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Numerical dynamics of integrodifference equations

Periodic solutions and invariant manifolds in $C^\alpha(\Omega)$

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Abstract Integrodifference equations are versatile models in theoretical ecology for the spatial dispersal of species evolving in non-overlapping generations. The dynamics of these infinite-dimensional discrete dynamical systems is often illustrated using computational simulations. This paper studies the effect of Nyström discretization to the local dynamics of periodic integrodifference equations with Hölder continuous functions over a compact domain as state space. We prove persistence and convergence for hyperbolic periodic solutions and their associated stable and unstable manifolds respecting the convergence order of the quadrature/cubature method.

Keywords Integrodifference equations · Numerical dynamics · Urysohn operator · Nyström method · Hölder continuity

Mathematics Subject Classification (2010) 65P40 · 45G15 · 65R20 · 37L45

1 Introduction

Integrodifference equations (IDEs for short) are infinite-dimensional discrete-time dynamical systems. They became popular tools in theoretical ecology over the recent years modelling the temporal evolution and spatial dispersal of species having non-overlapping generations [15, 18]. Furthermore, IDEs canonically arise as time discretizations of integrodifferential equations, as time-1-map of evolutionary partial differential equations or in the iterative solution of (nonlinear) boundary value problems [19, p. 190]. It is understood that IDEs involve an integral operator which is typically of Hammerstein- or more general Urysohn-type. Indeed, for our purposes a sufficiently flexible class are recursions of the form

$$u_{t+1}(x) = \int_\Omega f_t(x, y, u_t(y))\,\mathrm{d}y \quad \text{for all } x \in \Omega, \tag{I_0}$$

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whose natural state spaces consists of continuous or integrable functions over a compact subset $\Omega \subset \mathbb{R}^\kappa$ (the habitat in applications from ecology). In applied sciences the long-term behavior of IDEs is willingly illustrated using numerical simulations. For this purpose, [18, pp. 112–113] suggests to replace the integral in (I_0) by the trapezoidal or the Simpson rule. Both are special cases of general Nyström methods

$$u_{t+1}(x) = \sum_{\eta \in \Omega_n} w_\eta f_t(x, \eta, u_t(\eta)) \quad \text{for all } x \in \Omega \tag{I_n}$$

based on convergent quadrature or cubature rules with weights $w_\eta \geq 0$ and nodes $\eta \in \Omega_n$ over a finite grid $\Omega_n \subset \Omega$. Here, $n \in \mathbb{N}$ is related to the number of nodes in Ω_n and therefore the accuracy of the approximation, see [8]. We point out that Nyström methods yield full discretizations of (I_0) and can be evaluated immediately.

While the numerical analysis of integral equations is a well-established field, e.g. [4, 13], this paper enriches it by a dynamical aspect: We study and relate the long-term behavior of the iterates u_t generated by an IDE (I_0) to those of a Nyström discretization (I_n). This brings us to the area of numerical dynamics [17, 26, 27] addressing the following questions:

- Which dynamical or asymptotic properties of an IDE (I_0) as $t \to \infty$ are preserved when passing to its Nyström discretizations (I_n)?
- What can be said about convergence as $n \to \infty$ when the approximations become increasingly more accurate? In particular, are convergence rates of the integration rules preserved?

Concerning the qualitative behavior of autonomous ODEs, such problems originate in [6] and are surveyed in [27]. In between various contributions to continuous-time infinite-dimensional dynamical systems generated by functional differential equations [12] or evolutionary (e.g. parabolic) partial differential equations [26, 17] arose, both for spatial, as well as for full discretizations. IDEs merely require spatial discretization, but have in common with these problems that conventional error estimates fail to describe asymptotic behavior. In fact, bounds for the global discretization error typically grow exponentially in time and therefore establish convergence as $n \to \infty$ only over compact time intervals [21, Thm. 4.1]. Thus, techniques extending those of classical numerical analysis are required to tackle the above problems.

Previous contributions to the numerical dynamics of IDEs address basics and error estimates [21], as well as the persistence/convergence of globally asymptotically stable solutions [22]. This paper focusses on an another important aspect, namely the local saddle-point structure near periodic solutions to (I_0). Related work, but for autonomous evolutionary differential equations near equilibria, is due to [6, 11] (ODEs), [1] (parabolic PDEs) and [9] (retarded FDEs).

In contrast, we study time-periodic IDEs (I_0) in the vicinity of periodic solutions. We stress that periodic time-dependence is strongly motivated by applications to incorporate seasonal influences. While [21, 22] apply to semi-discretizations of (I_0) of collocation- or degenerate kernel-type [4, 13], which act between finite-dimensional function spaces, but still contain integrals, we tackle Nyström discretizations (I_n), because they can be evaluated immediately. At this point the question for an ambient state space of (I_0) arises. A natural choice are the continuous functions $C(\Omega)$

over a compact $\Omega \subset \mathbb{R}^\kappa$. Here however, already for linear integral operators, the discretization error under Nyström methods converges only in the strong, but not in the uniform topology as $n \to \infty$, see [13, pp. 130–131, Lemma 4.7.6]. Using the theory of collectively compact operators [2] one can still establish that fixed-points to (I_0) (and their stability properties) persist [3,28]. Nonetheless, it is not clear how to establish convergence of the associated stable and unstable manifolds of (I_n) to those of the original problem (I_0). For this reason we retreat to the Hölder continuous functions $C^\alpha(\Omega)$ as state space. This set-up is sufficiently general to capture most relevant applied problems [15, 18] and has the advantage that a more conventional perturbation theory (see App. A) applies to realize our goals. It should not be concealed, though, that the prize for this endeavor are more involved assumptions and technical preliminaries on Urysohn operators (well-definedness, complete continuity, differentiability). For the sake of a brief presentation they are outsourced to [23,24].

The structure of our presentation is as follows: In Sec. 2 we introduce a flexible framework for general periodic difference equations in Banach spaces and their linearization. Perturbation results for the Floquet spectrum of linear periodic equations are given in Thm. 2.1, while Thm. 2.2 addresses persistence and convergence of hyperbolic solutions and Thm. 2.3 the associated stable and unstable manifolds – when dealing with periodic equations we speak of fiber bundles. Although tailor-made for Nyström discretizations of IDEs, these results also apply to collocation- or degenerate kernel-discretizations, as well as when studying time-periodic evolutionary differential equations via their time-h-maps. The concrete case of Urysohn IDEs (I_0) is saved for Sec. 3 and illustrates how Thms. 2.1–2.3 apply. One obtains convergence of both hyperbolic solutions, and of the functions parametrizing their invariant fiber bundles with a rate given by the Hölder exponent $\alpha \in (0, 1]$ of the kernel functions f_t in the first variable. Nevertheless, for smooth f_t the higher-order convergence rates inherited from the particular quadrature/cubature rules are established.

Notation We write $\mathbb{R}_+ := [0, \infty)$ for the nonnegative reals, $\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$ for the unit circle in $\mathbb{C}$, $[\cdot] : \mathbb{R} \to \mathbb{Z}$ is the integer function and $|\cdot|$ denotes norms on finite-dimensional spaces. On the Cartesian product $X \times Y$ of normed spaces X, Y,

$$\|(x, y)\| := \max\{\|x\|_X, \|y\|_Y\} \tag{1.1}$$

is the product norm and we proceed accordingly on products of more than two spaces. The open resp. closed balls in X with radius $r \ge 0$ and center $x \in X$ are

$$B_r(x, X) := \{y \in X : \|y - x\| < r\}, \quad \bar{B}_r(x, X) := \{y \in X : \|y - x\| \le r\};$$

on a finite-dimensional X we write $B_r(x)$ and $\bar{B}_r(x)$. For nonempty $A \subseteq X$, $\operatorname{diam} A$ denotes the diameter of A, $\operatorname{dist}_A(x) := \inf_{a \in A} \|x - a\|$ the distance of a point $x \in X$ from A, $\operatorname{dist}(B, A) := \sup_{b \in B} \operatorname{dist}_A(b)$ the Hausdorff semidistance of $B \subseteq X$ from A and we set $B_r(A) := \{x \in X : \operatorname{dist}_A(x) < r\}$. We denote a subset $\mathcal{A} \subseteq \mathbb{Z} \times X$ as *nonautonomous set* with fibers $\mathcal{A}(t) := \{x \in X : (t, x) \in \mathcal{A}\}$, $t \in \mathbb{Z}$ and write $\mathcal{B}_r(\phi) := \{(t, u) \in \mathbb{Z} \times X : \|u - \phi_t\| < r\}$ for the r-neighborhood of a sequence $\phi = (\phi_t)_{t \in \mathbb{Z}}$ in X.

The bounded k-linear maps from the Cartesian product X^k to Y are denoted by $L_k(X,Y)$, $L(X,Y) := L_1(X,Y)$ and $L_0(X,Y) := Y$. Moreover, we abbreviate $L_k(X) := L_k(X,X)$, $L(X) := L(X,X)$, $GL(X)$ are the invertible maps in $L(X)$ and I_X is the identity on X. Furthermore, $N(T)$ is the kernel and $R(T)$ the range of $T \in L(X,Y)$; $\sigma(S)$ is the spectrum and $\sigma_p(S)$ the point spectrum of $S \in L(X)$.

2 Difference equations and perturbation

Let $(X, \|\cdot\|)$ denote a Banach space.

2.1 Periodic difference equations

Abstractly, we are interested in a family of nonautonomous difference equations

$$\boxed{u_{t+1} = \mathcal{F}_t^n(u_t)} \qquad (\Delta_n)$$

with right-hand sides $\mathcal{F}_t^n : U_t \to X$ on open sets $U_t \subseteq X$, $t \in \mathbb{Z}$, parametrized by $n \in \mathbb{N}_0$. In the following, $n \in \mathbb{N}$ is a discretization parameter such that $\mathcal{F}_t^n$ are understood as approximations converging to the original problem $\mathcal{F}_t^0$ as $n \to \infty$ in a sense to be defined below. A nonautonomous set $\mathcal{A} \subseteq \mathbb{Z} \times X$ with fibers $\mathcal{A}(t) \subseteq U_t$ for all $t \in \mathbb{Z}$ is called *forward invariant* or *invariant* (w.r.t. (Δ_n)), provided

$$\mathcal{F}_t^n(\mathcal{A}(t)) \subseteq \mathcal{A}(t+1), \qquad \mathcal{F}_t^n(\mathcal{A}(t)) = \mathcal{A}(t+1) \quad \text{for all } t \in \mathbb{Z}$$

resp., holds. Given an initial time $\tau \in \mathbb{Z}$, a *forward solution* to (Δ_n) is a sequence $\phi = (\phi_t)_{\tau \le t}$ satisfying $\phi_t \in U_t$ and the *solution identity* $\phi_{t+1} = \mathcal{F}_t^n(\phi_t)$ for all $\tau \le t$, a *backward solution* fulfills the solution identity for $t < \tau$ and for an *entire solution* $(\phi_t)_{t \in \mathbb{Z}}$ one has $\phi_{t+1} \equiv \mathcal{F}_t^n(\phi_t)$ on $\mathbb{Z}$. The forward solution starting at τ in the initial state $u_\tau \in U_\tau$ is uniquely determined as composition

$$\varphi^n(t; \tau, u_\tau) := \begin{cases} \mathcal{F}_{t-1}^n \circ \ldots \circ \mathcal{F}_\tau^n(u_\tau), & \tau < t, \\ u_\tau, & t = \tau \end{cases}$$

and denoted as the *general solution* to (Δ_n); it is defined as long as the compositions stay in U_t. A difference equation (Δ_n) is called θ_0-*periodic*, if both $\mathcal{F}_{t+\theta_0}^n = \mathcal{F}_t^n$ and $U_{t+\theta_0} = U_t$ hold for all $t \in \mathbb{Z}$ with some *basic period* $\theta_0 \in \mathbb{N}$; an *autonomous* equation is 1-periodic, i.e. there exists a $\mathcal{F}^n : U \to X$ with $\mathcal{F}_t^n \equiv \mathcal{F}^n$, $U_t \equiv U$ on $\mathbb{Z}$. A θ_1-*periodic solution* to (Δ_n) is an entire solution satisfying $\phi_t \equiv \phi_{t+\theta_1}$ on $\mathbb{Z}$.

Given a fixed $\theta \in \mathbb{N}$ and a sequence $u = (u_t)_{t \in \mathbb{Z}}$ with $u_t \in U_t$, $t \in \mathbb{Z}$, let us introduce the open product $\hat{U} := U_0 \times \ldots \times U_{\theta-1}$ and

$$\hat{u} := (u_0, \ldots, u_{\theta-1}) \in \hat{U}, \qquad (\overline{u_0, \ldots, u_{\theta-1}}) := (u_{t \bmod \theta})_{t \in \mathbb{Z}}.$$

In order to characterize and compute periodic solutions to $(\varDelta_n)$, $n \in \mathbb{N}_0$, we introduce the nonlinear operators

$$\hat{\mathfrak{F}}^n : \hat{U} \to X^\theta, \qquad \hat{\mathfrak{F}}^n(\hat{u}) := \begin{pmatrix} \mathcal{F}^n_{\theta-1}(u_{\theta-1}) \\ \mathcal{F}^n_0(u_0) \\ \mathcal{F}^n_1(u_1) \\ \vdots \\ \mathcal{F}^n_{\theta-2}(u_{\theta-2}) \end{pmatrix} \tag{2.1}$$

and use the norm induced via (1.1) on the Cartesian product X^θ.

The next two results are immediate:

Lemma 2.1 *Let $n \in \mathbb{N}_0$, $(\varDelta_n)$ be θ_0-periodic and θ be a multiple of θ_0:*

(a) If $(\phi_t)_{t\in\mathbb{Z}}$ is a θ-periodic solution to $(\varDelta_n)$, then $\hat{\phi} \in \hat{U}$ is a fixed point of $\hat{\mathfrak{F}}^n$.
(b) Conversely, if $\hat{\phi} \in \hat{U}$ is a fixed point of $\hat{\mathfrak{F}}^n$, then $\overline{(\phi_0, \dots, \phi_{\theta-1})}$ is a θ-periodic solution to $(\varDelta_n)$.

This characterization of periodic solutions to $(\varDelta_n)$ via the mapping $\hat{\mathfrak{F}}$ has the numerical advantage to avoid the computation of compositions $\varphi^n(\theta+\tau;\tau,\cdot) : U_\tau \to X$, $\tau \in \mathbb{Z}$, and therefore preserves (numerical) backward stability (see [10]).

Lemma 2.2 *Let $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, $(\varDelta_n)$ be θ_0-periodic and θ be a multiple of θ_0. If every $\mathcal{F}^n_t : U_t \to X$, $0 \le t < \theta_0$, is m-times continuously (Fréchet) differentiable, then $\hat{\mathfrak{F}}^n : \hat{U} \to X^\theta$ is of class C^m and for every $\hat{u} \in \hat{U}$ one has*

$$D\hat{\mathfrak{F}}^n(\hat{u}) = \begin{pmatrix} 0 & 0 & \cdots & \cdots & D\mathcal{F}^n_{\theta-1}(u_{\theta-1}) \\ D\mathcal{F}^n_0(u_0) & 0 & \cdots & \cdots & 0 \\ 0 & D\mathcal{F}^n_1(u_1) & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & D\mathcal{F}^n_{\theta-2}(u_{\theta-2}) & 0 \end{pmatrix}.$$

2.2 Linear periodic difference equations

Suppose that $\mathcal{K}^n_t \in L(X)$, $t \in \mathbb{Z}$, and consider a family of linear difference equations

$$\boxed{u_{t+1} = \mathcal{K}^n_t u_t} \tag{L_n}$$

in X parametrized by $n \in \mathbb{N}_0$. As above we understand (L_n), $n \in \mathbb{N}$, as perturbations of an initial problem (L_0). The *transition operator $\varPhi^n(t,\tau) \in L(X)$* of (L_n) is

$$\varPhi^n(t,\tau) := \begin{cases} \mathcal{K}^n_{t-1} \cdots \mathcal{K}^n_\tau, & \tau < t, \\ I_X, & t = \tau. \end{cases} \tag{2.2}$$

We are interested in θ-periodic equations (L_n), that is

$$\mathcal{K}^n_t = \mathcal{K}^n_{t+\theta} \quad \text{for all } t \in \mathbb{Z}, \tag{2.3}$$

allowing us to introduce the *period operator* $\Xi_\theta^n := \Phi^n(\theta, 0) \in L(X)$ of (L_n). Its eigenvalues are the *Floquet multipliers* and $\sigma_p(\Xi_\theta^n)$ is the *Floquet spectrum* of (L_n).

One says a linear difference equation (L_n) is *weakly hyperbolic*, if $1 \notin \sigma(\Xi_\theta^n)$, and *hyperbolic*, if $\mathbb{S}^1 \cap \sigma(\Xi_\theta^n) = \emptyset$ holds. In the hyperbolic situation, the spectrum can be decomposed as $\sigma(\Xi_\tau^n) = \sigma_+ \dot\cup \sigma_-$ with spectral sets

$$\sigma_+ \subseteq B_1(0), \qquad\qquad \sigma_- \subseteq \mathbb{C} \setminus \bar{B}_1(0).$$

With the spectral projections $P_+^n := \frac{1}{2\pi\imath} \int_{\mathbb{S}^1} (zI_X - \Xi_\theta^n)^{-1} \,\mathrm{d}z$, $P_-^n := I_X - P_+^n$ we introduce the fibers $\mathcal{V}_+^n(t) := \Phi^n(t,0)R(P_+^n)$ and $\mathcal{V}_-^n(t) := \Phi^n(t,0)R(P_-^n)$, first for $t \geq 0$ and then by θ-periodic continuation on $\mathbb{Z}$. This yields θ-periodic nonautonomous sets $\mathcal{V}_+^n \subseteq \mathbb{Z} \times X$ (*stable vector bundle*) and $\mathcal{V}_-^n \subseteq \mathbb{Z} \times X$ (*unstable vector bundle*) of (L_n). Then $\mathcal{V}_+^n$ is forward invariant, while $\mathcal{V}_-^n$ is invariant w.r.t. (L_n).

For compact operators $\Xi_\theta^n \in L(X)$ the Riesz-Schauder theory [16, pp. 428ff] applies: Every Floquet multiplier $\lambda \in \sigma_p(\Xi_\theta^n)$ possesses a minimal $\iota(\lambda) \in \mathbb{N}$ so that $N(\lambda I_X - \Xi_\theta^n)^j = N(\lambda I_X - \Xi_\theta^n)^{j+1}$ for all $j \geq \iota(\lambda)$ leading to finite-dimensional *generalized eigenspaces* $N(\lambda I_X - \Xi_\theta^n)^{\iota(\lambda)}$. All unstable fibers $\mathcal{V}_-^n(t)$, $t \in \mathbb{Z}$, have a constant finite dimension, which is denoted as the *Morse index* of (L_n) and equals the finite sum $\sum_{\lambda \in \sigma_-} \dim N(\lambda I_X - \Xi_\theta^n)^{\iota(\lambda)}$ of algebraic multiplicities.

We begin with a perturbation result for hyperbolic linear systems (L_n) under uniform convergence:

Theorem 2.1 (perturbed hyperbolicity) *Suppose that the θ-periodic linear difference equations (L_n), $n \in \mathbb{N}_0$, fulfill:*

(i) $\lim_{n\to\infty} \left\|\mathcal{K}_t^n - \mathcal{K}_t^0\right\|_{L(X)} = 0$ *for all* $0 \leq t < \theta$,
(ii) $\Xi_\theta^n \in L(X)$ *is compact for all* $n \in \mathbb{N}$.

Then also the period operator $\Xi_\theta^0 \in L(X)$ of (L_0) is compact and there exists a $N \in \mathbb{N}$ such that the following holds for all $n \geq N$ or $n = 0$:

(a) *With (L_0) also the perturbed equation (L_n) is weakly hyperbolic,*
(b) *with (L_0) also the perturbed equation (L_n) is hyperbolic. In particular, for reals $\beta \in \left(\max\left\{0, 1 - \frac{1}{2}\operatorname{dist}(\sigma(\Xi_\theta^0), \mathbb{S}^1)\right\}, 1\right)$, there exists a θ-periodic sequence $(P_t^n)_{t\in\mathbb{Z}}$ of* invariant projectors *in $L(X)$ with $\mathcal{K}_t^n P_t^n = P_{t+1}^n \mathcal{K}_t^n$ for all $t \in \mathbb{Z}$, so that the transition operators $\Phi^n(t,s)$ satisfy* $\dim \mathcal{V}_-^n = \dim \mathcal{V}_-^0$ *and the estimates*

$$\begin{aligned} \|\Phi^n(t,s)P_s^n\|_{L(X)} &\leq K\beta^{t-s} \quad \text{for all } s \leq t, \\ \|\Phi^n(t,s)[I_X - P_s^n]\|_{L(X)} &\leq K\beta^{s-t} \quad \text{for all } t \leq s, \end{aligned} \tag{2.4}$$

(c) $\lim_{n\to\infty} \left\|P_t^n - P_t^0\right\|_{L(X)} = 0$ *for all* $t \in \mathbb{Z}$.

Proof Let $0 \leq s < \theta$. Due to (i) the sequence $\left(\left\|\mathcal{K}_s^n - \mathcal{K}_s^0\right\|\right)_{n\in\mathbb{N}}$ is bounded and consequently we obtain from $\|\mathcal{K}_s^n\| \leq \left\|\mathcal{K}_s^0\right\| + \left\|\mathcal{K}_s^n - \mathcal{K}_s^0\right\|$ and the periodicity condition (2.3) that $c_t := \sup_{n\in\mathbb{N}_0} \|\mathcal{K}_t^n\| < \infty$ for all $t \in \mathbb{Z}$.

(I) *Claim*: $\lim_{n\to\infty} \|\Phi^n(t,0) - \Phi^0(t,0)\| = 0$ for all $0 \leq t$.
We proceed by mathematical induction. Thanks to (2.2), for $t = 0$ the assertion is trivial and for $t = 1$ it results from (i). In the induction step $t \to t+1$ we obtain

$$\|\Phi^n(t+1,0) - \Phi^0(t+1,0)\| \stackrel{(2.2)}{=} \|\mathcal{K}_t^n \Phi^n(t,0) - \mathcal{K}_t^0 \Phi^0(t,0)\|$$

$$\begin{aligned}
&\le \|\mathcal{K}_t^n\|\|\Phi^n(t,0)-\Phi^0(t,0)\|+\|\mathcal{K}_t^n-\mathcal{K}_t^0\|\|\Phi^0(t,0)\| \\
&\le c_t\|\Phi^n(t,0)-\Phi^0(t,0)\|+\|\mathcal{K}_t^n-\mathcal{K}_t^0\|\prod_{r=0}^{t-1}c_r \xrightarrow[n\to\infty]{(i)} 0
\end{aligned}$$

from the induction hypothesis and the triangle inequality, yielding the claim.

(II) *Claim*: $\Xi_\theta^0 \in L(X)$ is compact.
If we set $t=\theta$ in claim (I), then the period operators satisfy

$$\lim_{n\to\infty}\left\|\Xi_\theta^n-\Xi_\theta^0\right\|=0. \tag{2.5}$$

Hence, Ξ_θ^0 is the uniform limit of by (ii) compact operators Ξ_θ^n, $n\in\mathbb{N}$, and consequently compact [16, p. 416, Thm. 1.1].

(III) *Claim*: For every nonempty closed $S\subseteq\mathbb{C}$ with $\sigma(\Xi_\theta^0)\cap S=\emptyset$ there exists a $n_1\in\mathbb{N}$ such that $\sigma(\Xi_\theta^n)\cap S=\emptyset$ for all $n\ge n_1$.
Since the closed S and the compact $\sigma(\Xi_\theta^0)$ are disjoint they have a positive distance. Therefore, there is an $\varepsilon>0$ so that $S\cap B_\varepsilon(\sigma(\Xi_\theta^0))=\emptyset$. By the upper semicontinuity of the spectrum [5, p. 80, Lemma 3] and relation (2.5) there is a $n_1\in\mathbb{N}$ with $\sigma(\Xi_\theta^n)\subset B_\varepsilon(\sigma(\Xi_\theta^0))$ and consequently $\sigma(\Xi_\theta^n)$ stays disjoint from S for all $n\ge n_1$.

(a) If (L_0) is weakly hyperbolic, then $\sigma(\Xi_\theta^0)\cap\{1\}=\emptyset$ and (III) applied to the singleton $S=\{1\}$ yields the assertion.

(b) The hyperbolicity of (L_n) results as above in (a) with $S=\mathbb{S}^1$. Furthermore, then [25, p. 44, Prop. 3.13] implies that (L_n) possess an exponential dichotomy on $\mathbb{Z}$ as claimed with the θ-periodic invariant projectors P_t^n satisfying

$$\mathcal{K}_t^n|_{N(P_t^n)}\in GL(N(P_t^n),N(P_{t+1}^n))\quad\text{for all } t\in\mathbb{Z} \tag{2.6}$$

and $I_X-P_0^n=P_-^n$. In particular, by (2.5) and [5, p. 80, Cor. 1] the spectral projections P_-^n associated to the unstable spectral parts of Ξ_θ^n, $n\in\mathbb{N}_0$, fulfill that $\dim R(P_-^n)=\dim R(P_-^0)$ for large n, say for $n\ge n_2$. Thanks to (2.6) this extends to the dimension of the unstable bundles $\mathcal{V}_-^n$. Finally, we set $N:=\max\{n_1,n_2\}$.

(c) Combining (2.5) with [5, p. 80, Lemma 4] yields that the spectral projections satisfy $\lim_{n\to\infty}\left\|P_-^n-P_-^0\right\|=0$. Together with claim (I) we obtain for $t\in\mathbb{Z}$ that

$$\begin{aligned}
&\left\|P_t^n-P_t^0\right\|=\left\|[I_X-P_t^n]-[I_X-P_t^0]\right\| \\
&\overset{(2.6)}{\le}\|\Phi^n(t,0)\|\left\|P_-^n-P_-^0\right\|\left\|\Phi^0(0,t)\right\|+\left\|\Phi^n(t,0)-\Phi^0(t,0)\right\|\left\|P_-^0\Phi^0(0,t)\right\| \\
&\le\left(\prod_{r=0}^{t-1}c_r\right)\left\|P_-^n-P_-^0\right\|\left\|\Phi^0(0,t)\right\|+\left\|\Phi^n(t,0)-\Phi^0(t,0)\right\|\left\|P_-^0\Phi^0(0,t)\right\|
\end{aligned}$$

from the triangle inequality, whose right-hand side converges to 0 as $n\to\infty$. □

2.3 Perturbation of hyperbolic solutions and invariant bundles

We next address the robustness of θ_1-periodic solutions ϕ^0 to general θ_0-periodic difference equations (Δ_0), as well as their nearby saddle-point structure consisting of stable and unstable bundles (see [14, pp. 143ff, Chap. 6], [20, pp. 256ff, Sect. 4.6]) under perturbation. By imposing a natural hyperbolicity condition on the solution ϕ^0 it is shown that also the perturbations (Δ_n) have (locally unique) periodic solutions ϕ^n for sufficiently large n, which converge to ϕ^0 in the limit $n \to \infty$.

Let $\theta := \operatorname{lcm}\{\theta_0, \theta_1\}$. We suppose that the right-hand sides $\mathcal{F}_t^n$ of (Δ_n) are continuously differentiable. Our endeavor is based on the *variational equations*

$$\boxed{v_{t+1} = D\mathcal{F}_t^n(\phi_t^n)v_t} \tag{V_n}$$

associated to θ-periodic solutions ϕ^n of (Δ_n), $n \in \mathbb{N}_0$. Since the linear equations (V_n) are θ-periodic, the terminology and results from Sec. 2.2 apply to (V_n) with $\mathcal{K}_t^n = D\mathcal{F}_t^n(\phi_t^n)$ and the period operator Ξ_θ^n, $n \in \mathbb{N}_0$. In this context, we understand a solution ϕ^n of (Δ_n) as *(weakly) hyperbolic*, if (V_n) has the corresponding property.

Lemma 2.3 *Let $n \in \mathbb{N}_0$. If $\mathcal{F}_t^n : U_t \to X$ is continuously differentiable for all $0 \le t < \theta_0$, then the derivatives of the mappings $\hat{\mathcal{F}}^n : \hat{U} \to X^\theta$ defined in* (2.1) *satisfy $\sigma(\Xi_\theta^n)\setminus\{0\} = \sigma(D\hat{\mathcal{F}}^n(\hat{\phi}^n))^\theta\setminus\{0\}$ and $\sigma_p(\Xi_\theta^n)\setminus\{0\} = \sigma_p(D\hat{\mathcal{F}}^n(\hat{\phi}^n))^\theta\setminus\{0\}$.*

Based on this result, the (Floquet) spectrum of (V_n) can be computed from the (point) spectrum of the cyclic block operator given in Lemma 2.2. This has the numerical advantage of avoiding to evaluate the compositions (matrix products) Ξ_θ^n.

Proof Keeping $n \in \mathbb{N}_0$ fixed, we abbreviate $\mathcal{K}_t = D\mathcal{F}_t^n(\phi_t^n)$, $t \in \mathbb{Z}$, and observe that the θth power of $D\hat{\mathcal{F}}(\hat{\phi}^n)$ given in Lemma 2.2 becomes a block diagonal operator

$$D\hat{\mathcal{F}}(\hat{\phi}^n)^\theta = \begin{pmatrix} \mathcal{K}_{\theta-1}\mathcal{K}_{\theta-2}\cdots\mathcal{K}_0 & & & \\ & \mathcal{K}_0\mathcal{K}_{\theta-1}\cdots\mathcal{K}_1 & & \\ & & \ddots & \\ & & & \mathcal{K}_{\theta-2}\cdots\mathcal{K}_0\mathcal{K}_{\theta-1} \end{pmatrix}.$$

Referring to [25, p. 42, Prop. 3.11(a)] one has $\sigma(\Xi_\theta^n)\setminus\{0\} = \sigma(\mathcal{K}_{t+\theta-1}\cdots\mathcal{K}_t)\setminus\{0\}$ for all $t \in \mathbb{Z}$ and therefore $\sigma(\Xi_\theta^n) \setminus \{0\} = \sigma(D\hat{\mathcal{F}}(\hat{\phi}^n)^\theta) \setminus \{0\}$. Now the Spectral Mapping Theorem [5, p. 65, Thm. 2] yields the assertion for the spectra. Concerning the point spectrum the claim follows directly from the corresponding eigenvalue-eigenvector relations and the solution identity for (V_n). □

Our next result establishes persistence of hyperbolic periodic solutions to (Δ_0):

Theorem 2.2 (perturbed periodic solutions) *Let $\theta = \operatorname{lcm}\{\theta_0, \theta_1\}$. Suppose that the θ_0-periodic difference equations (Δ_n), $n \in \mathbb{N}_0$, fulfill:*

(i) $\mathcal{F}_t^n : U_t \to X$ *are continuously differentiable for all* $0 \le t < \theta_0$ *and* $n \in \mathbb{N}_0$,
(ii) $D\mathcal{F}_t^n : U_t \to L(X)$, $n \in \mathbb{N}$, *are uniformly continuous on bounded sets uniformly in* $n \in \mathbb{N}$, *the family* $\{D\mathcal{F}_t^n\}_{n\in\mathbb{N}}$ *is equicontinuous for all* $0 \le t < \theta_0$ *and for every* $n \in \mathbb{N}$ *there exists a* $0 \le t < \theta_0$ *such that* $D\mathcal{F}_t^n$ *has compact values.*

If ϕ^0 is a weakly hyperbolic θ_1-periodic solution to (Δ_0) and there exists a function $\Gamma_0 : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{\varrho\searrow 0}\Gamma_0(\varrho) = 0$ satisfying for all $0 \le t < \theta$ that

$$\left\|\mathcal{F}_t^n(\phi_t^0) - \mathcal{F}_t^0(\phi_t^0)\right\|_X \le \Gamma_0(\tfrac{1}{n}), \tag{2.7}$$

$$\lim_{n\to\infty} \left\|D\mathcal{F}_t^n(\phi_t^0) - D\mathcal{F}_t^0(\phi_t^0)\right\|_{L(X)} = 0, \tag{2.8}$$

then there exist reals $\rho_0 > 0$ and $N_0 \in \mathbb{N}$ such that the following hold for all $n \geq N_0$:

(a) There is a unique θ-periodic solution ϕ^n to (Δ_n) in the neighborhood $\mathcal{B}_{\rho_0}(\phi^0)$, it is weakly hyperbolic and there exists a constant $K_0 \geq 0$ such that

$$\sup_{t\in\mathbb{Z}} \left\|\phi_t^n - \phi_t^0\right\|_X \leq K_0 \Gamma_0(\tfrac{1}{n}), \tag{2.9}$$

(b) with ϕ^0 also the solution ϕ^n to (Δ_n) is hyperbolic with the same Morse index.

As the subsequent proof and Lemma 2.3 reveal, the constant $K_0 \geq 0$ essentially depends on the distance of the Floquet spectrum of ϕ^0 to the point $1 \in \mathbb{C}$. The value of K_0 blows up as this distance shrinks to 0, i.e. when (weak) hyperbolicity is lost.

Proof Let I denote the identity mapping on the Cartesian product X^θ. Our aim is to apply the Implicit Function Thm. A.1 with the open set $\Omega = \hat{U}$, Banach spaces $\mathcal{X} = \mathcal{Y} = X^\theta$, the parameter space $\Lambda := \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subseteq \mathbb{R}$ with metric $d(\lambda_1, \lambda_2) := |\lambda_1 - \lambda_2|$, $\lambda_0 := 0$, $x_0 := \hat{\phi}^0$, $y_0 := 0$ and the mapping

$$T : \hat{U} \times \Lambda \to X^\theta, \qquad T(x,\lambda) := \begin{cases} \hat{\mathcal{F}}^n(\hat{\phi}) - \hat{\phi}, & \lambda = \frac{1}{n}, \\ \hat{\mathcal{F}}^0(\hat{\phi}) - \hat{\phi}, & \lambda = 0 \end{cases}$$

with $x = \hat{\phi}$. Let us first verify the assumptions of Thm. A.1. It follows from (2.1) that the mapping T is well-defined.

ad (i'): Thanks to Lemma 2.1(a), for the θ-periodic solution ϕ^0 of (Δ_0) the resulting tuple $\hat{\phi}^0$ is a fixed point of $\hat{\mathcal{F}}^0$ and therefore $T(x_0, \lambda_0) = \hat{\mathcal{F}}^0(\hat{\phi}^0) - \hat{\phi}^0 = 0$.

ad (ii'): Referring to Lemma 2.2 and assumption (i) every mapping $\hat{\mathcal{F}}^n$ is continuously differentiable and so is each $T(\cdot, \lambda)$, $\lambda \in \Lambda$. Moreover, the partial derivative $D_1T(x_0, \lambda_0) = D\hat{\mathcal{F}}^0(\hat{\phi}^0) - I$ is invertible, because otherwise $1 \in \sigma(D\hat{\mathcal{F}}^0(\hat{\phi}^0))$ and thus $1 \in \sigma(D\hat{\mathcal{F}}^0(\hat{\phi}^0))^\theta \setminus \{0\} = \sigma(\Xi_\theta^0) \setminus \{0\}$ by Lemma 2.3. This contradicts the weak hyperbolicity assumption on the solution ϕ^0.

ad (iii'): First, we obtain (A.1) from the estimates

$$\|T(x_0,\lambda) - T(x_0,\lambda_0)\| = \left\|\hat{\mathcal{F}}^n(\hat{\phi}^0) - \hat{\mathcal{F}}^0(\hat{\phi}^0)\right\| \stackrel{(2.1)}{=} \max_{t=1}^{\theta} \left\|\mathcal{F}_t^n(\phi_t^0) - \mathcal{F}_t^0(\phi_t^0)\right\|$$

and thus $\|T(x_0,\lambda) - T(x_0,\lambda_0)\| \leq \Gamma_0(\lambda)$ (cf. (2.7)) for all $\lambda = \frac{1}{n} \in \Lambda$. Second, by assumption (ii) the derivatives $D\mathcal{F}_t^n : U_t \to L(X)$ are uniformly continuous on bounded sets, uniformly in $n \in \mathbb{N}$, and consequently there exist moduli of continuity $\omega_t : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{\varrho\searrow 0} \omega_t(\varrho) = 0$ and

$$\left\|D\mathcal{F}_t^n(\phi_t) - D\mathcal{F}_t^n(\phi_t^0)\right\| \leq \omega_t(\left\|\phi_t - \phi_t^0\right\|) \quad \text{for all } n \in \mathbb{N},\ 1 \leq t \leq \theta,$$

where $\hat{\phi} \in \hat{U}$. By the triangle inequality this results in

$$\begin{aligned} & \|D_1T(x,\lambda) - D_1T(x_0,\lambda_0)\| \\ \leq\ & \left\|D\hat{\mathcal{F}}^n(\hat{\phi}) - D\hat{\mathcal{F}}^n(\hat{\phi}^0)\right\| + \left\|D\hat{\mathcal{F}}^n(\hat{\phi}^0) - D\hat{\mathcal{F}}^0(\hat{\phi}^0)\right\| \end{aligned}$$

$$\stackrel{(2.1)}{=} \max_{t=1}^{\theta} \left\| D\mathcal{F}_t^n(\phi_t) - D\mathcal{F}_t^n(\phi_t^0) \right\| + \max_{t=1}^{\theta} \left\| D\mathcal{F}_t^n(\phi_t^0) - D\mathcal{F}_t^0(\phi_t^0) \right\|$$
$$\leq \max_{t=1}^{\theta} \omega_t(\left\| \phi_t - \phi_t^0 \right\|) + \max_{t=1}^{\theta} \left\| D\mathcal{F}_t^n(\phi_t^0) - D\mathcal{F}_t^0(\phi_t^0) \right\|$$

for all $\lambda = \frac{1}{n} \in \Lambda$. Now, with $\Omega'(\varrho) := \max_{t=1}^{\theta} \left\| D\mathcal{F}_t^{[1/\varrho]}(\phi_t^0) - D\mathcal{F}_t^0(\phi_t^0) \right\|$ satisfying $\lim_{\varrho \searrow 0} \Omega'(\varrho) = 0$ due to (2.8), this gives for all $\lambda = \frac{1}{n} \in \Lambda$ that

$$\|D_1 T(x, \lambda) - D_1 T(x_0, \lambda_0)\| \stackrel{(2.7)}{\leq} \max_{t=1}^{\theta} \omega_t(\left\| \phi_t - \phi_t^0 \right\|) + \Omega'(\tfrac{1}{n}) \leq \Gamma(\left\| \hat{\phi} - \hat{\phi}^0 \right\|, \lambda),$$

with the function $\Gamma(\varrho_1, \varrho_2) := \max_{t=1}^{\theta} \omega_t(\varrho_1) + \Omega'(\varrho_2)$, which clearly satisfies the limit relation $\lim_{\varrho_1, \varrho_2 \searrow 0} \Gamma(\varrho_1, \varrho_2) = 0$, i.e. (A.2) holds.

(a) Because the assumptions (i'–iii') of Thm. A.1 hold, we can choose $\rho, \delta > 0$ so small that (A.3) holds for e.g. $q := \frac{1}{2}$. Moreover, there exists a unique fixed point function $\hat{\phi} : B_\delta(\lambda_0) \to \bar{B}_{\rho_0}(\hat{\phi}^0, X^\theta)$ with $\hat{\mathcal{F}}^n(\hat{\phi}(\frac{1}{n})) = \hat{\phi}(\frac{1}{n})$ for all $n > \frac{1}{\delta}$. Then Lemma 2.1(b) guarantees that $\phi^n := \overline{(\phi_0(\frac{1}{n}), \ldots, \phi_{\theta-1}(\frac{1}{n}))}$ is the desired θ-periodic solution to (Δ_n) whenever $n \geq N_0 := [\frac{1}{\delta}] + 1$. We establish that the solutions ϕ^n are weakly hyperbolic. For this purpose, let $\varepsilon > 0$. First, thanks to (2.8) there exists a $n_1 \in \mathbb{N}$ such that

$$\left\| D\mathcal{F}_t^0(\phi_t^0) - D\mathcal{F}_t^n(\phi_t^0) \right\| \leq \tfrac{\varepsilon}{3} \quad \text{for all } t \in \mathbb{Z},\ n \geq n_1.$$

We know from Thm. A.1(c) that $\lim_{n\to\infty} \sup_{t\in\mathbb{Z}} \left\| \phi_t^n - \phi_t^0 \right\| = 0$ and since $D\mathcal{F}_t^n$ is equicontinuous by assumption (ii), there exists a $n_2 \in \mathbb{N}$ such that

$$\left\| D\mathcal{F}_t^n(\phi_t^0) - D\mathcal{F}_t^n(\phi_t^n) \right\| \leq \tfrac{\varepsilon}{3} \quad \text{for all } t \in \mathbb{Z},\ n \geq n_2.$$

Combining the last two inequalities readily yields $\left\| D\mathcal{F}_t^0(\phi_t^0) - D\mathcal{F}_t^n(\phi_t^n) \right\| < \varepsilon$ for all $t \in \mathbb{Z}$ and $n \geq \max\{n_1, n_2\}$, which establishes the limit relation

$$\lim_{n\to\infty} \left\| D\mathcal{F}_t^n(\phi_t^n) - D\mathcal{F}_t^0(\phi_t^0) \right\| = 0 \quad \text{for all } t \in \mathbb{Z}. \tag{2.10}$$

Second, assumption (ii) implies that the period operators Ξ_θ^n of (V_n), $n \in \mathbb{N}$, contain a compact factor and hence are compact [16, p. 417, Thm. 1.2]. Thus, Thm. 2.1(a) applies to $\mathcal{K}_t^n := D\mathcal{F}_t^n(\phi_t^n)$, $n \in \mathbb{N}_0$, and shows that ϕ^n are weakly hyperbolic. Finally, given N_0 and $\hat{\phi}$ as in (a) one has

$$\|\phi_t^n - \phi_t^n\| \stackrel{(1.1)}{\leq} \left\| \hat{\phi}^n - \hat{\phi}^0 \right\| \leq K_0 \Gamma_0(\tfrac{1}{n}) \quad \text{for all } n \geq N_0,\ t \in \mathbb{Z}$$

with $K_0 := 2\left\| [D\hat{\mathcal{F}}^0(\hat{\phi}^0) - I]^{-1} \right\|$, which concludes the proof of (a).

(c) In case the solution ϕ^0 is hyperbolic, then due to (2.10) and the compactness of the period operators Ξ_θ^n (see above), Thm. 2.1(b) applies to $\mathcal{K}_t^n := D\mathcal{F}_t^n(\phi_t^n)$, $n \in \mathbb{N}_0$. It follows that the solutions ϕ^n are hyperbolic as well. □

The dynamics of difference equations (Δ_n) in the vicinity of hyperbolic solutions ϕ^n is determined by a saddle-point structure consisting of local stable and unstable manifolds resp. fiber bundles [20, p. 256ff, Sect. 4.6] (in the periodic case). These sets allow a dynamical characterization and, given some $r_0 > 0$, we define the *local stable fiber bundle*

$$\mathcal{W}_+^n := \left\{(\tau, u_\tau) \in \mathcal{B}_{r_0}(\phi^n) : \varphi^n(t;\tau,u_\tau) - \phi_t^n \xrightarrow[t\to\infty]{} 0\right\}$$

and the *local unstable fiber bundle*

$$\mathcal{W}_-^n := \left\{(\tau, u_\tau) \in \mathcal{B}_{r_0}(\phi^n) : \begin{array}{l}\text{there exists a solution } (\phi_t)_{t\le\tau} \text{ of } (\Delta_n) \\ \text{with } \phi_\tau = u_\tau \text{ and } \phi_t - \phi_t^n \xrightarrow[t\to-\infty]{} 0\end{array}\right\}$$

associate to ϕ^n. The following result relates the fiber bundles of the perturbed equations (Δ_n), $n \in \mathbb{N}$, to that of the initial problem (Δ_0). It requires that $\{\mathcal{F}_t^n\}_{n\in\mathbb{N}}$ is *equidifferentiable* in each $u \in U_t$, that is there exists a $D\mathcal{F}_t^n(u) \in L(X)$ such that

$$\lim_{h\to 0} \tfrac{1}{\|h\|_X} \left\|\mathcal{F}_t^n(u+h) - \mathcal{F}_t^n(u) - D\mathcal{F}_t^n(u)h\right\|_X = 0 \quad \text{for all } t \in \mathbb{Z}$$

holds uniformly in $n \in \mathbb{N}$. We can now show that the saddle-point structure near

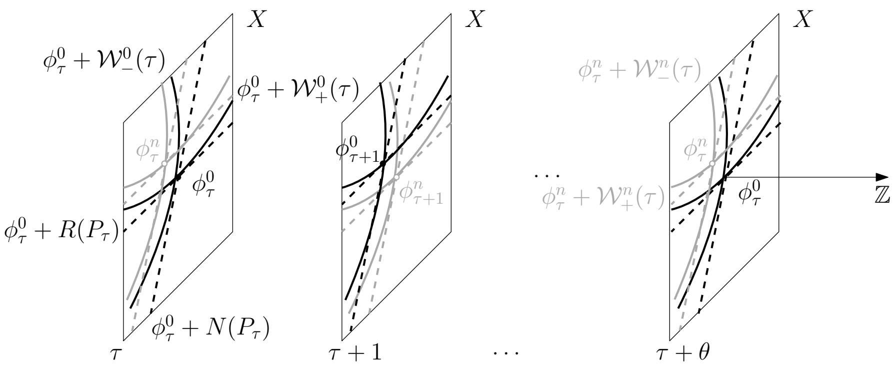

Fig. 2.1 Persistence of the saddle-point structure near a hyperbolic solution: A θ_1-periodic solution ϕ^0 (• black) of a θ_0-periodic equation (Δ_0) persists as as θ-periodic hyperbolic solution ϕ^n (∘ grey) to (Δ_n), $n \ge N_1$ (cf. Thm. 2.2). The corresponding stable bundle $\phi^0 + \mathcal{W}_+^0$ (black fibers) persists as $\phi^n + \mathcal{W}_+^n$ (grey fibers), both are locally graphs over $R(P_\tau)$ (dashed), while the unstable bundle $\phi^0 + \mathcal{W}_-^0$ (black fibers) persists as $\phi^n + \mathcal{W}_-^n$ (grey fibers), being locally graphs over $N(P_\tau)$ (dashed, cf. Thm. 2.3).

hyperbolic periodic solutions to (Δ_0) is preserved under perturbation (see Fig. 2.1).

Theorem 2.3 (perturbed stable and unstable fiber bundles) *Let* $\theta = \operatorname{lcm}\{\theta_0, \theta_1\}$ *and* $m \in \mathbb{N}$*. Suppose that the* θ_0*-periodic difference equations* (Δ_n), $n \in \mathbb{N}_0$*, fulfill:*

(i) $\mathcal{F}_t^n : U_t \to X$ *are* m*-times continuously differentiable for all* $n \in \mathbb{N}_0$ *on a convex, open set* U_t *and* $\{\mathcal{F}_t^n\}_{n\in\mathbb{N}}$ *is equidifferentiable for all* $0 \le t < \theta_0$*,*

(ii) $D\mathfrak{F}_t^n : U_t \to L(X)$, $n \in \mathbb{N}$, *are uniformly continuous on bounded sets uniformly in* $n \in \mathbb{N}$, *the family* $\{D\mathfrak{F}_t^n\}_{n\in\mathbb{N}}$ *is equicontinuous for all* $0 \le t < \theta_0$ *and for every* $n \in \mathbb{N}$ *there exists a* $0 \le t < \theta_0$ *such that* $D\mathfrak{F}_t^n$ *has compact values.*

If ϕ^0 *is a hyperbolic* θ_1*-periodic solution to* (Δ_0) *satisfying* (2.7), (2.8) *and* $(P_t)_{t\in\mathbb{Z}}$ *denotes the invariant projector onto the stable vector bundle* $\mathcal{V}_+^0$ *of* (V_0) *(cf. Thm. 2.1), then there exist* $\rho_1 > 0$ *and integers* $N_1 \ge N_0$ *so that the following holds for* $n \ge N_1$ *or* $n = 0$, *and the* θ*-periodic hyperbolic solutions* ϕ^n *ensured by Thm. 2.2:*

(a) The local stable fiber bundle $\mathcal{W}_+^n$ *of* (Δ_n) *allows the representation*

$$\mathcal{W}_+^n = \phi^n + \left\{(\tau, v + w_+^n(\tau, v)) \in \mathbb{Z} \times X : v \in B_{\rho_1}(0, R(P_\tau))\right\}$$

as graph of a mapping $w_+^n : \mathbb{Z} \times X \to X$ *with*

$$w_+^n(\tau + \theta, u) = w_+^n(\tau, u) = w_+^n(\tau, P_\tau u) \in N(P_\tau) \quad \textit{for all } \tau \in \mathbb{Z}$$

and $u \in X$. *Moreover,* $w_+^n(\tau, 0) \equiv 0$ *on* $\mathbb{Z}$, *the Lipschitz mappings* $w_+^n(\tau, \cdot)$ *are of class* C^m *and the stable fiber bundles of* (Δ_n) *and* (Δ_0) *are related via*

$$\begin{aligned}&\left\|w_+^n(\tau, v) - w_+^0(\tau, v)\right\|_X \\ &\le \frac{4K}{1-\beta} \sup_{\tau \le t} \left\| \int_0^1 \left[D\mathfrak{F}_t^0(\phi_t^0 + \vartheta\phi_t) - D\mathfrak{F}_t^n(\phi_t^n + \vartheta\phi_t)\right] \phi_t \, \mathrm{d}\vartheta \right\|_X \end{aligned} \tag{2.11}$$

for all $\tau \in \mathbb{Z}$, $v \in B_{\rho_1}(0, R(P_\tau))$, *where* $\phi_t = \varphi^0(t; \tau, \phi_\tau^0 + v + w_+^0(\tau, v)) - \phi_t^0$ *whenever* $\tau \le t$.

(b) The local unstable fiber bundle $\mathcal{W}_-^n$ *of* (Δ_n) *allows the representation*

$$\mathcal{W}_-^n = \phi^n + \left\{(\tau, v + w_-^n(\tau, v)) \in \mathbb{Z} \times X : v \in B_{\rho_1}(0, N(P_\tau))\right\}$$

as graph of a mapping $w_-^n : \mathbb{Z} \times X \to X$ *with*

$$w_-^n(\tau + \theta, u) = w_-^n(\tau, u) = w_-^n(\tau, [I_X - P_\tau]u) \in R(P_\tau) \quad \textit{for all } \tau \in \mathbb{Z}$$

and $u \in X$. *Moreover,* $w_-^n(\tau, 0) \equiv 0$ *on* $\mathbb{Z}$, *the Lipschitz mappings* $w_-^n(\tau, \cdot)$ *are of class* C^m *and the unstable fiber bundles of* (Δ_n) *and* (Δ_0) *are related via*

$$\begin{aligned}&\left\|w_-^n(\tau, v) - w_-^0(\tau, v)\right\|_X \\ &\le \frac{4K}{1-\beta} \sup_{t \le \tau} \left\| \int_0^1 \left[D\mathfrak{F}_t^0(\phi_t^0 + \vartheta\phi_t) - D\mathfrak{F}_t^n(\phi_t^n + \vartheta\phi_t)\right] \phi_t \, \mathrm{d}\vartheta \right\|_X \end{aligned} \tag{2.12}$$

for all $\tau \in \mathbb{Z}$, $v \in B_{\rho_1}(0, N(P_\tau))$, *where* $(\phi_t)_{t\le\tau}$ *is the (unique) backward solution to* (Δ_0) *starting in* $(\tau, v + w_-^0(\tau, v))$, *and have the same finite dimension.*

(c) $\mathcal{W}_+^n \cap \mathcal{W}_-^n = \phi^n$,

with the constants $\beta \in (0, 1)$, $K \ge 1$ *from Thm. 2.1 applied to* (V_0).

In order to achieve convergence as $n \to \infty$ via (2.11) and (2.12) one needs the derivatives $D\mathfrak{F}_t^n$ to tend to $D\mathfrak{F}_t^0$ on bounded sets and, thanks to Thm. 2.2, continuity of the derivative $D\mathfrak{F}_t^0$, $0 \le t < \theta_0$. A concrete illustration follows in Sect. 3.

Remark 2.1 (alternative representation of $\mathcal{W}_+^n$ and $\mathcal{W}_-^n$) With some $\tilde{\rho}_1 > 0$ the local stable and unstable fiber bundles of ϕ^n allow the alternative characterization

$$\begin{aligned}\mathcal{W}_+^n &= \phi^n + \left\{(\tau, v + \tilde{w}_+^n(\tau, v)) \in \mathbb{Z} \times X : v \in B_{\tilde{\rho}_1}(0, R(P_\tau^n))\right\},\\ \mathcal{W}_-^n &= \phi^n + \left\{(\tau, v + \tilde{w}_-^n(\tau, v)) \in \mathbb{Z} \times X : v \in B_{\tilde{\rho}_1}(0, N(P_\tau^n))\right\}\end{aligned}$$

as graphs over the vector bundles $\mathcal{V}_+^n$ resp. $\mathcal{V}_-^n$ of the variational equations (V_n), rather than over the vector bundles $\mathcal{V}_+^0$ resp. $\mathcal{V}_-^0$ of (V_0) (cf. [20, pp. 256ff, Sect. 4.6]) as in Thm. 2.3. In addition, then the associate mappings $\tilde{w}_+^n(\tau, \cdot), \tilde{w}_-^n(\tau, \cdot)$ possess values in $N(P_\tau^n)$ resp. in $R(P_\tau^n)$ for all $\tau \in \mathbb{Z}$. According to Thm. 2.1(c) the corresponding invariant projectors for (V_n) satisfy $\lim_{n\to\infty} \left\|P_t^n - P_t^0\right\| = 0$ for all $t \in \mathbb{Z}$. Therefore, $\mathcal{W}_-^n$ and $\mathcal{W}_-^0$ share their finite dimension.

Proof Since the existence of $\mathcal{W}_+^0, \mathcal{W}_-^0$ and their properties are well-established in the literature [20, pp. 187ff], we focus on their persistence and the convergence estimates (2.11) and (2.12). Let $\phi^n = (\phi_t^n)_{t\in\mathbb{Z}}$ denote the θ-periodic solutions of (Δ_n) guaranteed by Thm. 2.2 for $n \geq N_0$. The associate equations of perturbed motion

$$u_{t+1} = \bar{\mathcal{F}}_t^n(u_t), \qquad \bar{\mathcal{F}}_t^n(u) := \mathcal{F}_t^n(u + \phi_t^n) - \mathcal{F}_t^n(\phi_t^n) \qquad (\bar{\Delta}_n)$$

are θ-periodic and have the trivial solution. The general solutions φ^n of (Δ_n) and $\bar{\varphi}^n$ to $(\bar{\Delta}_n)$ are related by $\bar{\varphi}^n(t; \tau, u) = \varphi^n(t; \tau, u + \phi_\tau^n) - \phi_t^n$ for all $\tau \leq t$.

(a) For each fixed $\tau \in \mathbb{Z}$ the sequence space

$$\ell_\tau^+ := \left\{(\phi_t)_{\tau\leq t} : \phi_t \in X \text{ and } \lim_{t\to\infty} \|\phi_t\| = 0\right\},$$

is complete w.r.t. the sup-norm $\|\phi\|_\infty := \sup_{\tau\leq t} \|\phi_t\|$. For $\bar{\rho} > 0$ so small that $\|\phi_t\| < \bar{\rho}$ implies $\phi_t + \phi_t^n \in U_t$ for all $t \in \mathbb{Z}$ and $n \geq N_0$ we introduce the operator

$$T_+^n : B_{\bar{\rho}}(0, \ell_\tau^+) \to R(P_\tau) \times \ell_\tau^+, \qquad T_+^n(\phi)_t := \left(P_\tau\phi_\tau, \phi_{t+1} - \bar{\mathcal{F}}_t^n(\phi_t)\right)$$

for all $\tau \leq t$. Then $u_\tau = P_\tau u_\tau + [I_X - P_\tau]u_\tau \in X$ is contained in the stable bundle of $(\bar{\Delta}_n)$ if and only if $\phi := \bar{\varphi}^n(\cdot; \tau, u_\tau)$ satisfies (cf. [6, proof of Thm. 3.1])

$$T_+^n(\phi) = \left(P_\tau u_\tau, 0\right). \tag{2.13}$$

Our approach to (2.13) using the Lipschitz inverse function Thm. A.2 is based on the representation $T_+^n = A_+ + G_+^n$ with

$$\begin{aligned}A_+ &\in L(\ell_\tau^+, R(P_\tau) \times \ell_\tau^+), &\quad (A_+\phi)_t &:= \left(P_\tau\phi_\tau, \phi_{t+1} - D\mathcal{F}_t^0(\phi_t^0)\phi_t\right),\\ G_+^n &: \ell_\tau^+ \to R(P_\tau) \times \ell_\tau^+, &\quad G_+^n(\phi)_t &:= \left(0, D\mathcal{F}_t^0(\phi_t^0)\phi_t - \bar{\mathcal{F}}_t^n(\phi_t)\right)\end{aligned}$$

for all $\tau \leq t$. Note that the derivatives $D\mathcal{F}_t^0 : U_t \to L(X)$ exist by assumption (i).

(I) Claim: $A_+ \in GL(\ell_\tau^+, R(P_\tau) \times \ell_\tau^+)$ *with* $\left\|A_+^{-1}\right\| \leq \frac{2K}{1-\beta}$.

First of all, the sequence $(D\mathcal{F}_t^0(\phi_t^0))_{t\in\mathbb{Z}}$ in $L(X)$ is θ-periodic and therefore A_+ is bounded. In order to show that A_+ is invertible, given $v_\tau \in R(P_\tau)$ and a sequence $\psi \in \ell_\tau^+$, we observe that $A_+\phi = (v_\tau, \psi)$ has the unique solution

$$\phi_t = \Phi^0(t, \tau)P_\tau v_\tau + \sum_{s=\tau}^{t-1} \Phi^0(t, s+1)P_s\psi_s - \sum_{s=t}^{\infty} \Phi^0(t, s+1)[I_X - P_s]\psi_s$$

in ℓ_τ^+ (a proof can be modelled after e.g. [20, pp. 151–152, Thm. 3.5.3(a)]). Using the dichotomy estimates (2.4) it is not hard to show $\|\phi_t\| \leq K\|v_\tau\| + K\frac{1+\beta}{1-\beta}\|\psi\|_\infty$ for all $\tau \leq t$ and therefore $\|A_+^{-1}\| \leq K + K\frac{1+\beta}{1-\beta} = \frac{2K}{1-\beta}$.

(II) Claim: *There exist* $\rho \in (0, \bar\rho]$, $N_1 \geq N_0$ *such that* $\operatorname{lip} G_+^n|_{B_\rho(0)} \leq \frac{1-\beta}{4K}$ *holds for all* $n \geq N_1$.
Due to the limit relation (2.10) in the proof of Thm. 2.2 there is an $N_1 \geq N_0$ with

$$\left\|D\mathcal{F}_t^0(\phi_t^0) - D\mathcal{F}_t^n(\phi_t^n)\right\| \leq \frac{1-\beta}{8K} \quad \text{for all } t \in \mathbb{Z},\ n \geq N_1.$$

We next abbreviate $\mathcal{H}_t^n(u) := D\mathcal{F}_t^n(\phi_t^n)u - \bar{\mathcal{F}}_t^n(u)$. This function is continuously differentiable $D\mathcal{H}_t^n(u) = D\mathcal{F}_t^n(\phi_t^n) - D\bar{\mathcal{F}}_t^n(u) = D\mathcal{F}_t^n(\phi_t^n) - D\mathcal{F}_t^n(u + \phi_t^n)$. The Mean Value Inequality [16, p. 342, Cor. 4.3] and the fact that $\mathcal{F}_t^n$ is equidifferentiable by assumption (i) with continuous derivative thus implies that there exists a $\rho \in (0, \bar\rho]$ such that $\|\mathcal{H}_t^n(u) - \mathcal{H}_t^n(\bar u)\| \leq \frac{1-\beta}{8K}\|u - \bar u\|$ for all $t \in \mathbb{Z}$, $u, \bar u \in B_\rho(0, X)$ and $n \geq N_1$. In combination, due to the representation

$$G_+^n(\phi)_t = \left(0, D\mathcal{F}_t^0(\phi_t^0)\phi_t - D\mathcal{F}_t^n(\phi_t^n)\phi_t\right) + \left(0, \mathcal{H}_t^n(\phi_t)\right)$$

we finally obtain for all $\phi, \bar\phi \in B_\rho(0, \ell_\tau^+)$ that

$$\left\|G_+^n(\phi) - G_+^n(\bar\phi)\right\|_\infty \leq \tfrac{1-\beta}{4K}\left\|\phi - \bar\phi\right\|_\infty \quad \text{for all } n \geq N_1.$$

(III) In this step we apply the Lipschitz inverse function Thm. A.2 to solve the nonlinear equation (2.13) in the Banach spaces $\mathcal{X} = \ell_\tau^+$, $\mathcal{Y} = R(P_\tau) \times \ell_\tau^+$, points $x_0 := 0$, $y_0 := (P_\tau u_\tau, 0)$, the Lipschitz constant $l := \frac{1-\beta}{4K}$ and $\sigma := \frac{1-\beta}{2K}$. Therefore, for every $u_\tau \in B_{\frac{1-\beta}{4K^2}\rho}(0, X)$ one has

$$\|(P_\tau u_\tau, 0)\| \stackrel{(1.1)}{=} \|P_\tau u_\tau\| < \frac{1-\beta}{4K}\rho =: \rho_1$$

and there exists a unique solution $\phi_+^n(u_\tau) \in B_\rho(0, \ell_\tau^+)$ to (2.13). Then the function w_+^n parametrizing the stable bundle of $(\bar\Delta_n)$ is $w_+^n(\tau, v_\tau) := [I_X - P_\tau]\phi_+^n(v_\tau)_\tau$, where $v_\tau = P_\tau u_\tau$. We define $\phi := \bar\varphi^0(\cdot; \tau, v_\tau + w_+^0(\tau, v_\tau))$, $\bar\phi := \phi_+^n(v_\tau)$ and obtain

$$\begin{aligned}
&\left\|w_+^n(\tau, v_\tau) - w_+^0(\tau, v_\tau)\right\| = \left\|\bar\phi_\tau - \phi_\tau\right\| \leq \left\|\bar\phi - \phi\right\|_\infty \\
\stackrel{(A.4)}{\leq}\ & \frac{4K}{1-\beta}\left\|T_+^n(\bar\phi) - T_+^n(\phi)\right\|_\infty \\
\stackrel{(1.1)}{=}\ & \frac{4K}{1-\beta}\max\left\{\left\|P_n[\bar\phi_\tau - \phi_\tau]\right\|, \sup_{\tau \leq t}\left\|\bar\phi_{t+1} - \bar{\mathcal{F}}_t^n(\bar\phi_t) - [\phi_{t+1} - \bar{\mathcal{F}}_t^n(\phi_t)]\right\|\right\} \\
=\ & \frac{4K}{1-\beta}\sup_{\tau \leq t}\left\|\bar\phi_{t+1} - \bar{\mathcal{F}}_t^n(\bar\phi_t) - [\phi_{t+1} - \bar{\mathcal{F}}_t^n(\phi_t)]\right\|.
\end{aligned}$$

Because $\bar\phi$ solves $(\bar\Delta_n)$ and ϕ solves $(\bar\Delta_0)$, this simplifies to

$$\begin{aligned}
&\left\|w_+^n(\tau, v_\tau) - w_+^0(\tau, v_\tau)\right\| \\
\leq\ & \frac{4K}{1-\beta}\sup_{\tau \leq t}\left\|\phi_{t+1} - \bar{\mathcal{F}}_t^n(\phi_t)]\right\| = \tfrac{4K}{1-\beta}\sup_{\tau \leq t}\left\|\bar{\mathcal{F}}_t^0(\phi_t) - \bar{\mathcal{F}}_t^n(\phi_t)]\right\|
\end{aligned}$$

$$\stackrel{(\bar{\Delta}_n)}{=} \frac{4K}{1-\beta} \sup_{\tau \le t} \left\| \mathcal{F}_t^0(\phi_t + \phi_t^0) - \mathcal{F}_t^0(\phi_t^0) - [\mathcal{F}_t^n(\phi_t + \phi_t^n) + \mathcal{F}_t^n(\phi_t^n)] \right\|$$

and it remains to estimate the right-hand side in this inequality. Since U_t is assumed to be convex, we apply the Mean Value Theorem [16, p. 341, Thm. 4.2] and arrive at

$$\begin{aligned} &\left\| w_+^n(\tau, v_\tau) - w_+^0(\tau, v_\tau) \right\| \\ \le\ & \frac{4K}{1-\beta} \sup_{\tau \le t} \left\| \int_0^1 [D\mathcal{F}_t^0(\phi_t^0 + \vartheta\phi_t) - D\mathcal{F}_t^n(\phi_t^n + \vartheta\phi_t)]\phi_t \,\mathrm{d}\vartheta \right\| \end{aligned}$$

for all $v_\tau \in R(P_\tau)$. Here, for $\tau \le t$ one has the relation $\phi_t = \bar{\varphi}^0(t; \tau, v_\tau + w_+^0(\tau, v_\tau)) = \varphi^0(t; \tau, \phi_\tau^0 + v_\tau + w_+^0(\tau, v_\tau)) - \phi_t^0$.

(b) The argument is dual to the proof of (a), but now one works in the sequence space $\ell_\tau^- := \left\{ (\phi_t)_{t \le \tau} : \phi_t \in X \text{ and } \lim_{t \to -\infty} \|\phi_t\| = 0 \right\}$ being complete in the sup-norm. One applies Thm. A.2 with $\mathcal{X} = \ell_\tau^-$, $\mathcal{Y} = N(P_\tau) \times \ell_\tau^-$ and $x_0 := 0$, $y_0 := (u_\tau - P_\tau u_\tau, 0)$, $l := \frac{1-\beta}{4K}$, $\sigma := \frac{1-\beta}{2K}$ to the nonlinear operator

$$T_-^n : B_\rho(0, \ell_\tau^-) \to N(P_\tau) \times \ell_\tau^-, \quad T_-^n(\phi)_t := \left(\phi_\tau - P_\tau \phi_\tau, \phi_t - \bar{\mathcal{F}}_{t-1}^n(\phi_{t-1}) \right)$$

for all $t \le \tau$. If the unique solution to $T_-^n(\phi) = (P_\tau u_\tau, 0)$ is denoted by $\phi_-^n(v_\tau) \in \ell_\tau^-$ such that $v_\tau = u_\tau - P_\tau u_\tau$, then $w_-^n(\tau, v_\tau) := P_\tau \phi_-^n(v_\tau)_\tau$ has the claimed properties.

(c) is a consequence of [20, pp. 259–260, Thm. 4.6.4]. □

3 Urysohn integrodifference equations

Let us now illustrate the applicability of our abstract perturbation results from Sect. 2, when the initial problem (Δ_0) is an integrodifference equation

$$u_{t+1} = \mathcal{F}_t^0(u_t), \qquad \mathcal{F}_t^0(u) := \int_\Omega f_t(\cdot, y, u(y)) \,\mathrm{d}y, \tag{I_0}$$

whose right-hand side is an Urysohn operator over a compact nonempty $\Omega \subset \mathbb{R}^\kappa$. For the sake of having well-defined and smooth mappings $\mathcal{F}_t^0$, $t \in \mathbb{Z}$, in an ambient setting, several assumptions on the *kernel functions* f_t are due:

Hypothesis 3.1 *Let $m \in \mathbb{N}$ and $\alpha \in (0, 1]$. Suppose there exists a $\theta_0 \in \mathbb{N}$ and open, convex sets $Z_t \subseteq \mathbb{R}^d$ such that the kernel functions*

$$f_t = f_{t+\theta_0} : \Omega^2 \times \overline{Z_t} \to \mathbb{R}^d, \qquad Z_t = Z_{t+\theta_0} \quad \text{for all } t \in \mathbb{Z} \tag{3.1}$$

fulfill the following assumptions for all $0 \le t < \theta_0$ and $0 \le k \le m$:

(i) The derivative $D_3^k f_t : \Omega^2 \times \overline{Z_t} \to L_k(\mathbb{R}^d)$ exists as continuous function,
(ii) for all $r > 0$ there exists a continuous function $h_r : \Omega \to \mathbb{R}_+$ such that

$$\left| D_3^k f_t(x, y, z) - D_3^k f_t(\bar{x}, y, z) \right|_{L_k(\mathbb{R}^d)} \le h_r(y) \left| x - \bar{x} \right|^\alpha \tag{3.2}$$

for all $x, \bar{x}, y \in \Omega$, $z \in \overline{Z_t} \cap \bar{B}_r(0)$, *(iii) for all $r > 0$ there exists a function $c_r : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ satisfying the limit relation $\lim_{\delta \searrow 0} \sup_{y \in \Omega} c_r(\delta, y) = 0$, such that $|z - \bar{z}| \le \delta$ implies*

$$\begin{aligned}&\left|D_3^k f_t(x,y,z) - D_3^k f_t(x,y,\bar{z}) - \left[D_3^k f_t(\bar{x},y,z) - D_3^k f_t(\bar{x},y,\bar{z})\right]\right|_{L_k(\mathbb{R}^d)}\\ &\qquad \le c_r(\delta,y)\,|x-\bar{x}|^\alpha \quad \text{for all } x,\bar{x},y \in \Omega,\ \bar{z} \in \overline{Z_t} \cap \bar{B}_r(0). \qquad (3.3)\end{aligned}$$

Let $C(\Omega, \mathbb{R}^d)$ denote the set of continuous functions $u : \Omega \to \mathbb{R}^d$ equipped with the norm $\|u\|_0 := \sup_{x \in \Omega} |u(x)|$. If $\alpha \in (0, 1]$, then functions $u : \Omega \to \mathbb{R}^d$ having a bounded Hölder constant

$$[u]_\alpha := \sup_{\substack{x,\bar{x} \in \Omega \\ x \ne \bar{x}}} \frac{|u(x) - u(\bar{x})|}{|x - \bar{x}|^\alpha} < \infty$$

are called *α-Hölder* (*Lipschitz* in case $\alpha = 1$) and $C^\alpha(\Omega, \mathbb{R}^d) \subset C(\Omega, \mathbb{R}^d)$ denotes the entity of all such functions. It is a Banach space when equipped with the norm

$$\|u\|_\alpha := \begin{cases} \|u\|_0, & \alpha = 0, \\ \max\{\|u\|_0, [u]_\alpha\}, & \alpha \in (0,1]. \end{cases} \qquad (3.4)$$

Since the compact domain Ω is fixed throughout, we conveniently abbreviate

$$C_d^\alpha := C^\alpha(\Omega, \mathbb{R}^d), \qquad\qquad C_d^0 := C(\Omega, \mathbb{R}^d)$$

and obtain the open sets $U_t := \{u \in C_d^\alpha : u(\Omega) \subset Z_t\}$ for all $t \in \mathbb{Z}$.

For our subsequent analysis it is important to note that Hyp. 3.1 implies the corresponding assumptions made in [24, Sect. 2]. In detail, one has:

Proposition 3.1 (properties of (I_0)) *Let $t \in \mathbb{Z}$. If Hyp. 3.1 holds, then the Urysohn operator $\mathcal{F}_t^0 = \mathcal{F}_{t+\theta_0}^0 : U_t \to C_d^\alpha$ is well-defined, completely continuous and of class C^m with compact derivative*

$$D\mathcal{F}_t^0(u)v = \int_\Omega D_3 f_t(\cdot, y, u(y))v(y)\,\mathrm{d}y \quad \text{for all } u \in U_t,\ v \in C_d^\alpha. \qquad (3.5)$$

Combined with the solution identity this shows that entire solutions ϕ to (I_0) inherit the smoothness of the kernel function, i.e. $\phi_t \in C_d^\alpha$, $t \in \mathbb{Z}$. Yet for kernel functions of convolution type a higher smoothness can be expected (cf. [24, Sect. 2.3]).

Proof Above all, (I_0) and (3.1) show that $\mathcal{F}_t^0$ is θ_0-periodic in t. The results from [24] formulated in an abstract measure-theoretical set-up apply to $\mathcal{F}_t^0$ with the κ-dimensional Lebesgue measure $\mu = \lambda_\kappa$. By [24, Thm. 2.6], $\mathcal{F}_t^0$ is well-defined and due to [24, Cor. 2.7(i)] also completely continuous. In [24, Thm. 2.12] it is shown that $\mathcal{F}_t^0$ is of class C^m and [19, p. 89, Prop. 6.5] implies that $D\mathcal{F}_t^0(u)$, $u \in U_t$, is compact. □

Corollary 3.1 *Let $t \in \mathbb{Z}$ and $2 \le m$. If for every $r > 0$ there exists a continuous function $l_r : \Omega^2 \to \mathbb{R}_+$ with*

$$|D_3 f_t(x,y,z) - D_3 f_t(x,y,\bar{z})|_{L(\mathbb{R}^d)} \le l_r(x,y)\,|z - \bar{z}| \quad \text{for all } x, y \in \Omega$$

and $z, \bar z \in Z_t \cap \bar B_r(0)$*, then* $D\mathcal{F}_t^0 : U_t \to L(C_d^\alpha)$ *is Lipschitz on* C_d^0*-bounded sets, that is, for each* $r > 0$ *there exists a* $L_r \geq 0$ *such that*

$$\left\|D\mathcal{F}_t^0(u) - D\mathcal{F}_t^0(\bar u)\right\|_{L(C_d^\alpha)} \leq L_r \left\|u - \bar u\right\|_\alpha \quad \text{for all } u, \bar u \in U_t \cap \bar B_r(0, C_d^0) \tag{3.6}$$

with the Lipschitz constant $L_r := \max\left\{\sup_{\xi\in\Omega} \int_\Omega l_r(\xi, y)\,\mathrm{d}y, \int_\Omega h_r(y)\,\mathrm{d}y\right\}$.

Proof Let $v \in C_d^\alpha$, $r > 0$ and $u, \bar u \in U_t \cap \bar B_r(0, C_d^0)$.

(I) We derive that

$$\begin{aligned}
\left|[D\mathcal{F}_t^0(u) - D\mathcal{F}_t^0(\bar u)]v(x)\right| &\overset{(3.5)}{\leq} \int_\Omega \left|D_3 f_t(x, y, u(y)) - D_3 f_t(x, y, \bar u(y))\right| |v(y)| \,\mathrm{d}y \\
&\leq \int_\Omega l_r(x, y) \left|u(y) - \bar u(y)\right| |v(y)| \,\mathrm{d}y \\
&\overset{(3.4)}{\leq} \sup_{\xi\in\Omega} \int_\Omega l_r(\xi, y)\,\mathrm{d}y \left\|u - \bar u\right\|_\alpha \left\|v\right\|_\alpha \quad \text{for all } x \in \Omega.
\end{aligned}$$

Thus, $\left\|[D\mathcal{F}_t^0(u) - D\mathcal{F}_t^0(\bar u)]v\right\|_0 \leq \sup_{\xi\in\Omega} \int_\Omega l_r(\xi, y)\,\mathrm{d}y \left\|u - \bar u\right\|_\alpha \left\|v\right\|_\alpha$ after passing to the least upper bound over all $x \in \Omega$.

(II) With $Z_t \subseteq \mathbb{R}^d$ also $U_t \subseteq C_d^\alpha$ is convex. Therefore, the Mean Value Theorem [16, p. 341, Thm. 4.2] applies and shows for $x, \bar x \in \Omega$ that

$$\begin{aligned}
&[D\mathcal{F}_t^0(u) - D\mathcal{F}_t^0(\bar u)]v(x) - \left[D\mathcal{F}_t^0(u) - D\mathcal{F}_t^0(\bar u)\right] v(\bar x) \\
\overset{(3.5)}{=} &\int_\Omega \big[D_3 f_t(x, y, u(y)) - D_3 f_t(x, y, \bar u(y)) \\
&\quad - \big(D_3 f_t(\bar x, y, u(y)) - D_3 f_t(\bar x, y, \bar u(y))\big)\big] v(y)\,\mathrm{d}y \\
= &\int_\Omega \int_0^1 \big[D_3^2 f_t(x, y, \bar u(y) + \vartheta(u(y) - \bar u(y))) \\
&\quad - D_3^2 f_t(\bar x, y, \bar u(y) + \vartheta(u(y) - \bar u(y)))\,\mathrm{d}\vartheta\,[u(y) - \bar u(y)]\big] v(y)\,\mathrm{d}y.
\end{aligned}$$

Consequently Hyp. 3.1(ii) leads to

$$\begin{aligned}
&\left|[D\mathcal{F}_t^0(u) - D\mathcal{F}_t^0(\bar u)]v(x) - \left[D\mathcal{F}_t^0(u) - D\mathcal{F}_t^0(\bar u)\right] v(\bar x)\right| \\
\overset{(3.4)}{\leq} &\int_\Omega \int_0^1 \big|D_3^2 f\big(x, y, \bar u(y) + \vartheta(u(y) - \bar u(y))\big) \\
&\quad - D_3^2 f\big(\bar x, y, \bar u(y) + \vartheta(u(y) - \bar u(y))\big)\big|\,\mathrm{d}\vartheta\,\mathrm{d}y \left\|u - \bar u\right\|_\alpha \left\|v\right\|_\alpha \\
\overset{(3.2)}{\leq} &\int_\Omega h_r(y)\,\mathrm{d}y \left\|u - \bar u\right\|_\alpha \left\|v\right\|_\alpha |x - \bar x|^\alpha \quad \text{for all } x, \bar x \in \Omega,
\end{aligned}$$

which guarantees $[[D\mathcal{F}_t^0(u) - D\mathcal{F}_t^0(\bar u)]v]_\alpha \leq \int_\Omega h_r(y)\,\mathrm{d}y \left\|u - \bar u\right\|_\alpha \left\|v\right\|_\alpha$. Referring to (3.4) this implies the local Lipschitz estimate (3.6). □

Along with IDEs (I_0) we now consider their Nyström discretizations. They are based on *quadrature* ($\kappa = 1$) or *cubature rules* ($\kappa > 1$), i.e. a family of mappings

$$Q^n : C_d^0 \to \mathbb{R}^d, \qquad Q^n u := \sum_{\eta\in\Omega_n} w_\eta u(\eta) \quad \text{for all } n \in \mathbb{N} \tag{Q_n}$$

determined by a *grid* $\Omega_n \subset \Omega$ of finitely many *nodes* $\eta \in \Omega_n$ and *weights* $w_\eta \geq 0$; the dependence of w_η on $n \in \mathbb{N}$ is suppressed here. A rule (Q_n) is called (cf. [13])

- *convergent*, if $\lim_{n\to\infty} Q^n u = \int_\Omega u(y)\,\mathrm{d}y$ holds for all $u \in C_d^0$,
- *stable*, provided the weights satisfy

$$W := \sup_{n\in\mathbb{N}} W_n < \infty, \qquad W_n := \sum_{\eta\in\Omega_n} w_\eta. \tag{3.7}$$

Thanks to [13, p. 20, Thm. 1.4.17], convergence implies stability.

In order to evaluate the right-hand side of (I_0) approximately, we replace the integral by a convergent integration rule (Q_n), $n \in \mathbb{N}$. The resulting *Nyström method* (see [4, 13] for integral equations) yields the family of difference equations

$$u_{t+1} = \mathcal{F}_t^n(u_t), \qquad \mathcal{F}_t^n(u) := \sum_{\eta\in\Omega_n} w_\eta f_t(\cdot, \eta, u(\eta)). \tag{I_n}$$

Proposition 3.2 (properties of (I_n)) *Let $t \in \mathbb{Z}$. If Hyp. 3.1 holds, then the discrete Urysohn operator $\mathcal{F}_t^n = \mathcal{F}_{t+\theta_0}^n : U_t \to C_d^\alpha$, $n \in \mathbb{N}$, is well-defined, completely continuous and of class C^m with compact derivative*

$$D\mathcal{F}_t^n(u)v = \sum_{\eta\in\Omega_n} w_\eta D_3 f_t(\cdot, \eta, u(\eta))v(\eta) \quad \textit{for all } u \in U_t,\ v \in C_d^\alpha. \tag{3.8}$$

Moreover, if (Q_n) is stable, then $\{\mathcal{F}_t^n\}_{n\in\mathbb{N}}$ is equidifferentiable, $D\mathcal{F}_t^n$ are uniformly continuous on bounded sets uniformly in $n \in \mathbb{N}$ and $\{D\mathcal{F}_t^n\}_{n\in\mathbb{N}}$ is equicontinuous.

Proof The grids Ω_n, $n \in \mathbb{N}$, are a family of compact and discrete subsets of Ω. If we equip them with the measure $\mu(\Omega_n) := \sum_{\eta\in\Omega_n} w_\eta$, then due to [24, Ex. 2.2 and Rem. 2.5] the abstract measure-theoretical integral from [24] becomes

$$\int_{\Omega_n} f_t(x, y, u(y))\,\mathrm{d}\mu(y) = \sum_{\eta\in\Omega_n} w_\eta f_t(x, \eta, u(\eta)) \quad \text{for all } x \in \Omega$$

and leads to the discrete integral operators in (I_n). Given this, well-definedness, complete continuity and smoothness of $\mathcal{F}_t^n$ result from [24] as in the proof of Prop. 3.1. From now on, assume that (Q_n) is stable and choose $u \in U_t$.

(I) *Claim*: $\{\mathcal{F}_t^n\}_{n\in\mathbb{N}}$ is equidifferentiable.
For functions $h \in C_d^\alpha$ the remainder terms [24, (16) resp. (18)] become

$$r_0(h) = \sup_{\vartheta\in[0,1]} \left\| \sum_{\eta\in\Omega_n} w_\eta \left[D_3 f_t(\cdot, \eta, (u + \vartheta h)(\eta)) - D_3 f_t(\cdot, \eta, u(\eta))\right] \right\|_0,$$

$$\rho_0(h) = \int_0^1 \sum_{\eta\in\Omega_n} w_\eta \bar{c}_r^1(\vartheta \|h\|_0, y)\,\mathrm{d}\vartheta \leq \sum_{\eta\in\Omega_n} w_\eta \bar{c}_r^1(\|h\|_0, \eta).$$

Now it follows from (3.7) that $\lim_{h\to\infty} r_0(h) = \lim_{h\to\infty} \rho_0(h) = 0$ hold uniformly in $n \in \mathbb{N}$. This yields the claimed equidifferentiability.

(II) *Claim*: $D\mathcal{F}_t^n$ are uniformly continuous on bounded sets uniformly in $n \in \mathbb{N}$ (and thus $\{D\mathcal{F}_t^n\}_{n\in\mathbb{N}}$ is equicontinuous).

Let $\varepsilon > 0$, $v \in C_d^\alpha$ and given $u, \bar{u} \in U_t$ choose $r > 0$ so large that $\|u\|_0, \|\bar{u}\|_0 \le r$ holds. Because the (extended) derivative $D_3 f_t : \Omega^2 \times \overline{Z_t} \to L(\mathbb{R}^d)$ is uniformly continuous on the compact set $\Omega^2 \times (\overline{Z_t} \cap \bar{B}_r(0))$, there exists a $\delta_1 > 0$ such that

$$|z - \bar{z}| < \delta_1 \Rightarrow |D_3 f_t(x, y, z) - D_3 f_t(x, y, \bar{z})| < \tfrac{\varepsilon}{2W} \quad \text{for all } z, \bar{z} \in Z_t \cap \bar{B}_r(0)$$

and $x, y \in \Omega$. If $u, \bar{u} \in U_t$ satisfy $\|u - \bar{u}\|_0 < \delta_1$, then we obtain $|u(y) - \bar{u}(y)| < \delta_1$ for all $y \in \Omega$. First, this implies

$$\begin{aligned} &|[D\mathcal{F}_t^n(u) - D\mathcal{F}_t^n(\bar{u})]v(x)| \\ \overset{(3.8)}{\le} &\sum_{\eta \in \Omega_n} w_\eta \, |D_3 f_t(x, \eta, u(\eta)) - D_3 f_t(x, \eta, \bar{u}(\eta))| \, |v(\eta)| \\ \overset{(3.4)}{\le} &\sum_{\eta \in \Omega_n} w_\eta \tfrac{\varepsilon}{2W} \|v\|_\alpha \le \tfrac{\varepsilon}{2} \|v\|_\alpha \quad \text{for all } x \in \Omega \end{aligned}$$

and passing to the supremum over $x \in \Omega$ yields $\|[D\mathcal{F}_t(u) - D\mathcal{F}_t(\bar{u})]v\|_0 \le \frac{\varepsilon}{2} \|v\|_\alpha$. Second, from Hyp. 3.1(iii) there exists a $\delta_2 > 0$ such that $\sup_{y \in \Omega} c_r(\delta, y) < \frac{\varepsilon}{2W}$ for every $\delta \in (0, \delta_2]$ and consequently $\|u - \bar{u}\|_0 < \delta_2$ guarantees for all $x, \bar{x} \in \Omega$ that

$$\begin{aligned} &|[D\mathcal{F}_t^n(u) - D\mathcal{F}_t^n(\bar{u})]v(x) - [D\mathcal{F}_t^n(u) - D\mathcal{F}_t^n(\bar{u})]v(\bar{x})| \\ \overset{(3.8)}{\le} &\sum_{\eta \in \Omega_n} w_\eta |D_3 f_t(x, \eta, u(\eta)) - D_3 f_t(x, \eta, \bar{u}(\eta)) \\ &\qquad - [D_3 f_t(\bar{x}, \eta, u(\eta)) - D_3 f_t(\bar{x}, \eta, \bar{u}(\eta))]| \, |v(\eta)| \\ \overset{(3.3)}{\le} &\sum_{\eta \in \Omega_n} w_\eta c_r(\delta, \eta) \, |x - \bar{x}|^\alpha \, \|v\|_\alpha \overset{(3.7)}{\le} W \sup_{y \in \Omega} c_r(\delta, \eta) \, |x - \bar{x}|^\alpha \, \|v\|_\alpha \end{aligned}$$

and therefore $[[D\mathcal{F}_t^n(u) - D\mathcal{F}_t^n(\bar{u})]v]_\alpha \le \frac{\varepsilon}{2} \|v\|_\alpha$. Referring to (3.4) this results in

$$\|u - \bar{u}\|_0 < \min\{\delta_1, \delta_2\} \quad \Rightarrow \quad \|[D\mathcal{F}_t^n(u) - D\mathcal{F}_t^n(\bar{u})]v\|_\alpha \le \tfrac{\varepsilon}{2} \|v\|_\alpha$$

for all $n \in \mathbb{N}$. Since $v \in C_d^\alpha$ was arbitrary, this readily implies the claim. □

3.1 Hölder continuous kernel functions

We say an integration rule (Q_n) has *consistency order* $\alpha \in (0, 1]$ (cf. [13, p. 21, Def. 1.4.19]), if there exists a $c_0 \ge 0$ with

$$\left| \int_\Omega u(y) \, \mathrm{d}y - Q^n u \right| \le \frac{c_0}{n^\alpha} \|u\|_\alpha \quad \text{for all } u \in C_d^\alpha.$$

Example 3.1 (quadrature rules) Let $\Omega = [a, b]$ and $n \in \mathbb{N}$. The *(left)* resp. *(right) rectangular rules* $Q_{LR}^n u := \frac{b-a}{n} \sum_{j=0}^{n-1} u(a + j \frac{b-a}{n})$ and $Q_{RR}^n u := \frac{b-a}{n} \sum_{j=1}^{n} u(a + j \frac{b-a}{n})$ are convergent and satisfy the quadrature error (cf. [8, p. 52, Theorem])

$$\left| \int_a^b u(y) \, \mathrm{d}y - Q_i^n u \right| \le \frac{(b-a)^{\alpha+1}}{n^\alpha} [u]_\alpha \quad \text{for } i \in \{LR, RR\}.$$

Also the *midpoint rule* $Q_M^n u := \frac{b-a}{n}\sum_{j=0}^{n-1} u(a+(j+\frac{1}{2})\frac{b-a}{n})$ is convergent and as in [8, p. 52, Theorem] one derives the quadrature error

$$\left|\int_a^b u(y)\,\mathrm{d}y - Q_M^n u\right| \le \frac{(b-a)^{\alpha+1}}{2^\alpha n^\alpha}[u]_\alpha.$$

The *trapezoidal rule* $Q_T^n u := \frac{1}{2}(Q_{LR}^n u + Q_{RR}^n u)$ is convergent with the same quadrature error as for the rectangular rules. Finally, let $n \in \mathbb{N}$ be even. Representing the *Simpson rule* as convex combination $Q_S^n u := \frac{2}{3}Q_M^{n/2}u + \frac{1}{3}Q_T^{n/2}u$, one obtains

$$\left|\int_a^b u(y)\,\mathrm{d}y - Q_S^n u\right| \le \frac{2+2^\alpha}{3}\frac{(b-a)^{\alpha+1}}{n^\alpha}[u]_\alpha.$$

The next two results provide sufficient conditions on the kernel functions f_t such that the assumptions (2.7) or (2.8) are satisfied for Nyström discretizations (I_n).

Proposition 3.3 (convergence of $\mathcal{F}_t^n$) *Let $t \in \mathbb{Z}$. Suppose Hyp. 3.1 holds and that for every $r > 0$ there exists a $l_r^0 \ge 0$ such that*

$$|f_t(x,y,z) - f_t(x,\bar{y},\bar{z})| \le l_r^0 \max\{|y-\bar{y}|^\alpha, |z-\bar{z}|\} \tag{3.9}$$

for all $x, y, \bar{y} \in \Omega$ and $z, \bar{z} \in Z_t \cap \bar{B}_r(0)$. If (Q_n) has consistency order α, then for every $r > 0$ there exists a $c_r^0 \ge 0$ such that

$$\left\|\mathcal{F}_t^n(u) - \mathcal{F}_t^0(u)\right\|_\alpha \le \frac{c_0 c_r^0}{n^\alpha} \quad \textit{for all } n \in \mathbb{N},\ u \in U_t \cap \bar{B}_r(0, C_d^\alpha). \tag{3.10}$$

The magnitude of the constant c_r^0 is increasing in the Hölder norm of $u \in U_t$.

Proof Let $t \in \mathbb{Z}$, $r > 0$ and $u \in U_t \cap \bar{B}_r(0, C_d^\alpha)$. Because (Q_n) has consistency order α, there exists a $c_0 \ge 0$ such that $\left|\mathcal{F}_t^0(u)(x) - \mathcal{F}_t^n(u)(x)\right| \le \frac{c_0}{n^\alpha}\|f_t(x,\cdot,u(\cdot))\|_\alpha$ for all $x \in \Omega$. First, one has $\|f_t(x,\cdot,u(\cdot))\|_0 \le \sup_{\xi,y\in\Omega}|f_t(\xi,y,u(y))| =: b_t$ for every $x \in \Omega$. Second, due to the assumption (3.9) we conclude

$$|f_t(x,y,u(y)) - f_t(x,\bar{y},u(\bar{y}))| \le l_r^0 \max\{1,[u]_\alpha\}|y-\bar{y}|^\alpha \quad \text{for all } y,\bar{y} \in \Omega$$

and therefore $[f_t(x,\cdot,u(\cdot))]_\alpha \le l_r^0 \max\{1,[u]_\alpha\}$ holds. In conclusion, because of (3.4) we arrive at $\|f_t(x,\cdot,u(\cdot))\|_\alpha \le \max\left\{b_t, l_r^0\max\{1,[u]_\alpha\}\right\}$ for every $x \in \Omega$ and consequently choose $c_r^0 := \max_{t=1}^{\theta_0}\left\{b_t, l_r^0\max\{1,r\}\right\}$. □

Proposition 3.4 (convergence of $D\mathcal{F}_t^n$) *Let $t \in \mathbb{Z}$. Suppose Hyp. 3.1 holds and that for every $r > 0$ there exist constants*

(iv) $l_r^1 \ge 0$ *such that for all $x, y, \bar{y} \in \Omega$ and $z \in Z_t \cap \bar{B}_r(0)$ one has*

$$|D_3 f_t(x,y,z) - D_3 f_t(x,\bar{y},\bar{z})|_{L(\mathbb{R}^d)} \le l_r^1 \max\{|y-\bar{y}|^\alpha, |z-\bar{z}|\},$$

(v) $\gamma_r \ge 0$ *such that for all $x, \bar{x}, y, \bar{y} \in \Omega$ and $u \in U_t \cap \bar{B}_r(0, C_d^\alpha)$ one has*

$$\begin{aligned}|D_3 f_t(x,y,u(y)) - D_3 f_t(\bar{x},y,u(y)) - [D_3 f_t(x,\bar{y},u(\bar{y})) - D_3 f_t(\bar{x},\bar{y},u(\bar{y}))]|_{L(\mathbb{R}^d)}\\ \le \gamma_r |x-\bar{x}|^\alpha |y-\bar{y}|^\alpha.\end{aligned} \tag{3.11}$$

If (Q_n) has consistency order α, then for every $r > 0$ there exists a $c_r^1 \geq 0$ such that

$$\left\|D\mathcal{F}_t^n(u) - D\mathcal{F}_t^0(u)\right\|_{L(C_d^\alpha)} \leq \frac{c_0 c_r^1}{n^\alpha} \quad \text{for all } n \in \mathbb{N},\ u \in U_t \cap \bar{B}_r(0, C_d^\alpha). \tag{3.12}$$

Sufficient conditions for (3.11) to hold were given in [23, Rem. 1] on convex $\Omega \subset \mathbb{R}^\kappa$. Furthermore, the explicit form of the constant c_r^1 can be obtained from [23, (11)].

Proof Let $t \in \mathbb{Z}$, $r > 0$ and $u \in U_t \cap \bar{B}_r(0, C_d^\alpha)$ be fixed. By Prop. 3.1 the derivative of $\mathcal{F}_t^0$ is $D\mathcal{F}_t^0(u)v = \int_\Omega D_3 f_t(\cdot, y, u(y))v(y)\,\mathrm{d}y$ for all $v \in C_d^\alpha$. Given this, our goal is to apply the convergence result [23, Thm. 2] with the corresponding kernel $k_t(x, y) := D_3 f_t(x, y, u(y))$, whose assumptions are verified next:

ad (i): Thanks to $|u(y)| \leq r$ it holds $|k_t(x, y) - k_t(\bar{x}, y)| \leq h_r(y)\,|x - \bar{x}|^\alpha$ for all $x, \bar{x} \in \Omega$ due to (3.2) and therefore $[k_t(\cdot, y)]_\alpha \leq \sup_{\eta\in\Omega} h_r(\eta)$ for all $y \in \Omega$.

ad (ii): The assumption (iv) and $[u]_\alpha \leq r$ yield

$$|k_t(x, y) - k_t(x, \bar{y})| \leq l_r^1 \max\{1, [u]_\alpha\}\,|y - \bar{y}|^\alpha \quad \text{for all } y, \bar{y} \in \Omega$$

and thus $[k_t(x, \cdot)]_\alpha \leq l_r^1 \max\{1, r\}$ for all $x \in \Omega$.

ad (iii): As consequence of our assumption (3.11) one obtains for $x, \bar{x}, y, \bar{y} \in \Omega$ that $|k_t(x, y) - k_t(\bar{x}, y) - [k_t(x, \bar{y}) - k_t(\bar{x}, \bar{y})]| \leq \gamma_r\,|x - \bar{x}|^\alpha\,|y - \bar{y}|^\alpha$.

Finally, combining (i–iii) with the consistency order α of (Q_n) shows (3.12). □

Combining the assumptions of Prop. 3.1–3.4 and Cor. 3.1 yields

Corollary 3.2 (saddle-point structure of (I_0), C^α-case) *Suppose Hyp. 3.1 holds with $2 \leq m$ and ϕ^0 is a weakly hyperbolic θ_1-periodic solution to (I_0). If*

(iv) (Q_n) has consistency order $\alpha \in (0, 1]$,

then there exist constants $K_*, K_+, K_- \geq 0$ and $N_1 \in \mathbb{N}$ such that the following holds for all $n \geq N_1$: The associate weakly hyperbolic and θ-periodic solutions ϕ^n to (I_n) satisfy

$$\sup_{t\in\mathbb{Z}} \left\|\phi_t^n - \phi_t^0\right\|_\alpha \leq \frac{K_* c_0}{n^\alpha}. \tag{3.13}$$

If ϕ^0 is even hyperbolic, then for each $\tau \in \mathbb{Z}$ one has the estimates

(a) $\left\|w_+^n(\tau, v) - w_+^0(\tau, v)\right\|_\alpha \leq \frac{4K}{1-\beta}\frac{K_+}{n^\alpha} \sup_{\tau\leq t} \|\phi_t\|_\alpha$ for all $v \in B_{\rho_1}(0, R(P_\tau))$,

(b) $\left\|w_-^n(\tau, v) - w_-^0(\tau, v)\right\|_\alpha \leq \frac{4K}{1-\beta}\frac{K_-}{n^\alpha} \sup_{t\leq\tau} \|\phi_t\|_\alpha$ for all $v \in B_{\rho_1}(0, N(P_\tau))$,

with the forward resp. backward solution ϕ to the IDE (I_0) from Thm. 2.3.

Proof Let $t \in \mathbb{Z}$ and $r := \max_{t=1}^{\theta_1} \left\|\phi_t^0\right\|_\alpha$. It results from Prop. 3.1 and 3.2 that (I_n), $n \in \mathbb{N}_0$, satisfy the assumptions (i), (ii) of Thm. 2.2. Moreover, Prop. 3.3 implies (2.7) with $\Gamma_0(\varrho) := c_0 c_r^0 \varrho^\alpha$, while Prop. 3.4 guarantees that (2.8) holds. Hence, Thm. 2.2 applies and yields (3.13) with $K_* := K_0 c_r^0$. In particular, for $N_0 \in \mathbb{N}$ and $\rho_0 > 0$ from Thm. 2.2 there is a $N_1 \geq N_0$ so that $\sup_{t\in\mathbb{Z}} \left\|\phi_t^n - \phi_t^0\right\|_\alpha < \frac{\rho_0}{2}$ for all $n \geq N_1$.

(a) Let $\rho_1 > 0$ be so small that the sequence $(\phi_t)_{\tau\leq t}$ from Thm. 2.3(a) satisfies $\|\phi_t\|_\alpha < \frac{\rho_0}{2}$ for all $\tau \leq t$; such a ρ_1 exists since the sequence is contained in the stable fiber bundle of ϕ^0. Furthermore, for each $\vartheta \in [0, 1]$ we obtain

$$\left|\phi_t^n(y) + \vartheta\phi_t(y) - \phi_t^0(y)\right| \leq \left\|\phi_t^n + \vartheta\phi_t - \phi_t^0\right\|_0 < \rho_0 \quad \text{for all } y \in \Omega.$$

Now set $\bar{r} := r + \rho_0$. Combining the triangle inequality, Cor. 3.1 and Prop. 3.4 yields that there exist $L_{\bar{r}} \geq 0$ such that

$$\begin{aligned} &\left\|[D\mathcal{F}_t^n(\phi_t^n + \vartheta\phi_t) - D\mathcal{F}_t^0(\phi_t^0 + \vartheta\phi_t)]\phi_t\right\|_\alpha \\ \stackrel{(3.6)}{\leq}\; &\left\|[D\mathcal{F}_t^n(\phi_t^n + \vartheta\phi_t) - D\mathcal{F}_t^0(\phi_t^n + \vartheta\phi_t)]\phi_t\right\|_\alpha + L_{\bar{r}}\left\|\phi_t^n - \phi_t^0\right\|_\alpha \|\phi_t\|_\alpha \\ \stackrel{(3.12)}{\leq}\; &\frac{c_0 c_{\bar{r}}^1}{n^\alpha}\|\phi_t\|_\alpha + L_{\bar{r}}\left\|\phi_t^n - \phi_t^0\right\|_\alpha \|\phi_t\|_\alpha \stackrel{(2.9)}{\leq} \frac{c_0 c_{\bar{r}}^1}{n^\alpha}\|\phi_t\|_\alpha + \frac{L_{\bar{r}} K_* c_0}{n^\alpha}\|\phi_t\|_\alpha \end{aligned}$$

and with $K_+ := c_0 c_{\bar{r}}^1 + L_{\bar{r}} K_* c_0$ we obtain

$$\left\|[D\mathcal{F}_t^n(\phi_t^n + \vartheta\phi_t) - D\mathcal{F}_t^0(\phi_t^0 + \vartheta\phi_t)]\phi_t\right\|_\alpha \leq \frac{K_+}{n^\alpha}\sup_{\tau \leq s}\|\phi_s\|_\alpha \quad \text{for all } \vartheta \in [0,1],$$

$n \geq N_1$ and $\tau \leq t$. Hence, the claimed estimate follows from (2.11).

(b) As in (a), applying (2.12) rather than (2.11) leads to the assertion. □

3.2 Differentiable kernel functions

Convergence rates improving the consistency order $\alpha \in (0,1]$ obtained in Cor. 3.2 can be expected for integrands in (I_0) being differentiable in $y \in \Omega$. Here we follow the convention to consider a function on a not necessarily open set $\Omega \subset \mathbb{R}^\kappa$ as differentiable, if it allows a differentiable extension to an open superset of Ω.

Given p-times continuously differentiable functions $u : \Omega \to \mathbb{R}^d$ assume that (Q_n) allows a *quadrature* or *cubature error* of the form (see [8])

$$\left|\int_\Omega u(y)\,\mathrm{d}y - Q^n u\right| \leq \frac{c_p}{n^p}\sup_{x\in\Omega}|D^p u(x)| \quad \text{for all } n \in \mathbb{N} \tag{3.14}$$

with constants $c_p \geq 0$.

A smooth framework allows the following improvement of Prop. 3.3:

Proposition 3.5 (higher order convergence of $\mathcal{F}_t^n$) *Let $t \in \mathbb{Z}$, $p \in \mathbb{N}$ and Ω be convex. Suppose the kernel function $f_t : \Omega^2 \times Z_t \to \mathbb{R}^d$ fulfills:*

(iv) The partial derivative $D_1 f_t : \Omega^2 \times Z_t \to L(\mathbb{R}^\kappa, \mathbb{R}^d)$ exists,
(v) both $f_t, D_1 f_t$ are of class $C^p_{(2,3)}$.

If (Q_n) satisfies (3.14)*, then for every $r > 0$ there exists a $\bar{c}_r^0 \geq 0$ such that*

$$\left\|\mathcal{F}_t^n(u) - \mathcal{F}_t^0(u)\right\|_\alpha \leq \frac{c_p \bar{c}_r^0}{n^p} \quad \textit{for all } n \in \mathbb{N} \tag{3.15}$$

and p-times continuously differentiable functions $u \in U_t$.

Proof Let $t \in \mathbb{Z}$ and with $u \in U_t$ of class C^p it is convenient to define

$$F_t^{(1)} : \Omega^2 \to L(\mathbb{R}^\kappa, \mathbb{R}^d), \qquad F_t^{(1)}(x,y) := D_1 f_t(x, y, u(y)).$$

The estimate (3.15) for the $\|\cdot\|_0$-norm is an immediate consequence of the error estimate (3.14) and the higher-order chain rule. Let $x, \bar{x} \in \Omega$ and the Mean Value Theorem [16, p. 341, Thm. 4.2] gives

$$\begin{aligned}
& [\mathcal{F}_t^n(u) - \mathcal{F}_t^0(u)](x) - [\mathcal{F}_t^n(u) - \mathcal{F}_t^0(u)](\bar{x}) \\
= & \sum_{\eta \in \Omega_n} w_\eta f_t(x, \eta, u(\eta)) - \int_\Omega f_t(x, y, u(y))\,\mathrm{d}y \\
& - \sum_{\eta \in \Omega_n} w_\eta f_t(\bar{x}, \eta, u(\eta)) + \int_\Omega f_t(\bar{x}, y, u(y))\,\mathrm{d}y \\
= & \int_\Omega \int_0^1 D_1 f_t(\bar{x} + \vartheta(x - \bar{x}), y, u(y))\,\mathrm{d}\vartheta\,\mathrm{d}y\,(x - \bar{x}) \\
& - \sum_{\eta \in \Omega_n} w_\eta \int_0^1 D_1 f_t(\bar{x} + \vartheta(x - \bar{x}), \eta, u(\eta))\,\mathrm{d}\vartheta\,(x - \bar{x}),
\end{aligned}$$

from which Fubini's theorem [16, p. 162, Thm. 8.4] yields

$$\begin{aligned}
& [\mathcal{F}_t^n(u) - \mathcal{F}_t^0(u)](x) - [\mathcal{F}_t^n(u) - \mathcal{F}_t^0(u)](\bar{x}) \\
= & \int_0^1 \Big(\int_\Omega F_t^{(1)}(\bar{x} + \vartheta(x - \bar{x}), y)\,\mathrm{d}y - \sum_{\eta \in \Omega_n} w_\eta F_t^{(1)}(\bar{x} + \vartheta(x - \bar{x}), \eta) \Big)\,\mathrm{d}\vartheta\,(x - \bar{x})
\end{aligned}$$

and passing to the norm implies

$$\begin{aligned}
& \left| [\mathcal{F}_t^n(u) - \mathcal{F}_t^0(u)](x) - [\mathcal{F}_t^n(u) - \mathcal{F}_t^0(u)](\bar{x}) \right| \\
\overset{(3.14)}{\leq} & \frac{c_p}{n^p} \int_0^1 \sup_{y \in \Omega} \left| D_2^p F_t^{(1)}(\bar{x} + \vartheta(x - \bar{x}), y) \right| \,\mathrm{d}\vartheta\,|x - \bar{x}| \\
\leq & \frac{c_p}{n^p} (\operatorname{diam} \Omega)^{1-\alpha} \sup_{x, y \in \Omega} \left| D_2^p F_t^{(1)}(x, y) \right| |x - \bar{x}|^\alpha .
\end{aligned}$$

Hence, $[\mathcal{F}_t^n(u) - \mathcal{F}_t^0(u)]_\alpha \leq \frac{c_p}{n^p} (\operatorname{diam} \Omega)^{1-\alpha} \sup_{x,y \in \Omega} \left| D_2^p F_t^{(1)}(x, y) \right|$ and if we abbreviate $\bar{c}_r^0 := \max\left\{1, (\operatorname{diam} \Omega)^{1-\alpha}\right\} \max_{i=0}^1 \sup_{x,y \in \Omega} \left| D_2^p F_t^{(i)}(x, y) \right|$, then (3.4) implies the claimed estimate (3.15). □

Smooth functions f_t and reference solutions ϕ^0 allow better convergence rates. Indeed under the assumptions of Prop. 3.1, 3.2 and 3.4, 3.5, as well as Cor. 3.1 results:

Corollary 3.3 (saddle-point structure of (I_0), C^p-case) *Let $\Omega \subset \mathbb{R}^\kappa$ be convex. Suppose Hyp. 3.1 holds with* $\max\{2, p\} \leq m$ *and ϕ^0 is a weakly hyperbolic θ_1-periodic solution to (I_0). If*

(iv) (Q_n) is stable, has consistency order $\alpha \in (0, 1]$ and satisfies (3.14),
(v) the partial derivatives $D_1^k f_t : \Omega^2 \times Z_t \to L_k(\mathbb{R}^\kappa, \mathbb{R}^d)$ exists for $0 \leq k \leq p$,
(vi) both f_t and $D_1 f_t$ are of class $C^p_{(2,3)}$,

then there exist constants $K_*, K_+, K_- \geq 0$ *and* $N_1 \in \mathbb{N}$ *such that the following holds for all* $n \geq N_1$*: The associate weakly hyperbolic and* θ*-periodic solutions* ϕ^n *to* (I_n) *satisfy*

$$\sup_{t\in\mathbb{Z}} \left\|\phi_t^n - \phi_t^0\right\|_\alpha \leq \frac{K_* c_p}{n^p}. \tag{3.16}$$

If ϕ^0 *is even hyperbolic, then for each* $\tau \in \mathbb{Z}$ *one has the estimates*

(a) $\left\|w_+^n(\tau, v) - w_+^0(\tau, v)\right\|_\alpha \leq \frac{4K}{1-\beta}\frac{K_+}{n^p}(1 + \sup_{\tau\leq t}\|\phi_t\|_\alpha)$ *for all* p*-times continuously differentiable* $v \in B_{\rho_1}(0, R(P_\tau))$*,*
(b) $\left\|w_-^n(\tau, v) - w_-^0(\tau, v)\right\|_\alpha \leq \frac{4K}{1-\beta}\frac{K_-}{n^p}(1+\sup_{t\leq\tau}\|\phi_t\|_\alpha)$ *for all* $v \in B_{\rho_1}(0, N(P_\tau))$

with the forward resp. backward solution ϕ *to the IDE* (I_0) *from Thm. 2.3.*

Proof Let $t \in \mathbb{Z}$. Above all, as entire solutions to (I_0) the functions ϕ_t^0 are of class C^p due to (v) and [16, p. 355, Thm. 8.1]. By means of Prop. 3.5 the estimate (3.16) results as in the above proof of Cor. 3.2, with (3.10) replaced by (3.15).

(a) As in the above proof of Cor. 3.2 one obtains

$$\begin{aligned} &\left\|[D\mathfrak{F}_t^n(\phi_t^n + \vartheta\phi_t) - D\mathfrak{F}_t^0(\phi_t^0 + \vartheta\phi_t)]\phi_t\right\|_\alpha \\ \overset{(3.15)}{\leq}\; &\left\|[D\mathfrak{F}_t^n(\phi_t^n + \vartheta\phi_t) - D\mathfrak{F}_t^0(\phi_t^n + \vartheta\phi_t)]\phi_t\right\|_\alpha + \frac{L_{\bar{r}} c_p \bar{c}_{\bar{r}}^0}{n^p}\|\phi_t\|_\alpha \end{aligned} \tag{3.17}$$

for all $\vartheta \in [0, 1]$ and $n \geq N_1$. Prop. 3.1 yields the explicit derivative

$$D\mathfrak{F}_t^0(\phi_t^n + \vartheta\phi_t)\phi_t \overset{(3.5)}{=} \int_\Omega D_3 f_t(\cdot, y, \phi_t^n(y) + \vartheta\phi_t(y))\phi_t(y)\,\mathrm{d}y$$

and for the integrand on the right-hand side we observe: Thanks to (v) the periodic solution ϕ^n consists of C^p-functions $\phi_t^n : \Omega \to \mathbb{R}^d$ and also forward solutions to the IDE (I_0) are of class C^p, i.e. ϕ_t is a C^p-function for all $t > \tau$. For $t = \tau$ we have $\phi_\tau = v + w_+^0(\tau, v)$ and because $w_+^0(\tau, \cdot)$ is of class C^m by Thm. 2.3(a) and $p \leq m$, with v also the initial function ϕ_τ is p-times continuously differentiable. Due to (vi) this yields that the integrand $D_3 f_t(x, \cdot, \phi_t^n(\cdot) + \vartheta\phi_t(\cdot))\phi_t(\cdot) : \Omega \to \mathbb{R}^d$ is of class C^p and the estimate (3.14) applies. Hence, as in the proof of Prop. 3.5 one shows that there exists a $\tilde{C} \geq 0$ so that $\left\|[D\mathfrak{F}_t^n(\phi_t^n + \vartheta\phi_t) - D\mathfrak{F}_t^0(\phi_t^n + \vartheta\phi_t)]\phi_t\right\|_\alpha \leq \frac{\tilde{C}}{n^p}$ and whence (3.17) yields for all $\vartheta \in [0, 1]$, $n \geq N_1$ and $\tau \leq t$ that

$$\left\|[D\mathfrak{F}_t^n(\phi_t^n + \vartheta\phi_t) - D\mathfrak{F}_t^0(\phi_t^0 + \vartheta\phi_t)]\phi_t\right\|_\alpha \leq \left(\tilde{C} + L_r C \sup_{\tau\leq s}\|\phi_s\|_\alpha\right)\frac{1}{n^p}.$$

Therefore, the estimate (a) follows from (2.11).

(b) As above in (a), applying (2.12) rather than (2.11) leads to the claimed estimate. Note here that $(\phi_t)_{t\leq\tau}$ is a backward solution to (I_0) and consequently consists of C^p-solutions. Whence, also the initial value $\phi_\tau = v + w_-^0(\tau, v)$ is of class C^p and it is not necessary to assume v to be smooth. □

A Qualitative implicit and Lipschitz inverse function theorem

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. We formulate an abstract, but tailor-made implicit function theorem, whose parameter set is merely supposed to be a metric space (Λ, d):

Theorem A.1 (qualitative implicit function theorem) *Let $\Omega \subseteq \mathcal{X}$ be nonempty open, $x_0 \in \Omega$, $\lambda_0 \in \Lambda$, $y_0 \in \mathcal{Y}$, $q \in [0,1)$, and suppose $T : \Omega \times \Lambda \to \mathcal{Y}$ satisfies*

(i') $T(x_0, \lambda_0) = y_0$,
(ii') the partial derivative $D_1 T : \Omega \times \Lambda \to L(\mathcal{X}, \mathcal{Y})$ exists with $D_1 T(x_0, \lambda_0) \in GL(\mathcal{X}, \mathcal{Y})$,
(iii') there exist functions $\Gamma_0 : \mathbb{R}_+ \to \mathbb{R}_+$ and $\Gamma : \mathbb{R}_+^2 \to \mathbb{R}_+$ which satisfy $\lim_{\varrho \searrow 0} \Gamma_0(\varrho) = 0$, $\lim_{\varrho_1, \varrho_2 \searrow 0} \Gamma(\varrho_1, \varrho_2) = 0$, such that for all $x \in \Omega$, $\lambda \in \Lambda$ it holds

$$\|T(x_0, \lambda) - T(x_0, \lambda_0)\| \le \Gamma_0(d(\lambda, \lambda_0)), \tag{A.1}$$
$$\|D_1 T(x, \lambda) - D_1 T(x_0, \lambda_0)\| \le \Gamma(\|x - x_0\|, d(\lambda, \lambda_0)). \tag{A.2}$$

If $K := \left\|D_1 T(x_0, \lambda_0)^{-1}\right\|$ and $\rho_0, \delta > 0$ are chosen so small that

$$\Gamma_0(\delta) \le \tfrac{1-q}{K} \rho_0, \qquad \Gamma(\rho_0, \delta) \le \tfrac{q}{K}, \tag{A.3}$$

then there exists a function $\phi : B_\delta(\lambda_0, \Lambda) \to \bar{B}_{\rho_0}(x_0, \mathcal{X})$ satisfying

(a) $\phi(\lambda_0) = x_0$,
(b) $T(x, \lambda) = y_0$ in $\bar{B}_{\rho_0}(x_0, \mathcal{X}) \times B_\delta(\lambda_0, \Lambda)$ if and only if $x = \phi(\lambda)$,
(c) $\|\phi(\lambda) - x_0\| \le \frac{K}{1-q} \Gamma_0(d(\lambda, \lambda_0))$ for all $\lambda \in B_\delta(\lambda_0, \Lambda)$.

Proof The proof is similar to the one of [21, Thm. A.1]. □

Theorem A.2 (Lipschitz inverse function theorem) *Let $x_0 \in \mathcal{X}$ and $\rho > 0$ be given. If a mapping $T : \bar{B}_\rho(x_0, \mathcal{X}) \to \mathcal{Y}$ is of the form $T = A + G$ with*

(i) $A \in GL(\mathcal{X}, \mathcal{Y})$,
(ii) $G : \bar{B}_\rho(x_0, \mathcal{X}) \to \mathcal{Y}$ is Lipschitz with Lipschitz constant $l < \left\|A^{-1}\right\|^{-1}$,

then the following holds with $\sigma \in \left(l, \left\|A^{-1}\right\|^{-1}\right]$:

(a) For all $x, \bar{x} \in \bar{B}_\rho(x_0, \mathcal{X})$ one has

$$(\sigma - l) \|x - \bar{x}\| \le \|T(x) - T(\bar{x})\| \le (\|A\| + l) \|x - \bar{x}\|, \tag{A.4}$$

(b) for all $y \in \bar{B}_{(\sigma-l)\rho}(T(x_0), \mathcal{Y})$ the equation $T(x) = y$ has a unique solution $x^*(y) \in \bar{B}_\rho(x_0, \mathcal{X})$,
(c) with $G|_{B_\rho(x_0, \mathcal{X})}$ also the function $x^* : B_{(\sigma-l)\rho}(T(x_0), \mathcal{Y}) \to \mathcal{X}$ is of class C^m, $m \in \mathbb{N}_0$.

Proof See [14, p. 224, (C.11)], with the smoothness assertion resulting from the Uniform Contraction Principle [7, p. 25, Thm. 2.2]. □

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