $\lambda(t,p) := \varphi(t+p,p)$ *on* X *over the time shift* $\theta_t : P \to P$, $\theta_t p := t+p$, $t \in \mathbb{Z}_+$*, with noncompact base space* $P = \mathbb{Z}$*. On ambient spaces (topological, Banach), smoothness properties of* f_t *carry over to* λ*.*

Examples of processes on infinite-dimensional Banach spaces are the general (forward) solutions of nonautonomous functional differential equations (with state space $X = C[-r,0]$) or of evolutionary partial differential equations.

1.2.2 Nonautonomous differential and difference equations

Besides the process formulation, there is another possibility for (1.2d) or (1.2e) to generate a NDS. The corresponding construction is denoted as *Bebutov flow* and based on the fact that whenever x is a solution, then the shifted solution $x^\tau := x(\cdot+\tau)$ resp. $x^\tau := x_{\cdot+\tau}$ for some fixed $\tau \in \mathbb{T}$, satisfies the respective nonautonomous equation

$$\dot{x}^\tau = f^\tau(t,x_\tau) := f(t+\tau, x(t+\tau)), \qquad x^\tau_{t+1} = f^\tau(t,x_\tau) := f_{t+\tau}(x_{t+\tau}).$$

With given continuous right hand sides $f : \mathbb{T} \times \mathbb{R}^d \to \mathbb{R}^d$ (in case $\mathbb{T} = \mathbb{Z}$ we write $f(t,x) := f_t(x)$) we define the *hull* of f as follows

$$H(f) := \mathrm{cl}\left\{ f(t+\cdot,\cdot) : t \in \mathbb{T} \right\} \subseteq C(\mathbb{T} \times \mathbb{R}^d, \mathbb{R}^d),$$

where the closure is taken w.r.t. an ambient topology; a comparison of different appropriate topologies is given in [Sel71]. One example is the uniform convergence on compact sets given by the metric

$$d(f,g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(f,g)}{1+p_n(f,g)}, \quad p_n(f,g) := \sup_{(t,x)\in[-n,n]\times B_n(0)} \left| f(t,x) - g(t,x) \right|.$$

Choosing $P := H(f)$ the *Bebutov flow* reads as $\theta_t : H(f) \to H(f)$,

$$\theta_t p := p(t + \cdot, \cdot)$$

and can be shown to be continuous. If $\lambda(\cdot, p, x)$ denotes the solution to $\dot{x} = p(t, x)$ resp. $x_{t+1} = p_t(x_t)$ with $\lambda(0, p, x) = x$ and $p \in H(f)$, then (θ, λ) defines a NDS.

For right hand sides f being continuous and bounded, uniformly continuous on every set $\mathbb{T} \times K$, $K \subseteq \mathbb{R}^d$ compact, one obtains the *additional structure*: P is compact, θ is continuous and λ is continuous. Moreover, $\lambda(t, p, \cdot)$ inherits its smoothness from the mapping f.

Example 1.2.8 (periodic difference equations)**.** *For T-periodic difference equations* (1.2e)*, $T \in \mathbb{N}$, i.e.*

$$f_{t+T} = f_t \quad \textit{for all } t \in \mathbb{Z}$$

the hull $H(f)$ is finite $H(f) = \{f_0, \dots, f_{T-1}\}$ and the base space therefore compact.

If more is known about the explicit time-dependence of a nonautonomous difference equation (1.2e), a compact base space can be obtained as follows:

Example 1.2.9 (finitely many right hand sides)**.** *Suppose that the right hand sides f_t in* (1.2e) *are chosen in some (random) way from a finite number of continuous mappings $g_i : \mathbb{R}^d \to \mathbb{R}^d$, $i \in \{1, \dots, r\}$. Then* (1.2e) *can be represented as*

$$x_{t+1} = g_{i_t}(x_t)$$

with $i_t \in \{1, \dots, r\}$ for all $t \in \mathbb{Z}$. This difference equation generates a discrete NDS over the base space $P := \{1, \dots, r\}^{\mathbb{Z}}$ of bi-infinite sequences $p = (p_t)_{t \in \mathbb{Z}}$ in $\{1, \dots, r\}$. The base flow is given by

$$\theta_s p = \theta_s (p_t)_{t \in \mathbb{Z}} = (p_{t+s})_{t \in \mathbb{Z}}$$

and the cocycle mapping reads as

$$\lambda(t, p, x) := \begin{cases} x & t = 0, \\ g_{p_{t-1}} \circ \ldots \circ g_{p_0}(x) & t > 0 \end{cases}$$

for all $t \in \mathbb{Z}_+$, $x \in \mathbb{R}^d$ and sequences $p = (p_t)_{t \in \mathbb{Z}} \in P$. The base space P is a compact metric space w.r.t.

$$d(p) := \sum_{t \in \mathbb{Z}} (r+1)^{-|t|} \left| p_t - q_t \right|$$

and (θ, λ) is continuous.

1.2.3 Delay differential equations

Let $r \geq 0$. We consider a delay differential equation

$$\dot{x}(t) = f(t, x(t), x(t - r))$$

in $\mathbb{R}^d$ with a continuous right hand side being of class C^2 in the second and third argument. If we assume that f is bounded and uniformly continuous on every set of the form $\mathbb{R}\times K\times K$, $K\subseteq\mathbb{R}^d$ compact, then the hull $P := H(f)$ becomes compact w.r.t. the topology of uniform convergence on compact sets.

Let $X := C([-r,0],\mathbb{R}^d)$. By the standard theory of delay differential equations (DDEs for short, cf. [HVL93]), for every initial function $u_0\in X$ and every $g\in H(f)$ a DDE $\dot{x}(t) = g(t,x(t),x(t-r))$ admits a unique solution $\varphi(\cdot,g,u_0):\mathbb{R}\to X$ satisfying $\varphi(t,g,u_0)\equiv u_0(t)$ on $[-r,0]$. We now define

$$\varphi_t(g,u_0)(s) := \varphi(t+s,g,u_0) \quad \text{for all } t\geq 0,\ s\in[-r,0]$$

and obtain that φ_t is Lipschitz in the first and C^1 in the second argument. With the cocycle $\lambda(t,p,u_0) := \varphi_t(p,u_0)$ and the base flow

$$\theta_t : H(f)\to H(f), \qquad\qquad \theta_t g := g(t+\cdot,\cdot,\cdot);$$

this yields a nonautonomous dynamical system (θ,λ) on X.

Additional structure: P is compact, θ is continuous and the cocycle λ is continuous and continuously differentiable in the second argument.

1.2.4 Random differential and difference equations

Suppose that $(\Omega,\mathscr{F},\mathbb{P})$ is a probability space with sample space Ω, σ-algebra $\mathscr{F}$ and probability measure $\mathbb{P}:\mathscr{F}\to[0,1]$. A *metric dynamical system* is a family of mappings $\theta_t:\Omega\to\Omega$, $t\in\mathbb{T}$, with the properties

- $\theta_0 = \mathrm{id}$ and $\theta_t\circ\theta_s = \theta_{s+t}$ for all $s,t\in\mathbb{R}$,
- $(t,\omega)\mapsto\theta_t\omega$ is measurable,
- θ_t is $\mathbb{P}$-invariant, i.e., $\mathbb{P}(\theta_t B) = \mathbb{P}(B)$ for all $t\in\mathbb{T}$ and $B\in\mathscr{F}$.

With a metric dynamical system θ_t one denotes

$$\dot{x} = f(\theta_t\omega,x), \qquad\qquad x_{t+1} = f(\theta_t\omega,x_t)$$

as random differential resp. random difference equation. The corresponding solution satisfying the initial condition $x(0)=\xi$ resp. $x_0=\xi$ will be denoted by $t\mapsto\lambda(t,\omega)\xi$ and this defines a NDS (θ,λ) with the base space $P=\Omega$.

Additional structure: P is a probability space, the cocycle λ is measurable and $\lambda(\cdot,p)\xi:\mathbb{T}\to\mathbb{R}^d$ is absolutely continuous (cf. [Arn98]).

1.2.5 Control systems

A continuous *control system* is an ODE of the form

$$\dot{x} = f_0(x) + \sum_{j=1}^{n} u_j(t) f_j(x) \tag{1.2f}$$

with C^∞-functions $f_0, \dots, f_n : \mathbb{R}^d \to \mathbb{R}^d$ and control functions

$$u = (u_1, \dots, u_n) \in \mathscr{U} := \left\{u : \mathbb{R} \to U \text{ locally integrable}\right\} \subseteq L^\infty,$$

where $U \subseteq \mathbb{R}^n$ is compact and convex. For every control function $u \in \mathscr{U}$ and every initial value $\xi \in \mathbb{R}^d$ let $\lambda(\cdot, u, \xi)$ denote the unique solution of (1.2f) satisfying the initial condition $\lambda(0, u, \xi) = \xi$.

We define $\mathbb{T} = \mathbb{R}$, equip the base space $P = \mathscr{U}$ with the weak*-topology[4] and obtain a continuous base flow $\theta_t u := u(t + \cdot)$. Then (θ, λ) becomes a NDS on $\mathbb{R}^d$.

Additional structure: $P = \mathscr{U}$ is a compact separable metric space, θ is continuous and topologically transitive, and the cocycle λ is continuous (cf. [CK99]).

Exercises

Exercise 1.2.10. *Verify Rem. 1.2.2(3).*

1.3 Invariant and limit sets

In this section we introduce a number of notions which are important to describe and understand the asymptotic behavior of 2-parameter semigroups (processes). Such notions include invariant and limit sets. Let X denote a nonempty set and $\mathbb{I}$ be a $\mathbb{T}$-interval with associated extended state space $\mathscr{X} = \mathbb{I} \times X$. Throughout the section we assume that φ is a 2-parameter semigroup.

Definition 1.3.1 (invariant)**.** A nonautonomous set $\mathscr{A} \subseteq \mathscr{X}$ is called

(a) *forward invariant*, if and only if $\varphi(t,\tau)\mathscr{A}(\tau) \subseteq \mathscr{A}(t)$ for all $\tau < t$,
(b) *backward invariant*, if and only if $\mathscr{A}(t) \subseteq \varphi(t,\tau)\mathscr{A}(\tau)$ for all $\tau < t$,
(c) *invariant*, if and only if $\varphi(t,\tau)\mathscr{A}(\tau) = \mathscr{A}(t)$ for all $\tau < t$.

Example 1.3.2. *(1) The empty set $\mathbb{I} \times \emptyset$ and $\mathscr{X}$ are (forward, backward) invariant.*

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[4] this topology is metrizable with the metric

$$d(u,v) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\int_{\mathbb{R}} \langle u(t) - v(t), x_n(t)\rangle\, dt}{1 + \int_{\mathbb{R}} \langle u(t) - v(t), x_n(t)\rangle\, dt},$$

where $\{x_n\}_{n \in \mathbb{N}}$ is a countable dense subset of $L^1(\mathbb{R}, \mathbb{R}^d)$

(2) A function $\phi : \mathbb{I} \to X$ in $\mathcal{X}$ is called entire motion *(for φ), if one has the relation $\phi(t) = \varphi(t,\tau)\phi(\tau)$ for all $t \geq \tau$. Then the nonautonomous set ϕ is invariant. Since we did not assume $\varphi(t,\tau)$ to be onto, there might not exist an entire motion satisfying $\phi(\tau) = \xi$ for given $(\tau,\xi) \in \mathcal{X}$. Moreover, without the assumption that $\varphi(t,\tau)$ is one-to-one, there can exist more than one entire motion with $\phi(\tau) = \xi$.*

(3) Let $T \in \mathbb{T}_+$, $\mathbb{I}$ unbounded below and $\mathcal{A} \subseteq \mathcal{X}$ be a nonautonomous set. The nonautonomous set $\gamma^T_{\mathcal{A}} \subseteq \mathcal{X}$ given by the fibers

$$\gamma^T_{\mathcal{A}}(t) := \bigcup_{s \geq T} \varphi(t,t-s)\mathcal{A}(t-s) \subseteq X \quad \text{for all } t \in \mathbb{I}$$

*is called T-*truncated orbit *and $\gamma^0_{\mathcal{A}}$ the* orbit *of $\mathcal{A}$. One has the embedding*

$$\mathcal{A} \subseteq \gamma^0_{\mathcal{A}}, \qquad \gamma^{T_2}_{\mathcal{A}} \subseteq \gamma^{T_1}_{\mathcal{A}} \quad \text{for all } T_1 \leq T_2$$

and it is easily seen using Def. 1.3.1 that $\gamma^T_{\mathcal{A}}$ is (forward, backward) invariant, if $\mathcal{A}$ has the corresponding property.

The next result states that invariant sets consist of entire motions, which is basically due to the fact that 2-parameter semigroups are onto between the fibers of an invariant set. These motions, however, need not to be uniquely determined, since the semigroups are not assumed to be one-to-one.

Proposition 1.3.3. *The following assertions are equivalent:*

(a) A nonautonomous set $\mathcal{A} \subseteq \mathcal{X}$ is invariant,
(b) $\mathcal{A}$ is forward and backward invariant,
(c) for every pair $(\tau,\xi) \in \mathcal{A}$ there exists an entire motion $\phi : \mathbb{I} \to X$ for φ such that $\phi(\tau) = \xi$ and $\phi \subseteq \mathcal{A}$.

The entire motion ϕ is uniquely determined, provided every mapping

$$\varphi(t,s) : X \to X \quad \text{is one-to-one for all } s < t. \tag{1.3a}$$

Proof. The implication (b)⇒(a) is clear from Def. 1.3.1.

(a)⇒(c) Let $\mathcal{A}$ be invariant and choose $(\tau,\xi) \in \mathcal{A}$. For $t \geq \tau$ we define the function $\phi(t) := \varphi(t,\tau)\xi$ and the invariance of $\mathcal{A}$ yields $\phi(t) \in \mathcal{A}(t)$. On the other hand, for $\tau \geq t$ we have $\mathcal{A}(\tau) = \varphi(\tau,t)\mathcal{A}(t)$ and consequently there exist points $x_t \in \mathcal{A}(t)$ with $\xi = \varphi(\tau,t)x_t$. Thus, we define $\phi(t) := x_t$ for $t < \tau$ and $\phi : \mathbb{I} \to X$ is an entire motion with the desired properties. Under (1.3a) the sequence x_t is uniquely given.

(c)⇒(b) For arbitrary pairs $(\tau,\xi) \in \mathcal{A}$ there is an entire motion $\phi : \mathbb{I} \to X$ with $\phi(\tau) = \xi$ in $\mathcal{A}$. Hence, one has $(t,\varphi(t,\tau)\xi) = (t,\varphi(t,\tau)\phi(\tau)) = (t,\phi(t)) \in \mathcal{A}$ and thus the inclusion $\varphi(t,\tau)\xi \in \mathcal{A}(t)$ for $t \geq \tau$. So $\mathcal{A}$ is forward invariant. The backward invariance of $\mathcal{A}$ follows from $\xi = \varphi(\tau,t)\phi(t) \in \varphi(\tau,t)\mathcal{A}(t)$ for $t \leq \tau$. □

Proposition 1.3.4. *Let $\{\mathscr{A}_i\}_{i\in I}$ be a family of nonautonomous sets $\mathscr{A}_i \subseteq \mathscr{X}$, where I is an index set.*

(a) If each $\mathscr{A}_i$, $i \in I$, is forward invariant, then also the union $\bigcup_{i\in I} \mathscr{A}_i$ and the intersection $\bigcap_{i\in I} \mathscr{A}_i$ are forward invariant,
(b) if each $\mathscr{A}_i$, $i \in I$, is backward invariant, then also $\bigcup_{i\in I} \mathscr{A}_i$ is backward invariant; moreover, under (1.3a) *also $\bigcap_{i\in I} \mathscr{A}_i$ is backward invariant.*

Proof. The whole proof is based on the elementary relations

$$\varphi(t,s)\left(\bigcap_{i\in I} \mathscr{A}_i(t)\right) \subseteq \bigcap_{i\in I} \varphi(t,s)\mathscr{A}_i(t), \qquad \varphi(t,s)\left(\bigcup_{i\in I} \mathscr{A}_i(t)\right) = \bigcup_{i\in I} \varphi(t,s)\mathscr{A}_i(t)$$

for all $s < t$, with equality in the first case, if $\varphi(t,s)$ is one-to-one. □

We now postulate that (X,d) is a metric space and define the *distance* of a point $x \in A$ from a subset $A \subseteq X$ by

$$\operatorname{dist}(x,A) := \inf_{a\in A} d(x,a),$$

as well as the *Hausdorff semidistance* of two subsets $A, B \subset X$ by

$$\operatorname{dist}(A,B) := \sup_{a\in A} \operatorname{dist}(a,B) = \sup_{a\in A} \inf_{b\in B} d(a,b).$$

Lemma 1.3.5. *Let X be a complete metric space. If $(B_s)_{s\geq 0}$ is a decreasing nested family of nonempty compact subsets, then their intersection*

$$B_* := \bigcap_{s\geq 0} B_s$$

is nonempty, compact and satisfies $\lim_{s\to\infty} \operatorname{dist}(B_s, B_*) = 0$.

Proof. See [Zei93, p. 495, Prop. 11.4]. □

Moreover, which might be surprising at first reading, we assume the $\mathbb{T}$-interval $\mathbb{I}$ is unbounded below, i.e. of the form $\mathbb{I} = (-\infty, \tau] \cap \mathbb{T}$ with some $\tau \in \mathbb{T}$ or $\mathbb{I} = \mathbb{R} \cap \mathbb{T}$.

Theorem 1.3.6. *Suppose that X is a complete metric space, φ is continuous and let $\mathscr{A} \subseteq \mathscr{X}$ be nonempty compact. If $\mathscr{A}$ is forward invariant, then there exists a nonempty compact and invariant subset $\mathscr{A}_* \subseteq \mathscr{A}$.*

Proof. We keep $t \in \mathbb{I}$ fixed. Let $\mathscr{A}$ be forward invariant. Since $\mathscr{A}$ is compact, the continuity of φ shows that the images $\varphi(t,s)\mathscr{A}(s)$, $s \leq t$, are compact. Moreover, thanks to (cf. Def. 1.3.1(a))

$$\varphi(t,\tau)\mathscr{A}(\tau) \overset{(1.2c)}{=} \varphi(t,r)\varphi(r,\tau)\mathscr{A}(\tau) \subseteq \varphi(t,r)\mathscr{A}(r) \subseteq \mathscr{A}(t) \quad \text{for all } \tau \le r \le t \quad (1.3b)$$

the sets $\varphi(t,t-s)\mathscr{A}(t-s)$, $s \in \mathbb{T}_+$, form a nested family of nonempty compact subsets of $\mathscr{A}(t)$. Referring to Lemma 1.3.5 their intersection fiber-wise given by $\mathscr{A}_*(t) := \bigcap_{s\ge 0} \varphi(t,t-s)\mathscr{A}(t-s)$ is nonempty and compact as well, and it remains to show that $\mathscr{A}_*$ is also invariant:
($\subseteq$) For $r \le t$ we choose $a \in \mathscr{A}_*(r)$ and get $a \in \varphi(r,r-s)\mathscr{A}(r-s)$, $s \ge 0$. Thus,

$$\varphi(t,r)\{a\} \subseteq \varphi(t,r)\varphi(r,r-s)\mathscr{A}(r-s) \overset{(1.2c)}{=} \varphi(t,r-s)\mathscr{A}(r-s) \quad \text{for all } s \ge 0$$

and (1.3b) imply the inclusion

$$\varphi(t,r)\{a\} \subseteq \bigcap_{0\le s} \varphi(t,r-s)\mathscr{A}(r-s) = \bigcap_{0\le s} \varphi(t,t-s)\mathscr{A}(t-s) = \mathscr{A}_*(t)$$

for all $r \le t$ and we deduce the desired relation $\varphi(t,r)\mathscr{A}_*(r) \subseteq \mathscr{A}_*(t)$ for all $r \le t$.
($\supseteq$) Given an arbitrary $a \in \mathscr{A}_*(t)$ and $r \le t$ there exists a real sequence $r_n \le r$ with $\lim_{n\to\infty} r_n = -\infty$ and $a \in \varphi(t,r_n)\mathscr{A}(r_n) = \varphi(t,r)\varphi(r,r_n)\mathscr{A}(r_n)$. Therefore, we can choose $b_n \in \varphi(r,r_n)\mathscr{A}(r_n) \subseteq \mathscr{A}(r)$ such that $\varphi(t,r)b_n = a$. Since $\mathscr{A}(r)$ is compact there exists a convergent subsequence $(b_{n_j})_{j\in\mathbb{N}}$ with limit $b \in \mathscr{A}(r)$. By

$$\begin{aligned}\operatorname{dist}(b,\mathscr{A}_*(r)) &\le d(b,b_{n_j}) + \operatorname{dist}(b_{n_j},\mathscr{A}_*(r))\\ &\le d(b,b_{n_j}) + \operatorname{dist}(\varphi(r,r_{n_j})\mathscr{A}(r_{n_j}),\mathscr{A}_*(r)) \xrightarrow[j\to\infty]{} 0\end{aligned}$$

(cf. Lemma 1.3.5) we deduce $b \in \mathscr{A}_*(r)$ and the continuity of φ shows the limit relation $a = \lim_{j\to\infty} \varphi(t,r)b_{n_j} = \varphi(t,r)b$. This means $\mathscr{A}_*(t) \subseteq \varphi(t,r)\mathscr{A}_*(r)$. □

Now we are interested in the asymptotic behavior of 2-parameter semigroups.

Definition 1.3.7 (limit set)**.** Let $\mathscr{A} \subseteq \mathscr{X}$ be nonempty. The *ω-limit set* $\omega_{\mathscr{A}}$ of $\mathscr{A}$ is the nonautonomous set given by the fibers

$$\omega_{\mathscr{A}}(t) := \bigcap_{T\ge 0} \operatorname{cl} \bigcup_{s\ge T} \varphi(t,t-s)\mathscr{A}(t-s) = \bigcap_{T\ge 0} \operatorname{cl} \gamma^T_{\mathscr{A}}(t) \quad \text{for all } t \in \mathbb{I}.$$

Remark 1.3.8. *(1) In a dual fashion using pre-images, the* α-limit set $\alpha_{\mathscr{A}} \subseteq \mathscr{X}$ *of* $\mathscr{A}$ *can be defined as nonautonomous set given by the fibers*

$$\alpha_{\mathscr{A}}(t) := \bigcap_{T\ge 0} \operatorname{cl} \bigcup_{s\ge T} \varphi(t+s,t)^{-1}\mathscr{A}(t+s) \quad \textit{for all } t \in \mathbb{I}$$

with $\varphi(t+s,t)^{-1}\mathscr{A}(t+s) = \left\{x \in X : \varphi(t+s,t)x \in \mathscr{A}(t+s)\right\}$*. Without surjectivity assumptions on* $\varphi(t,r)$*,* $r < t$*,* α*-limit sets can be empty.*

(2) If $\mathscr{A}$ *is a forward invariant (backward invariant resp. invariant) set, then* $\omega_{\mathscr{A}} \subseteq \operatorname{cl}\mathscr{A}$ $(\operatorname{cl}\mathscr{A} \subseteq \omega_{\mathscr{A}}$ *resp.* $\omega_{\mathscr{A}} = \operatorname{cl}\mathscr{A})$*. For an entire motion* ϕ *one has* $\phi = \omega_\phi$*.*

The following characterization is helpful to derive topological and dynamical properties of $\omega_{\mathscr{A}}$. Namely, each point of an ω-limit set can be approximated by points on orbits starting in their defining sets $\mathscr{A}$.

Lemma 1.3.9. *Let $\mathscr{A} \subseteq \mathscr{X}$. Then $(t,x) \in \omega_{\mathscr{A}}$ if and only if there exist sequences $(t_n)_{n\geq 0}$ in $\mathbb{T}_+$ and $x_n \in \mathscr{A}(t-t_n)$ such that $\lim_{n\to\infty} t_n = \infty$ and*

$$\lim_{n\to\infty} \varphi(t,t-t_n)x_n = x.$$

Proof. Let $t \in \mathbb{I}$. We have to show two directions:

($\Rightarrow$) Assume first that $x \in \omega_{\mathscr{A}}(t)$. Then for all $T \geq 0$ one has $x \in \operatorname{cl}\gamma^T_{\mathscr{A}}(t)$ and there exist sequences x^T_k in $\gamma^T_{\mathscr{A}}(t) \subseteq X$ converging to x as $k \to \infty$. This means, for each $\varepsilon > 0$ there exists an integer $K = K(\varepsilon, T) > 0$ such that $d(x, x^T_k) < \varepsilon$ for all $k \geq K$. In particular, choosing $\varepsilon = \frac{1}{m}$ for some $m \in \mathbb{N}$ and setting $y_m := x^m_{K(1/m,m)}$ this yields $d(x, y_m) \leq \frac{1}{m}$ and, thus, $x = \lim_{m\to\infty} y_m$. Now one has

$$y_m \in \gamma^m_{\mathscr{A}}(t) = \bigcup_{s\geq m} \varphi(t,t-s)\mathscr{A}(t-s) \quad \text{for all } m \in \mathbb{N}$$

and there exist sequences $t_m \geq m$ in $\mathbb{N}$ and $x_m \in \mathscr{A}(t-t_m)$ such that one has $y_m = \varphi(t,t-t_m)x_m$. Thus, $\lim_{m\to\infty} \varphi(t,t-t_m)x_m = \lim_{m\to\infty} y_m = x$ holds and $t_m \geq m \to \infty$ for $m \to \infty$.

($\Leftarrow$) Conversely, let $x \in X$ be the limit of a sequence $\varphi(t,t-t_n)x_n$ as above. Due to our assumption, for every $T \geq 0$ there exists an $m \in \mathbb{N}$ such that $t_m \geq T$ and $\varphi(t,t-t_n)x_n \in \varphi(t,t-t_n)\mathscr{A}(t-t_n) \subseteq \gamma^T_{\mathscr{A}}(t)$ and $x \in \operatorname{cl}\gamma^T_{\mathscr{A}}(t)$. Since T was arbitrary, we deduce $x \in \bigcap_{T\geq 0} \operatorname{cl}\gamma^T_{\mathscr{A}}(t) = \omega_{\mathscr{A}}(t)$. ◻

Corollary 1.3.10. *For subsets $\mathscr{B} \subseteq \mathscr{A}$ one has $\omega_{\mathscr{B}} \subseteq \omega_{\mathscr{A}}$.*

Proof. Let $t \in \mathbb{I}$. The claim readily follows from Lemma 1.3.9, since for $(t,x) \in \omega_{\mathscr{B}}$ the corresponding sequence $x_n \in \mathscr{B}(t-t_n)$ also satisfies $x_n \in \mathscr{A}(t-t_n)$. ◻

Theorem 1.3.11. *For every $\mathscr{A} \subseteq \mathscr{X}$ the ω-limit set $\omega_{\mathscr{A}}$ is closed. Moreover, if φ is continuous, then $\omega_{\mathscr{A}}$ is forward invariant.*

Proof. Let $\tau \in \mathbb{I}$. As intersection of closed sets, the fibers $\omega_{\mathscr{A}}(\tau)$ are closed. Now assume φ is continuous and $\tau \leq t$. In order to show that $\omega_{\mathscr{A}}$ is forward invariant, we pick $(\tau,x) \in \omega_{\mathscr{A}}$ and show the inclusion $\varphi(t,\tau)x \in \omega_{\mathscr{A}}(t)$. From $x \in \omega_{\mathscr{A}}(\tau)$ we know by Lemma 1.3.9 that there exist sequences $\tau_n \to \infty$, $x_n \in \mathscr{A}(\tau-\tau_n)$ such that $x = \lim_{n\to\infty} \varphi(\tau,\tau-\tau_n)x_n$. By continuity of φ, one arrives at

$$\varphi(t,t-t_n)x_n = \varphi(t,\tau-\tau_n)x_n \overset{(1.2c)}{=} \varphi(t,\tau)\varphi(\tau,\tau-\tau_n)x_n \xrightarrow[n\to\infty]{} \varphi(t,\tau)x$$

with $t_n := \tau_n + t - \tau \to \infty$ for $n \to \infty$ and $x_n \in \mathscr{A}(\tau - \tau_n) = \mathscr{A}(t - t_n)$. Therefore, Lemma 1.3.9 implies $\varphi(t,\tau)x \in \omega_{\mathscr{A}}(t)$. □

For the remainder of this section we suppose that $\hat{\mathscr{B}}$ is a family of nonempty nonautonomous subsets of $\mathscr{X}$. The following attraction concept for nonautonomous sets $\mathscr{A}$ essentially means that the fibers $\mathscr{A}(t)$, $t \in \mathbb{I}$, attract particular nonautonomous sets from $\hat{\mathscr{B}}$ coming from $-\infty$.

Definition 1.3.12 (attracting)**.** A nonautonomous set $\mathscr{A} \subseteq \mathscr{X}$ is called $\hat{\mathscr{B}}$-*attracting*, if

$$\lim_{s\to\infty} \operatorname{dist}\big(\varphi(t,t-s)\mathscr{B}(t-s), \mathscr{A}(t)\big) = 0 \quad \text{for all } t \in \mathbb{I},\, \mathscr{B} \in \hat{\mathscr{B}} \tag{1.3c}$$

and we denote the family $\hat{\mathscr{B}}$ as *attraction universe*.

Remark 1.3.13. *From elementary properties of the Hausdorff semidistance one can deduce the following properties of attracting sets:*
(1) Supersets of $\hat{\mathscr{B}}$-attracting sets are $\hat{\mathscr{B}}$-attracting.
(2) Finite unions of $\hat{\mathscr{B}}$-attracting sets are $\hat{\mathscr{B}}$-attracting.

At first glance, the above attraction concept seems counter-intuitive, since it not necessarily implies the familiar forward convergence:

Example 1.3.14. *Suppose $\mathscr{X} = \mathbb{Z} \times \mathbb{R}$ and define the 2-parameter semigroup*

$$\varphi(t,\tau) = \begin{cases} \alpha_+^{t-\tau} & \textit{for } 0 \le \tau \le t, \\ \alpha_+^{t}\alpha_-^{-\tau} & \textit{for } \tau \le 0 \le t, \\ \alpha_-^{t-\tau} & \textit{for } \tau \le t \le 0 \end{cases}$$

with nonzero $\alpha_-, \alpha_+ \in \mathbb{R}$ satisfying $|\alpha_-| < 1 < |\alpha_+|$. Obviously, the nonautonomous set $\mathscr{A} := \mathbb{Z} \times \{0\}$ is invariant and due to

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(t,\tau)\xi, \mathscr{A}(t)) = 0 \quad \Leftrightarrow \quad \xi = 0 \quad \textit{for all } \tau \in \mathbb{Z}$$

we have no pointwise forward convergence towards $\mathscr{A}$. Nevertheless, $\mathscr{A}$ attracts all uniformly bounded subsets of $\mathscr{X}$. In order to show this, suppose $\mathscr{B} \subseteq \mathscr{X}$ is uniformly bounded with $\mathscr{B}(t) \subseteq [-R,R]$, $t \in \mathbb{Z}$, for some $R > 0$. The claim follows from

$$\begin{aligned} 0 &\le \operatorname{dist}(\varphi(t,t-s)\mathscr{B}(t-s), \mathscr{A}(t-s)) \le R\,\big|\varphi(t,t-s)\big| \\ &\le R\,|\alpha_-|^s \begin{cases} \left|\frac{\alpha_+}{\alpha_-}\right|^t & \textit{for } t \ge 0, \\ 1 & \textit{for } t < 0 \end{cases} \xrightarrow[s\to\infty]{} 0 \quad \textit{for all } t \in \mathbb{Z}. \end{aligned}$$

The motions of φ are illustrated in Fig. 1.4.

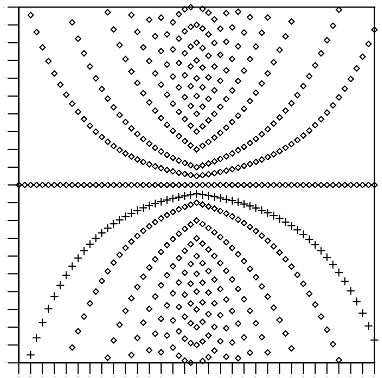

Fig. 1.4 Motions of the semiflow from Ex. 1.3.14

Proposition 1.3.15. *If $\mathscr{A} \subseteq \mathscr{X}$ is closed and $\hat{\mathscr{B}}$-attracting, then $\omega_{\mathscr{B}} \subseteq \mathscr{A}$ for all nonempty $\mathscr{B} \in \hat{\mathscr{B}}$.*

Proof. Let $\mathscr{B} \in \hat{\mathscr{B}}$ and $t \in \mathbb{I}$. For every $x \in \omega_{\mathscr{B}}(t)$ we obtain from Lemma 1.3.9 that there exist sequences $t_n \to \infty$, $x_n \in \mathscr{B}(t-t_n)$ such that $x = \lim_{n\to\infty} \varphi(t, t-t_n)x_n$. If we assume $x \notin \mathscr{A}(t)$, then $\varepsilon := \operatorname{dist}(x, \mathscr{A}(t)) > 0$ since $\mathscr{A}(t)$ is closed. Hence, for sufficiently large n we deduce $\operatorname{dist}(\varphi(t, t-t_n)x_n, \mathscr{A}(t)) \geq \frac{\varepsilon}{2}$ and therefore the contradiction $\operatorname{dist}(\varphi(t, t-t_n)\mathscr{B}(t-t_n), \mathscr{A}(t)) \geq \frac{\varepsilon}{2}$ to (1.3c). ◻

Proposition 1.3.16. *Let φ be continuous and suppose X is a Banach space. A compact, invariant and $\hat{\mathscr{B}}$-attracting nonautonomous set $\mathscr{A} \subseteq \mathscr{X}$ is connected, if one of the conditions holds:*

(a) $\hat{\mathscr{B}}$ contains all compact nonautonomous sets,
(b) $\mathscr{A}$ is uniformly bounded and $\hat{\mathscr{B}}$ consists of all uniformly bounded compact nonautonomous sets.

Proof. (a) For every $t \in \mathbb{I}$ we know from Mazur's theorem (cf. [AB99, p. 175, Thm. 5.20]) that the closed convex hull $\overline{\operatorname{co}}\mathscr{A}(t) \subseteq X$ of each fiber $\mathscr{A}(t)$ is compact and connected. Thus, $\mathscr{A}$ attracts $\overline{\operatorname{co}}\mathscr{A}$.

Suppose $\mathscr{A}$ is not connected. Then there exists a time $t_0 \in \mathbb{I}$ and open disjoint sets $U, V \subseteq X$ such that

$$\mathscr{A}(t_0) \subseteq U \cup V, \qquad U \cap \mathscr{A}(t_0) \neq \emptyset, \qquad V \cap \mathscr{A}(t_0) \neq \emptyset$$

hold. Yet, by continuity of φ we know that $C_n := \varphi(t_0, t_0 - s)\overline{\mathrm{co}}\mathscr{A}(t_0 - s)$ is a connected set for all $s \geq 0$. From the invariance of $\mathscr{A}$ one can deduce the relation $\mathscr{A}(t_0) = \varphi(t_0, t_0 - s)\mathscr{A}(t_0 - s) \subseteq C_s$ and therefore

$$U \cap \varphi(t_0, t_0 - s)\mathscr{A}(t_0 - s) \neq \emptyset, \qquad V \cap \varphi(t_0, t_0 - s)\mathscr{A}(t_0 - s) \neq \emptyset \quad \text{for all } s \geq 0.$$

Because each $C_s \subseteq X$ is connected, we can choose a sequence $x_n \in C_n \setminus (U \cup V)$. Since $\mathscr{A}(t_0)$ attracts every compact C_n in the sense of (1.3c), $x_n \in C_n$ implies that the set $\{x_n\}_{n \geq 0}$ is relatively compact and there exists a convergent subsequence $(x_{m_n})_{n \geq 0}$ with limit $x \in X$. Moreover, because the difference $C_n \setminus (U \cup V)$ is closed, one has $x \notin U \cup V$.

On the other hand, by construction there exist $y_{m_n} \in \overline{\mathrm{co}}\mathscr{A}(t_0 - m_n)$ satisfying $x_{m_n} = \varphi(t_0, t_0 - m_n)y_{m_n}$ and since $\mathscr{A}$ attracts compact sets, we get

$$0 \leq \operatorname{dist}\left(\left\{x_{m_n}\right\}, \mathscr{A}(t_0)\right) = \operatorname{dist}\left(\varphi(t_0, t_0 - m_n)\left\{y_{m_n}\right\}, \mathscr{A}(t_0)\right) \xrightarrow[n \to \infty]{(1.3c)} 0.$$

Therefore, $x \in \mathscr{A}(t_0) \subseteq U \cup V$. This is a contradiction.

(b) It is easy to see that the uniform boundedness of $\mathscr{A}$ carries over to the closed convex hull $\overline{\mathrm{co}}\mathscr{A}$. Then the assertion follows as above. □

The general theory of topological dynamics deals with semigroups on metric spaces and most results are based only on continuity properties of $\varphi(t, \tau)$. Particularly in an infinite-dimensional setting it is natural to discuss additional features that may be obtained, if we assume some degree of compactness. For instance, 2-parameter semigroups with relatively compact orbits yield nonempty limit sets (cf. Def. 1.3.7). Indeed, many of the applications occur in a setting wherein the given semigroup has a smoothing property. Next we examine three key concepts related to compactness and playing a pivotal role in the theory of attractors.

Definition 1.3.17 (compact)**.** A 2-parameter semigroup φ is called

(i) $\hat{\mathscr{B}}$-*compact*, if there exists a so-called *compactification time* $T \in \mathbb{T}_+$ such that for all $\mathscr{B} \in \hat{\mathscr{B}}$ the orbit $\gamma^T_{\mathscr{B}}$ is relatively compact,

(ii) $\hat{\mathscr{B}}$-*eventually compact*, if for all $t \in \mathbb{I}$, $\mathscr{B} \in \hat{\mathscr{B}}$ there exists an $T = T_t(\mathscr{B}) \in \mathbb{T}_+$ such that the set $\gamma^T_{\mathscr{B}}(t)$ is relatively compact,

(iii) $\hat{\mathscr{B}}$-*asymptotically compact*, if for all $t \in \mathbb{I}$, $\mathscr{B} \in \hat{\mathscr{B}}$, and all sequences $t_n \to \infty$ in $\mathbb{T}_+$, $x_n \in \mathscr{B}(t - t_n)$, the sequence $(\varphi(t, t - t_n)x_n)_{n \geq 0}$ in X possesses a convergent subsequence.

Remark 1.3.18. *If X is a compact metric space, then 2-parameter semigroups are $\hat{\mathscr{B}}$-compact with compactification time 0.*

Corollary 1.3.19. *A $\hat{\mathscr{B}}$-compact 2-parameter semigroup φ is $\hat{\mathscr{B}}$-eventually compact, and a $\hat{\mathscr{B}}$-eventually compact φ is $\hat{\mathscr{B}}$-asymptotically compact.*

Proof. Let $t \in \mathbb{I}$ and $\mathscr{B} \in \hat{\mathscr{B}}$. Since the first assertion is clear by definition we restrict to the second one. We know that there exists an $T \in \mathbb{T}_+$ such that $\gamma^T_{\mathscr{B}}(t)$ is relatively compact. Now choose sequences $t_n \to \infty$, $x_n \in \mathscr{B}(t-t_n)$, where w.l.o.g. we can assume $t_n \geq T$. Then $u_n := \varphi(t, t-t_n)x_n$ defines a sequence in the relatively compact fiber $\gamma^T_{\mathscr{B}}(t)$ and consequently there exists a convergent subsequence of $(u_n)_{n\in\mathbb{N}}$; thus, φ is $\hat{\mathscr{B}}$-asymptotically compact. □

We arrive at the main result of this section:

Theorem 1.3.20. *Let $\mathscr{A} \in \hat{\mathscr{B}}$. If φ is $\hat{\mathscr{B}}$-asymptotically compact, then*

(a) $\omega_{\mathscr{A}}$ is nonempty,
(b) $\omega_{\mathscr{A}}$ is compact,
(c) $\omega_{\mathscr{A}}$ is $\{\mathscr{A}\}$-attracting.

Moreover, provided φ is continuous, then $\omega_{\mathscr{A}}$ is invariant.

Proof. Let $\mathscr{A} \in \hat{\mathscr{B}}$ and $t \in \mathbb{I}$ be fixed.

(a) Since φ is $\hat{\mathscr{B}}$-asymptotically compact, for arbitrary sequences $t_n \to \infty$ and $x_n \in \mathscr{A}(t-t_n)$, we can extract a convergent subsequence $(\varphi(t, t-t_{n_l})x_{n_l})_{l\geq 0}$ from $(\varphi(t, t-t_n)x_n)_{n\geq 0}$. By construction, the limit of $\varphi(t, t-t_{n_l})x_{n_l}$ belongs to the fiber $\omega_{\mathscr{A}}(t)$ and we have $\omega_{\mathscr{A}}(t) \neq \emptyset$.

(b) We already know from Thm. 1.3.11 that $\omega_{\mathscr{A}}(t)$ is closed. Thus, in order to show its compactness, it suffices to see that for any sequence $(y_n)_{n\geq 0}$ in $\omega_{\mathscr{A}}(t)$ we can extract a convergent subsequence. Due to $y_n \in \omega_{\mathscr{A}}(t)$ there exist $t_n \geq n$ and $x_n \in \mathscr{A}(t-t_n)$ such that

$$d(\varphi(t, t-t_n)x_n, y_n) \leq \tfrac{1}{n} \quad \text{for all } n \in \mathbb{N}. \tag{1.3d}$$

Keeping in mind that φ is $\hat{\mathscr{B}}$-asymptotically compact, we obtain a subsequence $(t_{n_l})_{l\geq 0}$ such that $y := \lim_{l\to\infty} \varphi(t, t-t_{n_l})x_{n_l}$ exists and thus

$$d(y, y_{n_l}) \leq d(y, \varphi(t, t-t_{n_l})x_{n_l}) + d(\varphi(t, t-t_{n_l})x_{n_l}, y_{k_{n_l}}) \xrightarrow[l\to\infty]{(1.3d)} 0.$$

Hence, $\omega_{\mathscr{A}}$ has compact fibers.

(c) For our given nonautonomous set $\mathscr{A} \in \hat{\mathscr{B}}$ we have to show the limit relation $\lim_{s\to\infty} \operatorname{dist}(\varphi(t, t-s)\mathscr{A}(t-s), \omega_{\mathscr{A}}(t)) = 0$ (cf. (1.3c)). We proceed indirectly and suppose this relation does not hold. Then there exists an $\varepsilon > 0$ and sequences $t_n \to \infty$, $x_n \in \mathscr{A}(t-t_n)$ so that

$$\operatorname{dist}(\varphi(t,t-t_n)x_n,\omega_{\mathscr{A}}(t)) \geq \varepsilon. \tag{1.3e}$$

However, from the sequence $(\varphi(t,t-t_n)x_n)_{n\geq 0}$ we can extract a convergent subsequence with limit $y \in \omega_{\mathscr{A}}(t)$; this contradicts (1.3e).

Let φ be continuous and we get from Thm. 1.3.11 that $\omega_{\mathscr{A}}$ is forward invariant. It remains to prove the inclusion $\omega_{\mathscr{A}}(t) \subseteq \varphi(t,\tau)\omega_{\mathscr{A}}(\tau)$ for $\tau < t$. For $(t,y) \in \omega_{\mathscr{A}}$ there exist sequences $t_n \to \infty$, $x_n \in \mathscr{A}(t-t_n)$ with $y = \lim_{n\to\infty}\varphi(t,t-t_n)x_n$ (cf. Lemma 1.3.9). For reals $t_n \geq t-\tau$ and $\tau_n := t_n+\tau-t$ we have

$$\varphi(t,t-t_n)x_n \overset{(1.2c)}{=} \varphi(t,\tau)\varphi(\tau,\tau-\tau_n)x_n \tag{1.3f}$$

and as the 2-parameter semigroup φ is $\hat{\mathscr{B}}$-asymptotically compact, $\tau_n \to \infty$ and $x_n \in \mathscr{A}(\tau-\tau_n)$ holds, there exist subsequences $\tau_{n_m} \to \infty$, $x_{n_m} \in \mathscr{A}(\tau-\tau_{n_m})$ with $z := \lim_{m\to\infty}\varphi(\tau,\tau-\tau_{n_m})x_{n_m} \in \omega_{\mathscr{A}}(\tau)$. Hence, by (1.3f) and the continuity of φ we obtain $y = \varphi(t,\tau)z \in \varphi(t,\tau)\omega_{\mathscr{A}}(\tau)$. □

Exercises

Exercise 1.3.21. *Let A,B,C be (nonempty) subsets of a metric space (X,d) with $C \subseteq A$. Prove the monotonicity relations*

$$\operatorname{dist}(C,B) \leq \operatorname{dist}(A,B), \qquad \operatorname{dist}(B,A) \leq \operatorname{dist}(B,C) \tag{1.3g}$$

and find an example illustrating that the Hausdorff semidistance is not symmetric.

Exercise 1.3.22. *Suppose that $\theta_t : X \to X$ is a semidynamical system. Then the ω-limit sets of a point $\xi \in X$ resp. of a set $B \subseteq X$ are defined by*

$$\omega(\xi) := \bigcap_{r\geq 0} \operatorname{cl} \bigcup_{s\geq r} \{\theta_s(\xi)\}, \qquad \omega(B) := \bigcap_{r\geq 0} \operatorname{cl} \bigcup_{s\geq r} \theta_s(B).$$

Verify the inclusion $\bigcup_{\xi\in B}\omega(\xi) \subseteq \omega(B)$. Does equality hold in general?

1.4 Attractors and global attractors

We continue our studies of the asymptotic behavior of 2-parameter semigroups. Our primary interest is the nonautonomous set which consists of all bounded entire motions, the so-called global attractor. In this section we present results that construct attractors as ω-limit sets of absorbing sets. This construction requires at least asymptotical compactness of the 2-parameter semigroup.

In general, the global attractor has a complicated geometry reflecting the complexity of the longtime behavior of a given system. Yet, in our abstract set-up we desist from tackling such delicate issues and focus on existence issues. For this, assume the $\mathbb{T}$-interval $\mathbb{I}$ is unbounded below and X is a metric space. In addition, suppose throughout that $\hat{\mathscr{B}}$ is a family of nonempty subsets of $\mathscr{X}$ and φ is a 2-parameter semigroup on $\mathscr{X}$.

Definition 1.4.1 (attractor)**.** A compact nonautonomous set $\mathscr{A}^* \subseteq \mathscr{X}$ is said to be a $\hat{\mathscr{B}}$-*attractor*, if it is invariant and $\hat{\mathscr{B}}$-attracting. In case the attraction universe $\hat{\mathscr{B}}$ consists of all uniformly bounded subsets of $\mathscr{X}$, a $\hat{\mathscr{B}}$-attractor is called *global attractor*.

Remark 1.4.2. *For a global attractor $\mathscr{A}^*$ the elements of $\hat{\mathscr{B}}$ form a cover of the extended state space $\mathscr{X}$. We speak of a* local attractor*, whenever $\hat{\mathscr{B}}$ consists only of neighborhoods of $\mathscr{A}^*$.*

Proposition 1.4.3. *A global attractor $\mathscr{A}^* \subseteq \mathscr{X}$ admits the (incomplete) dynamical characterization*

$$\mathscr{A}^* \subseteq \left\{(\tau,\xi) \in \mathscr{X} : \textit{there exists an entire motion through } (\tau,\xi)\right\},$$
$$\left\{(\tau,\xi) \in \mathscr{X} : \textit{there exists a bounded entire motion through } (\tau,\xi)\right\} \subseteq \mathscr{A}^*.$$

Proof. Concerning the first inclusion, pick $(\tau,\xi) \in \mathscr{A}^*$ arbitrarily. Due to the invariance of the global attractor $\mathscr{A}^*$, Prop. 1.3.3 yields that there exists an entire motion ϕ through (τ,ξ).

Concerning the second inclusion, if there exists a bounded entire motion ϕ for φ, then the nonautonomous set ϕ is invariant and uniformly bounded. Since $\mathscr{A}^*$ attracts uniformly bounded nonautonomous sets, we have

$$0 = \lim_{s\to\infty} \operatorname{dist}(\varphi(t,t-s)\left\{\phi(t-s)\right\}, \mathscr{A}^*(t)) = \operatorname{dist}(\left\{\phi(t)\right\}, \mathscr{A}^*(t)) \quad \text{for all } t \in \mathbb{I},$$

thus $\phi(t) \in \operatorname{cl}\mathscr{A}^*(t)$. Hence, the closedness of $\mathscr{A}^*$ implies $\phi \subseteq \mathscr{A}^*$. □

Corollary 1.4.4. *A uniformly bounded global attractor $\mathscr{A}^*$ admits the complete dynamical characterization*

$$\mathscr{A}^* = \left\{(\tau,\xi) \in \mathscr{X} : \textit{there exists an entire bounded motion through } (\tau,\xi)\right\}$$

and is therefore uniquely determined.

Proof. Since $\mathscr{A}^*$ is uniformly bounded, the entire solution ϕ from the first inclusion in the above proof of Prop. 1.4.3 is clearly bounded. □

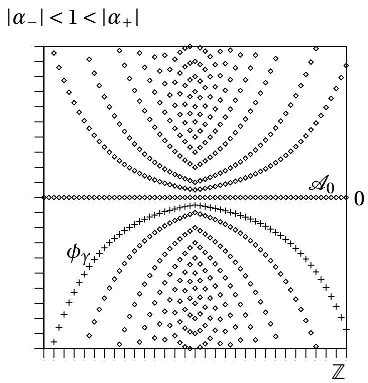

Fig. 1.5 Motions from Ex. 1.4.5, complete motion ϕ_γ, uniformly bounded global attractor $\mathscr{A}_0$

As demonstrated by Ex. 1.3.14, in general one cannot expect forward convergence towards global attractors. The following example shows that attractors need not to be unique, and without uniform boundedness assumptions also global attractors are not uniquely determined.

Example 1.4.5. *Suppose $\mathscr{X} = \mathbb{Z} \times \mathbb{R}$ and consider the 2-parameter group φ from the above Ex. 1.3.14. Given an arbitrary $\gamma \in \mathbb{R}$, it is easy to see that*

$$\phi_\gamma(t) := \gamma \begin{cases} \alpha_+^t & \text{for } t \geq 0, \\ \alpha_-^t & \text{for } t < 0 \end{cases}$$

defines an entire motion for φ. Hence, the nonautonomous sets $\mathbb{Z} \times \{0\}$ and ϕ_γ are invariant, as well as (cf. Prop. 1.3.4) their union $\mathscr{A}_\gamma := (\mathbb{Z} \times \{0\}) \cup \phi_\gamma$. In Ex. 1.3.14 we saw that $\mathbb{Z} \times \{0\}$ attracts uniformly bounded subsets of $\mathscr{X}$ and Rem. 1.3.13(1) guarantees that also $\mathscr{A}_\gamma$ has this property. Since all the fibers $\mathscr{A}_\gamma(\tau)$, $\tau \in \mathbb{Z}$, are compact (in fact finite), each $\mathscr{A}_\gamma$ is a global attractor for φ, among which $\mathscr{A}_0$ is the unique uniformly bounded global attractor. An illustration is given in Fig. 1.5.

Definition 1.4.6 (absorbing)**.** A nonempty nonautonomous set $\mathscr{A} \subseteq \mathscr{X}$ is called

(a) $\hat{\mathscr{B}}$*-absorbing*, if for all $t \in \mathbb{I}$, $\mathscr{B} \in \hat{\mathscr{B}}$ there exists an $T = T_t(\mathscr{B}) \geq 0$ with

$$\varphi(t, t-s)\mathscr{B}(t-s) \subseteq \mathscr{A}(t) \quad \text{for all } s \geq T,$$

(b) $\hat{\mathscr{B}}$*-uniformly absorbing*, if for all $\mathscr{B} \in \hat{\mathscr{B}}$ there exists an $T = T(\mathscr{B}) \geq 0$ with

$$\varphi(t, t-s)\mathscr{B}(t-s) \subseteq \mathscr{A}(t) \quad \text{for all } t \in \mathbb{I},\ s \geq T$$

and we denote the family $\hat{\mathscr{B}}$ as *absorption universe*. A 2-parameter semigroup is called $\hat{\mathscr{B}}$-*dissipative*, if it has a bounded $\hat{\mathscr{B}}$-absorbing set. Moreover, a $\hat{\mathscr{B}}$-dissipative 2-parameter semigroup is called *uniformly bounded (bounded, compact) dissipative*, if $\hat{\mathscr{B}}$ consists of all uniformly bounded (bounded, compact) subsets of $\mathscr{X}$.

Remark 1.4.7. *(1) Any $\hat{\mathscr{B}}$-uniformly absorbing set is $\hat{\mathscr{B}}$-absorbing.*

(2) Bounded dissipative 2-parameter semigroups are uniformly bounded dissipative and compact dissipative.

Example 1.4.8. *We consider a nonautonomous ODE*

$$\dot{x} = f(t,x) \tag{1.4a}$$

with a right-hand side $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ guaranteeing existence and uniqueness of solutions, and satisfies the estimate

$$\langle f(t,x), x\rangle \le \tfrac{\beta}{2} - \tfrac{\alpha}{2}|x|^2 \quad \text{for all } t \in \mathbb{R},\ x \in \mathbb{R}^d. \tag{1.4b}$$

with reals $\alpha > 0$, $\beta \ge 0$. Given fixed initial data $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}^d$ and the general solution $\varphi(\cdot;\tau,\xi)$ to (1.4a), we abbreviate $\varphi(t) := \varphi(t;\tau,\xi)$ and $\rho(t) := \langle \varphi(t), \varphi(t)\rangle$, which implies

$$\dot{\rho}(t) = 2\langle\dot{\varphi}(t),\varphi(t)\rangle = 2\langle f(t,\varphi(t)),\varphi(t)\rangle \overset{(1.4b)}{\le} \beta - \alpha\left|\varphi(t)\right|^2 = \beta - \alpha\rho(t).$$

We multiply this inequality with $e^{\alpha t}$ to obtain

$$\frac{d}{dt}\left(e^{\alpha t}\rho(t)\right) = \alpha e^{\alpha t}\rho(t) + e^{\alpha t}\dot{\rho}(t) \le \beta e^{\alpha t}$$

and integration between τ and t yields $e^{\alpha t}\rho(t) - e^{\alpha\tau}\rho(\tau) \le \frac{\beta}{\alpha}\left(e^{\alpha t} - e^{\alpha\tau}\right)$, hence

$$\left|\varphi(t;\tau,\xi)\right| \le \sqrt{e^{\alpha(\tau-t)}|\xi|^2 + \tfrac{\beta}{\alpha}\left(1 - e^{\alpha(\tau-t)}\right)}.$$

Now let $\mathscr{B} \subseteq \mathbb{R}\times\mathbb{R}^d$ be uniformly bounded, i.e. $\mathscr{B}(t) \subseteq \bar{B}_R(0)$ for all $t \in \mathbb{R}$. We choose $\rho_0 > \frac{\beta}{\alpha}$ and from the limit relation $\lim_{s\to\infty}\sqrt{e^{-\alpha s}R^2 + \frac{\beta}{\alpha}(1 - e^{-\alpha s})} = \sqrt{\frac{\beta}{\alpha}}$ we see that there exists a $T = T(\mathscr{B}) \ge 0$ such that

$$\varphi(t; t-s, \mathscr{B}(t-s)) \subseteq B_{\rho_0}(t) \quad \text{for all } t \in \mathbb{R},\ s \ge T.$$

Hence, the nonautonomous set $\mathscr{A} := \left\{(t,x) \in \mathbb{R}\times\mathbb{R}^d : |x| \le \rho_0\right\}$ uniformly absorbs all uniformly bounded subsets of $\mathbb{R}\times\mathbb{R}^d$.

Lemma 1.4.9. *If $\mathscr{A}$ is a $\hat{\mathscr{B}}$-absorbing nonautonomous set, then $\omega_{\mathscr{B}} \subseteq \omega_{\mathscr{A}}$ for all $\mathscr{B} \in \hat{\mathscr{B}}$ holds and in case $\mathscr{A} \in \hat{\mathscr{B}}$ one additionally obtains*

$$\omega_{\mathscr{A}} \subseteq \operatorname{cl}\mathscr{A}, \qquad \operatorname{cl}\bigcup_{\mathscr{B}\in\hat{\mathscr{B}}} \omega_{\mathscr{B}}(t) = \omega_{\mathscr{A}}(t) \quad \textit{for all } t \in \mathbb{I}. \tag{1.4c}$$

Proof. We arbitrarily fix $\mathscr{B} \in \hat{\mathscr{B}}$ and a pair $(t, y) \in \omega_{\mathscr{B}}$. By Lemma 1.3.9 there exist sequences $t_n \to \infty$ and points $x_n \in \mathscr{B}(t-t_n)$ such that

$$\lim_{n\to\infty} \varphi(t, t-t_n)x_n = y. \tag{1.4d}$$

Since the nonautonomous set $\mathscr{A}$ is $\hat{\mathscr{B}}$-absorbing, for each $m \in \mathbb{N}$ there exists a $t_{n_m} \geq m$ such that one has $\varphi(t-m, t-t_{n_m})\mathscr{B}(t-t_{n_m}) \subseteq \mathscr{A}(t-m)$ and we define points $y_m := \varphi(t-m, t-t_{n_m})x_{n_m} \in \mathscr{A}(t-m)$. This yields

$$\varphi(t, t-t_{n_m})x_{n_m} \overset{(1.2c)}{=} \varphi(t, t-m)\varphi(t-m, t-t_{n_m})x_{n_m} = \varphi(t, t-m)y_m$$

and thus $\lim_{m\to\infty} \varphi(t, t-m)y_m = y$ by (1.4d). By the inclusion $y_m \in \mathscr{A}(t-m)$, our Lemma 1.3.9 implies $y \in \omega_{\mathscr{A}}(t)$; since $(t, y) \in \omega_{\mathscr{B}}$ was arbitrary, we have $\omega_{\mathscr{B}} \subseteq \omega_{\mathscr{A}}$. For the remaining part of the proof suppose $\mathscr{A} \in \hat{\mathscr{B}}$. Since $\mathscr{A}$ is $\hat{\mathscr{B}}$-absorbing, there exists a real $T = T_t(\mathscr{B}) \geq 0$ with $\varphi(t, t-t_l)x_l \in \mathscr{A}(t)$ for all $l \in \mathbb{N}$ such that $l \geq T$ and therefore we have $y \in \operatorname{cl}\mathscr{A}(t)$. This guarantees $\omega_{\mathscr{B}}(t) \subseteq \operatorname{cl}\mathscr{A}(t)$ and the remaining relation is a straight forward consequence of the above. □

Under the assumption of continuity, we arrive at the first main theorem.

Theorem 1.4.10. *Let a 2-parameter semigroup be continuous and $\hat{\mathscr{B}}$-asymptotically compact with $\hat{\mathscr{B}}$-absorbing set $\mathscr{A}$. If $\mathscr{A} \in \hat{\mathscr{B}}$, then:*

(a) The nonautonomous set $\mathscr{A}^* := \omega_{\mathscr{A}}$ is a $\hat{\mathscr{B}}$-attractor,
(b) $\mathscr{A}^* \subseteq \operatorname{cl}\mathscr{A}$,
(c) $\mathscr{A}^*$ is minimal in the following sense: Every closed and $\hat{\mathscr{B}}$-attracting set $\mathscr{A}_* \subseteq \mathscr{X}$ satisfies $\mathscr{A}^* \subseteq \mathscr{A}_*$,
(d) every fiber of $\mathscr{A}^*$ allows the characterization

$$\mathscr{A}^*(t) = \operatorname{cl}\bigcup_{\mathscr{B}\in\hat{\mathscr{B}}} \omega_{\mathscr{B}}(t) \quad \textit{for all } t \in \mathbb{I}.$$

Remark 1.4.11. *If a 2-parameter semigroup is uniformly bounded dissipative with uniformly bounded absorbing set, then its global attractor $\mathscr{A}^* \subseteq \mathscr{X}$ is uniformly bounded and has the complete characterization from Cor. 1.4.4.*

Proof. Let $t \in \mathbb{I}$ and we define $\mathscr{A}^* := \omega_{\mathscr{A}}$.

(a) The compactness of $\mathscr{A}^*$ follows readily from Thm. 1.3.20(b). Thanks to the inclusion $\bigcup_{\mathscr{B}\in\hat{\mathscr{B}}} \omega_{\mathscr{B}}(t) \subseteq \mathscr{A}^*(t)$ (see (1.4c) in Lemma 1.4.9) we obtain

$$\operatorname{dist}(\varphi(t, t-s)\mathscr{B}(t-s), \mathscr{A}^*(t)) \overset{(1.3g)}{\leq} \operatorname{dist}(\varphi(t, t-s)\mathscr{B}(t-s), \omega_{\mathscr{B}}(t))$$

for $s \geq 0$, where the right-hand side tends to 0 for $s \to \infty$ by Thm. 1.3.20(c), and the relation $\lim_{s\to\infty} \operatorname{dist}(\varphi(t, t-s)\mathscr{B}(t-s), \mathscr{A}^*(t)) = 0$ for all $\mathscr{B} \in \hat{\mathscr{B}}$ follows. It remains to show the invariance of attractors. For this, we observe from Thm. 1.3.20 that

$$\mathscr{A}^*(t) \overset{(1.4c)}{=} \operatorname{cl} \bigcup_{\mathscr{B}\in\hat{\mathscr{B}}} \omega_{\mathscr{B}}(t) = \operatorname{cl} \bigcup_{\mathscr{B}\in\hat{\mathscr{B}}} \varphi(t,\tau)\omega_{\mathscr{B}}(\tau) = \operatorname{cl} \varphi(t,\tau) \bigcup_{\mathscr{B}\in\hat{\mathscr{B}}} \omega_{\mathscr{B}}(\tau)$$

for $\tau \leq t$. Since φ is continuous and $\operatorname{cl}\bigcup_{\mathscr{B}\in\hat{\mathscr{B}}} \omega_{\mathscr{B}}(\tau)$ compact (cf. Lemma 1.4.9),

$$\operatorname{cl} \varphi(t,\tau) \bigcup_{\mathscr{B}\in\hat{\mathscr{B}}} \omega_{\mathscr{B}}(\tau) = \varphi(t,\tau) \operatorname{cl} \bigcup_{\mathscr{B}\in\hat{\mathscr{B}}} \omega_{\mathscr{B}}(\tau);$$

thus, we have $\varphi(t,\tau)\mathscr{A}^*(\tau) = \mathscr{A}^*(t)$ for all $\tau \leq t$.

(b) We give another proof of the left relation in (1.4c). Since $\mathscr{A}$ is $\hat{\mathscr{B}}$-absorbing and $\mathscr{A} \in \hat{\mathscr{B}}$, there exists a real $T_t(\mathscr{A}) \geq 0$ with $\varphi(t, t-s)\mathscr{A}(t-s) \subseteq \mathscr{A}(t)$ for all $s \geq T_t(\mathscr{A})$ and consequently, from Def. 1.4.6 and assertion (a) follows

$$\begin{aligned} \mathscr{A}^*(t) = \omega_{\mathscr{A}}(t) &= \bigcap_{r\geq 0} \operatorname{cl} \bigcup_{s\geq r} \varphi(t, t-s)\mathscr{A}(t-s) \\ &\subseteq \bigcap_{r\geq T_t(\mathscr{A})} \operatorname{cl} \bigcup_{s\geq r} \varphi(t, t-s)\mathscr{A}(t-s) \\ &\subseteq \bigcap_{r\geq T_t(\mathscr{A})} \operatorname{cl} \mathscr{A}(t) = \operatorname{cl} \mathscr{A}(t). \end{aligned}$$

(c) Now, let $\mathscr{A}_* \subseteq \mathscr{X}$ be a closed and $\hat{\mathscr{B}}$-attracting nonautonomous set, i.e. in particular $\lim_{s\to\infty} \operatorname{dist}(\varphi(t, t-s)\mathscr{A}(t-s), \mathscr{A}_*(t)) = 0$. For each pair $(t, y) \in \mathscr{A}^*$ one has $y = \lim_{n\to\infty} \varphi(t, t-t_n)x_n$ with sequences $t_n \to \infty$, $x_n \in \mathscr{A}(t-t_n)$ (cf. Lemma 1.3.9), and consequently we derive $y \in \operatorname{cl}\mathscr{A}_*(t) = \mathscr{A}_*(t)$. This yields the inclusion $\mathscr{A}^*(t) \subseteq \mathscr{A}_*(t)$.

(d) See Lemma 1.4.9. □

Uniformly bounded global attractors are topologically connected:

Corollary 1.4.12. *Let X be a Banach. If one of the conditions (a) or (b) from Prop. 1.3.16 holds, then $\mathscr{A}^* \subseteq \mathscr{X}$ is connected.*

Proof. On the basis of Thm. 1.4.10 we can deduce that $\omega_{\mathscr{A}} = \mathscr{A}^*$ is compact, invariant and also $\hat{\mathscr{B}}$-attracting. Then Prop. 1.3.16 implies our claim. □

We round off this section with another central result on the attractors of parameter-dependent 2-parameter semigroups.

Theorem 1.4.13 (upper-semicontinuity of attractors)**.** *Let X, P be metric spaces and P be complete. If $\varphi(\cdot; p)$, $p \in P$, is a continuous, $\hat{\mathscr{B}}$-asymptotically* *compact and $\hat{\mathscr{B}}$-dissipative parameter-dependent 2-parameter semigroup with $\hat{\mathscr{B}}$-absorbing set $\mathscr{A} \in \hat{\mathscr{B}}$ (uniformly in $p \in P$), then:*

(a) Every $\varphi(\cdot; p)$ has a $\hat{\mathscr{B}}$-attractor $\mathscr{A}_p^*$,

(b) if $\mathscr{A} \in \hat{\mathscr{B}}$ is uniformly bounded and $\hat{\mathscr{B}}$ contains all uniformly bounded finite sets, then $\mathscr{A}_p^*$ is upper-semicontinuous, i.e.

$$\lim_{p \to p_0} \operatorname{dist}(\mathscr{A}_p^*(t), \mathscr{A}_{p_0}^*(t)) = 0 \quad \textit{for all } t \in \mathbb{I},\ p_0 \in P.$$

Proof. Let $t \in \mathbb{I}$, $p \in P$ and we restrict to the discrete time case.

(a) The existence of $\hat{\mathscr{B}}$-attractors $\mathscr{A}_p^*$ follows from Thm. 1.4.10(a). Here, $\mathscr{A}_p^*$ is given as ω-limit set of $\mathscr{A}$ w.r.t. the 2-parameter semigroup $\varphi(\cdot; p)$.

(b) W.l.o.g. we can assume that $\mathscr{A} \subseteq \mathscr{X}$ is closed. From Thm. 1.4.10(b) we obtain $\mathscr{A}_p^* \subseteq \mathscr{A}$ and consequently the nonautonomous set $\mathscr{V}$ given by the fibers

$$\mathscr{V}(t) := \bigcup_{p \in P} \mathscr{A}_p^*(t) \quad \text{for all } t \in \mathbb{I}$$

is bounded. Due to the compactness of each $\mathscr{A}_p^*(t)$ (cf. Thm. 1.3.20(b)) the bounded union $\mathscr{V}$ is relatively compact. In order to prove the upper semicontinuity of a fiber $\mathscr{A}_p^*(t)$, we must show the following: Provided $(p_j)_{j\in\mathbb{N}}$, $(x_j)_{j\in\mathbb{N}}$ denote convergent sequences in the spaces P, X, resp., with $x_j \in \mathscr{A}_{p_j}^*$ and respective limits $p_0 \in P$, $x_0 \in X$, then one has $x_0 \in \mathscr{A}_{p_0}^*(t)$.

Since the nonautonomous set $\mathscr{A}_{p_j}^*$ is invariant (cf. Thm. 1.3.20), there exists a point $y_j^1 \in \mathscr{A}_{p_j}^*(t-1)$ with $x_j = \varphi(t, t-1; p_j) y_j^1$ (cf. Prop. 1.3.3). Because the fiber $\mathscr{V}(t-1)$ is relatively compact and $\mathscr{A}(t-1)$ is closed, we may assume that the bounded sequence $(y_j^1)_{j\in\mathbb{N}}$ has a limit $y_0^1 \in \mathscr{A}(t-1)$; the continuity of φ guarantees $x_0 \in \varphi(t, t-1; p_0) y_0^1$. In the same fashion, for all $n \in \mathbb{N}$ there exists a convergent sequence $(y_j^n)_{j\in\mathbb{N}}$ in $\mathscr{A}_{p_j}^*(t-n)$ with limit $y_0^n \in \mathscr{A}(t-n)$ and $\varphi(t, t-n; p_0) y_0^n = x_0$. Having this at hand, introduce a sequence ϕ in X,

$$\phi(t+n) := y_0^n \in \mathscr{A}(t-n) \quad \text{for all } n < 0,$$
$$\phi(t+n) := \varphi(t+n, t; p_0) x_0 \quad \text{for all } n \geq 0;$$

by definition, ϕ represents an entire motion for $\varphi(\cdot; p_0)$. Due to the fact that $\mathscr{A}$ is a $\hat{\mathscr{B}}$-absorbing set, for every $m \in \mathbb{Z}_0^+$ there exists an $N(m) \in \mathbb{Z}_0^+$ with

$$\phi(t+m) = \varphi(t+m, t-n; p_0) y_0^n \in \mathscr{A}(t+m) \quad \text{for all } n \geq N(m).$$

Therefore, the sequence ϕ is a bounded entire motion for $\varphi(\cdot; p_0)$. Because, by assumption, $\mathscr{A}_{p_0}^*$ attracts uniformly bounded finite sets, we can show as in the proof of Prop. 1.4.3 that $\phi(t) \in \mathscr{A}_{p_0}^*(t)$, i.e. $x_0 \in \mathscr{A}_{p_0}^*(t)$. □

Exercises

Exercise 1.4.14. *Let $\theta > 0$. A 2-parameter semigroup φ is called θ*-periodic, *if the relation $\varphi(t+\theta, \tau+\theta) = \varphi(t,\tau)$ holds for all $\tau \le t$. What can you say about the periodicity properties of the resulting ω-limit sets?*

Exercise 1.4.15. *Determine the global attractor of the autonomous scalar differential equation*

$$\dot{x} = -x(x^4 - 2x^2 + 1 - p)$$

depending on the parameter $p \in \mathbb{R}$.

Exercise 1.4.16. *Determine the global attractor of the following difference equations depending on a parameter $p > 0$:*

(a) For $p > 0$:

$$x_{k+1} = \frac{p x_k}{1 + |x_k|}$$

(b) For $p > 1$:

$$x_{k+1} = \frac{p_k x_k}{1 + |x_k|}, \qquad \lambda_k := \begin{cases} p, & k \ge 0, \\ p^{-1}, & k < 0 \end{cases}$$

Chapter 2
Linear differential equations

…dichotomies, rather than Lyapunov's characteristic exponents, are the key to questions of asymptotic behaviour for nonautonomous differential equations.
W.A. Coppel [Cop78]

Our previous considerations dealt with difference and differential equations, i.e. discrete and continuous time, simultaneously. Now we restrict to finite-dimensional differential equations and follow [SS78, Sie02], but point out that a corresponding discrete theory can be found in [AS01].

In Ex. 0.0.2 we have illustrated that even for periodic equations, eigenvalues of coefficient matrices do not yield stability properties. To circumvent this problem, several more adequate spectral notions have been developed. Among them, exponential dichotomies and the related dichotomy spectrum are the most appropriate for us.

2.1 Preliminaries

In this chapter we deal with linear ordinary differential equations

$$\boxed{\dot{x} = A(t)x} \tag{L}$$

with a continuous coefficient matrix $A : \mathbb{R} \to \mathbb{R}^{d\times d}$. We point out that our subsequent theory can be extended to the situation that A is piecewise continuous, or even $A \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{d\times d})$ (cf. [AW96, Kur86]). The natural set to describe the dynamics of (L) is the extended state space $\mathscr{X} := \mathbb{R} \times \mathbb{R}^d$.

Let $\Phi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{d\times d}$ denote the *transition operator* or *transition matrix* for (L), i.e. $\Phi(\cdot,\tau)\xi$ solves the initial value problem $\dot{x} = A(t)x$, $x(\tau) = \xi$. It is a 2-parameter semigroup satisfying $\Phi(t,s) \in GL(\mathbb{R}^d)$ and

$$D_1\Phi(t,s) = A(t)\Phi(t,s), \qquad \Phi(t,s)\Phi(s,\tau) = \Phi(t,\tau) \quad \text{for all } \tau,s,t \in \mathbb{R}. \tag{2.1a}$$

In order to classify the solutions to (L) according to their exponential growth, we introduce the notions:

Definition 2.1.1 ($\gamma^{\pm}$-boundedness)**.** Let $\gamma,\tau\in\mathbb{R}$. A continuous function $\phi:\mathbb{R}\to\mathbb{R}^d$ is called

(a) γ^+-*bounded*, if $\sup_{\tau\le t} e^{\gamma(\tau-t)}\left|\phi(t)\right|<\infty$,
(b) γ^--*bounded*, if $\sup_{t\le\tau} e^{\gamma(\tau-t)}\left|\phi(t)\right|<\infty$,
(c) γ-*bounded*, if $\sup_{t\in\mathbb{R}} e^{\gamma(\tau-t)}\left|\phi(t)\right|<\infty$.

Remark 2.1.2. *(1) Note that* 0*-boundedness corresponds to the conventional notion of boundedness. The zero function is* $\gamma^{\pm}$*-bounded for every* $\gamma\in\mathbb{R}$*.*

(2) The sets of $\gamma^{\pm}$*-bounded functions*

$$X^+_{\tau,\gamma}:=\left\{\phi:[\tau,\infty)\to\mathbb{R}^d\,|\,\sup_{\tau\le t}e^{\gamma(\tau-t)}\left|\phi(t)\right|<\infty\right\},\qquad \|\phi\|^+_{\tau,\gamma}:=\sup_{\tau\le t}e^{\gamma(\tau-t)}\left|\phi(t)\right|,$$
$$X^-_{\tau,\gamma}:=\left\{\phi:(-\infty,\tau]\to\mathbb{R}^d\,|\,\sup_{t\le\tau}e^{\gamma(\tau-t)}\left|\phi(t)\right|<\infty\right\},\qquad \|\phi\|^-_{\tau,\gamma}:=\sup_{t\le\tau}e^{\gamma(\tau-t)}\left|\phi(t)\right|$$

are Banach spaces. Indeed, the form a scale of Banach spaces in form of the implications

$$\gamma\le\bar\gamma\Rightarrow X^+_{\tau,\gamma}\subseteq X^+_{\tau,\bar\gamma},\qquad\qquad \bar\gamma\le\gamma\Rightarrow X^-_{\tau,\gamma}\subseteq X^-_{\tau,\bar\gamma}.$$

In classical autonomous dynamical systems, a particular important class are the so-called *hyperbolic matrices* $A\in\mathbb{R}^{d\times d}$ resp. *hyperbolic differential equations*

$$\boxed{\dot x = Ax,}\tag{2.1b}$$

which are characterized by the fact to have no eigenvalues on the imaginary axis in $\mathbb{C}$. Their importance is due to the fact that also a perturbed matrix B is hyperbolic, as long as $B-A\in\mathbb{R}^{d\times d}$ has a small norm, as well as their prototypical dynamical behavior:

Indeed, for a hyperbolic matrix $A\in\mathbb{R}^{d\times d}$ one has the spectral decomposition

$$\sigma(A)=\{\lambda_1,\dots,\lambda_n\}\,\dot\cup\,\{\lambda_{n+1},\dots,\lambda_m\}$$

with integers $0\le m\le n\le d$, and

$$\Re\lambda_j<0\quad\text{for all } 1\le j\le m,\qquad\qquad \Re\lambda_j>0\quad\text{for all } m<j\le n.$$

Moreover, if $\operatorname{Eig}_j A\subseteq\mathbb{R}^d$ denotes the generalized eigenspace[1] corresponding to the eigenvalue λ_j, it is not difficult to deduce the dynamical characterization

[1] a generalized eigenspace is defined by the relation $\operatorname{Eig}_j A:=\oplus_{k=1}^d N(A-\lambda_j\,\mathrm{id})^k$

$$S := \left\{\xi \in \mathbb{R}^d : \lim_{t\to\infty} e^{At}\xi = 0\right\} = \left\{\xi \in \mathbb{R}^d : \sup_{t\ge 0}\left|e^{At}\xi\right| < \infty\right\} = \oplus_{j=1}^{m} \operatorname{Eig}_j A,$$

$$U := \left\{\xi \in \mathbb{R}^d : \lim_{t\to\infty} e^{-At}\xi = 0\right\} = \left\{\xi \in \mathbb{R}^d : \sup_{t\le 0}\left|e^{-At}\xi\right| < \infty\right\} = \oplus_{j=m+1}^{1} \operatorname{Eig}_j A.$$

For obvious reasons, S is called the *stable* and U the *unstable subspace* of (2.1b) (see Fig. 2.1); one has $\mathbb{R}^d = S \oplus U$.

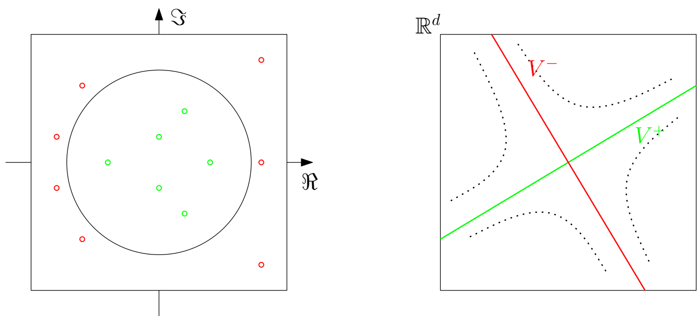

Fig. 2.1 WARNING!!! Not correct Hyperbolic spectral decomposition for A (left). Stable (green) subspace S and unstable (red) subspace U

Our next goal is to obtain a similar dynamical characterization for nonautonomous difference equations

$$x_{k+1} = A_k x_k, \tag{L}$$

where $A_k \in \mathbb{R}^{d\times d}$, $k \in \mathbb{I}$, is assumed to be a sequence of invertible matrices; the latter assumption is for the sake of a simple presentation only.

An *invariant projector* for (L) is a function $P : \mathbb{R} \to \mathbb{R}^{d\times d}$ of projections $P(t)$, $t \in \mathbb{R}$, satisfying

$$P(t)\Phi(t,s) = \Phi(t,s)P(s) \quad \text{for all } s,t \in \mathbb{R}. \tag{2.1c}$$

Due to the relation $P = \Phi(\cdot,s)P(s)\Phi(s,\cdot)$ an invariant projector is continuous and the spaces $P(t)$, $t \in \mathbb{R}$, have the same dimensions. This also holds for the fibers of the associated vector bundles

$$\mathcal{N}(P) := \{(\tau,\xi) \in \mathcal{X} : \xi \in N(P(\tau))\}, \qquad \mathcal{R}(P) := \{(\tau,\xi) \in \mathcal{X} : \xi \in R(P(\tau))\}.$$

For later use, we also define *Green's function* $\Gamma : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{d\times d}$ by

$$\Gamma(t,s) := \begin{cases} \Phi(t,s)P(s), & s \le t, \\ -\Phi(t,s)[\mathrm{id} - P(s)], & t < s. \end{cases}$$

Definition 2.1.3 (exponential dichotomy)**.** A linear differential equation (L) is said to possess an *exponential dichotomy* (ED for short), if there exists an invariant projector $P : \mathbb{R} \to \mathbb{R}^{d\times d}$ and reals $\alpha > 0$, $K \geq 1$ such that

$$|\Phi(t,s)P(s)| \leq Ke^{-\alpha(t-s)} \quad \text{for all } s \leq t,$$
$$|\Phi(t,s)[\mathrm{id} - P(s)]| \leq Ke^{\alpha(t-s)} \quad \text{for all } t \leq s.$$

Remark 2.1.4. *Let $\gamma \in \mathbb{R}$ be given.*

(1) In the following the shifted system

$$\boxed{\dot{x} = [A(t) - \gamma\,\mathrm{id}]x} \tag{L_γ}$$

will play an important role; its transition operator reads as $\Phi_\gamma(t,s) = e^{\gamma(s-t)}\Phi(t,s)$ for all $s,t \in \mathbb{R}$.

(2) If (L_γ) admits an ED with projector $P(t) \equiv \mathrm{id}$ (resp. $P(t) \equiv 0$), then also (L_ζ) also has an ED with the same projector for $\gamma \leq \zeta$ (resp. $\zeta \leq \gamma$).

(3) We define the invariant vector bundles

$$\mathscr{S}_\gamma := \left\{(\tau,\xi) \in \mathscr{X} : \Phi(\cdot,\tau)\xi \in X^+_{\tau,\gamma}\right\}, \qquad \mathscr{U}_\gamma := \left\{(\tau,\xi) \in \mathscr{X} : \Phi(\cdot,\tau)\xi \in X^-_{\tau,\gamma}\right\},$$

and observe their monotonicity properties:

$$\gamma \leq \zeta \qquad \Rightarrow \qquad \mathscr{S}_\gamma \subseteq \mathscr{S}_\zeta \text{ and } \mathscr{U}_\gamma \supseteq \mathscr{U}_\zeta.$$

Example 2.1.5. *The linear autonomous equation*

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} x \tag{2.1d}$$

has the invariant vector bundles $\mathbb{R} \times \{(0,0)\}$, $\mathscr{S} := \mathbb{R} \times \mathbb{R} \times \{0\}$, $\mathscr{U} := \mathbb{R} \times \{0\} \times \mathbb{R}$ and $\mathbb{R}\times\mathbb{R}^2$. Their projections $\{(0,0)\}$, $S := \mathbb{R}\times\{0\}$, $U := \{0\}\times\mathbb{R}$ and $\mathbb{R}\times\mathbb{R}^2$ are usually called invariant subspaces. Note that S and U are the eigenspaces corresponding to the eigenvalues -1 and 1 of the coefficient matrix in (2.1d). *We deduce the dynamical characterization*

$$\mathscr{S}_0 = \left\{(\tau,\xi) \in \mathbb{R}\times\mathbb{R}^2 : (e^{-(\cdot-\tau)}\xi_1, e^{\cdot-\tau}\xi_2) \in X^+_{\tau,0}\right\},$$
$$\mathscr{U}_0 = \left\{(\tau,\xi) \in \mathbb{R}\times\mathbb{R}^2 : (e^{-(\cdot-\tau)}\xi_1, e^{\cdot-\tau}\xi_2) \in X^-_{\tau,0}\right\}.$$

Proposition 2.1.6. *Let $\gamma \in \mathbb{R}$. If (L_γ) has an ED with projector P, then*

$$\mathscr{S}_\gamma = \mathscr{R}(P), \qquad \mathscr{U}_\gamma = \mathscr{N}(P), \qquad \mathscr{S}_\gamma \oplus \mathscr{U}_\gamma = \mathscr{X}.$$

Proof. We only prove $\mathscr{S}_\gamma = \mathscr{R}(P)$, since the proof of $\mathscr{U}_\gamma = \mathscr{N}(P)$ is dual and the claimed Whitney sum is a direct consequence. Let $\tau \in \mathbb{R}$.

($\subseteq$) Given $\xi \in \mathscr{S}_\gamma(\tau)$ we obtain the existence of a $C \ge 0$ such that

$$|\Phi(t,\tau)\xi| \le Ce^{\gamma(t-\tau)} \quad \text{for all } \tau \le t$$

and consequently $\left|\Phi_\gamma(t,\tau)\xi\right| \le C$. Let us decompose $\xi = \xi_1 + \xi_2$ with $\xi_1 \in R(P(\tau))$, $\xi_2 \in N(P(\tau))$ and verify $\xi_2 = 0$. Due to (2.1a) one has

$$\xi_2 = \Phi_\gamma(\tau,t)\Phi_\gamma(t,\tau)[\mathrm{id} - P(\tau)]\xi = \Phi_\gamma(\tau,t)[\mathrm{id} - P(\tau)]\Phi_\gamma(t,\tau)\xi \quad \text{for all } t \in \mathbb{R}$$

and the ED of (L_γ) implies

$$|\xi_2| \le Ke^{\alpha(\tau-t)}\left|\Phi_\gamma(t,\tau)\xi\right| \le CKe^{\alpha(\tau-t)} \quad \text{for all } \tau \le t.$$

Since $\alpha > 0$ we obtain in the limit $t \to \infty$ that $\xi_2 = 0$.

($\supseteq$) Conversely, suppose $\xi \in R(P(\tau))$, i.e. $\xi = P(\tau)\xi$. Our first dichotomy estimate ensures

$$e^{\gamma(\tau-t)}|\Phi(t,\tau)\xi| = \left|\Phi_\gamma(t,\tau)\xi\right| \le Ke^{-\alpha(t-\tau)}|\xi| \le K|\xi| \quad \text{for all } \tau \le t.$$

Thus, $\Phi(\cdot,\tau)\xi \in X^+_{\tau,\gamma}$ and we have $\xi \in \mathscr{S}_\gamma(\tau)$. □

Corollary 2.1.7. *(a) If $\delta < \gamma + \alpha$, then (L_γ) has a unique δ^+-bounded solution in $\mathscr{N}(P)$, namely the trivial one.*

(b) If $\gamma - \alpha < \delta$, then (L_γ) has a unique δ^--bounded solution in $\mathscr{R}(P)$, namely the trivial one.

(c) If $\gamma - \alpha < \delta < \gamma + \alpha$, then the unique δ-bounded solution of (L_γ) is the trivial one.

Proof. (a) Let $\phi \in X^+_{\tau,\delta}$ be a solution of (L_γ) in $\mathscr{N}(P)$. Then Prop. 2.1.6 implies $\phi(\tau) \in R(P(\tau))$ and consequently $\phi(\tau) \in R(P(\tau)) \cap N(P(\tau))$, i.e. $\phi(\tau) = 0$.

(b) can be shown analogously, while (c) follows from the above. □

Next we consider linear inhomogeneous differential equations

$$\boxed{\dot{x} = [A(t) - \gamma\,\mathrm{id}]x + r(t)} \tag{2.1e}$$

with a continuous inhomogeneity $r : \mathbb{R} \to \mathbb{R}^d$.

Theorem 2.1.8 (linear-inhomogeneous perturbations)**.** *Let $\gamma, \tau \in \mathbb{R}$. If (L_γ) admits an ED and $\delta \in (\gamma - \alpha, \gamma + \alpha)$, then the following holds true:*

(a) For every $x_0 \in \mathbb{R}^d$ *and every inhomogeneity* $r \in X^+_{\tau,\delta}$ *there exists exactly one solution* $\phi^* \in X^+_{\tau,\delta}$ *of* (2.1e) *with* $(\tau, \phi^*(\tau) - x_0) \in \mathcal{N}(P)$*; one has*

$$\left\|\phi^*\right\|^+_{\tau,\delta} \le K|x_0| + C_1(\delta)\|r\|^+_{\tau,\delta}. \tag{2.1f}$$

(b) For every $x_0 \in \mathbb{R}^d$ *and every inhomogeneity* $r \in X^-_{\tau,\delta}$ *there exists exactly one solution* $\phi^* \in X^-_{\tau,\delta}$ *of* (2.1e) *with* $(\tau, \phi^*(\tau) - x_0) \in \mathcal{R}(P)$*; one has*

$$\left\|\phi^*\right\|^-_{\tau,\delta} \le K|x_0| + C_1(\delta)\|r\|^-_{\tau,\delta},$$

with the real constant $C_1(\delta) := \frac{K}{\delta+\alpha-\gamma} + \frac{K}{\alpha+\gamma-\delta}$.

Proof. (a) Let $\tau \in \mathbb{R}$ be fixed, $x_0 \in \mathbb{R}^d$, $r \in X^+_{\tau,\delta}$ be arbitrary and let Γ_γ denote Green's function associate to (L_γ). Then the function $\phi^* : [\tau, \infty) \to \mathbb{R}^d$,

$$\phi^*(t) := \Phi_\gamma(t,\tau)P(\tau)x_0 + \int_\tau^\infty \Gamma_\gamma(t,s)r(s)\,ds \tag{2.1g}$$

is well-defined and using the decomposition

$$\phi^*(t) = \Phi_\gamma(t,\tau)P(\tau)x_0 + \int_\tau^t \Phi_\gamma(t,s)P(s)r(s)\,ds - \int_t^\infty \Phi_\gamma(t,s)\,[\mathrm{id} - P(s)]\,r(s)\,ds$$

we deduce

$$\begin{aligned}
\left|\phi^*(t)\right| \le\ & \left|\Phi_\gamma(t,\tau)P(\tau)\right||x_0| + \int_\tau^t \left|\Phi_\gamma(t,s)P(s)\right||r(s)|\,ds \\
& + \int_t^\infty \left|\Phi_\gamma(t,s)\,[\mathrm{id}-P(s)]\right||r(s)|\,ds \\
\le\ & Ke^{(\gamma-\alpha)(t-\tau)}|x_0| + K\int_\tau^t e^{(\gamma-\alpha)(t-s)}|r(s)|\,ds + K\int_t^\infty e^{(\gamma+\alpha)(t-s)}|r(s)|\,ds \\
\le\ & Ke^{(\gamma-\alpha)(t-\tau)}|x_0| \\
& + \left[K\int_\tau^t e^{(\gamma-\alpha)(t-s)}e^{\delta(s-\tau)}\,ds + K\int_t^\infty e^{(\gamma+\alpha)(t-s)}e^{\delta(s-\tau)}\,ds\right]\|r\|^+_{\tau,\delta} \\
\le\ & Ke^{(\gamma-\alpha)(t-\tau)}|x_0| + \left[\frac{K}{\delta-\gamma+\alpha}\left(e^{\delta(t-\tau)} - e^{(\gamma-\alpha)(t-\tau)}\right) + \frac{K}{\gamma+\alpha-\delta}e^{\delta(t-\tau)}\right]\|r\|^+_{\tau,\delta}
\end{aligned}$$

for all $\tau \le t$. From this we obtain

$$\begin{aligned}
\left|\phi^*(t)\right|e^{\delta(\tau-t)} \le\ & K\underbrace{e^{(\gamma-\delta-\alpha)(t-\tau)}}_{\le 1}|x_0| + \left[\frac{K}{\delta-\gamma+\alpha}\underbrace{\left(1 - e^{(\gamma-\delta-\alpha)(t-\tau)}\right)}_{\in(0,1]} + \frac{K}{\gamma+\alpha-\delta}\right]\|r\|^+_{\tau,\delta} \\
\le\ & K|x_0| + C_1(\delta)\|r\|^+_{\tau,\delta} \quad \text{for all } \tau \le t,
\end{aligned}$$

and passing over to the least upper bound over $\tau \le t$ we deduce $\phi^* \in X^+_{\tau,\delta}$, as well as the estimate (2.1f). Furthermore, we define $A_\gamma(t) := A(t) - \gamma\,\mathrm{id}$ and compute

$$\begin{aligned}\dot{\phi}^*(t) &\overset{(2.1g)}{\equiv} A_\gamma(t)\Phi_\gamma(t,\tau)P(\tau)x_0 + P(t)r(t) + A_\gamma(t)\int_\tau^t \Phi_\gamma(t,s)P(s)r(s)\,ds \\ &\quad + [\mathrm{id} - P(t)]\,r(t) - A_\gamma(t)\int_t^\infty \Phi_\gamma(t,s)\,[\mathrm{id} - P(s)]\,r(s)\,ds \\ &\equiv A_\gamma(t)\left(\Phi_\gamma(t,\tau)P(\tau)x_0 + \int_\tau^\infty \Gamma_\gamma(t,s)r(s)\,ds\right) + r(t) \overset{(2.1g)}{\equiv} A_\gamma(t)\phi^*(t) + r(t)\end{aligned}$$

on $\mathbb{R}$ and thus ϕ^* is a solution to (2.1e). It remains to show $(\tau, \phi^*(\tau) - x_0) \in \mathcal{N}(P)$, which follows from

$$\begin{aligned}P(\tau)\phi^*(\tau) &\overset{(2.1g)}{=} P(\tau)\Phi_\gamma(\tau,\tau)P(\tau)x_0 + P(\tau)\int_\tau^\infty \Gamma_\gamma(\tau,s)r(s)\,ds \\ &= P(\tau)x_0 + \int_\tau^\infty P(\tau)\Gamma_\gamma(\tau,s)r(s)\,ds = P(\tau)x_0.\end{aligned}$$

Finally, ϕ^* is uniquely determined. Indeed, if we suppose that $\psi_* \in X^+_{\tau,\delta}$ is another solution to (2.1e) satisfying $(\tau, \psi_*(\tau) - x_0) \in \mathcal{N}(P)$, then the difference $\phi^* - \psi_* \in X^+_{\tau,\delta}$ is a solution of the homogeneous system (L_γ) and we obtain

$$P(\tau)\left[\phi^*(\tau) - \psi_*(\tau)\right] = P(\tau)x_0 - P(\tau)x_0 = 0.$$

Hence, $\phi^* - \psi_*$ is a δ^+-bounded solution of (L_γ) in $\mathcal{N}(P)$, and Cor. 2.1.7(a) implies $\phi^* - \psi_* = 0$.

(b) Completely analogous to step (a) one shows that $\phi^* : (-\infty, \tau] \to \mathbb{R}^d$,

$$\phi^*(t) := \Phi_\gamma(t,\tau)\,[\mathrm{id} - P(\tau)]\,x_0 + \int_{-\infty}^\tau \Gamma_\gamma(t,s)r(s)\,ds \tag{2.1h}$$

is a δ^--bounded solution of (2.1e) satisfying the assertions. In particular, its uniqueness follows from Cor. 2.1.7(b). □

Theorem 2.1.9 (linear-inhomogeneous perturbations)**.** *Let $\gamma \in \mathbb{R}$. If (L_γ) admits an ED and $\delta \in (\gamma - \alpha, \gamma + \alpha)$, then for every δ-bounded inhomogeneity $r : \mathbb{R} \to \mathbb{R}^d$ the equation* (2.1e) *has a unique δ-bounded solution. It is given by*

$$\phi^*(t) := \int_{\mathbb{R}} \Gamma_\gamma(t,s)r(s)\,ds \tag{2.1i}$$

and satisfies for every $\tau \in \mathbb{R}$ that

$$\sup_{t\in\mathbb{R}} e^{\delta(\tau - t)}\left|\phi^*(t)\right| \le C_2(\delta)\sup_{t\in\mathbb{R}} e^{\delta(\tau - t)}\,|r(t)| \tag{2.1j}$$

with the real constant $C_2(\delta) := C_1(\delta) + \max\left\{\frac{K}{\delta-\alpha-\gamma}, \frac{K}{\gamma+\alpha-\delta}\right\}$ *and* $C_1(\delta) > 0$ *from Thm. 2.1.8.*

Proof. The proof is similar to the one of Thm. 2.1.8 and omitted. □

Theorem 2.1.10 (roughness theorem)**.** *Suppose that* (L) *admits an ED with constants* K, α*. If* $B : \mathbb{R} \to \mathbb{R}^{d\times d}$ *is a continuous function satisfying*

$$\sup_{t\in\mathbb{R}} |B(t) - A(t)| \le \frac{\alpha}{4K^2},$$

then also the perturbed equation $\dot{x} = B(t)x$ *has an ED.*

Proof. We refer to [Cop78, p. 34] for a proof. Note that more recent proofs of Thm. 2.1.10 use very similar methods like the ones we are about to apply in Chapt. 3 for the construction of integral manifolds. □

Exercise 2.1.11. *Show the relation*

$$|\Phi(t,s)| \le \exp\left(\int_s^t |A(r)|\, dr\right) \quad \text{for all } s,t \in \mathbb{R}.$$

Exercise 2.1.12. *Prove that an invariant projector* $P : \mathbb{R} \to \mathbb{R}^{d\times d}$ *for a linear differential equation* (L) *with a continuous coefficient mapping* $A : \mathbb{R} \to \mathbb{R}^{d\times d}$ *is continuously differentiable.*

Exercise 2.1.13. *Find the invariant vector bundles of*

$$\dot{x} = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} x$$

and determine the sets $\mathscr{S}_\gamma, \mathscr{U}_\gamma$ *for* $\gamma \in \mathbb{R}$.

Exercise 2.1.14. *Verify that an autonomous linear equation* (2.1b) *has an ED, if and only if* A *has no eigenvalue on the imaginary axis.*

2.2 Dichotomy spectrum

In this section we investigate a spectral concept which is appropriate to establish a qualitative theory for nonautonomous differential equations. The following notion is crucial:

Definition 2.2.1 (dichotomy spectrum)**.** The *dichotomy spectrum* of (L) is the set

$$\Sigma(A) := \left\{\gamma \in \mathbb{R} : (L_\gamma) \text{ admits no ED}\right\}$$

and its complement $\rho(A) := \mathbb{R} \setminus \Sigma(A)$ is called *resolvent set.*

Lemma 2.2.2. *The resolvent set $\rho(A)$ is open. This means for every $\gamma \in \rho(A)$ exists an $\varepsilon = \varepsilon(\gamma) > 0$ such that $(\gamma - \varepsilon, \gamma + \varepsilon) \subset \rho(A)$ and moreover*

$$\mathscr{S}_\zeta = \mathscr{S}_\gamma, \qquad \mathscr{U}_\zeta = \mathscr{U}_\gamma \quad \text{for all } \zeta \in (\gamma - \varepsilon, \gamma + \varepsilon). \tag{2.2a}$$

Proof. For each $\gamma \in \rho(A)$ the shifted equation (L_γ) admits an ED, i.e.

$$\left|\Phi_\gamma(t,s)P(s)\right| \le Ke^{-\alpha(t-s)} \quad \text{for all } s \le t,$$
$$\left|\Phi_\gamma(t,s)[\mathrm{id} - P(s)]\right| \le Ke^{\alpha(t-s)} \quad \text{for all } t \le s$$

with an invariant projector $P : \mathbb{R} \to \mathbb{R}^{d\times d}$ and the reals $K \ge 1$, $\alpha > 0$. For $\varepsilon := \frac{\alpha}{2} > 0$ and $\zeta \in (\gamma - \varepsilon, \gamma + \varepsilon)$ we have $\Phi_\zeta(t,s) = e^{(\gamma-\zeta)(t-s)}\Phi_\gamma(t,s)$. Now P is also an invariant projector for (L_ζ) satisfying the estimates

$$\left|\Phi_\zeta(t,s)P(s)\right| \le Ke^{(\gamma-\zeta-\alpha)(t-s)} \quad \text{for all } s \le t,$$
$$\left|\Phi_\zeta(t,s)[\mathrm{id} - P(s)]\right| \le Ke^{(\gamma-\zeta+\alpha)(t-s)} \quad \text{for all } t \le s,$$

hence, $\zeta \in \rho(A)$. Moreover, since the EDs involve the same projector P, the dynamical characterization from Lemma 2.1.6 yields (2.2a). □

Lemma 2.2.3. *Let $\gamma_1, \gamma_2 \in \rho(A)$ with $\gamma_1 < \gamma_2$. Then $\mathcal{V} := \mathscr{U}_{\gamma_1} \cap \mathscr{S}_{\gamma_2}$ is an invariant vector bundle for (L) satisfying exactly one of the following two alternatives and the statements in each alternative are equivalent:*

Alternative I	*Alternative II*
(A) $\mathcal{V} = \mathcal{O}$	*(A')* $\mathcal{V} \ne \mathcal{O}$
(B) $[\gamma_1, \gamma_2] \subset \rho(A)$	*(B') there is a* $\zeta \in (\gamma_1, \gamma_2) \cap \Sigma(A)$
(C) $\mathscr{S}_{\gamma_1} = \mathscr{S}_{\gamma_2}$ *and* $\mathscr{U}_{\gamma_1} = \mathscr{U}_{\gamma_2}$	*(C')* $\dim \mathscr{S}_{\gamma_1} < \dim \mathscr{S}_{\gamma_2}$
(D) $\mathscr{S}_\gamma = \mathscr{S}_{\gamma_2}$ *and* $\mathscr{U}_\gamma = \mathscr{U}_{\gamma_2}$ *for all* $\gamma \in [\gamma_1, \gamma_2]$	*(D')* $\dim \mathscr{U}_{\gamma_1} > \dim \mathscr{U}_{\gamma_2}$

Proof. $(B) \Rightarrow (D)$ Arguing indirectly, we assume there exists a $\gamma \in [\gamma_1, \gamma_2]$ such that $\mathscr{S}_\gamma \ne \mathscr{S}_{\gamma_2}$, or $\mathscr{U}_\gamma \ne \mathscr{U}_{\gamma_2}$. W.l.o.g. we restrict to $\mathscr{S}_\gamma \ne \mathscr{S}_{\gamma_2}$ and define

$$\zeta_0 := \inf\left\{\zeta \in [\gamma, \gamma_2] : \mathscr{S}_\zeta = \mathscr{S}_{\gamma_2}\right\}.$$

The inequality $\mathscr{S}_\gamma \ne \mathscr{S}_{\gamma_2}$ implies $\zeta_0 \in [\gamma, \gamma_2]$ and therefore $\zeta_0 \in \rho(A)$. There are two cases to consider:
(i) $\mathscr{S}_{\zeta_0} = \mathscr{S}_{\gamma_2}$, for which Lemma 2.2.2 implies $\mathscr{S}_\zeta = \mathscr{S}_{\zeta_0}$ for $\zeta \in (\zeta_0 - \varepsilon, \zeta_0 + \varepsilon)$

(ii) $\mathscr{S}_{\zeta_0} \neq \mathscr{S}_{\gamma_2}$, for which Lemma 2.2.2 guarantees $\mathscr{S}_\zeta \neq \mathscr{S}_{\zeta_0}$ for $\zeta \in (\zeta_0 - \varepsilon, \zeta_0 + \varepsilon)$ with some $\varepsilon > 0$. Both conclusions contradict the definition of ζ_0.

$(D) \Rightarrow (C)$ is obvious.

$(C) \Rightarrow (B)$ The two shifted equations (L_{γ_1}) and (L_{γ_2}) admit EDs with constants $K_i \geq 1$, $\alpha_i > 0$ for $i = 1, 2$. Since $\mathscr{S}_{\gamma_1} = \mathscr{S}_{\gamma_2}$ and $\mathscr{U}_{\gamma_1} = \mathscr{U}_{\gamma_2}$, Prop. 2.1.6 ensures that both EDs have the same invariant projector P. We define $K := \max\{K\}$, $\alpha := \min\{\alpha_1, \alpha_2\}$ and deduce

$$\begin{aligned} \left|\Phi_{\gamma_i}(t,s)P(s)\right| &\leq K e^{-\alpha(t-s)} \quad \text{for all } s \leq t, \\ \left|\Phi_{\gamma_i}(t,s)[\mathrm{id} - P(s)]\right| &\leq K e^{\alpha(t-s)} \quad \text{for all } t \leq s \end{aligned}$$

and $i = 1, 2$. The first inequality for $i = 1$ and the second one for $i = 2$ imply

$$\begin{aligned} \left|\Phi_{\gamma}(t,s)P(s)\right| &\leq K e^{-\alpha(t-s)} \quad \text{for all } s \leq t, \\ \left|\Phi_{\gamma}(t,s)[\mathrm{id} - P(s)]\right| &\leq K e^{\alpha(t-s)} \quad \text{for all } t \leq s \end{aligned}$$

for every $\gamma \in [\gamma_1, \gamma_2]$ and consequently $[\gamma_1, \gamma_2] \subset \rho(A)$.

$(C) \Rightarrow (A)$ Prop. 2.1.6 implies $\mathscr{V} = \mathscr{U}_{\gamma_1} \cap \mathscr{S}_{\gamma_2} = \mathscr{U}_{\gamma_1} \cap \mathscr{S}_{\gamma_1} = \mathscr{O}$ and we have established the implications $(B) \Leftrightarrow (C) \Leftrightarrow (D) \Rightarrow (A)$.

$(C') \Leftrightarrow (D')$ Prop. 2.1.6 yields $\dim \mathscr{S}_{\gamma_i} + \dim \mathscr{U}_{\gamma_i} = d$ for $i = 1, 2$, and the equivalences $\dim \mathscr{S}_{\gamma_1} < \dim \mathscr{S}_{\gamma_2} \Leftrightarrow d - \dim \mathscr{U}_{\gamma_1} < d - \dim \mathscr{U}_{\gamma_2} \Leftrightarrow \dim \mathscr{U}_{\gamma_1} > \dim \mathscr{U}_{\gamma_2}$.

$(B') \Rightarrow (C'), (D')$ Since (B') is the opposite of (B), the established implication $(C) \Rightarrow (B)$ yields $\mathscr{S}_{\gamma_1} \neq \mathscr{S}_{\gamma_2}$ or $\mathscr{U}_{\gamma_1} \neq \mathscr{U}_{\gamma_2}$. Monotonicity guarantees $\mathscr{S}_{\gamma_1} \subsetneq \mathscr{S}_{\gamma_2}$ or $\mathscr{U}_{\gamma_1} \supsetneq \mathscr{U}_{\gamma_2}$, and w.l.o.g. we focus on the first inclusion. Then there exists an instant $t \in \mathbb{R}$ such that $\mathscr{S}_{\gamma_1}(t) \subsetneq \mathscr{S}_{\gamma_2}(t)$. For subspaces, however, this is possible only if $\dim \mathscr{S}_{\gamma_1}(t) < \dim \mathscr{S}_{\gamma_2}(t)$.

$(C'), (D') \Rightarrow (A')$ Using $\dim \mathscr{S}_{\gamma_1} < \dim \mathscr{S}_{\gamma_2}$ and $\dim \mathscr{S}_{\gamma_1} + \dim \mathscr{U}_{\gamma_1} = d$ we obtain

$$\dim \mathscr{V} = \dim\left(\mathscr{U}_{\gamma_1} \cap \mathscr{S}_{\gamma_2}\right) \geq \dim \mathscr{U}_{\gamma_1} + \dim \mathscr{S}_{\gamma_2} - d > \dim \mathscr{U}_{\gamma_1} + \dim \mathscr{S}_{\gamma_1} - d = 0$$

and therefore $\mathscr{V}$ is not the trivial invariant vector bundle, which is 0-dimensional.

$(A') \Rightarrow (B')$ Since (A') is the opposite of (A), the proved implication $(B) \Rightarrow (A)$ implies the opposite of (B), which is (B'). Thus, we have shown that the assertions in Alternative II are equivalent.

$(A) \Rightarrow (B)$ remains to show, but this is equivalent to the proved $(B') \Rightarrow (A')$. □

Before stating the main result of this section, we observe the relation

$$X_1 \supseteq X_2 \quad \Rightarrow \quad X_1 \cap (Y + X_2) = (X_1 \cap Y) + X_2 \tag{2.2b}$$

for subspaces X_1, X_2, Y of a linear space X.

Theorem 2.2.4 (spectral theorem)**.** *The dichotomy spectrum* $\Sigma(A)$ *of* (L) *is the disjoint union of* $n \in \{0, \ldots, d\}$ *closed intervals, so-called* spectral intervals, *i.e. one has* $\Sigma(A) = \emptyset$, $\Sigma(A) = \mathbb{R}$ *or one of the four cases*

$$\Sigma(A) = \begin{cases} [a_1, b_1] \\ or \\ (-\infty, b_1] \end{cases} \cup [a_2, b_2] \cup \ldots \cup [a_{n-1}, b_{n-1}] \cup \begin{cases} [a_n, b_n] \\ or \\ [a_n, \infty) \end{cases} \tag{2.2c}$$

with reals $a_1 \le b_1 < a_2 \le b_2 < \ldots < a_n \le b_n$. *Moreover, if we choose*

- $\gamma_0 \in \rho(A)$ *with* $(-\infty, \gamma_0) \subseteq \rho(A)$ *if possible and otherwise set* $\mathscr{U}_{\gamma_0} := \mathscr{X}$, $\mathscr{S}_{\gamma_0} := \mathscr{O}$,
- $\gamma_n \in \rho(A)$ *with* $(\gamma_n, \infty) \subseteq \rho(A)$ *if possible and otherwise set* $\mathscr{U}_{\gamma_n} := \mathscr{O}$, $\mathscr{S}_{\gamma_n} := \mathscr{X}$,

then the sets $\mathscr{V}_0 := \mathscr{S}_{\gamma_0}$, $\mathscr{V}_{n+1} := \mathscr{U}_{\gamma_n}$ *are invariant vector bundles of* (L). *If* $n \ge 2$ *and* $\gamma_i \in (b_i, a_{i+1}) \subseteq \rho(A)$, *then the intersections*

$$\mathscr{V}_i := \mathscr{U}_{\gamma_{i-1}} \cap \mathscr{S}_{\gamma_i} \quad \textit{for all } 1 \le i < n$$

are called spectral manifolds *and have the following properties:*

(a) $\mathscr{V}_i$ *is an invariant vector bundle of* (L) *with* $\dim \mathscr{V}_i > 0$,
(b) $\mathscr{V}_i$, $0 \le i \le n+1$, *are independent of the choice of* γ_i *above,*
(c) *one has the Whitney sum* $\mathscr{V}_0 \oplus \ldots \oplus \mathscr{V}_{n+1} = \mathscr{X}$.

Remark 2.2.5. *The robustness Thm. 2.1.10 guarantees that for each sufficiently small* $\varepsilon > 0$ *and each* $\gamma \in \rho(A)$ *there exists a* $\delta = \delta(\varepsilon, \gamma) > 0$ *such that* γ *is also contained in the resolvent set* $\rho(B)$ *of a perturbed equation* $\dot{x} = B(t)x$, *provided* $B : \mathbb{R} \to \mathbb{R}^{d \times d}$ *is a continuous function satisfying*

$$\sup_{t \in \mathbb{R}} |B(t) - A(t)| \le \delta.$$

Nevertheless, under perturbation of (L) *two nearby spectral intervals in* (2.2c) *can melt together, or a spectral interval can break up into two new intervals. One says that* $\Sigma(A)$ *depends upper-semicontinuously on* A.

Proof. Above all, recall that the resolvent set $\rho(A)$ is open due to Lemma 2.2.2 and therefore the dichotomy spectrum $\Sigma(A) \subseteq \mathbb{R}$ is the disjoint union of closed intervals. Next we establish that the set $\Sigma(A)$ consists of at most d intervals. Indeed, if $\Sigma(A)$ contains $d+1$ components, then one can choose points $\zeta_1 < \ldots < \zeta_d$ in $\rho(A)$ such that each of the intervals $(-\infty, \zeta_1), (\zeta_1, \zeta_2), \ldots, (\zeta_d, \infty)$ has a nonempty intersection with $\Sigma(A)$. Now Alternative II of Lemma 2.2.3 implies

$$0 \le \dim \mathscr{S}_{\zeta_1} < \ldots < \dim \mathscr{S}_{\zeta_d} \le d$$

and therefore $\dim \mathscr{S}_{\zeta_1} = 0$ or $\dim \mathscr{S}_{\zeta_d} = d$ holds. W.l.o.g. we suppose $\dim \mathscr{S}_{\zeta_d} = d$, which means $\mathscr{S}_{\zeta_d} = \mathscr{X}$. Thanks to Prop. 2.1.6 the invariant projector P of the corresponding ED for (L_{ζ_d}) is the identity and Rem. 2.1.4(2) implies the contradiction $(\zeta_d,\infty)\subseteq\rho(A)$. This proves the above representations (2.2c) of $\Sigma(A)$.

Evidently, the sets $\mathcal{V}_i$, $0\le i\le n+1$, are invariant vector bundles (cf. Prop. 1.3.4). To show that $\dim\mathcal{V}_i>0$ for $i\ge 1$, let us assume that $\dim\mathcal{V}_1=0$ and therefore $\mathcal{U}_{\gamma_0}\cap\mathcal{S}_{\gamma_1}=\mathcal{O}$. If $(-\infty,b_1]$ is a spectral interval, then $\mathcal{S}_{\gamma_1}=\mathcal{O}$ and the projector of the corresponding ED for equation (L_{γ_1}) equals 0. Hence, Rem. 2.1.4(2) implies the contradiction $(-\infty,\gamma_1)\subseteq\rho(A)$. If $[a_1,b_1]$ is a spectral interval, then $[\gamma_0,\gamma_1]\cap\Sigma(A)\ne\emptyset$ and Alternative II of Lemma 2.2.3 yields a contradiction. Therefore, $\dim\mathcal{V}_1>0$ and similarly $\dim\mathcal{V}_n>0$. Furthermore for $n\ge 3$ and $i=2,\dots,n-1$ one has $(\gamma_{i-1},\gamma_i)\cap\Sigma(A)\ne\emptyset$ and Alternative II of Lemma 2.2.3 implies $\dim\mathcal{V}_i>0$.

For $i<j$ we have $\mathcal{V}_i\subseteq\mathcal{S}_{\gamma_i}$ and $\mathcal{V}_j\subseteq\mathcal{U}_{\gamma_{j-1}}\subseteq\mathcal{U}_{\gamma_i}$ and with Prop. 2.1.6 this gives $\mathcal{V}_i\cap\mathcal{V}_j\subseteq\mathcal{S}_{\gamma_i}\cap\mathcal{U}_{\gamma_i}=\mathcal{O}$, so $\mathcal{V}_i\cap\mathcal{V}_j=\mathcal{O}$ for $i\ne j$.

To show the representation $\mathcal{V}_0+\ldots+\mathcal{V}_{n+1}=\mathcal{X}$, we recall the monotonicity relations (cf. Rem. 2.1.4(3))

$$\mathcal{S}_{\gamma_0}\subseteq\ldots\subseteq\mathcal{S}_{\gamma_n},\qquad\qquad \mathcal{U}_{\gamma_0}\supseteq\ldots\supseteq\mathcal{U}_{\gamma_n}$$

and the identities $\mathcal{S}_{\gamma_i}+\mathcal{U}_{\gamma_i}=\mathbb{R}^d$ for $0\le i\le n$. Thus, $\mathcal{X}=\mathcal{V}_0+\mathcal{U}_{\gamma_0}$. Using the algebraic relation (2.2b) for $n\ge 1$, one has

$$\mathcal{X}=\mathcal{V}_0+\mathcal{U}_{\gamma_0}\cap\underbrace{(\mathcal{S}_{\gamma_1}+\mathcal{U}_{\gamma_1})}_{=\mathcal{X}}=\mathcal{V}_0+(\mathcal{U}_{\gamma_0}\cap\mathcal{S}_{\gamma_1})+\mathcal{U}_{\gamma_1}=\mathcal{V}_0+\mathcal{V}_1+\mathcal{U}_{\gamma_1},$$

$$\mathcal{X}=\mathcal{V}_0+\mathcal{V}_1+\mathcal{U}_{\gamma_1}\cap\underbrace{(\mathcal{S}_{\gamma_2}+\mathcal{U}_{\gamma_2})}_{=\mathcal{X}}=\mathcal{V}_0+\mathcal{V}_1+(\mathcal{U}_{\gamma_1}\cap\mathcal{S}_{\gamma_2})+\mathcal{U}_{\gamma_2}=\mathcal{V}_0+\mathcal{V}_1+\mathcal{V}_2+\mathcal{U}_{\gamma_2}$$

and mathematical induction yields $\mathcal{X}=\mathcal{V}_0+\ldots+\mathcal{V}_{n+1}$.

In order to finish the proof, let $\delta_0,\dots,\delta_n\in\rho(A)$ satisfy the same properties as the reals $\gamma_0,\dots,\gamma_n$ given above. Then Alternative I of Lemma 2.2.3 guarantees

$$\mathcal{S}_{\gamma_i}=\mathcal{S}_{\delta_i},\qquad\qquad \mathcal{U}_{\gamma_i}=\mathcal{U}_{\delta_i}\quad\text{for all } 0\le i\le n$$

and consequently the invariant vector bundles $\mathcal{V}_0,\dots,\mathcal{V}_{n+1}$ do not depend on the choice of $\gamma_0,\dots,\gamma_n$. □

Exercise 2.2.6. *Show that a scalar differential equation $\dot{x}=a(t)x$, $a:\mathbb{R}\to\mathbb{R}$ piecewise continuous, has the transition operator*

$$\Phi(t,s)=\exp\left(\int_s^t a(\tau)\,d\tau\right)\quad\textit{for all } s,t\in\mathbb{R}$$

and compute the dichotomy spectrum $\Sigma(a)$ for the following coefficient mappings:

(a) $a(t)=|t|$
(b) $a(t)=t$
(c) $a(t)=\begin{cases}\beta+t, & t<0,\\ \beta, & t\ge 0\end{cases}$

(d) $a(t) = \begin{cases} \alpha, & t < 0, \\ \alpha + t, & t \geq 0 \end{cases}$

(e) $a(t) = \begin{cases} \alpha, & t < 0, \\ \beta, & t \geq 0 \end{cases}$

with reals $\alpha \leq \beta$.

2.3 Bounded growth

In this section we investigate the dichotomy spectrum for a class of equations frequently met in applications. A linear differential equation (L) is said to possess *bounded growth*, if there exist constants $K \geq 1$, $a \geq 0$ such that

$$|\Phi(t,s)| \leq K e^{a|t-s|} \quad \text{for all } s, t \in \mathbb{R}. \tag{2.3a}$$

From Exercise 2.1.11 we see that systems (L) with a bounded coefficient function $A : \mathbb{R} \to \mathbb{R}^{d \times d}$, and in particular autonomous and periodic problems, have bounded growth. This class features a simple dichotomy spectrum:

Theorem 2.3.1. *A linear equation* (L) *has bounded growth, if and only if it possesses a nonempty and compact dichotomy spectrum*

$$\Sigma(A) = \bigcup_{i=1}^{n} [a_i, b_i] \quad \text{with } 1 \leq n \leq d,$$

the spectral manifolds $\mathcal{V}_0, \mathcal{V}_{n+1}$ *are trivial and therefore* $\mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_n = \mathscr{X}$.

Proof. $(\Rightarrow)$ We assume that (2.3a) is fulfilled and choose $\gamma > a$. For $\alpha := \gamma - a > 0$ the estimate (2.3a) implies

$$\left|\Phi_\gamma(t,s)\right| \leq K e^{-\alpha(t-s)} \quad \text{for all } s \leq t$$

and consequently (L_γ) admits an ED with projector $P(t) \equiv \mathrm{id}$. In particular, we have $\gamma \in \rho(A)$ and similarly for $\gamma < -a$ it is $\Sigma(A) \subseteq [-a, a]$, i.e. the dichotomy spectrum is bounded. In addition, Prop. 2.1.6 guarantees $\mathcal{V}_0 = \mathcal{V}_{n+1} = \mathcal{O}$, since

$$\begin{aligned} &\mathscr{S}_\gamma = \mathscr{X}, && \mathscr{U}_\gamma = \mathcal{O} \quad \text{for all } \gamma > a, \\ &\mathscr{S}_\gamma = \mathcal{O}, && \mathscr{U}_\gamma = \mathscr{X} \quad \text{for all } \gamma < -a. \end{aligned}$$

It remains to verify that $\Sigma(A)$ is nonempty. For this, $\gamma_0 := \inf\left\{\gamma \in \rho(A) : \mathscr{S}_\gamma = \mathscr{X}\right\}$ fulfills $\gamma_0 \in [-a, a]$. Arguing indirectly, let us assume $\gamma_0 \in \rho(A)$ and we distinguish two cases:

- $\mathscr{S}_{\gamma_0} = \mathscr{X}$: Then Lemma 2.2.2 implies $\mathscr{S}_\gamma = \mathscr{X}$ for $\gamma \in (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon)$
- $\mathscr{S}_{\gamma_0} = \mathscr{O}$: Here, Lemma 2.2.2 implies $\mathscr{S}_\gamma \neq \mathscr{X}$ for $\gamma \in (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon)$

for some $\varepsilon > 0$. Both conclusions contradict the definition of γ_0.

($\Leftarrow$) Choose points $\gamma_0, \dots, \gamma_n \in \rho(A)$ such that

$$\gamma_0 < a_1 \leq b_1 < \gamma_1 < \ldots < \gamma_{n-1} < a_n \leq b_n < \gamma_n.$$

Monotonicity implies the inclusion $\mathcal{V}_i = \mathscr{U}_{\gamma_{i-1}} \cap \mathscr{S}_{\gamma_i} \subseteq \mathscr{U}_{\gamma_0} \cap \mathscr{S}_{\gamma_n}$, $1 \leq i \leq n$, and therefore $\mathcal{V}_1 + \ldots + \mathcal{V}_n \subseteq \mathscr{U}_{\gamma_0} \cap \mathscr{S}_{\gamma_n}$. Since by assumption $\mathscr{X} = \mathcal{V}_1 + \ldots + \mathcal{V}_n$, one arrives at $\mathscr{U}_{\gamma_0} = \mathscr{X}$ and $\mathscr{S}_{\gamma_n} = \mathscr{X}$. Thus, Prop. 2.1.6 shows that (L_{γ_0}) admits an ED with constants $K \geq 1$, $\alpha_1 > 0$ and projector $P(t) \equiv 0$, i.e. the dichotomy estimate

$$|\Phi(t,s)| \leq K e^{(\gamma_0 + \alpha_1)(t-s)} \quad \text{for all } t \leq s$$

holds. Also the shifted equation (L_{γ_n}) has an ED with constants $K \geq 1$, $\alpha_2 > 0$, projector $P(t) \equiv \mathrm{id}$ and it is

$$|\Phi(t,s)| \leq K e^{(\gamma_n + \alpha_2)(t-s)} \quad \text{for all } s \leq t.$$

We finally combine the two above estimates with

$$K := \max\{K\}, \qquad a := \max\left\{0, -\gamma_0 - \alpha_1, \gamma_n - \alpha_2\right\}$$

in order to obtain $|\Phi(t,s)| \leq K e^{a|t-s|}$ for all $s, t \in \mathbb{R}$. □

Exercise 2.3.2. *Given $A \in \mathbb{R}^{d \times d}$ compute the dichotomy spectrum of an autonomous equation* (2.1b).

Chapter 3
Integral manifolds

Study the behavior of solutions of the linear equation $y' = A(t)y$ near $y = 0$, and then show (if possible) that the solutions of the nonlinear equation

$$y' = A(t)y + F(y,t)$$

near $y = 0$ inherit the same behavior.
G.R. Sell [Sel78]

In qualitative studies on nonlinear dynamical systems, invariant manifolds are omnipresent and play a crucial role in a variety of ways for local as well as global questions: For instance, local stable and unstable manifolds dictate the saddle-point behavior in the vicinity of hyperbolic solutions (or surfaces) of a system. Center manifolds are a primary tool to simplify given dynamical systems in terms of a reduction of their state space dimension — this manifests in the celebrated reduction principle of Pliss. Concerning a more global perspective, stable manifolds serve as separatrix between different domains of attractions and allow a classification of solutions with a specific asymptotic behavior. Systems with a gradient structure possess global attractors consisting of unstable manifolds (and equilibria). Finally, so-called inertial manifolds are global versions of the classical center-unstable manifolds and yield a global reduction principle for typically infinite-dimensional dissipative equations.

For nonautonomous differential equations, invariant manifolds are denoted as integral manifolds. More precisely, an *integral manifold* $\mathcal{W} \subseteq \mathbb{R} \times \mathbb{R}^d$ is an invariant nonautonomous set, where each fiber $\mathcal{W}(t)$ is a (smooth) manifold. For linear differential equations

$$\dot{x} = A(t)x$$

the spectral manifolds $\mathcal{V}_i$ constructed in Thm. 2.2.4 are examples of integral manifolds. In the present chapter we rigorously investigate how $\mathcal{V}_i$ persist under nonlinear perturbations.

3.1 Semilinear differential equations

In this section we deal with semilinear differential equation

$$\dot{x} = A(t)x + F(t,x) \tag{S}$$

in the state space $\mathbb{R}^d$. In this context the term "semilinear" means that the nonlinear term $F:\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ is so weak (in a sense to be specified) that the dynamics of (S) is largely determined by its linear part (L).

The transition matrix of (L) is denoted by $\Phi(t,s) \in \mathbb{R}^{d\times d}$ and we write $\varphi(t;\tau,\xi)$ for the general solution to (S).

We begin by stating some frequently used assumptions for our prototype system (S). From now on we assume:

Hypothesis 3.1.1. *(i) The linear part* (L) *has the dichotomy spectrum*

$$\Sigma(A) = \bigcup_{i=1}^{n} [a_i, b_i] \quad \textit{with } 1 \le n \le d.$$

(ii) One has the identity

$$F(t,0) \equiv 0 \quad \textit{on } \mathbb{R} \tag{3.1a}$$

and the continuous mapping $F:\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ *satisfies a global Lipschitz estimate*

$$|F(t,x) - F(t,\bar{x})| \le L|x-\bar{x}| \quad \textit{for all } t \in \mathbb{R},\ x,\bar{x} \in \mathbb{R}^d \tag{3.1b}$$

with $L \ge 0$. *Moreover, for some* $\delta_{\max} > 0$ *we require*

$$L < \frac{\delta_{\max}}{4K}, \tag{3.1c}$$

choose a fixed $\delta \in (4KL, \delta_{\max})$ *and abbreviate* $\Gamma_i := (b_i + \delta, a_{i+1} - \delta)$.

(iii) Assume the partial derivatives $D_2^k F(t,\cdot)$, $t \in \mathbb{R}$, *exist, are continuous on* $\mathbb{R}^d$ *up to order* $m \in \mathbb{N}$, *and suppose they are globally bounded, i.e. for* $2 \le k \le m$ *we have*

$$|F|_k := \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^d} \left\| D_2^k F(t,x) \right\| < \infty.$$

Remark 3.1.2. *(1) We choose* $\gamma \in \rho(A)$, *say* $\gamma \in (b_i, a_{i+1})$ *for* $0 \le i \le n$[1] *and obtain that the shifted differential equation* (L_γ) *admits an ED on* $\mathbb{R}$ *with invariant projector* Q_i *and complementary projector* $P_i := \mathrm{id} - Q_i$. *This means we can choose reals* $\alpha_i < \beta_i$ *such that*

$$|\Phi(t,s)Q_i(s)| \le K e^{\alpha_i(t-s)}, \qquad |\Phi(s,t)P_i(t)| \le K e^{\beta_i(s-t)} \quad \textit{for all } s \le t. \tag{3.1d}$$

It is easy to see that the existence of suitable values for δ *follows from* (3.1c). *Due to the inequality* $0 < \delta < \delta_{\max}$ *there exist functions* γ *such that* $\alpha_i + \delta < \gamma < \beta_i - \delta$.

[1] this includes the convention $b_0 = -\infty$ and $a_{n+1} = \infty$

(2) As a consequence of (3.1b)*, the partial derivatives D_2F are globally bounded on $\mathbb{R}\times\mathbb{R}^d$ by the Lipschitz constant L.*

(3) Under Hyp. 3.1.1(i)–(ii) the solutions $\varphi(\cdot;\tau,\xi)$ exist and are unique on $\mathbb{R}$ for arbitrary initial pairs $\tau\in\mathbb{R}$, $\xi\in\mathbb{R}^d$.

(4) There are two possible interpretations of the smallness condition (3.1c).

- *When interested in local results near the trivial solution of* (S)*, by means of a cut-off-technique using radial retractions, we can replace the strong assumption on the existence of $L<\infty$ and* (3.1c) *by*

$$\lim_{x,\bar{x}\to 0}\frac{F(t,x)-F(t,\bar{x})}{\|x-\bar{x}\|}=0 \quad \textit{uniformly in } t\in\mathbb{R}.$$

  Similarly, using C^m-bump functions (cf., for instance, Lemma 3.2.6), one substitutes the existence of $|F|_n$, $i=1,2$, $n\in\{2,\dots,m\}$ and (3.1c) *by*

$$\lim_{x\to 0} D_2F(t,x)=0 \quad \textit{uniformly in } t\in\mathbb{R}.$$

  As mentioned above, however, the obtained results hold only locally, i.e., as long as solutions stay in a neighborhood of zero.
- *On the other hand, for differential equations* (S) *possessing an absorbing ball $B_R(0)\subseteq\mathbb{R}^d$, $R>0$, it is sufficient to assume a Lipschitz condition on $B_R(0)$ only (uniformly in $t\in\mathbb{R}$). Provided the gap $\beta_i-\alpha_i$ in $\Sigma(A)$ is sufficiently wide, one can choose $\delta_{\max}$ so large that* (3.1c) *holds after the nonlinearities F have been restricted to $B_R(0)$ using the above cut-off approaches.*

At this point we transplant most of our technical preparations into an abstract lemma. It particularly allows to characterize the γ^+-solutions of (S) as fixed points of a suitable so-called *Lyapunov-Perron operator*.

Lemma 3.1.3. *Assume Hyp. 3.1.1(i)–(ii), choose $\tau\in\mathbb{R}$ fixed and set $\delta_{\max}:=\frac{\beta-\alpha}{2}$. Then for growth rates $\gamma\in(\alpha,\beta)$, the operator $T_\tau: X^+_{\tau,\gamma}\times\mathbb{R}^d\to X^+_{\tau,\gamma}$,*

$$\begin{aligned}T_\tau(v;x_0):=&\Phi(\cdot,\tau)Q(\tau)x_0+\int_\tau^\cdot \Phi(\cdot,s)Q(s)F(s,v(s))\,ds\\&-\int_\cdot^\infty \Phi(\cdot,s)P(s)F(s,v(s))\,ds\end{aligned}\tag{3.1e}$$

is well-defined and has, for fixed $x_0\in\mathbb{R}^d$, the following properties:

(a) $v:[\tau,\infty)\to\mathbb{R}^d$ is a γ^+-bounded solution of (S) *with $Q(\tau)v(\tau)=Q(\tau)x_0$, if and only if $v\in X^+_{\tau,\gamma}$ solves the fixed point problem*

$$v=T_\tau(v;x_0).\tag{3.1f}$$

Moreover, in case $\gamma\in[\alpha+\delta,\beta-\delta]$, we have:

(b) $T_\tau(\cdot;x_0)$ is a uniform contraction with Lipschitz condition

$$\|T_\tau(\nu;x_0)-T_\tau(\bar\nu;x_0)\|^+_{\tau,\gamma}\le \ell\,\|\nu-\bar\nu\|^+_{\tau,\gamma}, \tag{3.1g}$$

where $\ell:=\frac{2K}{\delta}L<1$,

(c) *the unique fixed point* $\nu^*_\tau(x_0)\in X^+_{\tau,\gamma}$ *of* $T_\tau(\cdot;x_0)$ *does not depend on the growth rate* γ*, it satisfies* $\nu^*_\tau(0)=0$, $\nu^*_\tau(x_0)=\nu^*_\tau(Q(\tau)x_0)$ *and we have*

$$\operatorname{lip}P(\tau)\nu^*_\tau(\cdot)(\tau)\le\frac{K^2L}{\delta-2KL}, \tag{3.1h}$$

(d) *for* $\gamma\in\Gamma$ *the mapping* $\nu^*_\tau:\mathbb{R}^d\to X^+_{\tau,\gamma}$ *is continuous.*

Proof. Let $\tau\in\mathbb{R}$ be arbitrarily fixed, and choose a growth rate $\gamma\in[\alpha+\delta,\beta-\delta]$. We show the well-definedness of the operator T_τ. Thereto, let $x_0\in\mathbb{R}^d$ be arbitrary. For $\nu,\bar\nu\in X^+_{\tau,\gamma}$ we obtain

$$\begin{aligned}
&|T_\tau(\nu;x_0)(t)-T_\tau(\bar\nu;x_0)(t)|\,e^{\gamma(\tau-t)}\\
&\overset{(3.1e)}{\le}\left|\int_\tau^t\Phi(t,s)Q(s)\,[F(s,\nu(s))-F(s,\bar\nu(s))]\,ds\right|e^{\gamma(\tau-t)}\\
&\quad+\left|\int_t^\infty\Phi(t,s)P(s)\,[F(s,\nu(s))-F(s,\bar\nu(s))]\,ds\right|e^{\gamma(\tau-t)}\\
&\overset{(3.1d)}{\le}K\int_\tau^t e^{\alpha(t-s)}\,|F(s,\nu(s))-F(s,\bar\nu(s))|\,dse^{\gamma(\tau-t)}\\
&\quad+K\int_t^\infty e^{\beta(t-s)}\,|F(s,\nu(s))-F(s,\bar\nu(s))|\,dse^{\gamma(\tau-t)}\\
&\overset{(3.1b)}{\le}\left(K\int_\tau^t e^{\alpha(t-s)}e^{\gamma(s-t)}\,ds+K\int_t^\infty e^{\beta(t-s)}e^{\gamma(s-t)}\,ds\right)L\,\|\nu-\bar\nu\|^+_{\tau,\gamma}\\
&\le\left(\frac{K}{\gamma-\alpha}+\frac{K}{\beta-\gamma}\right)L\,\|\nu-\bar\nu\|^+_{\tau,\gamma}\quad\text{for all }\tau\le t.
\end{aligned}\tag{3.1i}$$

To verify that T_τ is well-defined, we observe

$$\begin{aligned}
|T_\tau(\nu;x_0)(t)|\,e^{\gamma(\tau-t)}&\le|T_\tau(0;x_0)(t)|\,e^{\gamma(\tau-t)}+|T_\tau(\nu;x_0)(t)-T_\tau(0;x_0)(t)|\,e^{\gamma(\tau-t)}\\
&\overset{(3.1a)}{\le}|\Phi(t,\tau)Q(\tau)x_0|\,e^{\gamma(\tau-t)}+\|T_\tau(\nu;x_0)-T_\tau(0;x_0)\|^+_{\tau,\gamma}\\
&\overset{(3.1d)}{\le}K|x_0|+\left(\frac{K}{\gamma-\alpha}+\frac{K}{\beta-\gamma}\right)L\,\|\nu\|^+_{\tau,\gamma}\quad\text{for all }\tau\le t
\end{aligned}$$

and taking the supremum over $t\in[\tau,\infty)$ implies $T_\tau(\nu;x_0)\in X^+_{\tau,\gamma}$.

(a) Let $x_0\in\mathbb{R}^d$ be arbitrary.

($\Rightarrow$) If $\nu\in X^+_{\tau,\gamma}$ is a solution of (S) with $Q(\tau)\nu(\tau)=Q(\tau)x_0$, then ν also solves the linear-inhomogeneous differential equation

$$\dot x=A(t)x+F(t,\nu(t)) \tag{3.1j}$$

on $[\tau,\infty)$, where the inhomogeneous part satisfies

$$|F(t,v(t))|\,e^{\gamma(\tau-t)} \overset{(3.1a)}{=} |F(t,v(t))-F(t,0)|\,e^{\gamma(\tau-t)} \overset{(3.1a)}{\leq} L\,\|v\|^+_{\tau,\gamma} \quad \text{for all } \tau\leq t$$

and is therefore in $X^+_{\tau,\gamma}$. Then Prop. 2.1.8(a) implies that v is uniquely determined and given by the right hand side of (3.1e). So v satisfies (3.1f).
($\Leftarrow$) If $v\in X^+_{\tau,\gamma}$ solves the fixed point problem (3.1f), then a direct computation in (3.1e) yields that v solves the differential equation (3.1j) and thus also (S). Moreover, from (3.1f) and (3.1e) we have $Q(\tau)v(\tau)=Q(\tau)x_0$.

(b) Passing over to the least upper bound for $t\in[\tau,\infty)$ in (3.1i) yields

$$\|T_\tau(v;x_0)-T_\tau(\bar{v};x_0)\|^+_{\tau,\gamma}\leq \ell\,\|v-\bar{v}\|^+_{\tau,\gamma} \quad \text{for all } v,\bar{v}\in X^+_{\tau,\gamma}$$

and our choice of δ in Hyp. 3.1.1(ii) guarantees $\ell<1$. Therefore, the contraction mapping principle implies that there exists a unique fixed point $v^*_\tau(x_0)\in X^+_{\tau,\gamma}$ of $T_\tau(\cdot;x_0)$, which moreover satisfies

$$\left\|v^*_\tau(x_0)\right\|^+_{\tau,\gamma}\leq \frac{K}{1-\ell}\,|x_0| \quad \text{for all } v\in X^+_{\tau,\gamma}.$$

(c) The fixed point $v^*_\tau(x_0)\in X^+_{\tau,\gamma}$ is independent of $\gamma\in\left[\alpha+\delta,\beta-\delta\right]$, because we have the inclusion $X^+_{\tau,\alpha+\delta}\subseteq X^+_{\tau,\gamma}$, and thus, every $T_\tau(\cdot;x_0):X^+_{\tau,\gamma}\to X^+_{\tau,\gamma}$ has the same fixed point as its restriction $T_\tau(\cdot;x_0)|_{X^+_{\tau,\alpha+\delta}}$. Using the assumption (3.1a) and the uniqueness of solutions, we see $\varphi(t;\tau,0)\equiv 0$ on $[\tau,\infty)$ and since trivially $\varphi(\cdot;\tau,0)\in X^+_{\tau,\gamma}$ holds, the assertion (a) with $x_0=0$ implies that $\varphi(\cdot;\tau,0)$ solves the fixed point equation (3.1f). This fixed point, in turn, is unique and so we get $v^*_\tau(0)=\varphi(\cdot;\tau,0)=0$. Directly from (3.1e) we obtain the identity

$$v^*_\tau(Q(\tau)x_0)=T_\tau(v^*_\tau(Q(\tau)x_0);Q(\tau)x_0)=T_\tau(v^*_\tau(Q(\tau)x_0);x_0)$$

and therefore, $v^*_\tau(Q(\tau)x_0)$ is the unique fixed point of $T_\tau(\cdot;x_0)$, i.e., we have

$$v^*_\tau(x_0)=v^*_\tau(Q(\tau)x_0) \quad \text{for all } x\in\mathbb{R}^d. \tag{3.1k}$$

To prove the Lipschitz estimate (3.1h) consider $x_0,\bar{x}_0\in\mathbb{R}^d$ and the corresponding fixed points $v^*_\tau(x_0),v^*_\tau(\bar{x}_0)\in X^+_{\tau,\gamma}$ of $T_\tau(\cdot;x_0)$ and $T_\tau(\cdot;\bar{x}_0)$, respectively. We have

$$\begin{aligned}
&\left\|v^*_\tau(x_0)-v^*_\tau(\bar{x}_0)\right\|^+_{\tau,\gamma}\\
\overset{(3.1f)}{\leq}&\left\|T_\tau(v^*_\tau(x_0);x_0)-T_\tau(v^*_\tau(\bar{x}_0);x_0)\right\|^+_{\tau,\gamma}+\left\|T_\tau(v^*_\tau(\bar{x}_0);x_0)-T_\tau(v^*_\tau(\bar{x}_0);\bar{x}_0)\right\|^+_{\tau,\gamma}\\
\overset{(3.1g)}{\leq}&\,\ell\left\|v^*_\tau(x_0)-v^*_\tau(\bar{x}_0)\right\|^+_{\tau,\gamma}+\left\|T_\tau(v^*_\tau(\bar{x}_0);x_0)-T_\tau(v^*_\tau(\bar{x}_0);\bar{x}_0)\right\|^+_{\tau,\gamma}
\end{aligned}$$

and thus,

$$\begin{aligned}
\left\|v^*_\tau(x_0)-v^*_\tau(\bar{x}_0)\right\|^+_{\tau,\gamma} &\leq \frac{1}{1-\ell}\left\|T_\tau(v^*_\tau(\bar{x}_0);x_0)-T_\tau(v^*_\tau(\bar{x}_0);\bar{x}_0)\right\|^+_{\tau,\gamma}\\
\overset{(3.1e)}{=}\frac{1}{1-\ell}\sup_{t\in[\tau,\infty)}|\Phi(t,\tau)Q(\tau)\,(x_0-\bar{x}_0)|\,e^{\gamma(\tau-t)} &\overset{(3.1d)}{\leq} \frac{K}{1-\ell}\,|x_0-\bar{x}_0|\,.
\end{aligned} \tag{3.1l}$$

Moreover, directly from (3.1e) and (3.1f) we get the identity

$$P(\cdot)v_\tau^*(x_0) \overset{(2.1c)}{=} -\int_\cdot^\infty \Phi(\cdot,s)Q(s)F(s,v_\tau^*(x_0)(s))\,ds$$

and similarly to the proof of (b) this yields

$$\left\| P(\cdot)\left[v_\tau^*(x_0)-v_\tau^*(\bar{x}_0)\right]\right\|_{\tau,\gamma}^+ \le \frac{K}{\beta-\gamma} L \left\| v_\tau^*(x_0)-v_\tau^*(\bar{x}_0)\right\|_{\tau,\gamma}^+,$$

which, together with (3.1l), implies (3.1h). We have established the assertion (c).

(d) The continuity of $v_\tau^* : \mathbb{R}^d \to X_{\tau,\gamma}^+$ is clear due to (3.1l). □

Having collected the preparations in Lemma 3.1.3, we may now head for a general and quantitative version of the stable manifold theorem. It generalizes the classical theory in two directions: First, it holds for nonautonomous equations. Second, besides stable and unstable manifolds, we can additionally construct the whole hierarchy of invariant manifolds including strongly stable/unstable or center-stable/-unstable manifolds.

Theorem 3.1.4 (pseudo-stable and -unstable integral manifolds)**.** *Assume that Hyp. 3.1.1(i)–(ii) is fulfilled. If we choose* $1 \le i < n$ *such that*

$$\left(\alpha_i,\beta_i\right) \cap \Sigma(A) = \emptyset \tag{3.1m}$$

and $\delta_{\max} = \frac{\beta_i-\alpha_i}{2}$*, then the following statements are true:*

(a) The pseudo-stable integral manifold

$$\mathcal{W}_i^+ := \left\{(\tau,x_0)\in\mathbb{R}\times\mathbb{R}^d : \varphi(\cdot;\tau,x_0)\in X_{\tau,\gamma}^+ \text{ for all } \gamma\in\Gamma_i\right\} \tag{3.1n}$$

is an integral manifold of (S) *possessing the representation*

$$\mathcal{W}_i^+ = \left\{(\tau,\xi+w_i^+(\tau,\xi))\in\mathbb{R}\times\mathbb{R}^d : (\tau,\xi)\in\mathcal{R}(Q_i)\right\} \tag{3.1o}$$

with a unique continuous mapping $w_i^+ : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ *satisfying*

$$w_i^+(\tau,x_0) = w_i^+(\tau,Q_i(\tau)x_0) \in R(P_i(\tau)) \quad \text{for all } \tau\in\mathbb{R},\ x_0\in\mathbb{R}^d \tag{3.1p}$$

and the invariance equation

$$P_i(t)\varphi(t;\tau,x_0) = w_i^+(t,Q_i(t)\varphi(t;\tau,x_0)) \quad \text{for all } (\tau,x_0)\in\mathcal{W}_i^+ \tag{3.1q}$$

and $t\in\mathbb{R}$*. Furthermore, it holds:*

(a_1) $w_i^+(\tau,0)\equiv 0$ *on* $\mathbb{R}$*,*

(a_2) $w_i^+ : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ *is continuous and satisfies the Lipschitz estimate*

$$\operatorname{lip} w_i^+(\tau,\cdot) \le \frac{K^2 L}{\delta - 2KL} \quad \textit{for all } \tau \in \mathbb{R}, \tag{3.1r}$$

(a_3) *if additionally Hyp. 3.1.1(iii) and the gap condition*

$$m_i^+ \alpha_i < \beta_i$$

holds for some $m_i^+ \in \{1, \dots, m\}$*, and if we set*

$$\delta_{\max} := \min\left\{\frac{\beta_i - \alpha_i}{2}, \frac{\beta_i - m\alpha_i}{2m}\right\},$$

then the partial derivatives $D^j_{(2,3)} w_i^+$ *exist, are continuous up to order* m_i^+*, and there exist reals* $M_s^j > 0$*, such that*

$$\left| D_2^j w_i^+(\tau, x_0) \right| \le M_s^n \quad \textit{for all } 1 \le j \le m_i^+,\ \tau \in \mathbb{R},\ x_0 \in \mathbb{R}^d,$$

(a_4) *if the differential equation* (S) *is* T*-periodic for some* $T > 0$*, then* $w_i^+(\cdot, x_0)$ *is* T*-periodic for all* $x_0 \in \mathbb{R}^d$*.*

(b) The pseudo-unstable integral manifold

$$\mathscr{W}_i^- := \left\{(\tau, x_0) \in \mathbb{R} \times \mathbb{R}^d : \varphi(\cdot; \tau, x_0) \in X^-_{\tau,\gamma} \textit{ for all } \gamma \in \Gamma\right\}$$

is an integral manifold of (S) *possessing the representation*

$$\mathscr{W}_i^- = \left\{(\tau, \eta + w_i^-(\tau, \eta)) \in \mathbb{R} \times \mathbb{R}^d : (\tau, \eta) \in \mathscr{R}(P_i)\right\}$$

with a unique mapping $w_i^- : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ *satisfying*

$$w_i^-(\tau, x_0) = w_i^-(\tau, P_i(\tau)x_0) \in R(Q_i(\tau)) \quad \textit{for all } \tau \in \mathbb{R},\ x_0 \in \mathbb{R}^d$$

and the invariance equation

$$Q_i(t)\varphi(t; \tau, x_0) = w_i^-(t, P_i(t)\varphi(t; \tau, x_0)) \quad \textit{for all } (\tau, x_0) \in \mathscr{W}_i^-, \tag{3.1s}$$

and $t \in \mathbb{R}$*. Furthermore, it holds:*

(b_1) $w_i^-(\tau, 0) \equiv 0$ *on* $\mathbb{R}$*,*

(b_2) $w_i^- : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ *is continuous and satisfies the Lipschitz estimate*

$$\operatorname{lip} w_i^-(\tau,\cdot) \le \frac{K^2 L}{\delta - 2KL} \quad \textit{for all } \tau \in \mathbb{R},$$

(b_3) *if additionally Hyp. 3.1.1(iii) and the gap condition*

$$\alpha_i < m_i^- \beta_i$$

holds for some $m_i^- \in \{1,\dots,m\}$, *and if we set*

$$\delta_{\max} := \min\left\{\frac{\beta_i - \alpha_i}{2}, \frac{m\beta_i - \alpha_i}{2m}\right\},$$

then the partial derivatives $D^j_{(2,3)} w^-$ *exist, are continuous up to order* m_i^-, *and there exist reals* $M_r^j > 0$, *such that*

$$\left| D_2^j w_i^-(\tau, x_0) \right| \le M_r^j \quad \text{for all } 1 \le j \le m_i^-, \tau \in \mathbb{R}, x_0 \in \mathbb{R}^d,$$

(b_4) *if the differential equation* (S) *is* T*-periodic for some* $T > 0$, *then* $w_i^-(\cdot, x_0)$ *is* T*-periodic for all* $x_0 \in \mathbb{R}^d$.

(c) Under the condition $L < \frac{\delta}{6K}$ *only the zero solution of* (S) *is contained both in* $\mathcal{W}_i^+$ *and* $\mathcal{W}_i^-$, *i.e.*

$$\mathcal{W}_i^+ \cap \mathcal{W}_i^- = \mathbb{R} \times \{0\},$$

and hence the zero solution is the only γ*-bounded solution of* (S) *for any growth rate* $\gamma \in \Gamma_i$.

Remark 3.1.5 (gap condition)**.** *Under Hyp. 3.1.1(iii) the integral manifolds* $\mathcal{W}_i^+$ *and* $\mathcal{W}_i^-$ *are of class* C^1. *Higher order differentiability is more subtle: If* $\alpha_i \le 0$, *then the gap condition* $m\alpha_i < \beta_i$ *is always fulfilled and the integral manifolds* $\mathcal{W}_i^+$ *are as smooth as the function* F. *Dually, for* $0 \le \beta_i$ *one has* $\alpha_i < \beta_i$ *and also* $\mathcal{W}_i^-$ *inherit its smoothness from* F.

Remark 3.1.6 (global stable/unstable manifolds)**.** *In the hyperbolic case* $0 \notin \Sigma(A)$, *say for* $b_i < 0 < a_{i+1}$, *we can choose* $0 \in \Gamma_i$. *Then the set* $\mathcal{W}_i^+$ *is called the* global stable manifold, *while* $\mathcal{W}_i^-$ *is the* global unstable manifold; *both do not depend on the choice of the growth rate* $\gamma \in \Gamma_i$.

- *Choosing* $\gamma < 0$, *the global stable manifold* $\mathcal{W}_i^+$ *consists of all solutions which decay exponentially to* 0 *in forward time. It is as smooth as the nonlinearity* F.
- *For* $0 < \gamma$ *the global unstable manifold* $\mathcal{W}_i^-$ *consists of all solutions which decay exponentially to* 0 *for* $t \to -\infty$. *It has the same smoothness as* F.

Integral manifolds $\mathcal{W}_j^+$, $j < i$, *associated with spectral gaps left of* 0 *are denoted as* global strongly stable manifolds; *they share the smoothness with* F. *Accordingly,* $\mathcal{W}_j^-$, $i < j$, *corresponding to spectral gaps right of* 0 *are called* global strongly unstable manifolds.

Proof. Let $\tau \in \mathbb{R}$ be arbitrary, but fixed, and let us choose $\gamma \in \Gamma_i$. We keep $1 \le i < n$ fixed and suppress the dependence on i.

(a) First of all, we want to show that $\mathcal{W}^+$ is an integral manifold of (S). By definition, the solution $\varphi(\cdot; \tau, \xi_0)$ is γ^+-bounded for arbitrary pairs of initial values $(\tau, \xi_0) \in \mathcal{W}^+$. The 2-parameter group property (1.2c) implies for any instant

$t_0 \in [\tau,\infty)$ that $\varphi(t;t_0,\varphi(t_0;\tau,\xi_0)) \equiv \varphi(t;\tau,\xi_0)$ on $\mathbb{T}^+_{t_0}$. Hence $\varphi(\cdot;t_0,\varphi(t_0;\tau,\xi_0))$ is a γ^+-bounded function and this yields $(t_0,\varphi(t_0;\tau,\xi_0)) \in \mathscr{W}^+$ for $t_0 \in [\tau,\infty)$.

For $x_0 \in \mathbb{R}^d$, by Lemma 3.1.3(a), the unique fixed point $v^*_\tau(x_0) \in X^+_{\tau,\gamma}$ of $T_\tau(\cdot;x_0)$ is a solution of the differential equation (S) satisfying $Q(\tau)v^*_\tau(x_0)(\tau) = Q(\tau)x_0$. Now we define

$$w^+(\tau,x_0) := P(\tau)v^*_\tau(x_0)(\tau) \tag{3.1t}$$

and evidently have $w^+(\tau,x_0) \in R(P(\tau))$. In addition, Lemma 3.1.3(c) implies $w^+(\tau,x_0) = w^+(\tau,Q(\tau)x_0)$ (cf. (3.1k)). We postpone the continuity proof for w^+ to the end of part (a) and verify the representation (3.1o) and the invariance equation (3.1q) now.

($\subseteq$) Let $(\tau,x_0) \in \mathscr{W}^+$, i.e., $\varphi(\cdot;\tau,x_0)$ is γ^+-bounded. Then $\varphi(\cdot;\tau,x_0)$ trivially satisfies $Q(\tau)\varphi(\tau;\tau,x_0) = Q(\tau)x_0$ and is consequently the unique fixed point of (3.1e), i.e., we have $\varphi(\cdot;\tau,x_0) = v^*_\tau(x_0)$ (see Lemma 3.1.3(a)). This implies

$$x_0 = v^*_\tau(x_0)(\tau) = Q(\tau)v^*_\tau(x_0)(\tau) + P(\tau)v^*_\tau(x_0)(\tau) = Q(\tau)x_0 + P(\tau)v^*_\tau(Q(\tau)x_0)(\tau),$$

since $v^*_\tau(x_0) = v^*_\tau(Q(\tau)x_0)$ holds due to $T_\tau(\cdot;x_0) = T_\tau(\cdot;Q(\tau)x_0)$ (cf. (3.1e)). So, setting $\xi := Q(\tau)x_0 \in R(Q(\tau))$, we have $x_0 = \xi + P(\tau)v^*_\tau(\xi) = \xi + w^+(\tau,\xi)$ by (3.1t) and (3.1o) is verified.

($\supseteq$) Let $x_0 \in \mathbb{R}^d$ be of the form $x_0 = \xi + w^+(\tau,\xi)$, $\xi \in R(Q(\tau))$. Then

$$x_0 \overset{(3.1t)}{=} \xi + P(\tau)v^*_\tau(\xi)(\tau) = Q(\tau)v^*_\tau(\xi)(\tau) + P(\tau)v^*_\tau(\xi)(\tau) = v^*_\tau(\xi)(\tau)$$

and therefore, due to the uniqueness of solutions, one arrives at the identity $\varphi(\cdot;\tau,x_0) = \varphi(\cdot;\tau,v^*_\tau(\xi)(\tau)) = v^*_\tau(\xi) \in X^+_{\tau,\gamma}$.

With $(\tau,\xi_0) \in \mathscr{W}^+$ the invariance of $\mathscr{W}^+$ implies

$$\varphi(t;\tau,\xi_0) = Q(t)\varphi(t;\tau,\xi_0) + w^+(t,Q(t)\varphi(t;\tau,\xi_0))$$

for $\tau \le t$ and multiplication with $P(t)$ yields (3.1q).

(a_1) From Lemma 3.1.3(c) we get $w^+(\tau,0) = P(\tau)v^*_\tau(0)(\tau) = 0$ (cf. (3.1t)).

(a_2) To prove the claimed Lipschitz estimates consider $x_0, \bar{x}_0 \in \mathbb{R}^d$ and corresponding fixed points $v^*_\tau(x_0), v^*_\tau(\bar{x}_0) \in X^+_{\tau,\gamma}$ of $T_\tau(\cdot;x_0)$ and $T_\tau(\cdot;\bar{x}_0)$, respectively. One gets from Lemma 3.1.3(c)

$$\left|w^+(\tau,x_0) - w^+(\tau,\bar{x}_0)\right| \overset{(3.1t)}{=} \left|P(\tau)\left[v^*_\tau(x_0)(\tau) - v^*_\tau(\bar{x}_0)(\tau)\right]\right| \overset{(3.1h)}{\le} \frac{K^2L}{\delta - 2KL}\,|x_0 - \bar{x}_0|.$$

(a_3) Due to its technical complexity, we omit the differentiability proof for the mapping w^+. It is based on a "formal differentiation" of the fixed point identity (3.1f) w.r.t. the variable $x_0 \in \mathbb{R}^d$. Concerning the details, we refer to [PS04].

It remains to show the continuity statement for w^+. Thereto, let $\tau_0 \in \mathbb{R}$, $\xi_0 \in \mathbb{R}^d$. Then for arbitrary $\tau \in \mathbb{R}$, $x_0 \in \mathbb{R}^d$ we obtain the estimate

$$\left|w^+(\tau,x_0) - w^+(\tau,\xi_0)\right| \overset{(3.1r)}{\le} \frac{2K^2L}{\delta - 4KL}\,|x_0 - \xi_0| + \left|w^+(\tau,\xi_0) - w^+(\tau_0,\xi_0)\right|$$

and to verify the continuity of w^+ in (τ_0, ξ_0), it remains to prove the limit relation

$$\lim_{\tau \to \tau_0} w^+(\tau, \xi_0) = w^+(\tau_0, \xi_0). \tag{3.1u}$$

We abbreviate $\phi(\tau) := \varphi(\tau; \tau_0, Q(\tau_0)\xi_0 + w^+(\tau_0, \xi_0))$ and remark that the solution ϕ of (S) exists in a neighborhood of τ_0. Moreover, as a preparation we have the estimate (cf. (a_1))

$$\left|Q(\tau_0)\xi_0 + w^+(\tau_0, \xi_0)\right| \overset{(3.1d)}{\leq} K|\xi_0| + \left|w^+(\tau_0, \xi_0) - w^+(\tau_0, 0)\right| \overset{(3.1r)}{\leq} \left(K + \frac{2K^2L}{\delta - 4KL}\right)|\xi_0|$$

and we therefore obtain

$$\lim_{\tau \to \tau_0} \phi(\tau) = \phi(\tau_0). \tag{3.1v}$$

By definition of ϕ we get $P(\tau_0)\phi(\tau_0) = w^+(\tau_0, \xi_0)$, $Q(\tau_0)\phi(\tau_0) = Q(\tau_0)\xi_0$ from (3.1t) and (3.1q) implies $P(\tau)\phi(\tau) = w^+(\tau, Q(\tau)\phi(\tau))$. Hence, we arrive at

$$\begin{aligned} \left|w^+(\tau, \xi_0) - w^+(\tau_0, \xi_0)\right| &\overset{(3.1p)}{\leq} \left|w^+(\tau, Q(\tau)\xi_0) - w^+(\tau, Q(\tau)\phi(\tau))\right| \\ &\quad + \left|w^+(\tau, Q(\tau)\phi(\tau)) - w^+(\tau_0, \xi_0)\right| \\ &\overset{(3.1r)}{\leq} \frac{2K^3L}{\delta - 4KL}\left|\xi_0 - \phi(\tau)\right| + \left|P(\tau)\phi(\tau) - P(\tau_0)\phi(\tau_0)\right|, \end{aligned}$$

and so (3.1v) readily implies the desired limit relation (3.1u), because the invariant projectors $P, Q : \mathbb{R} \to \mathbb{R}^{d \times d}$ are continuous.

(a_4) Choose a growth rate $\gamma \in \Gamma$ and an arbitrary $\xi_0 \in R(Q(\tau))$. Then the solution $\nu := \varphi(\cdot; \tau, \xi_0 + w^+(\tau, \xi_0))$ of (S) is γ^+-bounded. Because of the T-periodicity of (S), we know that also $\tilde{\nu} := \nu(\cdot - T)$ is a γ^+-bounded solution. Hence, we have the inclusion $(\tau + T, \tilde{\nu}(\tau)) \in \mathscr{W}^+$ and consequently

$$\begin{aligned} w^+(\tau + T, \xi_0) &\overset{(3.1p)}{=} w^+(\tau + T, Q(\tau)\nu(-T + \tau + T)) = w^+(\tau + T, Q(\tau + T)\tilde{\nu}(\tau + T)) \\ &\overset{(3.1q)}{=} P(\tau + T)\tilde{\nu}(\tau + T) \overset{(3.1p)}{=} w^+(\tau, \xi_0), \end{aligned}$$

i.e., we established the T-periodicity of $w^+(\cdot, \xi_0)$ in case $\xi_0 \in R(Q(\tau))$. Now the T-periodicity of $w^+(\cdot, x_0)$ for general $x_0 \in \mathbb{R}^d$ follows from (3.1p).

(b) Since the present part (b) of Thm. 3.1.4 can be proved along the same lines as part (a), we present only a sketch of the proof. Analogously to (a), for $x_0 \in \mathbb{R}^d$, the γ^--bounded solutions ν the differential equation (S) with $P(\tau)\nu(\tau) = P(\tau)x_0$ may be characterized as fixed points of the operator $\bar{T}_\tau : X^-_{\tau,\gamma} \times \mathbb{R}^d \to X^-_{\tau,\gamma}$,

$$\begin{aligned} \bar{T}_\tau(\nu; x_0) :=& \Phi(\cdot, \tau)P(\tau)x_0 + \int_\tau^\cdot \Phi(\cdot, s)P(\tau)F(s, \nu(s))\,ds \\ &+ \int_{-\infty}^\cdot \Phi(\cdot, s)Q(\tau)F(s, \nu(s))\,ds. \end{aligned}$$

Here a counterpart to the above Lemma 3.1.3 holds true in the Banach space $X^-_{\tau,\gamma}$. It follows from the assumption (3.1c) that $\bar{T}_\tau(\cdot;x_0)$ is a uniform contraction on $X^-_{\tau,\gamma}$ and if $v^*_\tau(x_0) \in X^-_{\tau,\gamma}$ denotes its unique fixed point we define the mapping $w^- : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ by $w^-(\tau, x_0) := Q(\tau)v^*_\tau(x_0)(\tau)$. The claimed properties of w^- can be proved using dual arguments as (a).

(c) Let $v : \mathbb{R} \to \mathbb{R}^d$ be any γ-bounded solution of (S). By means of Hyp. 3.1.1(ii), the mapping $F(\cdot, v(\cdot))$ is γ-bounded and the unique γ-bounded solution of (3.1j), which additionally satisfies

$$\sup_{t\in\mathbb{R}} |v(t)|\, e^{\gamma(\tau-t)} \overset{(3.1b)}{\le} \frac{3K}{\delta} L \sup_{t\in\mathbb{R}} |v(t)|\, e^{\gamma(\tau-t)}.$$

Using $L < \frac{\delta}{6K}$, we thus obtain $v = 0$ and the proof of Thm. 3.1.4 is complete. □

The above Thm. 3.1.4 states that for each gap in the dichotomy spectrum $\Sigma(A)$ there exists a pair of integral manifolds $\mathcal{W}^+_i$ and $\mathcal{W}^-_i$ intersecting along the trivial solution. These integral manifolds are nested, i.e. ordered w.r.t. the set inclusion:

Corollary 3.1.7 (pseudo-stable and -unstable hierarchy)**.** *For each* $1 \le i < n$ *choose reals* $\alpha_i < \beta_i$ *satisfying* (3.1m). *If we define* $\delta_{\max} := \min_{i=1}^{n-1} \frac{\beta_i - \alpha_i}{2}$, *then one has*

(a) the pseudo-stable hierarchy

$$\mathbb{R} \times \{0\} =: \mathcal{W}^+_0 \subseteq \mathcal{W}^+_1 \subseteq \ldots \subseteq \mathcal{W}^+_{n-1} \subseteq \mathcal{W}^+_n := \mathbb{R} \times \mathbb{R}^d,$$

(b) the pseudo-unstable hierarchy

$$\mathbb{R} \times \{0\} =: \mathcal{W}^-_n \subseteq \mathcal{W}^-_{n-1} \subseteq \ldots \subseteq \mathcal{W}^-_1 \subseteq \mathcal{W}^-_0 := \mathbb{R} \times \mathbb{R}^d.$$

Proof. If we choose $\gamma_i \in \Gamma_i$ for $1 \le i < n$, then $\gamma_i < \gamma_{i+1}$ holds true. Then the inclusion $\mathcal{W}^+_i \subset \mathcal{W}^+_{i+1}$ follows from the dynamical characterization (3.1n) and Rem. 2.1.2(2). Thus, we have shown (a) and assertion (b) follows accordingly. □

In the linear case of Thm. 2.2.4 we investigated spectral manifolds constructed as intersection of a pseudo-stable spectral manifold $\mathcal{S}$ with an appropriate pseudo-unstable one $\mathcal{U}$. These integral manifolds persist in our semilinear situation. Thereto, we define the subspaces

$$\mathcal{P}^j_i := \left\{(\tau,\xi) \in \mathbb{R}^d \times \mathbb{R}^d : \xi \in R(Q_i(\tau)) \cap R(P_{j-1}(\tau))\right\},$$
$$\mathcal{Q}^j_i := \left\{(\tau,\xi) \in \mathbb{R}^d \times \mathbb{R}^d : \xi \in R(P_i(\tau)) + R(Q_{j-1}(\tau))\right\}.$$

Proposition 3.1.8 (intersection of integral manifolds)**.** *We assume that Hyp. 3.1.1(i)–(ii) is fulfilled. For each* $1 \le i < n$ *choose reals* $\alpha_i < \beta_i$ *satisfying* (3.1m) *and for* $1 \le j \le i < n$ *define* $\delta_{\max} := \min_{k \in \{i, j-1\}} \frac{\beta_k - \alpha_k}{2}$. *If the Lipschitz constant L is sufficiently small, then the nonautonomous set*

$$\mathscr{W}_i^j := \left\{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d \;\middle|\; \begin{array}{l} \text{there exists a solution } \phi : \mathbb{R} \to \mathbb{R}^d \text{ of (S)} \\ \text{with } \phi(\tau) = \xi \in \mathbb{R}^d \text{ and } \phi \in X^+_{\tau,\gamma}, \phi \in X^-_{\tau,\delta} \end{array} \right\}$$

is an integral manifold for (S)*, which is independent of* $\gamma \in \Gamma_i$, $\delta \in \Gamma_{j-1}$ *and possesses the representation as graph*

$$\mathscr{W}_i^j = \mathscr{W}_i^+ \cap \mathscr{W}_{j-1}^- = \left\{ (\tau, \eta + w_i^j(\tau, \eta)) \in \mathbb{R} \times \mathbb{R}^d : (\tau, \eta) \in \mathscr{P}_i^j \right\} \tag{3.1w}$$

of a uniquely determined mapping $w_i^j : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ *with*

$$w_i^j(\tau, \xi) = w_i^j(\tau, P_i^j(\tau)\xi) \in \mathscr{Q}_i^j(\tau) \quad \text{for all } (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d. \tag{3.1x}$$

Furthermore, it holds:

(a) $w_i^j : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ *is continuous with* $w_i^j(0, \eta) \equiv 0$ *on* $\mathbb{R}$,

(b) $w_i^j(\tau, \cdot)$ *is globally Lipschitz,*

(c) under additionally Hyp. 3.1.1(iii), and if the gap conditions

$$m_i^j \alpha_i < \beta_i, \qquad \alpha_{j-1} < m_i^j \beta_{j-1}$$

are fulfilled, then the partial derivatives $D_1^k w_i^j(\tau, \cdot)$ *exist, are continuous and globally bounded up to order* $m_i^j \le m$,

(d) if the differential equation (S) *is T-periodic for some* $T > 0$, *then* $w_i^j(\cdot, x_0)$ *is T-periodic for all* $x_0 \in \mathbb{R}^d$.

Proof. We only provide a sketch of the proof and omit the technical details. Suppose $1 \le j \le i < N$ and subdivide the proof into several steps:

(I) Our Thm. 3.1.4 guarantees the existence of two integral manifolds $\mathscr{W}_i^+$ and $\mathscr{W}_{j-1}^-$. For sufficiently small $L \ge 0$, one sees from Thm. 3.1.4(a_2) and (b_2) that the corresponding functions w_i^+ and w_{j-1}^- both satisfy

$$\operatorname{lip}_2 w_i^+ \le q < 1, \qquad \operatorname{lip}_2 w_{j-1} \le q < 1$$

for some $q \in [0, 1)$. Having this at our disposal, for every $\tau \in \mathbb{R}$ we define the operator $T_\tau : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ by

$$T_\tau(x, z; y) := \left(w_i^+(\tau, z + y), w_{j-1}^-(\tau, x + y) \right). \tag{3.1y}$$

Considering $y \in \mathbb{R}^d$ as a fixed parameter, thanks to the estimate

$$\begin{aligned}\left\|T_\tau(x,z;y)-T_\tau(\bar{x},\bar{z};y)\right\| &= \max\Big\{\left\|w_i^+(\tau,z+y)-w_i^+(\tau,\bar{z}+y)\right\|,\\ &\qquad\quad \left\|w_{j-1}^-(\tau,x+y)-w_{j-1}^-(\tau,\bar{x}+y)\right\|\Big\}\\ &\le q\left\|\left(x-\bar{x},z-\bar{z}\right)\right\| \quad \text{for all } x,\bar{x},z,\bar{z}\in\mathbb{R}^d,\end{aligned}$$

the operator $T_\tau(\cdot,y):\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d\times\mathbb{R}^d$ is a uniform contraction in $\tau\in\mathbb{R}$ and $y\in\mathbb{R}^d$. Similarly, we deduce from Thm. 3.1.4(a_2) and (b_2) that $\operatorname{lip}_3 T_\tau<\infty$ and the uniform contraction principle ensures that there exists a unique fixed point $\Upsilon_{i,j}(\tau,y)=(\Upsilon_{i,j}^+,\Upsilon_{i,j}^-)(\tau,y)\in\mathbb{R}^d\times\mathbb{R}^d$ of $T_\tau(\cdot,y)$.

(II) Now we infer the representation (3.1w) of $\mathscr{W}_i^j$ as graph of a function w_i^j over $\mathscr{P}_i^j$. From Thm. 3.1.4(a) we know that a point $(\tau,x_0)\in\mathbb{R}\times\mathbb{R}^d$ is contained in $\mathscr{W}_i^+$, if and only if there exists a $\xi_0\in R(Q_i(\tau))$ such that $x_0=\xi_0+w_i^+(\tau,\xi_0)$ and accordingly $Q_i(\tau)x_0=\xi_0+Q_i(\tau)w_i^+(\tau,x_0)=\xi_0$. This yields $(\tau,x_0)\in\mathscr{W}_i^+$ if and only if $x_0=Q_i(\tau)x_0+w_i^+(\tau,Q_i(\tau)x_0)$. Analogously from Thm. 3.1.4(b) we have the inclusion $(\tau,x_0)\in\mathscr{W}_{j-1}^-$ if and only if $x_0=P_{j-1}(\tau)x_0+w_{j-1}^-(\tau,P_{j-1}(\tau)x_0)$. The unique decomposition $x_0=\xi+\eta+\zeta$ into $\xi\in R(Q_i(\tau))$, $\eta\in\mathscr{P}_i^j(\tau)$, $\zeta\in R(P_{j-1}(\tau)$ leads to the equivalence

$$\begin{aligned}(\tau,x_0)\in\mathscr{W}_i^j \quad &\Leftrightarrow \quad x_0=Q_i(\tau)x_0+w_i^+(\tau,Q_i(\tau)x_0) \text{ and}\\ &\qquad\ \ x_0=P_{j-1}(\tau)x_0+w_{j-1}^-(\tau,P_{j-1}(\tau)x_0)\\ &\Leftrightarrow \quad \zeta=w_i^+(\tau,\xi+\eta) \text{ and } \xi=w_{j-1}^-(\tau,\eta+\zeta)\\ &\overset{(3.1y)}{\Leftrightarrow} \quad (\xi,\zeta)=T_\tau(\xi,\zeta;\eta),\end{aligned}$$

i.e. the pair $(\xi,\zeta)\in R(Q_i(\tau))\times R(P_{j-1}(\tau))$ is a fixed point of $T_\tau(\cdot;\eta)$; from the above step (I) it is uniquely determined by $\Upsilon_{i,j}(\tau,\eta)$. As a result, if we define $w_i^j(\tau,x_0):=\Upsilon_{i,j}^+(\tau,P_i^j(\tau)x_0)+\Upsilon_{i,j}^-(\tau,P_i^j(\tau)x_0)$ for $(\tau,x_0)\in\mathbb{R}\times\mathbb{R}^d$, then the representation (3.1w) holds. Moreover, by construction one has

$$w_i^j(\tau,P_i^j(\tau)x_0)=w_i^j(\tau,x_0)=w_i^+(\tau,x_0)+w_{j-1}^-(\tau,x_0)\in\mathscr{Q}_i^j(\tau).$$

(a) Since the contraction $T_\tau(\cdot;y)$ depends continuously on $(\tau,y)\in\mathbb{R}\times\mathbb{R}^d$, also its unique fixed point is continuous in these parameters due to the uniform contraction principle.

(b) The mapping $T_\tau(x,z;\cdot)$ fulfills a global Lipschitz estimate. This properties carries over to the fixed point mapping $\Upsilon_{i,j}$ and the claim follows.

(c) Due to Thm. 3.1.4(a_3) and (b_3) the mapping T_τ is of class $C^{m_i^j}$. By the uniform C^m-contraction principle, also the fixed point mapping has this property.

(d) follows directly from Thm. 3.1.4(a_4) and (b_4). □

Corollary 3.1.9 (extended hierarchy of integral manifolds)**.** *If we define define* $\delta_{\max} := \min_{i=1}^{n-1} \frac{\beta_i - \alpha_i}{2}$*, then one has the* extended hierarchy

$$\begin{array}{ccccccc}
\mathcal{W}_1^1 \subset \mathcal{W}_2^1 & \subset & \ldots & \subset & \mathcal{W}_{n-1}^1 & \subset & \mathbb{R}\times\mathbb{R}^d \\
\cup & & & & \cup & & \cup \\
\mathcal{W}_2^2 & \subset & \ldots & \subset & \mathcal{W}_{n-1}^2 & \subset & \mathcal{W}_n^2 \\
 & & & & \cup & & \cup \\
 & & \ddots & & \vdots & & \vdots \\
 & & & & \cup & & \cup \\
 & & & & \mathcal{W}_{n-1}^{n-1} & \subset & \mathcal{W}_n^{n-1} \\
 & & & & & & \cup \\
 & & & & & & \mathcal{W}_n^n .
\end{array}$$

Remark 3.1.10 (classical hierarchy)**.** *We suppose that* $0 \in \Sigma(A)$*, say* $0 \in [a_i, b_i]$*, and obtain intervals* $\Gamma_{i-1} \subseteq (-\infty, 0)$*,* $\Gamma_i \subseteq (0, \infty)$*. This spectral splitting yields the following integral manifolds:*

- $\mathcal{W}_i^1 = \mathcal{W}_i^+$ *is the* global center-stable manifold. *Thanks to* $0 \le b_i < \gamma_i \in \Gamma_i$ *it contains all solutions, which are not growing too fast as* $t \to \infty$ *in the sense of* γ_i^+*-boundedness. In particular, all forward bounded (or periodic or constant) solutions are contained in* $\mathcal{W}_i^+$*.*
- $\mathcal{W}_{i-1}^1 = \mathcal{W}_{i-1}^+$ *denotes the* global stable manifold. *Due to the inclusion* $\gamma_{i-1} \in \Gamma_{i-1}$ *with* $\gamma_{i-1} < a_i \le 0$ *it consists of all solutions decaying exponentially as* $t \to \infty$*, i.e. being* γ_{i-1}^+*-bounded.*
- $\mathcal{W}_n^i = \mathcal{W}_{i-1}^-$ *is the* global center-unstable manifold. *All solutions of* (S) *which are not growing too fast as* $t \to -\infty$ *(in the sense of* γ_{i-1}^-*-boundedness) are included in the center-unstable manifold. This time, all solutions which are bounded in backward time (or periodic or constant) lie on* $\mathcal{W}_{i-1}^-$*.*
- *The* global center manifold $\mathcal{W}_i^i = \mathcal{W}_i^+ \cap \mathcal{W}_{i-1}^-$ *consists of solutions both on the center-stable and -unstable manifold. Particularly, all bounded (or periodic or homoclinic/heteroclinic) solutions lie on this integral manifold.*
- $\mathcal{W}_n^{i+1} = \mathcal{W}_i^-$ *is the* global unstable manifold. *It consists of all solutions decaying to* 0 *exponentially as* $t \to -\infty$ *in the sense of* γ_i^-*-boundedness mit* $0 \le b_i < \gamma_i$*.*

In the autonomous case, thanks to Thm. 3.1.4(a_4)*,* (b_4) *and Prop. 3.1.8, these integral manifolds reduce to the classical five invariant manifolds.*

Proof. The cases $i = 1$ and $j = n$ have already been shown in Cor. 3.1.7 in form of the pseudo-stable and -unstable hierarchy $\mathcal{W}_i^1 = \mathcal{W}_i^+$ resp. $\mathcal{W}_n^j = \mathcal{W}_{j-1}^-$. We thus restrict to indices $1 < j \le i < n$. Above all, we choose growth rates $\gamma \in \Gamma_i$, $\delta \in \Gamma_{j-1}$ and point out that the sets $\mathcal{W}_i^j$ are dynamically characterized using solutions being both γ^+- and δ^--bounded. A growth rate $\bar{\gamma} \in \Gamma_{i+1}$ satisfies $\gamma < \bar{\gamma}$ and

Rem. 2.1.2 yields the inclusion $X^+_{\tau,\gamma} \subseteq X^+_{\tau,\bar{\gamma}}$ guaranteeing $\mathcal{W}_i^j \subset \mathcal{W}_{i+1}^j$. Analogously, for growth rates $\bar{\delta} \in \bar{\Gamma}_{j-2}$ one has $\bar{\delta} < \delta$, $X^-_{\tau,\delta} \subseteq X^-_{\tau,\bar{\delta}}$ and thus $\mathcal{W}_i^j \subset \mathcal{W}_i^{j-1}$. □

Theorem 3.1.11 (asymptotic phase)**.** *Assume that Hyp. 3.1.1(i)–(ii) is fulfilled. For $1 < i < n$ choose reals $\alpha_i < \beta_i$ satisfying* (3.1m)*, define $\delta_{\max} := \frac{\beta_i - \alpha_i}{2}$ and choose $\gamma \in \Gamma_i$. If L is sufficiently small, then the following holds:*

(a) The pseudo-unstable integral manifold $\mathcal{W}_i^-$ from Thm. 3.1.4(b) possesses an asymptotic forward phase*, i.e. there exists a mapping $\pi_i^+ : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ with the property*

$$\left|\varphi(t;\tau,\xi) - \varphi(t;\tau,\pi_i^+(\tau,\xi))\right| \leq C_i^+(|\xi|)e^{\gamma(t-\tau)} \quad \text{for all } \tau \leq t, \tag{3.1z}$$

where the function $C_i^+ : [0,\infty) \to [0,\infty)$ maps bounded sets into bounded sets. Moreover, $\pi_i^+(\tau,\cdot) : \mathbb{R}^d \to \mathcal{W}_i^-(\tau)$ is a continuous retraction onto the τ-fiber $\mathcal{W}_i^-(\tau)$, satisfying $\varphi(t;\tau,\cdot) \circ \pi_i^+(\tau,\cdot) = \pi_i^+(t,\cdot) \circ \varphi(t;\tau,\cdot)$ for all $\tau \leq t$.

(b) The pseudo-stable integral manifold $\mathcal{W}_i^+$ from Thm. 3.1.4(a) possesses an asymptotic backward phase*, i.e. there exists a mapping $\pi_i^- : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ with the property*

$$\left|\varphi(t;\tau,\xi) - \varphi(t;\tau,\pi_i^-(\tau,\xi))\right| \leq C_i^-(|\xi|)e^{\gamma(t-\tau)} \quad \text{for all } t \leq \tau,$$

where the function $C_i^- : [0,\infty) \to [0,\infty)$ maps bounded sets into bounded sets. Moreover, $\pi_i^-(\tau,\cdot) : \mathbb{R}^d \to \mathcal{W}_i^+(\tau)$ is a continuous retraction onto the τ-fiber $\mathcal{W}_i^+(\tau)$ satisfying $\varphi(t;\tau,\cdot) \circ \pi_i^-(\tau,\cdot) = \pi_i^-(t,\cdot) \circ \varphi(t;\tau,\cdot)$ for all $t \leq \tau$.

Proof. The proof is relatively involved and therefore omitted. However, it is based on the constructions of so-called invariant foliations; for details we refer the interested reader to [AW03]. □

Exercise 3.1.12. *Consider the semilinear differential equation* (S) *in $\mathbb{R}^2$ with*

$$A := \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \qquad F(x) := \begin{pmatrix} 0 & -\varepsilon \\ \varepsilon & 0 \end{pmatrix}$$

with fixed reals $\alpha < \beta$ and a parameter ε. Discuss the dynamics of $\dot{x} = Ax + F(x)$ for different values of ε and relate it to the spectral gap condition (3.1c).

3.2 Local integral manifolds

In this section, we make the first attempt to weaken the global assumptions in form of Hyp. 3.1.1(ii)–(iii). Thereto, we consider a general nonautonomous ODE

$$\boxed{\dot{x} = f(t,x)} \tag{O}$$

with a continuous right-hand side $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ being of class C^m, $m \geq 1$, in the state space variable with continuous partial derivatives $D_2 f$. For simplicity, we assume that f is defined on the whole set $\mathbb{R} \times \mathbb{R}^d$. The general solution is denoted by $\varphi(t;\tau,\xi)$.

We suppose that (O) has a bounded reference solution $\phi^* : \mathbb{R} \to \mathbb{R}^d$, which might be a constant, a periodic or a general bounded solution. We are interested in the behavior of (O) in the vicinity of ϕ^*. In particular, we want to provide a local description of the *stable set* corresponding to ϕ^*,

$$\mathcal{W}^+_{\phi^*} := \left\{ (\tau,\xi) \in \mathbb{R} \times \mathbb{R}^d \,\middle|\, \lim_{t\to\infty} \left|\varphi(t;\tau,\xi) - \phi^*(t)\right| = 0 \right\},$$

as well as of the *unstable set* corresponding to ϕ^*,

$$\mathcal{W}^-_{\phi^*} := \left\{ (\tau,\xi) \in \mathbb{R} \times \mathbb{R}^d \,\middle|\, \lim_{t\to-\infty} \left|\varphi(t;\tau,\xi) - \phi^*(t)\right| = 0 \right\}.$$

For our differential equation (O) we make the following assumptions:

Hypothesis 3.2.1. *Let $\rho_0 > 0$, $m, n \in \mathbb{N}$ and suppose that:*

(i) The variational equation

$$\boxed{\dot{x} = D_2 f(t,\phi^*(t))x} \tag{V}$$

has the dichotomy spectrum $\Sigma(\phi^*) = \bigcup_{i=1}^n [a_i, b_i]$ with $1 \leq n \leq d$.

(ii) The following limit relation holds uniformly in $t \in \mathbb{R}$,

$$\lim_{x\to 0} \int_0^1 \left[D_2 f(t,\phi^*(t) + hx) - D_2 f(t,\phi^*(t))\, dh \right] = 0. \tag{3.2a}$$

(iii) The partial derivatives $D_2^k f$, $1 \leq k \leq m$ are uniformly bounded, i.e. for each bounded $B \subseteq \mathbb{R}^d$ one has $\sup_{t\in\mathbb{R}} \sup_{x\in B} \left|D_2^k f(t,x)\right| < \infty$.

Remark 3.2.2. *The condition* (3.2a) *will guarantee (see below) that a nonlinearity is of order $o(x)$ as $x \to 0$ uniformly in $t \in \mathbb{R}$. This can be seen from the mean value theorem (see [Lan93, p. 341, Thm. 4.2])*

$$\begin{aligned} &f(t,x+\phi^*(t)) - f(t,\phi^*(t)) - D_2 f(t,\phi^*(t))x \\ &\qquad = \int_0^1 \left[D_2 f(t,x+h\phi^*(t)) - D_2 f(t,\phi^*(t)) \right] dh\, x \end{aligned}$$

Before we formulate our first result, a weaker version of the invariance notion from Def. 1.3.1 is due, which is tailor-made for the things to come. Given a vector bundle $\mathcal{V} \subseteq \mathbb{R} \times \mathbb{R}^d$ and an open neighborhood $\mathcal{U} \subseteq \mathbb{R} \times \mathbb{R}^d$ of ϕ^*, we say a graph

$$\mathcal{W} := \left\{ (\tau, \xi + w(\tau,\xi)) \in \mathbb{R} \times \mathbb{R}^d : \xi \in \mathcal{V}(\tau) \cap \mathcal{U}(\tau) \right\}$$

of a given mapping $w : \mathscr{V} \cap \mathscr{U} \to \mathbb{R}^d$ is a *local integral manifold* of eqn. (O), if

$$(t_0, x_0) \in \mathscr{W} \quad \Rightarrow \quad (t, \varphi(t; t_0, x_0)) \in \mathscr{W}$$

holds for all t as long as $\varphi(t; t_0, x_0) \in \mathscr{U}(t)$. In case $\mathscr{U} = \mathbb{R} \times \mathbb{R}^d$ we say $\mathscr{W}$ is a *global integral manifold* of (O), if the above conditions holds for all $t \in \mathbb{R}$. One speaks of a C^m*-integral manifold* of (O), provided the partial derivatives $D_2^n w$ exist and are continuous for $n \in \{1, \dots, m\}$.

Theorem 3.2.3 (local integral manifolds)**.** *Under Hyp. 3.2.1 there exist reals $\rho \in (0, \rho_0)$, $\gamma_0, \dots, \gamma_m \ge 0$ such that for all $1 \le i < n$ the following holds:*

(a) Equation (O) *has a local C^1-integral manifold*

$$\phi^* + \mathscr{W}_i^+ := \phi^* + \left\{(\tau, \eta + w_i^+(\tau, \eta)) \in \mathbb{R} \times \mathbb{R}^d : (\tau, \eta) \in \mathscr{B}_\rho(0)\right\}$$

with a C^1-mapping $w_i^+ : \mathscr{B}_\rho(0) \to \mathbb{R}^d$ satisfying (3.1q) *for all $\tau \in \mathbb{R}$, $\xi \in B_\rho(0)$. Moreover, for $(\tau, \xi) \in \mathscr{B}_\rho(0)$ one has:*

(a_1) *$w_i^+(\tau, 0) \equiv 0$ on $\mathbb{R}$ and $\left|w_i^+(\tau, \xi)\right| \le \rho$,*

(a_2) *$\lim_{x \to 0} D_2 w_i^+(t, x) = 0$ uniformly in $t \in \mathbb{R}$,*

(a_3) *under the gap condition*

$$m\alpha_i < \beta_i \tag{3.2b}$$

holds, then $\phi^* + \mathscr{W}_i^+$ is a C^m-integral manifold with

$$\left|D_2^n w_i^+(\tau, \xi)\right| \le \gamma_n \quad \text{for all } 0 \le n \le m. \tag{3.2c}$$

(b) Equation (O) *has a local C^1-integral manifold*

$$\phi^* + \mathscr{W}_i^- := \phi^* + \left\{(\tau, \eta + w_i^-(\tau, \eta)) \in \mathbb{R} \times \mathbb{R}^d : (\tau, \eta) \in \mathscr{B}_\rho(0)\right\}$$

with a C^1-mapping $w_i^- : \mathscr{B}_\rho(0) \to \mathbb{R}^d$ satisfying (3.1s) *for all $\tau \in \mathbb{R}$, $\xi \in B_\rho(0)$. Moreover, for $(\tau, \xi) \in \mathscr{B}_\rho(0)$ one has:*

(b_1) *$w_i^-(\tau, 0) \equiv 0$ on $\mathbb{R}$ and $\left|w_i^-(\tau, \xi)\right| \le \rho$,*

(b_2) *$\lim_{x \to 0} D_2 w_i^-(t, x) = 0$ uniformly in $t \in \mathbb{R}$,*

(b_3) *under the gap condition $\alpha_i < m\beta_i$, then $\phi^* + \mathscr{W}_i^-$ is a C^m-integral manifold with*

$$\left|D_2^n w_i^-(\tau, \xi)\right| \le \gamma_n \quad \text{for all } 0 \le n \le m.$$

(c) One has $(\phi^* + \mathscr{W}_i^+) \cap (\phi^* + \mathscr{W}_i^-) = \phi^*$.

The pseudo-stable and -unstable manifolds $\phi^* + \mathscr{W}_i^+$ and $\phi^* + \mathscr{W}_i^-$ intersect along the solution ϕ^*. Moreover, due to $D_2 w_i^\pm(\tau, 0) \equiv 0$ on $\mathbb{R}$, they are tangential to the invariant vector bundles $\mathscr{R}(Q_i)$ resp. $\mathscr{R}(P_i)$.

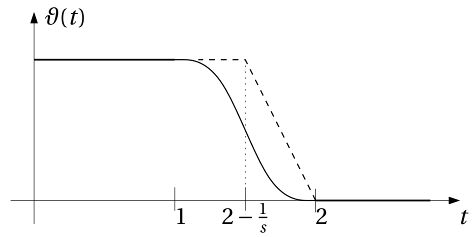

Fig. 3.1 The bump function from Lemma 3.2.5

Remark 3.2.4. *If both* (O) *and* ϕ^* *are* p*-periodic, then the integral manifolds* $\mathcal{W}_i^{\pm}$ *are also* p*-periodic. In particular, for autonomous eqns.* (O) *and constant solutions* ϕ^* *the fibers are constant and one calls* $\mathcal{W}_i^{+}(\tau)$ *or* $\mathcal{W}_i^{-}(\tau)$ *an* invariant manifold.

Before we can tackle the proof of Thm. 3.2.3, some preparations on smooth extensions of functions are due. They address bump functions and provide a kind of optimality in their Lipschitz constant (cf. Fig. 3.1).

Lemma 3.2.5 (bump functions)**.** *For every real* $s > 1$ *there exists a function* $\vartheta \in C^\infty(\mathbb{R})$ *such that* $\vartheta(t) \equiv 1$ *on* $(-\infty, 1]$, $\vartheta(t) \in [0,1]$ *for* $t \in [1,2]$, $\vartheta(t) \equiv 0$ *on* $[2,\infty)$ *and* $D\vartheta(t) \in [-s,0]$ *for* $t \in \mathbb{R}$*, as well as* $t\vartheta(t) \in [0,s]$ *for all* $t \geq 0$.

Proof. For reals $r > 0$ consider the bump function $\omega_r : \mathbb{R} \to \mathbb{R}$,

$$\omega_r(t) := \begin{cases} \exp\left(-\frac{r}{1-4t^2}\right) & \text{for } |t| < \frac{1}{2}, \\ 0 & \text{for } |t| \geq \frac{1}{2} \end{cases}$$

of class C^∞ (cf. [AMR88, p. 94]). Then $\vartheta_r : \mathbb{R} \to \mathbb{R}$, $\vartheta_r(t) := \int_{-\infty}^t \omega_r / \int_{-\infty}^\infty \omega_r$ is an increasing C^∞-function with $\vartheta_r(t) = 0$ for $t \leq -\frac{1}{2}$, $\vartheta_r(t) = 1$ for $t \geq \frac{1}{2}$ and the derivative $D\vartheta_r(t) = \omega_r(t)/\int_{-\infty}^\infty \omega_r$. From the properties of ω_r we see that $\min_{t\in\mathbb{R}} D\vartheta_r(t) = 0$ and $m(r) := \max_{t\in\mathbb{R}} D\vartheta_r(t) = \exp(-r)/\int_{-\infty}^\infty \omega_r$. It is not difficult to prove that $m : (0,\infty) \to \mathbb{R}$ is a strictly increasing continuous function with $\lim_{r\searrow 0} m(r) = 1$. Thus, for every $s > 1$ there exists a $r^* > 0$ such that $m(r^*) \leq s$, and therefore $D\vartheta_{r^*}(t) \in [0,s]$ for all $t \in \mathbb{R}$. In conclusion, the function ϑ given by $\vartheta(t) := \vartheta_{r^*}(\frac{3}{2} - t)$ satisfies the assertions.

Yet, it remains the establish the final estimate. By construction, the minimal slope of ϑ is greater or equal than $-s$. Due to $\vartheta(2) = 0$, this yields $\vartheta(t) \leq -s(t-2)$ for all $t \in \left[2 - \frac{1}{s}, 2\right]$ (see Fig. 3.1). Thus, it is $t\vartheta(t) = st(2-t) \leq 2 - \frac{1}{s}$ for all $t \in \left[2-\frac{1}{s}, 2\right]$ and since also $t\vartheta(t) \leq t \leq 2 - \frac{1}{s}$ holds for $t \in \left[0, 2-\frac{1}{s}\right]$, we have deduced the desired inequality $t\vartheta(t) \leq 2 - \frac{1}{s}$ for all $t \geq 0$. Having this at hand, the elementary estimate $2 - \frac{1}{s} \leq s$, $s > 1$, yields our claim. $\square$

Proposition 3.2.6 (cut-off functions)**.** *For all reals $\rho > 0$ and $s > 1$ there exists a C^m-function $\chi_\rho : \mathbb{R}^d \to \mathbb{R}$ such that $\chi_\rho(x) \equiv 1$ for $|x| \le \rho$, $\chi_\rho(x) \in (0,1)$ for $|x| \in (\rho, 2\rho)$, $\chi_\rho(x) \equiv 0$ for $|x| \ge 2$ and $\left|D\chi_\rho(x)\right| \le \frac{s}{\rho}$, as well as the inclusion $x\chi_\rho(x) \in \bar{B}_{s\rho}(0)$ for all $x \in \mathbb{R}^d$.*

Proof. Given $\rho > 0$ we define $\chi_\rho : \mathbb{R}^d \to \mathbb{R}$ by $\chi_\rho(x) := \vartheta\left(\frac{|x|}{\rho}\right)$ with a bump function ϑ from Lemma 3.2.5. In a neighborhood of 0 we have $\chi_\rho(x) \equiv 1$. Outside this set, by assumption, χ_ρ is the composition of C^m-mappings and therefore of class C^m. The bound for the derivative follows from the chain rule yielding

$$\left|D\chi_\rho(x)\right| \le \frac{1}{\rho}\left|D\vartheta\left(\tfrac{|x|}{\rho}\right)\right|\left|Dn\left(\tfrac{|x|}{\rho}\right)\right| \le \frac{s}{\rho} \quad \text{for all } x \in \mathbb{R}^d$$

using Lemma 3.2.5. It is a consequence of the final estimate in Lemma 3.2.5 that $\left|\chi_\rho(x)x\right| = \rho\vartheta\left(\frac{|x|}{\rho}\right)\frac{|x|}{\rho} \le \rho s$ for all $x \in \mathbb{R}^d$ holds and we are done. □

Proposition 3.2.7 (C^m-extension)**.** *If $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a C^m-mapping in the second variable with*

$$\ell(r) := \operatorname{lip}_2 F|_{\mathscr{B}_r(0)} < \infty \quad \textit{for all } r > 0,$$

then for every $s > 1$ and $\rho > 0$ there exists a C^m-mapping $F^\rho : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ with the following properties:

(a) $F^\rho(t,x) = F(t,x)$ for all $t \in \mathbb{R}$, $x \in B_\rho(0)$,
(b) one has the global Lipschitz estimate

$$\operatorname{lip}_2 F^\rho \le (1+2s)\ell(\rho),$$

(c) if the derivatives $D_2^n F$, $n \in \{0,\dots,m\}$, are uniformly bounded, then the same holds for F^ρ.

Proof. For a given $s > 1$ choose $\rho > 0$. We define the modification $F^\rho : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ by $F^\rho(t,y) := F(t,\chi_\rho(x)x)$, which is of class C^m in the second argument and satisfies assertions (a) and (c). In order to establish the remaining claim (b), we consider the C^m-function $\theta_\rho : \mathbb{R}^d \to \mathbb{R}^d$, $\theta_\rho(x) := \chi_\rho(x)x$. By the product rule (cf. [Lan93, p. 336]), Prop. 3.2.6 and Lemma 3.2.5 one has the estimate

$$\left|D\theta_\rho(x)\right| \le \left|D\chi_\rho(x)x\right| + \left|\chi_\rho(x)\right| \le \frac{s}{\rho}|x| + \left|\chi_\rho(x)\right| \le 2s+1 \quad \text{for all } x \in \mathbb{R}^d$$

and thus $\operatorname{lip}\theta_\rho \le 2s+1$ by the mean value inequality (cf. [Lan93, p. 342, Cor. 4.3]). For all $t \in \mathbb{R}$, $x, \bar{x} \in \mathbb{R}^d$ this yields the Lipschitz estimate

$$\left|F^{\rho}(t,x)-F^{\rho}(t,\bar{x})\right| \leq \ell_1(s\rho)\left|\theta_\rho(x)-\theta_\rho(\bar{x})\right| \leq (1+2s)\ell_1(s\rho)\,|x-\bar{x}|\,.$$

□

We introduce the *equation of perturbed motion*

$$\dot{x} = f(t,x+\phi^*(t)) - f(t,\phi^*(t)) \tag{P}$$

and observe that it has the trivial solution. Note that the behavior of (O) near ϕ^* is the same as the behavior of (P) near 0. Moreover, we point out that even if (O) does not depend on time, then (P) is still nonautonomous as long as ϕ^* varies in time. This emphasizes the role our nonautonomous theory even in the classical field of autonomous dynamical systems. An equivalent representation of (P) is

$$\dot{x} = A(t)x + F(t,x) \tag{3.2d}$$

with the functions $A:\mathbb{R}\to\mathbb{R}^{d\times d}$ and $F:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d$ given by

$$A(t) := D_2 f(t,\phi^*(t)), \quad F(t,x) := f(t,x+\phi^*(t)) - f(t,\phi^*(t)) - D_2 f(t,\phi^*(t))x.$$

Proof of Thm. 3.2.3. Let $\rho > 0$. Above all, we use Prop. 3.2.7 in order to obtain a C^m-smooth modification F^ρ. Due to (3.2a) one has the limit relation

$$\lim_{x\to 0}|D_2F(t,x)| = 0 \quad \text{uniformly in } t\in\mathbb{R}$$

and hence $\lim_{r\searrow 0}\operatorname{lip}_2 F|_{B_r(0)} = 0$. For $\rho > 0$ sufficiently small, by Prop. 3.2.7(b), we can make also $\operatorname{lip}_2 F^\rho$ as small as we want. Then the modified equation

$$\dot{x} = A(t)x + F^\rho(t,x) \tag{3.2e}$$

satisfies the assumptions of Thm. 3.1.4. Thus, there exist global integral manifolds $\tilde{\mathscr{W}}_i^\pm$ given as graph of a mapping $\tilde{w}_i^\pm$ over vector bundles $\mathscr{R}(P_i)$ resp. $\mathscr{R}(Q_i)$. Then the mappings $w_i^\pm := \tilde{w}_i^\pm|_{\mathscr{B}_\rho(0)}$ fulfill the assertions claimed in Thm. 3.2.3. □

The following example shows that the gap condition (3.2b) is sharp, i.e. the integral manifold $\mathscr{W}_i^+$ from Thm. 3.2.3(a) is not of class C^m in general, even if f is a C^∞-function.

Example 3.2.8. *Given an integer $m \geq 2$, let us consider the planar autonomous differential equation*

$$\begin{cases} \dot{x} = x \\ \dot{y} = my + x^m \end{cases}, \tag{3.2f}$$

with the trivial solution $\phi^* = 0$. It fulfills the assumptions of Thm. 3.2.3(a) in form of the dichotomy spectrum $\Sigma(\phi^*) = \{1, m\}$. Consequently, there exists an integral manifold $\mathscr{W}_1^+ \subseteq \mathbb{R}\times\mathbb{R}^2$ given as graph of a function $w_1^+ : \mathscr{B}_\rho(0) \to \mathbb{R}^2$ for some $\rho > 0$. On the other hand, for every $\gamma\in\mathbb{R}$ the sets

$$W_\gamma := \left\{(\xi,\eta)\in B_\rho(0)\setminus\{0\} : \eta = \tfrac{\xi^m}{2}\ln\xi^2 + \gamma\xi^m\right\} \cup \{0\}$$

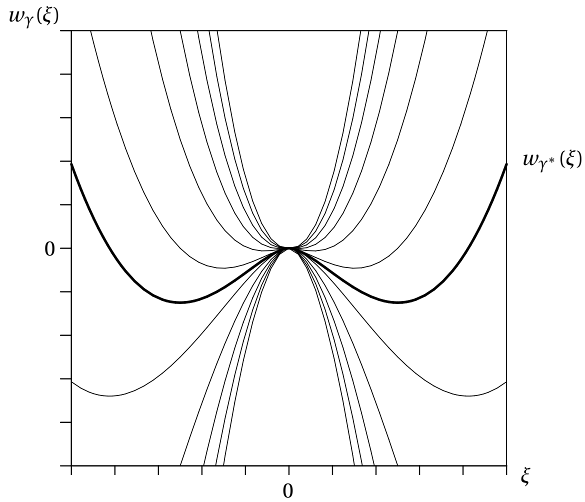

Fig. 3.2 Graphs of the functions w_γ from Exam. 3.2.13

contain the origin and are (locally) forward invariant w.r.t. (3.2f)*, i.e.* $\mathbb{R}\times W_\gamma$ *is a forward invariant integral manifold. Additionally, each pair* $(\xi,\eta)\in B_\rho(0)$, $\xi\neq 0$, *is contained in exactly one of the sets* W_γ*, namely for* $\gamma=\frac{\eta}{\xi^m}-\frac{\ln\xi^2}{2}$*. Thus, the integral manifold* $\mathscr{W}_1^+$ *from Thm. 3.2.3(a) has the form* $\mathbb{R}\times W_{\gamma^*}$ *for some* $\gamma^*\in\mathbb{R}$ *(see Fig. 3.2). Every fiber* W_γ *is graph of a* C^{m-1}*-function* $w_\gamma(\xi)=\eta$*, but* w_γ *fails to be* m*-times continuously differentiable. Note that in the present example the gap condition* $\alpha_1 < m_s\beta_1$ *is only fulfilled for* $1\le m_s<m$.

Corollary 3.2.9 (invariance equation)**.** *The mappings* $w_i^\pm$ *satisfy the* invariance equations

$$\begin{aligned}A(\tau)w_i^+(\tau,\xi)&+P_i(\tau)F(\tau,\xi+w_i^+(\tau,\xi))\\&=D_2w_i^+(\tau,\xi)\left(A(\tau)\xi+Q_i(\tau)F(\tau,\xi+w_i^+(\tau,\xi))\right)+D_1w_i^+(\tau,\xi)\end{aligned}$$

for all $\tau\in\mathbb{R}$, $\xi\in B_\rho(0)\cap R(Q_i(\tau))$ *and*

$$\begin{aligned}A(\tau)w_i^-(\tau,\xi)&+Q_i(\tau)F(\tau,\xi+w_i^-(\tau,\xi))\\&=D_2w_i^-(\tau,\xi)\left(A(\tau)\xi+P_i(\tau)F(\tau,\xi+w_i^-(\tau,\xi))\right)+D_1w_i^-(\tau,\xi)\end{aligned}$$

for all $\tau\in\mathbb{R}$, $\xi\in B_\rho(0)\cap R(P_i(\tau))$.

Proof. See Ex. 3.2.18 □

$$\begin{array}{ccccc} \mathscr{W}_u & \subset & \mathscr{W}_{cu} & \subset & \mathbb{R}\times\mathbb{R}^d \\ & & \cup & & \cup \\ & & \mathscr{W}_c & \subset & \mathscr{W}_{cs} \\ & & & & \cup \\ & & & & \mathscr{W}_s \end{array}$$

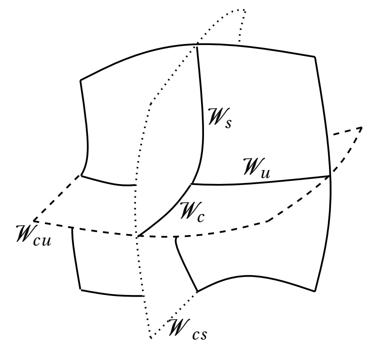

Fig. 3.3 Classical hierarchy of invariant manifolds (left) and classical invariant manifolds $\mathscr{W}_{cs}$ (dotted), $\mathscr{W}_{cu}$ (dashed) and $\mathscr{W}_u, \mathscr{W}_s, \mathscr{W}_c$ (right)

Proposition 3.2.10 (local center manifolds)**.** *Let $m \in \mathbb{N}$ and assume that Hyp. 3.2.1 are satisfied. If (i, j) is a pair satisfying $1 \le j \le i < n$, then there exists a $\rho \in (0, \rho_0)$ such that the intersection*

$$\phi^* + \mathscr{W}_i^j := \phi^* + \mathscr{W}_i^+ \cap \mathscr{W}_{j-1}^-$$

is a local C^1-integral manifold of (O)*, representable as graph*

$$\mathscr{W}_i^j = \left\{(\tau, \eta + w_i^j(\tau,\eta)) \in \mathbb{R}\times\mathbb{R}^d : (\tau,\eta) \in \mathscr{B}_\rho(0)\right\}$$

of a C^1-mapping $w_i^j : \mathscr{B}_\rho(0) \to \mathbb{R}^d$ satisfying (3.1x) *for all $(\tau,\xi) \in \mathscr{B}_\rho(0)$. Furthermore, for all $(\tau,\xi) \in \mathscr{B}_\rho(0)$ it holds:*

(a) $w_i^j(\tau, 0) = 0$ *on* $\mathbb{R}$ *and* $\left|w_i^j(\tau,\xi)\right| \le \rho$ *for all* $(\tau,\xi) \in \mathscr{B}_\rho(0)$*,*
(b) $\lim_{x\to 0} D_2 w_i^j(t,x) = 0$ *uniformly in* $t \in \mathbb{R}$*,*
(c) if additionally the gap conditions $m_i^j \alpha_i^m < \beta_i$ and $\alpha_{j-1} < m_i^j \beta_{j-1}$ hold, then $\phi^* + \mathscr{W}_i^j$ is a local $C^{m_i^j}$-integral manifold.

Remark 3.2.11 (classical hierarchy)**.** *For equations* (O) *with* $0 \in \Sigma(\phi^*)$*, for example* $0 \in [a_i, b_i]$*, we get the following* classical integral manifolds *associated to* ϕ^**:*

- Stable manifold $\phi^* + \mathscr{W}_s = \phi^* + \mathscr{W}_{i-1}^1$
- Center-stable manifold $\phi^* + \mathscr{W}_{cs} = \phi^* + \mathscr{W}_i^1$
- Center-unstable manifold $\phi^* + \mathscr{W}_{cu} = \phi^* + \mathscr{W}_n^i$

- Unstable manifold $\phi^* + \mathscr{W}_u = \phi^* + \mathscr{W}_n^{i+1}$
- Center manifold $\phi^* + \mathscr{W}_c := \phi^* + \mathscr{W}_i^i$

These integral manifolds form the classical hierarchy *depicted in Fig. 3.3, and* $\mathscr{W}_s$*,* $\mathscr{W}_u$ *inherit their smoothness from* (O)*. As long as solutions stay in the nonautonomous set* $\phi^* + B_\rho(0)$ *one can describe them asymptotically as in Rem. 3.1.10.*

Remark 3.2.12. *For a* p*-periodic eqn.* (O) *and solutions* ϕ^* *the integral manifolds* $\mathscr{W}_i^j$ *are also* p*-periodic. In particular, for autonomous eqns.* (O) *and constant* ϕ^* *one speaks of* invariant manifolds *and the above classical special cases are denoted as* stable, center-stable, center-unstable, unstable *resp.* center manifold.

Proof. The argument is parallel to Thm. 3.2.3 and we omit further details. □

Next we illustrate that the center-unstable manifolds as postulated above in Thm. 3.2.3 need not to be uniquely determined.

Example 3.2.13. *The* 2*-dimensional autonomous equation*

$$\begin{cases} \dot{x} = -x, \\ \dot{y} = y^2, \end{cases} \tag{3.2g}$$

with the trivial solution $\phi^* = 0$ *fulfills the assumptions of Thm. 3.2.3(b); the dichotomy spectrum reads as* $\Sigma(\phi^*) = \{-1, 0\}$*. For every* $\gamma \in \mathbb{R}$ *it is easy to verify that*

$$\mathscr{W}_\gamma := \left\{(\tau, \xi, \eta) \in \mathbb{R} \times \mathbb{R} \times (-\infty, 1) : \xi = \gamma e^{1/\eta} \text{ for } \eta < 0 \text{ and } \xi = 0 \text{ for } \eta \geq 0\right\}$$

is a local integral manifold of (3.2g) *in the sense that* $\mathscr{W}_\gamma$ *is a locally invariant graph containing the zero solution and being tangential.*

We continue with an asymptotic description of the stable and center-stable manifolds, as well as of their unstable counterparts.

Corollary 3.2.14. *Under Hyp. 3.2.1 the following holds:*

(a_1) *If* $\phi - \phi^*$ *decays exponentially in forward time, then there exists a* $t_0 \in \mathbb{R}$ *such that* $(t, \phi(t)) \in \phi^* + \mathscr{W}_s$ *for all* $t \geq t_0$*,*
(a_2) *there exists a* $\rho_1 \in (0, \rho)$ *such that every forward solution of* (O) *starting in* $\phi^* + (\mathscr{W}_s \cap \mathscr{B}_{\rho_1}(0))$ *decays exponentially to* ϕ^* *in forward time.*
(b_1) *If* $\phi - \phi^*$ *decays exponentially in backward time, then there exists a* $t_0 \in \mathbb{R}$ *so that* $(t, \phi(t)) \in \phi^* + \mathscr{W}_u$ *for all* $t \leq t_0$*,*
(b_2) *there exists a* $\rho_1 \in (0, \rho)$ *such that every backward solution of* (O) *starting in* $\phi^* + (\mathscr{W}_u \cap \mathscr{B}_{\rho_1}(0))$ *decays exponentially to* ϕ^* *in backward time.*

Proof. By passing over to the equation of perturbed equation (P) we can assume that ϕ^* is the trivial solution of (O).

(a) We choose $1 \le i < n$ with with $b_i < 0 \le a_{i+1}$ and growth rates α, β with

$$b_i < \alpha < \beta < a_{i+1}, \qquad \frac{\alpha + \beta}{2} < 0.$$

Thus, as in the proof of Thm. 3.2.3(a) there exists a global integral manifold $\tilde{\mathscr{W}}^+$ of (3.2e), consisting of forward solutions to (3.2e) in $X^+_{\tau,\gamma}$ with $\gamma < \frac{\alpha+\beta}{2}$.

(a_1) Since the solution ϕ is exponentially decaying, there exists a $\delta < 0$ such that $\phi \in X^+_{\tau,\delta}$, $\tau \in \mathbb{R}$; by an appropriate choice of α, β one has $\delta \le \gamma$. Thus, there exists an entry time $t_0 \in \mathbb{R}$ with $\phi(t) \in B_\rho(0)$ for $t \ge t_0$. Because the stable manifold $\mathscr{W}_s$ of (O) and $\tilde{\mathscr{W}}^+$ coincide on $\mathscr{B}_\rho(0)$, one has $(t, \phi(t)) \in \mathscr{W}_s$ for all $t \ge t_0$.

(a_2) Every initial pair $(\tau, \xi) \in \mathscr{W}_s \cap \mathscr{B}_\rho(0)$ is contained in an integral manifold $\tilde{\mathscr{W}}^+$ of the modified eqn. (3.2e) and moreover yields a γ^+-bounded solution $\tilde{\varphi}(\cdot; \tau, \xi)$ of (3.2e). Due to $\gamma < 0$ this solution decays exponentially in forward time. Accordingly, for sufficiently small $\rho_1 \in (0, \rho)$ one has $(t, \tilde{\varphi}(t; \tau, \xi)) \in \mathscr{W}_s \cap \mathscr{B}_\rho(0)$ and $\tilde{\varphi}(\cdot; \tau, \xi)$ coincides with a solution of (O) starting in (τ, ξ).

(b) can be shown analogously. □

Corollary 3.2.15. *Under Hyp. 3.2.1 the following holds: If there exists a $t_0 \in \mathbb{R}$ with $(t, \phi(t)) \in \mathscr{B}_\rho(\phi^*)$ for all*

(a) $t_0 \le t$, then $(t, \phi(t)) \in \phi^* + \mathscr{W}_{cs}$ for all $t_0 \le t$,
(b) $t \le t_0$, then $(t, \phi(t)) \in \phi^* + \mathscr{W}_{cu}$ for all $t \le t_0$.

Proof. W.l.o.g. we again suppose that ϕ^* is the trivial solution of (O).

(a) First, choose $1 \le i < m$ minimal with $0 \le b_i$ and growth rates α, β such that $b_i < \alpha < \beta < a_{i+1}$. The proof of Thm. 3.2.3(a) guarantees an integral manifold $\tilde{\mathscr{W}}^+$ of the modified system (3.2e). We know that $\tilde{\mathscr{W}}^+$ consists of γ^+-bounded solutions for some $0 \le \gamma$. If a solution $\phi : [\tau, \infty) \to \mathbb{R}^d$ of (O) stays in $\mathscr{B}_\rho(0)$ for all $t \ge t_0$, then it also solves (3.2e) and is γ^+-bounded. Hence, the solution is contained in $\tilde{\mathscr{W}}^+$ for $t \ge t_0$ and therefore on $\mathscr{W}_{cs} = \tilde{\mathscr{W}}^+ \cap \mathscr{B}_\rho(0)$.

(b) One proceeds analogously. □

In the remaining, we discuss a canonical application of local integral manifolds to stability theory. The simplest situation is given for solutions ϕ^* with a hyperbolic variational equation (V) with associated dichotomy spectrum $\Sigma(\phi^*)$.

Theorem 3.2.16 (principle of linearized stability)**.** *Under Hyp. 3.2.1 the following holds:*

(a) If $\Sigma(\phi^*) \subseteq (-\infty, 0)$, then ϕ^* is uniformly asymptotically stable.
(b) If there exists a spectral interval $[a_i, b_i]$ of $\Sigma(\phi^*)$ with $0 < a_i$, then the solution ϕ^* is unstable.

Proof. W.l.o.g. we can restrict to the equation of perturbed motion (P) in its semilinear form (3.2d).

(a) Thanks to our assumption $\Sigma(\phi^*) \subseteq (-\infty, 0)$ there exist $K \geq 1$ and $\alpha > 0$ such that the transition operator of (V) fulfills

$$|\Phi(t,s)| \leq K e^{-\alpha(t-s)} \quad \text{for all } s \leq t. \tag{3.2h}$$

Rather than (3.2d) we consider the modified equation

$$\dot{x} = A(t)x + F^\rho(t,x) \tag{3.2i}$$

and choose $\rho > 0$ so small that $\operatorname{lip}_2 F^\rho \leq \frac{\alpha}{K}$. Given $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}^d$ the general solution $\tilde{\varphi}$ of (3.2i) also solves the linear inhomogeneous equation

$$\dot{x} = A(t)x + F^\rho(t, \tilde{\varphi}(t;\tau,\xi))$$

and using the variation of constants formula we arrive at

$$\tilde{\varphi}(t;\tau,\xi) = \Phi(t,\tau)\xi + \int_\tau^t \Phi(t,s)F^\rho(s,\tilde{\varphi}(s;\tau,\xi))\,ds \quad \text{for all } \tau \leq t$$

Using the estimate (3.2h) this implies

$$\begin{aligned}
\left|\tilde{\varphi}(t;\tau,\xi)\right| &\leq |\Phi(t,\tau)||\xi| + \int_\tau^t |\Phi(t,s)| \left|F^\rho(s,\tilde{\varphi}(s;\tau,\xi)) - F^\rho(s,0)\right| ds \\
&\overset{(3.2h)}{\leq} K e^{-\alpha(t-\tau)}|\xi| + K\int_\tau^t e^{-\alpha(t-s)} \left|F^\rho(s,\tilde{\varphi}(s;\tau,\xi)) - F^\rho(s,0)\right| ds \\
&\leq K e^{-\alpha(t-\tau)}|\xi| + K \operatorname{lip}_2 F^\rho \int_\tau^t e^{-\alpha(t-s)} \left|\tilde{\varphi}(s;\tau,\xi)\right| ds
\end{aligned}$$

and consequently

$$\left|\tilde{\varphi}(t;\tau,\xi)\right| e^{\alpha(t-\tau)} \leq K|\xi| + K \operatorname{lip}_2 F^\rho \int_\tau^t \left|\tilde{\varphi}(s;\tau,\xi)\right| e^{\alpha(s-\tau)}\,ds \quad \text{for all } \tau \leq t.$$

Now the Gronwall inequality applies and leads to

$$\left|\tilde{\varphi}(t;\tau,\xi)\right| \leq K|\xi|\, e^{(K \operatorname{lip}_2 F^\rho - \alpha)(t-\tau)} \quad \text{for all } \tau \leq t;$$

our assumptions on ρ yield $K \operatorname{lip}_2 F^\rho - \alpha < 0$. Hence, the trivial solution to (3.2i) is uniformly asymptotically stable. Since the general solutions of (3.2i) and (3.2d) coincide on $\mathscr{B}_\rho(0)$, the solution ϕ^* of (O) inherits this property.

(b) Our assumptions guarantee that the trivial solution to (P) admits an unstable integral manifold. It contains all solutions leaving a sufficiently small neighborhood of 0 in forward time. □

In absence of an unstable integral manifold, the above principle of linearized stability yields (exponential) stability of ϕ^*. Conversely, if there is an unstable manifold, one obtains the instability of ϕ^*. Between these two cases is the situation of a nonhyperbolic variational equation (V), where a center-unstable integral manifold $\mathscr{W}_{cu}$ exists. Then stability properties are determined by the behavior on $\mathscr{W}_{cu}$ and therefore by a lower-dimensional equation (O) reduced to $\phi^* + \mathscr{W}_{cu}$:

Theorem 3.2.17 (reduction principle)**.** *Suppose Hyp. 3.2.1 is satisfied and choose $1 \le i < n$ with $b_i < 0$. The solution ϕ^* of* (O) *is stable (uniformly stable, asymptotically stable, uniformly asymptotically stable, exponentially stable, or unstable), if and only if the zero solution of the reduced equation*

$$\dot{y} = A(t)y + P_i(t)F(t, y + w_i^-(t,y)) \tag{3.2j}$$

in $\mathscr{R}(P_i)$ has the respective stability property.

Proof. First, we can restrict to the equation of perturbed motion (P).

Our assumptions guarantee that one can choose $\gamma \in \Gamma_i$ with $\gamma \le 0$. In addition, there exists a pseudo-unstable integral manifold $\mathscr{W}_i^-$ associated to the trivial solution of (P); it is graph of a function w_i^- defined on a neighborhood $\mathscr{B}_\rho(0)$ in $\mathscr{R}(P_i)$. By construction (cf. the proof of Thm. 3.2.3(b)), $\mathscr{W}_i^-$ is the restriction of global integral manifold $\tilde{\mathscr{W}}_i^-$ for the modified eqn. (3.2e) as in the proof of Thm. 3.2.3, which is graph of a mapping $\tilde{w}_i^-$ and $w_i^- = \tilde{w}_i^-|_{\mathscr{B}_\rho(0)}$ with

$$\tilde{w}_i^-(t,0) \equiv 0 \quad \text{on } \mathbb{R}, \qquad\qquad \operatorname{lip}_2 \tilde{w}_i^+ = \ell < \infty. \tag{3.2k}$$

Thanks to Thm. 3.1.11(a) the integral manifold $\mathscr{W}_i^-$ has an asymptotic forward phase π_i^+ satisfying (3.1z). More precisely, with the constant $C_i^+(|\xi|) > 0$ occurring in (3.1z) one has

$$C_i^+(|\xi|) = C_1|\xi|, \qquad \left|Q_i(\tau)\pi_i^+(\tau,\xi)\right| \le C_2|\xi| \quad \text{for all } \tau \in \mathbb{R},\ \xi \in \mathbb{R}^d \tag{3.2l}$$

with some reals $C_1, C_2 > 0$.

($\Rightarrow$) The reduced eqn. (3.2j) describes the dynamics of (P) on the locally invariant pseudo-unstable integral manifold $\mathscr{W}_i^-$. This local invariance yields that stability properties of the zero solution for (P) carry over to (3.2j).

($\Leftarrow$) Conversely, if the zero solution of the reduced eqn. (3.2j) is unstable, then by invariance of $\mathscr{W}_i^-$, also the zero solution of (P) is unstable.
Now, let $\varepsilon > 0$, $\tau \in \mathbb{R}$ be given. We suppose the zero solution of (3.2j) is stable, i.e. there exists a $\delta \in \left(0,\rho\right)$ so that

$$\left|\phi_0(t)\right| < \frac{\varepsilon}{2(1+\ell)} \quad \text{for all } t \ge \tau \tag{3.2m}$$

and any solution $\phi_0 : [\tau,\infty) \to \mathbb{R}^d$ of (3.2j) with $\phi_0(\tau) \in B_\delta(0) \cap \mathscr{R}(P_i)(\tau)$. In the following, let $\phi : [\tau,\infty) \to \mathbb{R}^d$ be an arbitrary solution of (P) with

$$\left|\phi(\tau)\right| < \min\left\{\frac{\varepsilon}{3C_1}, \frac{\delta}{2C_2}\right\},$$

Due to the asymptotic forward phase from Thm. 3.1.11(a), we establish that there exists a corresponding solution $\tilde{\phi}_0 : [\tau,\infty) \to \mathbb{R}^d$ of the global equation

$$\dot{y} = A(t)y + P_i(t)F^\rho(t, y + w_i^-(t, y))$$

(cf. (3.2j)) in the pseudo-unstable vector bundle $\mathscr{R}(P_i)$ with

$$\left|\tilde{\varphi}(t;\tau,\phi(\tau)) - \tilde{\varphi}\big(t;\tau,\tilde{\phi}_0(\tau) + \tilde{w}_i^-(\tau,\tilde{\phi}_0(\tau))\big)\right| \stackrel{(3.1z)}{\leq} C_1\left|\phi(\tau)\right| e^{\gamma(t-\tau)} \quad \text{for all } \tau \leq t,$$

where $\tilde{\varphi}$ is the general solution of (3.2e). We have from Thm. 3.1.11(a),

$$\left|\tilde{\phi}_0(\tau)\right| = \left|Q_i(\tau)\pi_i^+(\tau,\phi(\tau))\right| \stackrel{(3.2l)}{\leq} C_2\left|\phi(\tau)\right| < \delta$$

and thus (3.2m) gives us $\left|\tilde{\phi}_0(t)\right| < \frac{\varepsilon}{2\left(1+\tilde{\ell}_i^-(c)\right)}$ for all $t \geq \tau$. But this yields (note $e^{\gamma(t-\tau)} \leq 1$ for $\tau \leq t$) with the triangle inequality and Thm. 3.1.4(b_2),

$$\begin{aligned}
\left|\tilde{\varphi}(t;\tau,\phi(\tau))\right| &\leq \left|\tilde{\varphi}(t;\tau,\phi(\tau)) - \tilde{\varphi}\big(t;\tau,\pi_i^+(\tau,\phi(\tau))\big)\right| + \left|\tilde{\varphi}\big(t;\tau,\pi_i^+(\tau,\phi(\tau))\big)\right| \\
&\leq C_1\left|\phi(\tau)\right| e^{\gamma(t-\tau)} + \left|\tilde{\phi}_0(t) + \tilde{w}_i^-(t,\tilde{\phi}_0(t))\right| \\
&\stackrel{(3.2k)}{\leq} C_1\left|\phi(\tau)\right| + (1+\ell)\left|\tilde{\phi}_0(t)\right| < \varepsilon \quad \text{for all } \tau \leq t
\end{aligned}$$

and 0 is a stable solution of (3.2e). However, since (O) and (3.2e) coincide on the ball $\mathscr{B}_\rho(0)$, and due to $\tilde{\varphi}(t;\tau,\phi(\tau)) \in B_\rho(0)$ for all $\tau \leq t$, it is $\phi = \tilde{\varphi}(\cdot;\tau,\phi(\tau))$. Thus, the zero solution is also stable w.r.t. (P). Keeping in mind that $\mathscr{W}_i^-$ is uniformly exponentially attracting (cf. (3.1z)) with constants independent of $\tau \in \mathbb{R}$, a similar reasoning gives us the assertion on the remaining stability properties. ◻

Exercise 3.2.18. *Prove Cor. 3.2.9.*

References

[AB99] C.D. Aliprantis & K.C. Border, *Infinite dimensional analysis – A hitchhiker's guide*, 2nd ed., Springer, Berlin etc., 1999.

[AMR88] R.H. Abraham, J.E. Marsden & T. Ratiu, *Manifolds, tensor analysis, and applications*, Applied Mathematical Sciences 75, Springer, Berlin etc., 1988.

[Arn98] L. Arnold, *Random dynamical systems*, Monographs in Mathematics, Springer, Berlin etc., 1998.

[AS01] B. Aulbach & S. Siegmund, *The dichotomy spectrum for noninvertible systems of linear difference equations*, J. Difference Equ. Appl. **7** (2001), no. 6, 895–913.

[Aul04] B. Aulbach, *Gewöhnliche Differentialgleichungen*, 2nd ed., Spektrum Akademischer Verlag, Heidelberg, 2004.

[AW96] B. Aulbach & T. Wanner, *Integral manifolds for Carathéodory type differential equations in Banach spaces*, Six Lectures on Dynamical Systems (B. Aulbach and F. Colonius, eds.), World Scientific, Singapore etc., 1996, pp. 45–119.

[AW03] ———, *Invariant foliations for Carathéodory type differential equations in Banach spaces*, Advances of Stability Theory at the End of the XX. Century (V. Lakshmikantham and A.A. Martynyuk, eds.), Taylor & Francis, London, 2003, pp. 1–14.

[CK99] F. Colonius & W. Kliemann, *The dynamics of control*, Birkhäuser, Basel-Berlin-Boston, 1999.

[Cop78] W.A. Coppel, *Dichotomies in stability theory*, Lect. Notes Math. 629, Springer, Berlin etc., 1978.

[Hen81] D. Henry, *Geometric theory of semilinear parabolic equations*, Lect. Notes Math. 840, Springer, Berlin etc., 1981.

[HVL93] J.K. Hale & S.M. Verduyn Lunel, *Introduction to functional differential equations*, Applied Mathematical Sciences 99, Springer, Berlin etc., 1993.

[Kur86] J. Kurzweil, *Ordinary differential equations*, Studies in Applied Mathematics 13, Elsevier, Amsterdam etc., 1986.

[Lan93] S. Lang, *Real and functional analysis*, Graduate Texts in Mathematics 142, Springer, Berlin etc., 1993.

[PS04] C. Pötzsche & S. Siegmund, *C^m-smoothness of invariant fiber bundles for dynamic equations on measure chains*, Adv. Difference Equ. **2** (2004), 141–182.

[Sel71] G.R. Sell, *Topological dynamics and differential equations*, Van Nostrand Reinhold, London-New York, 1971.

[Sel78] ———, *The structure of a flow in the vicinity of an almost periodic motion*, J. Differ. Equations **27** (1978), no. 3, 359–393.

[Sie02] S. Siegmund, *Dichotomy spectrum for nonautonomous differential equations*, J. Dyn. Differ. Equations **14** (2002), no. 1, 243–258.

[SS78] R.J. Sacker & G.R. Sell, *A spectral theory for linear differential systems*, J. Differ. Equations **27** (1978), 320–358.

[Zei93] E. Zeidler, *Nonlinear functional analysis and its applications I (Fixed-points theorems)*, Springer, Berlin etc., 1993.

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