NUMERICAL DYNAMICS OF INTEGRODIFFERENCE EQUATIONS: FORWARD DYNAMICS AND PULLBACK ATTRACTORS

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ABSTRACT. In order to determine the dynamics of nonautonomous equations both their forward and pullback behavior need to be understood. For this reason we provide sufficient criteria for the existence of such attracting invariant sets in a general setting of nonautonomous difference equations in metric spaces. In addition it is shown that both forward and pullback attractors, as well as forward limit sets persist and that the latter two notions even converge under perturbation. As concrete application, we study integrodifference equation under spatial discretization of collocation type.

1. INTRODUCTION

1.1. **Perturbation of attractors.** Obtaining a precise insight into the structure of a global attractor can be seen as the holy grail in the field of dissipative dynamical systems. This compact, invariant set attracting each bounded subset of the state space contains all equilibria, periodic solutions, homo- or heteroclinic connections and overall the essential dynamics. In general the structure of a global attractor can be rather complex and therefore it is no surprise that such objects behave sensitively w.r.t. perturbations and parameter variations [11]. However, under mild conditions global attractors are upper-semicontinuous in parameters. Criteria for lower semicontinuity and hence for full continuity in the Hausdorff metric are much harder to prove and often involve conditions being difficult to verify. Nevertheless it was shown in [16] that global attractors are at least continuous on a residual subset of the parameter space.

For nonautonomous dynamical systems [3, 5, 25, 28] the range of possible long term behaviors, as well as the theory of attractors is understandably more involved. For instance, in order to obtain a full picture of the dynamics it might be necessary to consider different attractor concepts simultaneously. Although the notion of a pullback attractor shares several properties of the global attractor it is easy to construct systems with totally different forward dynamics but sharing the same pullback attractor [24]. This deficit stimulated the development of forward attractors [23]. While a perturbation theory for pullback attractors can be found in [3, 5] and their continuity on a residual set is due to [17], related results for forward

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attractors are sparse, which might be due to their nonuniqueness and recent origin. An exception is [27] addressing the upper-semicontinuity of forward limit sets under time discretizations of finite-dimensional nonautonomous ODEs.

Indeed, particularly important classes of perturbations originate in numerical analysis, when an equation generating a dynamical system is discretized. It actually is a key topic in Numerical Dynamics [37] to determine how attractors behave under discretization. This research was initiated in [21] for temporal discretizations of autonomous ODEs in $\mathbb{R}^d$, extensions to pullback attractors of nonautonomous ODEs can be found in [6], and related results for semiflows generated by PDEs are due to [12, 22] and [36].

Forward attractors and limit sets of discrete time nonautonomous dynamical systems in infinite dimensions were constructed and studied in [18] providing a theoretical foundation. As a continuation, this paper addresses the persistence of forward attractors and establishes the upper-semicontinuity of forward limit sets under perturbations. Remaining in the setting of [18] our criteria are formulated for abstract difference equations in metric spaces. We believe that they apply to a wide range of evolutionary equations and their spatial discretizations. Nevertheless, although convergence under perturbation is proven, at this level of generality we are not able to establish a particular convergence rate without imposing further assumptions.

1.2. **Integrodifference equations and spatial discretization.** As concrete application serve nonautonomous integrodifference equations (short IDEs)

$$u_{t+1} = \int_\Omega f_t(\cdot, y, u_t(y))\,\mathrm{d}y \tag{I_0}$$

(see Sect. 3). Such recursions in infinite-dimensional spaces originally stem from population genetics but gained popularity as models for the dispersal of species in theoretical ecology over the last decades [29, 30]. Recently also nonautonomous models were studied in [4, 19], which clearly motivates our general approach.

The several approaches to spatially discretize IDEs are based on related techniques for nonlinear integral equations [1]. Among them, in Sect. 3 we focus on collocation methods approximating continuous functions on a compact habitat. As concrete collocation spaces we exemplify splines of different order. In this framework we provide conditions that pullback and forward attractors, as well as forward limit sets persist under collocation, and establish convergence of pullback attractors and forward limit sets for increasingly more accurate discretizations. Convergence rates could not be derived using our proof techniques, but we experimentally demonstrate in Sect. 4 that quadratic convergence under piecewise linear collocation is preserved. We close with a construction of forward limit sets for asymptotically autonomous spatial Ricker equations.

1.3. **Notation.** Let $\mathbb{R}_+ := [0, \infty)$. A *discrete interval* $\mathbb{I}$ is the intersection of a real interval with the integers $\mathbb{Z}$, $\mathbb{I}' := \{t \in \mathbb{I} : t + 1 \in \mathbb{I}\}$ and in particular

$$\mathbb{I}_\tau^+ := \{t \in \mathbb{I} : \tau \le t\}, \qquad \mathbb{I}_\tau^- := \{t \in \mathbb{I} : t \le \tau\} \quad \text{for } \tau \in \mathbb{Z},$$

as well as $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

On a given metric space (X, d), $\overline{A}$ denotes the closure of a subset $A \subseteq X$, $B_r(x) := \{y \in X : d(x, y) < r\}$ is the open ball with center $x \in X$ and radius $r > 0$, and $\bar{B}_r(x)$ its closure. We write $\operatorname{dist}_A(x) := \inf_{a \in A} d(x, a)$ for the *distance* of a point $x \in X$ to a subset $A \subseteq X$ and $B_r(A) := \{x \in X : \operatorname{dist}_A(x) < r\}$ for its r-neighborhood. The *Hausdorff semidistance* of bounded and closed subsets $A, B \subseteq X$ is defined as

$$\operatorname{dist}(A, B) := \sup_{a \in A} \operatorname{dist}_B(a),$$

and, with a further closed and bounded subset $C \subseteq X$, one has the properties

$$\operatorname{dist}(A, B) \leq \operatorname{dist}(A, C) + \operatorname{dist}(C, B). \tag{1.1}$$

A mapping $\mathcal{F} : X \to X$ is called *bounded*, if it maps bounded subsets of X into bounded sets and *globally bounded*, if $\mathcal{F}(X)$ is bounded. A *completely continuous* map $\mathcal{F}$ is continuous and maps bounded sets into relatively compact sets.

Beyond this general terminology, we need to introduce terms commonly used in the area of nonautonomous dynamics [25]: A subset $\mathcal{A} \subseteq \mathbb{I} \times X$ with nonempty *fibers* $\mathcal{A}(t) := \{x \in X : (t, x) \in \mathcal{A}\}$, $t \in \mathbb{I}$, is called *nonautonomous set*. A nonautonomous set $\mathcal{A}$ is said to have some topological property if each fiber $\mathcal{A}(t)$, $t \in \mathbb{I}$, has this property. Furthermore, one speaks of a *bounded* nonautonomous set $\mathcal{A}$, if there exists real $R > 0$ and a point $x_0 \in X$ such that $\mathcal{A}(t) \subseteq B_R(x_0)$ holds for all $t \in \mathbb{I}$. If $\mathbb{I}$ is unbounded above, then we define

$$\limsup_{t \to \infty} \mathcal{A}(t) := \bigcap_{\tau \in \mathbb{I}} \overline{\bigcup_{\tau \leq t} \mathcal{A}(t)}.$$

2. Nonautonomous difference equations and attractors

Let (X, d) be a complete metric space and $U_t \neq \emptyset$, $t \in \mathbb{I}'$, be closed subsets of X. We consider nonautonomous difference equations

$$\boxed{u_{t+1} = \mathcal{F}_t(u_t)} \tag{Δ}$$

having continuous right-hand sides $\mathcal{F}_t : U_t \to X$, $t \in \mathbb{I}'$.

Given an *initial time* $\tau \in \mathbb{I}$, a *forward solution* $(\phi_t)_{\tau \leq t}$ to (Δ) is a sequence satisfying $\phi_t \in U_t$ and $\phi_{t+1} = \mathcal{F}_t(\phi_t)$ for all $\tau \leq t, t \in \mathbb{I}'$, whilst an *entire solution* $(\phi_t)_{t \in \mathbb{I}}$ meets the same identity for all $t \in \mathbb{I}'$. The *general solution* $\varphi(\cdot; \tau, u_\tau)$ of (Δ) is the unique forward solution starting at $\tau \in \mathbb{I}$ in the *initial state* $u_\tau \in U_\tau$, i.e.

$$\varphi(t; \tau, u_\tau) := \begin{cases} \mathcal{F}_{t-1} \circ \ldots \circ \mathcal{F}_\tau(u_\tau), & \tau < t, \\ u_\tau, & \tau = t, \end{cases} \tag{2.1}$$

as long as the iterates stay in U_t.

From now on, assume that $\mathcal{F}_t(U_t) \subseteq U_{t+1}$ holds for all $t \in \mathbb{I}'$ and abbreviate

$$\mathcal{U} := \{(t, u) \in \mathbb{I} \times X : u \in U_t\}.$$

Then φ satisfies the *process* (or *two-parameter semi-group*) *property*

$$\varphi(t; s, \varphi(s; \tau, u)) = \varphi(t; \tau, u) \quad \text{for all } \tau \leq s \leq t,\ u \in U_\tau.$$

A nonautonomous set $\mathcal{A} \subseteq \mathcal{U}$ is said to be *positively* or *negatively invariant*, if

$$\mathcal{F}_t(\mathcal{A}(t)) \subseteq \mathcal{A}(t+1), \qquad \mathcal{A}(t+1) \subseteq \mathcal{F}_t(\mathcal{A}(t)) \quad \text{for all } t \in \mathbb{I}'$$

holds, respectively. Accordingly, an *invariant* set $\mathcal{A}$ is both positively and negatively invariant, that is $\mathcal{F}_t(\mathcal{A}(t)) = \mathcal{A}(t+1)$ for all $t \in \mathbb{I}'$.

A bounded nonautonomous set $\mathcal{A} \subseteq \mathcal{U}$ is called *absorbing*, if for each $\tau \in \mathbb{I}$ and bounded $\mathcal{B} \subseteq \mathcal{U}$, there is an *absorption time* $T = T(\tau, \mathcal{B}) \in \mathbb{N}$ such that

$$\varphi(t; \tau, \mathcal{B}(\tau)) \subseteq \mathcal{A}(t) \quad \text{for all } t, \tau \in \mathbb{I},\ t - \tau \geq T.$$

In addition, for *pullback* and *forward absorbing sets*, the present time t and the initial time τ work similarly to those in the properties of pullback and forward attraction (one notion of time is fixed and the other tends to infinity or minus infinity, cf. [25, pp. 44–45, Def. 3.3]).

2.1. **Pullback attractors.** Let a discrete interval $\mathbb{I}$ be unbounded below. A *pullback attractor* $\mathcal{A}^* \subseteq \mathcal{U}$ of (Δ) is a compact and invariant nonautonomous set, which pullback attracts bounded sets $\mathcal{B} \subseteq \mathcal{U}$, i.e.

$$\lim_{s \to \infty} \operatorname{dist}\left(\varphi(\tau; \tau - s, \mathcal{B}(\tau - s)), \mathcal{A}^*(\tau)\right) = 0 \quad \text{for all } \tau \in \mathbb{I}.$$

With comprehensive information on pullback attractors given in [3, 5, 25, 33], our existence condition for pullback attractors is based on the following notion: A difference equation (Δ) is called *pullback asymptotically compact*, if for every $\tau \in \mathbb{I}$, every sequence $(s_k)_{k \in \mathbb{N}}$ in $\mathbb{N}$ with $s_k \to \infty$ as $k \to \infty$ and every bounded sequence $(a_k)_{k \in \mathbb{N}}$ with $a_k \in \mathcal{U}(\tau - s_k)$, the sequence $(\varphi(\tau; \tau - s_k, a_k))_{k \in \mathbb{N}}$ in $\mathcal{U}(\tau)$ has a convergent subsequence.

Theorem 2.1 (Existence of pullback attractors, cf. [33, p. 19, Thm. 1.3.9])**.** *Suppose a difference equation* (Δ) *has a pullback absorbing set* $\mathcal{A}$*. If* (Δ) *is pullback asymptotically compact, then it possesses a unique pullback attractor* $\mathcal{A}^* \subseteq \mathcal{A}$*.*

The importance of pullback attractors is due to their following dynamical characterization (see [33, p. 17, Cor. 1.3.4])

$$\mathcal{A}^* = \left\{ (t, u) \in \mathcal{U} : \begin{array}{l} \text{there exists a bounded entire} \\ \text{solution } \phi \text{ to } (\Delta) \text{ with } \phi_\tau = u \end{array} \right\} \tag{2.2}$$

motivating that pullback attractors are a suitable nonautonomous extension of global attractors [3, p. 20, Thm. 1.9] of autonomous difference equations.

2.2. **Forward dynamics.** On a discrete interval $\mathbb{I}$ unbounded above, we assume the right-hand sides $\mathcal{F}_t : U \to U$, $t \in \mathbb{I}$, of (Δ) act on and map into a common closed subset $U \neq \emptyset$ of X. A *forward attractor* $\mathcal{A}^+ \subseteq \mathcal{U}$ of (Δ) is a compact and invariant nonautonomous set, which forward attracts bounded sets $\mathcal{B} \subseteq \mathcal{U}$, i.e.

$$\lim_{s \to \infty} \operatorname{dist}\big(\varphi(\tau + s; \tau, \mathcal{B}(\tau)), \mathcal{A}^+(\tau + s)\big) = 0 \quad \text{for all } \tau \in \mathbb{I}.$$

In contrast to pullback attractors $\mathcal{A}^*$, forward attractors $\mathcal{A}^+$ need not to be unique (cf. [26, Sect. 4]) and further properties are given in [7, 18, 23, 25, 26]. Not surprisingly, the existence of forward attractors is based on suitable compactness properties. For nonautonomous sets $\mathcal{A} \subseteq \mathcal{U}$, a difference equation (Δ) is called

- $\mathcal{A}$*-asymptotically compact*, if there exists a nonempty, compact set $K \subseteq U$ such that K forward attracts $\mathcal{A}(\tau)$, i.e.,
$$\lim_{s \to \infty} \operatorname{dist}\left(\varphi(\tau + s; \tau, \mathcal{A}(\tau)), K\right) = 0 \quad \text{for all } \tau \in \mathbb{I},$$
- *strongly* $\mathcal{A}$*-asymptotically compact*, if there exists a nonempty, compact set $K \subseteq U$ such that every sequence $((s_k, \tau_k))_{k \in \mathbb{N}}$ in $\mathbb{N} \times \mathbb{I}$ with $s_k \to \infty$, $\tau_k \to \infty$ as $k \to \infty$ yields
$$\lim_{k \to \infty} \operatorname{dist}\left(\varphi(\tau_k + s_k; \tau_k, \mathcal{A}(\tau_k)), K\right) = 0.$$

For positively invariant sets $\mathcal{A}$, strong $\mathcal{A}$-asymptotic compactness implies $\mathcal{A}$-asymptotic compactness (cf. [18, Rem. 4.1]).

In preparation of our next results, let us introduce a set $\Omega_{\mathcal{A}} \subseteq \mathcal{U}$ as

$$\Omega_{\mathcal{A}}(\tau) := \bigcap_{0 \le s} \overline{\bigcup_{s \le t} \varphi(\tau + t; \tau, \mathcal{A}(\tau))} \quad \text{for all } \tau \in \mathbb{I}$$

and the *forward limit set* for the dynamics of (Δ) starting from $\Omega_{\mathcal{A}}$ is

$$\omega_{\mathcal{A}}^{+} := \overline{\bigcup_{\tau \in \mathbb{I}} \Omega_{\mathcal{A}}(\tau)} \subseteq U.$$

Theorem 2.2 (Forward limit sets, cf. [18, Thm. 4.6])**.** *Suppose a difference equation* (Δ) *has a bounded and positively invariant set* $\mathcal{A}$*. If* (Δ) *is* $\mathcal{A}$*-asymptotically compact with a compact* $K \subseteq U$*, then its forward limit set* $\omega_{\mathcal{A}}^{+} \subseteq K$ *is nonempty, compact and forward attracts* $\mathcal{A}$*.*

In comparison to Thm. 2.1 a result guaranteeing the existence of forward attractors is more involved. It requires the assumption $\mathbb{I} = \mathbb{Z}$ and to introduce a further set $\mathcal{A}^{\star} \subseteq \mathcal{U}$ by its fibers

$$\mathcal{A}^{\star}(\tau) := \bigcap_{0 \le s} \varphi(\tau; \tau - s, \mathcal{A}(\tau - s)) \quad \text{for all } \tau \in \mathbb{Z}.$$

Theorem 2.3 (Forward attractors)**.** *Let* $\mathbb{I} = \mathbb{Z}$*. Suppose a difference equation* (Δ) *has a closed, forward absorbing and positively invariant set* $\mathcal{A}$*. If* (Δ) *is pullback asymptotically compact, strongly* $\mathcal{A}$*-asymptotically compact with compact* $K \subseteq U$ *and*

(i) *for every sequence* $((s_k, \tau_k))_{k \in \mathbb{N}}$ *in* $\mathbb{N} \times \mathbb{Z}$ *with* $\tau_k \to \infty$ *as* $k \to \infty$ *one has*

$$\lim_{k \to \infty} \operatorname{dist}\left(\varphi(\tau_k + s_k; \tau_k, K), K\right) = 0,$$

(ii) *for all* $\varepsilon > 0$ *and* $S \in \mathbb{N}$*, there exists a* $\delta > 0$ *such that for all* $\tau \in \mathbb{Z}$ *one has the implication*

$$\left.\begin{array}{r} u, v \in \mathcal{A}(\tau) \cup K, \\ d(u, v) < \delta \end{array}\right\} \Rightarrow \max_{0 < s \le S} d(\varphi(\tau + s; \tau, u), \varphi(\tau + s; \tau, v)) < \varepsilon,$$

(iii) $\omega_{\mathcal{A}}^{+} = \limsup_{t \to \infty} \mathcal{A}^{\star}(t)$*, i.e. the forward limit sets for the dynamics starting from* $\Omega_{\mathcal{A}}$ *and from within* $\mathcal{A}^{\star}$ *coincide*

hold, then $\mathcal{A}^{\star} \subseteq \mathcal{A}$ *is a forward attractor of* (Δ)*.*

Remark 2.4. (1) A situation, where an assumption such as (i) trivially holds, is when the compact set $K \subseteq U$ is positively invariant w.r.t. (Δ).

(2) The assumption (ii) can be interpreted as uniform continuity of every mapping $\varphi(\tau + s; \tau, \cdot)$ on $\mathcal{A}(\tau) \cup K$ and finite discrete intervals $[\tau, \tau + s] \cap \mathbb{Z}$, $\tau \in \mathbb{I}$, $0 < s \le S$. Hence, if we assume that there exists a bounded $B \subseteq U$ satisfying

$$\bigcup_{s \in \mathbb{N}_0} \left(\mathcal{A}(\tau + s) \cup \varphi(\tau + s; \tau, K)\right) \subseteq B \quad \text{for all } \tau \in \mathbb{Z},$$

and a Lipschitz condition with uniform constant $\ell \ge 0$ for each $\mathcal{F}_t : U \to U$, $t \in \mathbb{I}$, on B, then $d(\varphi(\tau + s; \tau, u), \varphi(\tau + s; \tau, v)) \le \ell^s d(u, v)$ for all $u, v \in B$. From this we obtain that (ii) holds true.

Proof. Invariance, being nonempty and compactness of $\mathcal{A}^\star \subseteq \mathcal{A}$ are established in [18, Prop. 3.1], while the forward attraction is due to [18, Cor. 4.19]. □

2.3. **Perturbation of attractors.** Rather than sticking to a single problem (Δ), we now proceed to families of nonautonomous difference equations

$$\boxed{u_{t+1} = \mathcal{F}_t^n(u_t)} \tag{Δ_n}$$

having continuous right-hand sides $\mathcal{F}_t^n : U_t \to X$, $t \in \mathbb{I}'$, which are parameterized by $n \in \mathbb{N}_0$. The general solution of (Δ_n) is denoted by φ^n.

For $n \in \mathbb{N}$ the difference equations (Δ_n) may describe perturbations (concretely, numerical discretizations with accuracy increasing in n) of an initial problem (Δ_0). Hence, a crucial concept allowing us to compare the behavior of solutions to (Δ_n), $n \in \mathbb{N}$, to the solutions of (Δ_0) is the *local error*

$$e_t^n(u) := d(\mathcal{F}_t^n(u), \mathcal{F}_t(u)).$$

We say the difference equations (Δ_n), $n \in \mathbb{N}$, are *convergent*, if there exists a *convergence function* $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{\rho \searrow 0} \Gamma(\rho) = 0$ such that for every bounded $B \subseteq U_t$, there exists a $C(B) \geq 0$ with

$$\sup_{u \in B} e_t^n(u) \leq C(B)\Gamma(\tfrac{1}{n}) \quad \text{for all } t \in \mathbb{I}',\ n \in \mathbb{N}. \tag{2.3}$$

One speaks of a *convergence rate* $\mu > 0$, if $\Gamma(\rho) = \rho^\mu$.

The following result determines how the local error evolves over time. It is crucial in order to apply our perturbation results in Thm. 2.6 and 2.7:

Proposition 2.5 (Global discretization error)**.** *Let $\tau, T \in \mathbb{I}$ with $\tau \leq T$ be fixed, $u \in U_\tau$ and $\mathcal{B} \subseteq \mathcal{U}$ be a bounded nonautonomous set such that*

$$\varphi^0(t; \tau, u) \in \mathcal{B}(t) \quad \textit{for all } \tau \leq t \leq T. \tag{2.4}$$

If the difference equations (Δ_n), $n \in \mathbb{N}$, are convergent and for every $t \in \mathbb{I}'$ and bounded $B \subseteq X$ there exists a real $\ell_t(B) \geq 0$ such that

$$d(\mathcal{F}_t^n(u), \mathcal{F}_t^n(v)) \leq \ell_t(B) d(u, v) \quad \textit{for all } n \in \mathbb{N},\ u, v \in U_t \cap B, \tag{2.5}$$

then for every $\rho > 0$ there exists an $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$ one has the inclusion $\varphi^n(t; \tau, u) \in B_\rho(\mathcal{B}(t))$ and the global discretization error

$$d(\varphi^n(t; \tau, u), \varphi^0(t; \tau, u)) \leq \Gamma\left(\tfrac{1}{n}\right) \sum_{l_1=\tau}^{t-1} C(\mathcal{B}(l_1)) \prod_{l_2=l_1+1}^{t-1} \ell_{l_2}\big(B_\rho(\mathcal{B}(l_2))\big) \tag{2.6}$$

for all $\tau \leq t \leq T$.

Proof. Let $\rho > 0$ and choose $(\tau, u) \in \mathcal{B}$, $\tau \leq T$. By induction over $t \geq \tau$ we establish the existence of a $N_0 = N_0(T) \in \mathbb{N}$ such that both (2.6) and

$$\varphi^n(t; \tau, u) \in B_\rho(\mathcal{B}(t)) \quad \text{for all } \tau \leq t \leq T,\ n \geq N_0 \tag{2.7}$$

hold. This is trivially true for $t = \tau$. Assume that (2.6)–(2.7) are satisfied for some fixed $t < T$ and set $x_t := d(\varphi^n(t; \tau, u), \varphi^0(t; \tau, u))$. Using the triangle inequality,

$$x_{t+1} \overset{(2.1)}{\leq} d\big(\mathcal{F}_t^n(\underbrace{\varphi^n(t; \tau, u)}_{\overset{(2.7)}{\in} B_\rho(\mathcal{B}(t))}), \mathcal{F}_t^n(\underbrace{\varphi^0(t; \tau, u)}_{\overset{(2.4)}{\in} \mathcal{B}(t)})\big) + e_t^n(\varphi^0(t; \tau, u))$$

$$\begin{aligned}
&\overset{(2.5)}{\leq} \ell_t\big(B_\rho(\mathcal{B}(t))\big)x_t + e_t^n(\underbrace{\varphi^0(t;\tau,u)}_{\overset{(2.4)}{\in}\mathcal{B}(t)})\\
&\overset{(2.3)}{\leq} \ell_t\big(B_\rho(\mathcal{B}(t))\big)x_t + C(\mathcal{B}(t))\Gamma(\tfrac{1}{n})\\
&\overset{(2.6)}{\leq} \ell_t\big(B_\rho(\mathcal{B}(t))\big)\Gamma(\tfrac{1}{n})\sum_{l_1=\tau}^{t-1} C(\mathcal{B}(l_1)) \prod_{l_2=l_1+1}^{t-1} \ell_{l_2}\big(B_\rho(\mathcal{B}(l_2))\big) + C(\mathcal{B}(t))\Gamma(\tfrac{1}{n})\\
&= \Gamma(\tfrac{1}{n})\sum_{l_1=\tau}^{t} C(\mathcal{B}(l_1)) \prod_{l_2=l_1+1}^{t} \ell_{l_2}\big(B_\rho(\mathcal{B}(l_2))\big)
\end{aligned}$$

results. For a sufficiently large N_0 it holds $d(\varphi^n(t+1;\tau,u),\varphi^0(t+1;\tau,u)) < \rho$ and the desired inclusion $\varphi^n(t+1;\tau,u) \in B_\rho(\mathcal{B}(t+1))$ results. □

Rather than providing suitable conditions on (Δ_n) guaranteeing that the existence of a pullback attractor $\mathcal{A}_0^*$ for (Δ_0) implies that also the perturbations (Δ_n), $n \in \mathbb{N}$, have pullback attractors $\mathcal{A}_n^*$ (*persistence*), on the present abstract level we simply assume that such attractors $\mathcal{A}_n^*$, $n \in \mathbb{N}_0$, do exist. Then the next result is a discrete time version of [3, p. 152, Thm. 7.13].

Theorem 2.6 (Upper-semicontinuity of pullback attractors)**.** *Let $\mathbb{I}$ be unbounded below. If each difference equation (Δ_n), $n \in \mathbb{N}_0$, has a pullback attractor $\mathcal{A}_n^*$ and*

(i) $\bigcup_{n\in\mathbb{N}_0} \mathcal{A}_n^*(\tau)$ *is relatively compact for every $\tau \in \mathbb{I}$,*
(ii) $\bigcup_{n\in\mathbb{N}_0} \bigcup_{s\leq\tau} \mathcal{A}_n^*(s)$ *is bounded for every $\tau \in \mathbb{I}$,*
(iii) *(*convergence condition*) for all $\varepsilon > 0$, $S \in \mathbb{N}$ and every compact $K \subseteq X$ there exists an $n_0 \in \mathbb{N}$ such that for all $\tau \in \mathbb{I}$ one has the implication*

$$n > n_0 \Rightarrow \max_{0<s\leq S} \sup_{u\in U_{\tau-s}\cap K} d(\varphi^n(\tau;\tau-s,u),\varphi^0(\tau;\tau-s,u)) < \varepsilon$$

hold, then $\lim_{n\to\infty} \max_{t\in I} \operatorname{dist}(\mathcal{A}_n^*(t),\mathcal{A}_0^*(t)) = 0$ *for every bounded $I \subseteq \mathbb{I}$.*

Proof. Let $n \in \mathbb{N}_0$. Recall from (2.2) that each fiber $\mathcal{A}_n^*(t) \subseteq \mathcal{U}(t)$, $t \in \mathbb{I}$, consists of initial values for a bounded entire solutions ϕ^n to (Δ_n). The proof is then divided into two claims.

(I) Claim: *There is a bounded entire solution ϕ^0 to (Δ_0) and a subsequence $(\phi^{n_k})_{k\in\mathbb{N}}$ of $(\phi^n)_{n\in\mathbb{N}}$ such that $\phi_t^0 \in \mathcal{A}_0^*(t)$ for all $t \in \mathbb{I}$ and $(\phi^{n_k})_{k\in\mathbb{N}}$ converges to ϕ^0 uniformly on bounded subintervals of $\mathbb{I}$.*

In fact, $(\phi_\tau^n)_{n\in\mathbb{N}}$ can be seen as sequence in $\overline{\bigcup_{n\in\mathbb{N}_0} \mathcal{A}_n^*(\tau)}$ with $\tau \in \mathbb{I}$. Combining this with the fact from (i) that $\overline{\bigcup_{n\in\mathbb{N}_0} \mathcal{A}_n^*(\tau)}$ is compact yields the existence of an infinite subset $N_0 \subseteq \mathbb{N}$ so that the subsequence $(\phi_\tau^n)_{n\in N_0}$ of $(\phi_\tau^n)_{n\in\mathbb{N}}$ converges to a limit $u_\tau \in \overline{\bigcup_{n\in\mathbb{N}_0} \mathcal{A}_n^*(\tau)}$. If we set $\phi_t^0 := \varphi^0(t;\tau,u_\tau)$ for each $t \in \mathbb{I}_\tau^+$, then $(\phi^n)_{n\in N_0}$ converges to ϕ^0 uniformly on bounded subintervals of $\mathbb{I}_\tau^+$ due to (iii).

Now by mathematical induction, we suppose that there exists a bounded forward solution $(\phi_t^0)_{s\leq t}$, $s \leq \tau$, to (Δ_0) as well as infinite subsets $N_s \subset N_{s-1}$ such that $(\phi^n)_{n\in N_s}$ converges to ϕ^0 uniformly in bounded subintervals of $\mathbb{I}_s^+$. Similarly to the above, since the sequence $\big(\phi_{s-1}^n\big)_{n\in N_s}$ is contained in the compact set $\overline{\bigcup_{n\in\mathbb{N}_0} \mathcal{A}_n^*(s-1)}$, there is an infinite subset $N_{s+1} \subset N_s$ so that the subsequence

$\left(\phi^n_{s-1}\right)_{n\in N_{s+1}}$ of $\left(\phi^n_{s-1}\right)_{n\in N_s}$ is convergent to $u_{s-1}\in\overline{\bigcup_{n\in\mathbb{N}_0}\mathcal{A}^*_n(s-1)}$. Therefore, $(\phi^0_t)_{s-1\le t}$ is a bounded solution to (Δ_0) and again by (iii), the sequence $(\phi^n)_{n\in N_{s+1}}$ converges to ϕ^0 uniformly in bounded subintervals of $\mathbb{I}^+_{s-1}$.

Repeating this procedure eventually results in a bounded entire solution ϕ^0 to the equation (Δ_0) and thus, $\phi^0_t\in\mathcal{A}^*_0(t)$ for each $t\in\mathbb{I}$ due to (ii). Moreover, each sequence $(\phi^n_t)_{n\in\mathbb{N}}$ in $\mathcal{A}^*_n(t)$ has a uniformly convergent subsequence $(\phi^{n_s}_t)_{n_s\in N_s}$ converging to the limit $\phi^0_t\in\mathcal{A}^*_0(t)$ for each $t\in\mathbb{I}$ as n_s is the s-th element of $N_s\subseteq\mathbb{N}$, $s\in\mathbb{N}$. This completes the proof of the first claim.

(II) Claim: $\lim_{n\to\infty}\operatorname{dist}(\mathcal{A}^*_n(t),\mathcal{A}^*_0(t))=0$ *for all* $t\in\mathbb{I}$.
By means of contradiction, assume the limit in the claim fails for some instant $t\in\mathbb{I}$, i.e. there are $\varepsilon_0>0$ and $n_0\in\mathbb{N}$ such that $\operatorname{dist}\left(\mathcal{A}^*_{n_0}(t),\mathcal{A}^*_0(t)\right)\ge\varepsilon_0$ holds. Because $\mathcal{A}^*_{n_0}(t)$ is compact, there exists a bounded entire solution ϕ^{n_0} with $\phi^{n_0}_t\in\mathcal{A}^*_{n_0}(t)$ so that $\operatorname{dist}_{\mathcal{A}^*_0(t)}(\phi^{n_0}_t)\ge\varepsilon_0$, a contradiction to claim (I) as for every bounded entire solution ϕ^n, the values $\phi^n_t\in\mathcal{A}^*_n(t)$ have a convergent subsequence with limit in $\mathcal{A}^*_0(t)$.

Because bounded subsets $I\subseteq\mathbb{I}$ are finite, the assertion follows from (II). □

In contrast, a related result addressing the forward dynamics inherently yields the existence of limit sets:

Theorem 2.7 (Upper-semicontinuity of forward limit sets)**.** *Let $\mathbb{I}$ be unbounded above. Suppose a bounded set $\mathcal{A}\subseteq\mathcal{U}$ is positively invariant for every difference equation (Δ_n), $n\in\mathbb{N}_0$. If each (Δ_n), $n\in\mathbb{N}_0$, is strongly $\mathcal{A}$-asymptotically compact with a compact $K_n\subseteq U$, then its forward limit set $\omega^+_{\mathcal{A},n}\subseteq K_n$ is nonempty, compact and forward attracts $\mathcal{A}$. If $K:=\bigcup_{n\in\mathbb{N}_0}K_n$ and moreover*

(i) *$\omega^+_{\mathcal{A},0}$ attracts K uniformly in $\tau\in\mathbb{I}$, i.e.,*
$$\lim_{s\to\infty}\sup_{\tau\in\mathbb{I}}\operatorname{dist}\left(\varphi^0(\tau+s;\tau,K),\omega^+_{\mathcal{A},0}\right)=0,$$
(ii) *for every sequence $((s_k,\tau_k))_{k\in\mathbb{N}}$ in $\mathbb{N}\times\mathbb{I}$ with $\tau_k\to\infty$ as $k\to\infty$ one has*
$$\lim_{k\to\infty}\operatorname{dist}(\varphi^n(\tau_k+s_k;\tau_k,K_n),K_n)=0\quad\textit{for all } n\in\mathbb{N}_0,$$
(iii) *for all $\varepsilon>0$, $S\in\mathbb{N}$ and $n\in\mathbb{N}_0$, there exists a $\delta>0$ such that for all $\tau\in\mathbb{I}$ one has the implication*
$$\left.\begin{array}{r}u,v\in\mathcal{A}(\tau)\cup K_n,\\ d(u,v)<\delta\end{array}\right\}\Rightarrow\max_{0<s\le S}d(\varphi^n(\tau+s;\tau,u),\varphi^n(\tau+s;\tau,v))<\varepsilon,$$
(iv) (convergence condition) *for all $\varepsilon>0$, $S\in\mathbb{N}$, there exists an $n_0\in\mathbb{N}$ such that for all $\tau\in\mathbb{I}$ one has the implication*
$$\left.\begin{array}{r}u\in\mathcal{A}(\tau)\cup K,\\ n>n_0\end{array}\right\}\Rightarrow\max_{0<s\le S}d(\varphi^n(\tau+s;\tau,u),\varphi^0(\tau+s;\tau,u))<\varepsilon$$

hold, then $\lim_{n\to\infty}\operatorname{dist}\left(\omega^+_{\mathcal{A},n},\omega^+_{\mathcal{A},0}\right)=0$.

Proof. To begin with, recall from Thm. 2.2 that each (Δ_n), $n\in\mathbb{N}_0$, has a nonempty, compact, forward limit set $\omega^+_{\mathcal{A},n}\subseteq K_n$. Moreover, thanks to [18, Thm. 4.10], $\omega^+_{\mathcal{A},n}$ is *asymptotically negatively invariant*, i.e., for all $u\in\omega^+_{\mathcal{A},n}$, $\varepsilon>0$ and $T>0$, there is an integer S_ε satisfying $\tau+S_\varepsilon-T\in\mathbb{I}$ and a point $u_\varepsilon\in\omega^+_{\mathcal{A},n}$ such that
$$d(\varphi^n(\tau+S_\varepsilon;\tau+S_\varepsilon-T,u_\varepsilon),u)<\varepsilon\quad\text{for all }\tau\in\mathbb{I},\ n\in\mathbb{N}_0.$$

In order to establish $\lim_{n\to\infty} \operatorname{dist}\Big((\omega^+_{\mathcal{A},n}, \omega^+_{\mathcal{A},0})\Big) = 0$ we proceed by contradiction. Assume that the above limit does not hold. Then there is a sequence $(n_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ satisfying $n_j \to \infty$ as $j \to \infty$ and an $\varepsilon_0 > 0$ such that

$$4\varepsilon_0 \le \operatorname{dist}\Big(\omega^+_{\mathcal{A},n_j}, \omega^+_{\mathcal{A},0}\Big) \quad \text{for all } j \in \mathbb{N}.$$

By the compactness of $\omega^+_{\mathcal{A},n_j}$, the supremum in the definition of $\operatorname{dist}\Big(\omega^+_{\mathcal{A},n_j}, \omega^+_{\mathcal{A},0}\Big)$ is attained and so is the existence of a point $w_j \in \omega^+_{\mathcal{A},n_j}$ that satisfies

$$4\varepsilon_0 \le \operatorname{dist}\Big(\omega^+_{\mathcal{A},n_j}, \omega^+_{\mathcal{A},0}\Big) = \operatorname{dist}_{\omega^+_{\mathcal{A},0}}(w_j). \tag{2.8}$$

On the other hand, from (i), there is a time $T_0 \in \mathbb{N}$ such that

$$\operatorname{dist}\Big(\varphi^0(\tau+s;\tau,K), \omega^+_{\mathcal{A},0}\Big) < \varepsilon_0 \quad \text{for all } s \ge T_0,\ \tau \in \mathbb{I}. \tag{2.9}$$

Since $T_0 > 0$ and $w_j \in \omega^+_{\mathcal{A},n_j}$, the asymptotic negative invariance of $\omega^+_{\mathcal{A},n_j}$ yields that there is an $s_0 = s_0(\varepsilon_0) \in \mathbb{I}$ with $t_0 := \tau + s_0$, a $\tau_0 := t_0 - T_0 \in \mathbb{I}$ and a $u_j = u_j(n_j, T_0) \in \omega^+_{\mathcal{A},n_j} \subseteq K_{n_j}$ so that

$$d\left(w_j, \varphi^{n_j}(t_0;\tau_0,u_j)\right) < \varepsilon_0.$$

Now notice that as $u_j \in \mathcal{A}(\tau_0) \cup K$, with integers $N = N(\varepsilon_0, T_0) > 0$ independent of τ_0 and $n_j \in \mathbb{Z}^+_N$, our assumption (iv) yields

$$d(\varphi^{n_j}(t_0;\tau_0,u_j), \varphi^0(t_0;\tau_0,u_j)) < \varepsilon_0,$$

as well as (2.9) results in

$$\operatorname{dist}_{\omega^+_{\mathcal{A},0}}(\varphi^0(t_0;\tau_0,u_j)) < \operatorname{dist}\Big(\varphi^0(t_0;\tau_0,K), \omega^+_{\mathcal{A},0}\Big) < \varepsilon_0.$$

Combining the last three inequalities, we obtain

$$\begin{aligned}\operatorname{dist}_{\omega^+_{\mathcal{A},0}}(w_j) &\le d\left(w_j, \varphi^{n_j}(t_0;\tau_0,u_j)\right) + d\left(\varphi^{n_j}(t_0;\tau_0,u_j), \varphi^0(t_0;\tau_0,u_j)\right) \\ &\quad + \operatorname{dist}_{\omega^+_{\mathcal{A},0}}(\varphi^0(t_0;\tau_0,u_j)) < \varepsilon_0 + \varepsilon_0 + \varepsilon_0 = 3\varepsilon_0,\end{aligned}$$

a contradiction to (2.8). The proof is done. □

3. INTEGRODIFFERENCE EQUATIONS UNDER DISCRETIZATIONS

In this Section we apply our abstract perturbation theory for general difference equations (Δ_0) to nonautonomous integrodifference equations. They are recursions of the form

$$\boxed{u_{t+1} = \int_\Omega f_t(\cdot, y, u_t(y))\,\mathrm{d}y,} \tag{I_0}$$

where $\Omega \subset \mathbb{R}^\kappa$ is assumed to be nonempty and compact. As state space for (I_0) we consider an ambient subset of the $\mathbb{R}^d$-valued continuous functions over Ω, abbreviated by C_d and equipped with the maximum norm $\|u\| := \max_{x\in\Omega} |u(x)|$.

3.1. **Basics and collocation.** Throughout we suppose the following standing assumption in order to obtain well-definedness for (I_0):

Hypothesis. *Suppose for all $t \in \mathbb{I}'$ that a subset $Z_t \subseteq \mathbb{R}^d$ is nonempty and closed, while a kernel function $f_t : \Omega \times \Omega \times Z_t \to \mathbb{R}^d$ is continuous and satisfies*

(H_1) *there exist $\alpha_t, \beta_t : \Omega^2 \to \mathbb{R}_+$ measurable in the second argument with*

$$a_t := \sup_{x\in\Omega} \int_\Omega \alpha_t(x,y)\,\mathrm{d}y < \infty, \qquad b_t := \sup_{x\in\Omega} \int_\Omega \beta_t(x,y)\,\mathrm{d}y < \infty$$

and

$$|f_t(x,y,z)| \le \beta_t(x,y) + \alpha_t(x,y)|z| \quad \text{for all } x,y \in \Omega,\ z \in Z_t, \tag{3.1}$$

(H_2) $\int_\Omega f_t(x,y,u(y))\,\mathrm{d}y \in Z_{t+1}$ *for all $x \in \Omega$ and $u \in U_t$ with*

$$U_t := \{u \in C_d : u(x) \in Z_t \text{ for all } x \in \Omega\}.$$

The continuity of the kernel functions f_t guarantees that the Urysohn operators

$$\mathcal{F}_t : U_t \to C_d, \qquad \mathcal{F}_t(u) := \int_\Omega f_t(\cdot, y, u(y))\,\mathrm{d}y \tag{3.2}$$

are completely continuous on the closed sets U_t (cf. [31, 35]). Furthermore, Hypothesis (H_2) ensures the inclusion $\mathcal{F}_t(U_t) \subseteq U_{t+1}$ for all $t \in \mathbb{I}'$. In particular, (I_0) is a special case of (Δ_0) in the metric space $X = C_d$, $d(u,v) = \|u - v\|$.

In order to simulate IDEs (I_0) on a computer, finite-dimensional approximations of their right-hand sides (3.2) and of their state space C_d are due. For this purpose, choose linearly independent functions $\phi_1, \dots, \phi_{d_n} \in C_1$ yielding an *ansatz space* $X_n := \operatorname{span}\{\phi_1, \dots, \phi_{d_n}\}$ of finite dimension d_n. With a projector $\pi_n \in L(C_1)$ onto the space X_n it is clear that

$$\Pi_n \in L(C_d), \qquad \Pi_n(u_j)_{j=1}^d := (\pi_n u_j)_{j=1}^d$$

is a projector onto the Cartesian product X_n^d. Supplementing this, for convenience we define $\Pi_0 := \mathrm{id}_{C_d}$. In the following, we impose the following stability assumption (cf. [10, p. 50, Def. 4.8]):

Hypothesis. (H_3) *The projections Π_n are uniformly bounded, i.e.*

$$p := \sup_{n\in\mathbb{N}} \|\Pi_n\| \in [1,\infty), \tag{3.3}$$

(H_4) $\Pi_n U_t \subseteq U_t$ *for all $n \in \mathbb{N}$ and $t \in \mathbb{I}'$.*

A *collocation method* is a spatial discretization of (I_0) of the form

$$u_{t+1} = \mathcal{F}_t^n(u_t), \qquad \mathcal{F}_t^n := \Pi_n \mathcal{F}_t : U_t \to X_n^d. \tag{I_n}$$

Thanks to (H_4) these difference equations are well-defined and fit into the general framework of (Δ_n), $n \in \mathbb{N}$. Based on Hypothesis (H_1) we next identify absorbing sets for IDEs (I_0) and their spatial discretizations.

Proposition 3.1 (Absorbing sets for (I_n)). *Let $\rho > 1$ and suppose $(H_1$–$H_4)$ hold.*

(a) *If $\mathbb{I}$ is unbounded below, and for all $\tau \in \mathbb{I}$ one has*

$$\lim_{s\to\infty} \prod_{l=\tau-s}^{\tau-1} pa_l = 0, \qquad R_\tau := p \sum_{l_1=-\infty}^{\tau-1} b_{l_1} \prod_{l_2=l_1+1}^{\tau-1} pa_{l_2} \in (0,\infty) \tag{3.4}$$

and $\sup_{\tau\in\mathbb{I}} R_\tau < \infty$, then $\mathcal{A} := \{(\tau,u) \in \mathcal{U} : \|u\| \le \rho R_\tau\}$ is pullback absorbing for each collocation discretization (I_n), $n \in \mathbb{N}_0$.

(b) *If $\mathbb{I}$ is unbounded above, and for all $\tau \in \mathbb{I}$ one has*

$$\lim_{s\to\infty} \prod_{l=\tau}^{\tau+s-1} pa_l = 0, \qquad R_\tau := p \lim_{t\to\infty} \sum_{l_1=\tau}^{t-1} b_{l_1} \prod_{l_2=l_1+1}^{t-1} pa_{l_2} \in (0,\infty) \tag{3.5}$$

and $\sup_{\tau\in\mathbb{I}} R_\tau < \infty$, then $\mathcal{A} := \{(\tau,u) \in \mathcal{U} : \|u\| \le \rho R_\tau\}$ is forward absorbing for each collocation discretization (I_n), $n \in \mathbb{N}_0$.

Remark 3.2. (1) When dealing with the initial equation (I_0), i.e. in case $n = 0$, one can choose $p = 1$.

(2) If the reals $a_t =: a$, $b_t =: b$ obtained from the linear growth assumption (3.1) are constant in time, then the condition $a \in [0, \frac{1}{p})$ implies that both the pullback and forward absorbing sets have constant fibers

$$\mathcal{A} = \left\{(t,u) \in \mathcal{U} : \|u\| \le \tfrac{\rho pb}{1-pa}\right\}.$$

In particular, for constant functions $\alpha_t(x,y) \equiv: \alpha$, $\beta_t(x,y) \equiv: \beta$ in (3.1), then $a = \alpha\mu_\kappa(\Omega)$, $b = \beta\mu_\kappa(\Omega)$, where $\mu_\kappa(\Omega) > 0$ is the Lebesgue measure of $\Omega \subset \mathbb{R}^\kappa$.

Proof. Let $t \in \mathbb{I}'$, $u \in U_t$ and $n \in \mathbb{N}_0$. For every $x \in \Omega$ we obtain

$$\begin{aligned} |\mathcal{F}_t^n(u)(x)| &\overset{(3.3)}{\le} p\,|\mathcal{F}_t(u)(x)| \overset{(I_0)}{\le} p \int_\Omega |f_t(x,y,u(y))| \,\mathrm{d}y \\ &\overset{(3.1)}{\le} p \int_\Omega \beta_t(x,y)\,\mathrm{d}y + p \int_\Omega \alpha_t(x,y)\,|u(y)|\,\mathrm{d}y \le pb_t + pa_t\,\|u\| \end{aligned}$$

and passing to the supremum over $x \in \Omega$ yields $\|\mathcal{F}_t(u)\| \le pb_t + pa_t\,\|u\|$. Then both assertions (a) and (b) for (I_n) result from [18, Prop. 3.2 resp. 4.16], where the condition $\sup_{\tau\in\mathbb{I}} R_\tau < \infty$ guarantees boundedness of $\mathcal{A}$. $\square$

Hypothesis. (H_5) *For every $t \in \mathbb{I}'$ and $r > 0$, there exists a function $\lambda_t : \Omega \times \Omega \to \mathbb{R}_+$ measurable in the second argument with*

$$|f_t(x,y,z) - f_t(x,y,\bar z)| \le \lambda_t(x,y)\,|z - \bar z| \quad \text{for all } z, \bar z \in Z_t \cap \bar B_r(0) \tag{3.6}$$

and $x, y \in \Omega$, where $\ell_t(r) := \sup_{x\in\Omega} \int_\Omega \lambda_t(x,y)\,\mathrm{d}y < \infty$.

Lemma 3.3. *Suppose $(H_1$–$H_3)$ and (H_5) hold. For each $t \in \mathbb{I}'$ and $r > 0$ one has the Lipschitz estimate*

$$\|\mathcal{F}_t^n(u) - \mathcal{F}_t^n(\bar u)\| \le p\ell_t(r)\,\|u - \bar u\| \quad \text{for all } n \in \mathbb{N}_0,\ u, \bar u \in U_t \cap \bar B_r(0). \tag{3.7}$$

Proof. Using (3.3) the claim results from [35, Cor. B.6]. $\square$

The local discretization error in collocation discretizations (I_n) depends on the smoothness of the image functions $\mathcal{F}_t(u) : \Omega \to \mathbb{R}^d$. Thereto, given a subspace $X(\Omega) \subset C_d$ one denotes a right-hand side (3.2) as $X(\Omega)$-*smoothing*, if $\mathcal{F}_t(u) \in X(\Omega)$ for all $t \in \mathbb{I}'$ and $u \in U_t$. For instance, convolution kernels yield that $\mathcal{F}_t(u)$ have a higher order smoothness than the f_t itself (cf. [9, pp. 50ff, Sect. 3.4]).

Note that for collocation methods based on polynomial interpolation the stability condition (3.3) fails even with Chebyshev nodes as collocation points (see [10, p. 52]). Nevertheless, the stability condition (3.3) holds when working with spline functions as basis of X_n. For this purpose, we restrict to $\Omega = [a,b]$, choose a grid

$$a =: x_0 < x_1 < \ldots < x_n := b \tag{3.8}$$

and abbreviate $\overline{h}_n := \max_{j=0}^{n-1}(x_{j+1} - x_j)$, $\underline{h}_n := \min_{j=0}^{n-1}(x_{j+1} - x_j)$. Beyond that assume that the grid (3.8) is extended to a strictly increasing sequence $(x_j)_{j\in\mathbb{Z}}$.

Let $\beta_j^l : \mathbb{R} \to \mathbb{R}_+$, $j \in \mathbb{Z}$, denote the *B-splines* of degree $l \in \mathbb{N}$ introduced e.g. in [14, pp. 242ff] and consider the subspaces

$$X_n := \operatorname{span}\left\{\beta_{-l}^l, \dots, \beta_{n-1}^l\right\} \subset C[a, b].$$

Example 3.4 (linear splines)**.** The B-splines $\beta_j^1 : \mathbb{R} \to [0, 1]$, $-1 \le j \le n$ (the hat functions), form a basis of the ansatz space consisting of piecewise linear functions $X_n \subset C[a, b]$ having dimension $d_n = n + 1$. The spline projections

$$\pi_n u := \sum_{j=0}^{n} u(x_j)\beta_{j-1}^1$$

fulfill to the interpolation conditions $(\pi_n u)(x_j) = u(x_j)$, $0 \le j \le n$, satisfy $\|\pi_n\| = 1$ (see [2, p. 450]), hence $p = 1$ in (3.3), and the error estimate [14, p. 275]

$$\|u - \pi_n u\| \le \tfrac{\overline{h}_n^2}{8} \left\|u''\right\| \quad \text{for all } n \in \mathbb{N},\ u \in C^2[a, b].$$

Consequently, if (I_0) is $C^2[a, b]^d$-smoothing, then an application to (I_n) yields

$$e_t^n(u) \le \tfrac{\overline{h}_n^2}{8} \left\|\mathcal{F}_t(u)''\right\| \quad \text{for all } t \in \mathbb{I}',\ u \in U_t \tag{3.9}$$

for the local discretization error. The same quadratic convergence also holds, when multidimensional piecewise linear interpolation is used on domains $\Omega \subset \mathbb{R}^\kappa$ (see [35, Sect. 3.1.3]).

Example 3.5 (quadratic splines)**.** Given a grid (3.8) we introduce the $n + 2$ collocation points

$$\xi_0 := a, \qquad \xi_j := \tfrac{x_j + x_{j-1}}{2} \quad \text{for all } 0 \le j \le n, \qquad \xi_{n+1} =: b.$$

Then the quadratic splines $\beta_j^2 : \mathbb{R} \to [0, 1]$, $-2 \le j \le n$, yield an ansatz space $X_n \subset C^1[a, b]$ of dimension $d_n = n + 2$, where the interpolation conditions $(\pi_n u)(\xi_j) = u(\xi_j)$, $0 \le j \le n + 1$, result in a projection $\pi_n : C[a, b] \to X_n$ satisfying $\|\pi_n\| \le 2$ (cf. [20, Cor. 3.2]), thus $p = 2$ in (3.3) and

$$\|u - \pi_n u\| \le \tfrac{\overline{h}_n^3}{24} \left\|u^{(3)}\right\| \quad \text{for all } n \in \mathbb{N},\ u \in C^3[a, b]$$

as interpolation error (see [8, Thm. 6]). Hence, for $C^3[a, b]^d$-smoothing equations (I_0) the local discretization error becomes

$$e_t^n(u) \le \tfrac{\overline{h}_n^3}{24} \left\|\mathcal{F}_t(u)^{(3)}\right\| \quad \text{for all } t \in \mathbb{I}',\ u \in U_t.$$

Example 3.6 (cubic splines)**.** The cubic splines $\beta_j^3 : \mathbb{R} \to [0, 1]$, $-3 \le j \le n$, establish an ansatz space $X_n \subset C^2[a, b]$ of dimension $d_n = n + 3$. Supplementing the interpolation conditions $(\pi_n u)(x_j) = u(x_j)$, $0 \le j \le n$, by Hermite boundary conditions

$$(\pi_n u)''(a) = u''(a), \qquad (\pi_n u)''(b) = u''(b)$$

leads to a projection $\pi_n : C[a,b] \to X_n$ with $\|\pi_n\| \leq 1 + \frac{3}{2}\frac{\overline{h}_n}{\underline{h}_n}$ (cf. [32, Thm. 3.1]). Whence, $p := 1 + \frac{3}{2}\sup_{n\in\mathbb{N}}\frac{\overline{h}_n}{\underline{h}_n} < \infty$ implies stability. Furthermore, the interpolation error becomes

$$\|u - \pi_n u\| \leq \tfrac{5\overline{h}_n^4}{384}\left\|u^{(4)}\right\| \quad \text{for all } n \in \mathbb{N},\ u \in C^4[a,b]$$

(see [13, Thm. 4]). In case (I_0) is $C^4[a,b]$-smoothing one obtains

$$e_t^n(u) \leq \tfrac{5\overline{h}_n^4}{384}\left\|\mathcal{F}_t(u)^{(4)}\right\| \quad \text{for all } t \in \mathbb{I}',\ u \in U_t$$

as local discretization error.

In conclusion, collocation methods (I_n), $n \in \mathbb{N}$, based on splines having the order $l \in \{1,2,3\}$ are convergent, provided

- $\lim_{n\to\infty} \overline{h}_n = 0$ and
- the derivatives $\mathcal{F}_t(\cdot)^{(l+1)} : U_t \to C[a,b]^d$ are bounded functions, that is, they map bounded subsets of U_t to bounded sets uniformly in $t \in \mathbb{I}'$.

3.2. **Pullback attractors.** Let $\mathbb{I}$ be unbounded below. An immediate consequence of Prop. 3.1 is

Corollary 3.7. *The pullback absorbing set $\mathcal{A} \subseteq \mathcal{U}$ from Prop. 3.1(a) is positively invariant for every (I_n), $n \in \mathbb{N}_0$.*

Proof. Let $n \in \mathbb{N}_0$, $\tau \in \mathbb{I}'$ and $u \in u \cap \bar{B}_{\rho R_\tau}(0)$. Thanks to (H_2) and (H_4) it remains to establish the inclusion $\mathcal{F}_\tau^n(u) \subseteq \bar{B}_{\rho R_{\tau+1}}(0)$. Thereto, we obtain as in the proof of Prop. 3.1 that

$$\|\mathcal{F}_\tau^n(u)\| \leq pb_\tau + pa_\tau\|u\| \leq pb_\tau + \rho pa_\tau R_\tau.$$

In combination with

$$\begin{aligned} R_{\tau+1} &\overset{(3.4)}{=} p\sum_{l_1=-\infty}^{\tau} b_{l_1}\prod_{l_2=l_1+1}^{\tau} pa_{l_2} = pa_\tau p\sum_{l_1=-\infty}^{\tau-1} b_{l_1}\prod_{l_2=l_1+1}^{\tau-1} pa_{l_2} + pb_\tau\prod_{l_2=\tau+1}^{\tau} pa_{l_2} \\ &\overset{(3.4)}{=} pa_\tau R_\tau + pb_\tau \end{aligned}$$

this implies $\|\mathcal{F}_\tau^n(u)\| \leq \rho R_{\tau+1} + p(1-\rho)b_\tau \leq \rho R_{\tau+1}$. This is the assertion. □

Theorem 3.8 (Pullback attractors for (I_n))**.** *Let $(H_1$–$H_4)$ and the limit relations* (3.4) *hold for all $\tau \in \mathbb{I}$. If $\sup_{\tau\in\mathbb{I}} R_\tau < \infty$, then every (I_n), $n \in \mathbb{N}_0$, has a pullback attractor $\mathcal{A}_n^* \subseteq \mathbb{I} \times C_d$. If $\mathcal{F}_t : U_t \to U_{t+1}$ maps bounded subsets of U_t into bounded sets uniformly in $t \in \mathbb{I}'$ and moreover*

(i) *(H_5) is satisfied with $\ell(r) := \sup_{t\in\mathbb{I}'}\ell_t(r) < \infty$ for all $r > 0$,*
(ii) *the collocation discretizations (I_n), $n \in \mathbb{N}$, are convergent*

hold, then $\lim_{n\to\infty}\max_{t\in I}\operatorname{dist}(\mathcal{A}_n^*(t), \mathcal{A}_0^*(t)) = 0$ on each bounded $I \subseteq \mathbb{I}$.

Proof. Let $\rho > 1$. We apply Thm. 2.1 and 2.6 to the difference equations (I_n), $n \in \mathbb{N}_0$, in the space $X = C_d$ equipped with the metric $d(u,v) := \|u - v\|$.

By Prop. 3.1(a) the nonautonomous set $\mathcal{A} := \{(\tau,u) \in \mathcal{U} : \|u\| \leq \rho R_\tau\}$ is pullback absorbing for each (I_n), $n \in \mathbb{N}_0$. Furthermore, since the right-hand sides $\mathcal{F}_t : U_t \to C_d$ of (I_0) are completely continuous, also their discretizations $\mathcal{F}_t^n = \Pi_n\mathcal{F}_t$ share this property for all $t \in \mathbb{I}'$ and $n \in \mathbb{N}$. Therefore, (I_n) is pullback asymptotically compact (see [33, p. 13, Cor. 1.2.22]) and Thm. 2.1 guarantees that every IDE (I_n), $n \in \mathbb{N}_0$, has a pullback attractor $\mathcal{A}_n^*$ with the claimed properties.

We verify the assumptions of Thm. 2.6:

ad (i): By the Arzelà-Ascoli theorem [15, p. 44, Thm. 3.3] we have to show that the union $\bigcup_{n\in\mathbb{N}_0}\mathcal{A}_n^*(t)$ is bounded and equicontinuous for all $t\in\mathbb{I}$. Since boundedness is a consequence of the inclusion $\bigcup_{n\in\mathbb{N}_0}\mathcal{A}_n^*(t)\subseteq\mathcal{A}(t)$, it remains to establish the equicontinuity. Let $n\in\mathbb{N}_0$. Since pullback attractors $\mathcal{A}_n^*$ are invariant, one has

$$\mathcal{A}_n^*(t+1)=\mathcal{F}_t^n(\mathcal{A}_n^*(t))\subseteq\mathcal{F}_t^n(\mathcal{A}(t))\quad\text{for all } t\in\mathbb{I}'$$

and consequently, $\bigcup_{n\in\mathbb{N}_0}\mathcal{A}_n^*(t+1)\subseteq\bigcup_{n\in\mathbb{N}_0}\mathcal{F}_t^n(\mathcal{A}(t))$ for all $t\in\mathbb{I}'$. Because the subsets of equicontinuous sets are equicontinuous again, it suffices to establish equicontinuity of the union $\bigcup_{n\in\mathbb{N}_0}\mathcal{F}_t^n(\mathcal{A}(t))$ for any $t\in\mathbb{I}'$.

Thereto, let $\varepsilon>0$. Because (I_n), $n\in\mathbb{N}$, is assumed to be convergent, there exists an $N\in\mathbb{N}$ such that

$$e_t^n(u)=\|\mathcal{F}_t^n(u)-\mathcal{F}_t^0(u)\|<\tfrac{\varepsilon}{4}\quad\text{for all } u\in\mathcal{A}(t),\ n\geq N.$$

On the other hand, the image $\mathcal{F}_t^0(\mathcal{A}(t))$ due to [15, p. 43, Prop. 3.1] is even uniformly equicontinuous, i.e., there exists a $\delta>0$ such that

$$|x-y|<\delta\quad\Rightarrow\quad|\mathcal{F}_t^0(u)(x)-\mathcal{F}_t^0(u)(y)|<\tfrac{\varepsilon}{4}\quad\text{for all } x,y\in\Omega.$$

Combining both inequalities above, in case $|x-y|<\delta$ it results from the triangle inequality that

$$\begin{aligned}|\mathcal{F}_t^n(u)(x)-\mathcal{F}_t^n(u)(y)|&\leq|\mathcal{F}_t^n(u)(x)-\mathcal{F}_t^0(u)(x)|+|\mathcal{F}_t^0(u)(x)-\mathcal{F}_t^0(u)(y)|\\&\leq\tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{4}<\varepsilon\quad\text{for all } u\in\mathcal{A}(t),\ n\geq N.\end{aligned}$$

This yields that $\mathcal{F}_t^0(\mathcal{A}(t))\cup\bigcup_{n\geq N}\mathcal{F}_t^n(\mathcal{A}(t))$ is equicontinuous and additionally including the finitely many relatively compact sets $\mathcal{F}_t^n(\mathcal{A}(t))$, $1\leq n<N$, preserves equicontinuity. As a result, every $\bigcup_{n\in\mathbb{N}_0}\mathcal{A}_n^*(t)$, $t\in\mathbb{I}$, is relatively compact.

ad (ii): Due to $\bigcup_{n\in\mathbb{N}_0}\bigcup_{s\leq t}\mathcal{A}_n^*(s)\subseteq\mathcal{A}(t)$ for all $t\in\mathbb{I}$ we obtain the claim.

ad (iii): In order to apply Prop. 2.5 we note our assumption (H_5) and Lemma 3.3 imply that for each $r>0$ there exists a $\ell(r)\geq0$ such that

$$\|\mathcal{F}_t^n(u)-\mathcal{F}_t^n(v)\|\leq p\ell(r)\,\|u-v\|\quad\text{for all } n\in\mathbb{N},\ t\in\mathbb{I}'$$

and $u,v\in U_t\cap\bar{B}_r(0)$. Hence, the collocation discretizations (I_n) fulfill (2.5). Now let $\varepsilon>0$, $\tau\in\mathbb{I}'$, $S\in\mathbb{N}$ and $K\subseteq C_d$ be compact. Given $u\in U_{\tau-s}\cap K$ choose a nonautonomous set $\mathcal{B}\subseteq\mathcal{U}$ such that

$$\varphi^0(\tau;\tau-s,u)\in\mathcal{B}(\tau)\quad\text{for all } 0<s\leq S.$$

Because, by assumption, $\mathcal{F}_t:U_t\to U_{t+1}$ maps bounded subsets of U_t into bounded sets uniformly in $t\in\mathbb{I}'$, the set $\mathcal{B}$ can be chosen to be bounded. Hence, there exists a $r>0$ such that $\mathcal{B}(t)\subseteq\bar{B}_r(0)$ for all $t\in\mathbb{I}$. Now choose $\rho>r$ and due to Prop. 2.5 there exists an $N_0\in\mathbb{N}$ such that

$$\begin{aligned}\left\|\varphi^n(\tau;\tau-s,u)-\varphi^0(\tau;\tau-s,u)\right\|&\overset{(2.6)}{\leq}\Gamma\left(\tfrac{1}{n}\right)\sum_{l_1=\tau-s}^{\tau-1}C(\mathcal{B}(l_1))\prod_{l_2=l_1+1}^{\tau-1}p\ell(\rho)\\&\leq\Gamma\left(\tfrac{1}{n}\right)C(\bar{B}_r(0))\sum_{l_1=\tau-s}^{\tau-1}\prod_{l_2=l_1+1}^{\tau-1}p\ell(\rho)\end{aligned}$$

for all $0 < s \leq S$ and $n > N_0$. It is easy to see that the right-hand side in the above estimate does not depend on $\tau \in \mathbb{I}'$. Since our convergence assumption (ii) ensures $\lim_{n\to\infty} \Gamma(\frac{1}{n}) = 0$, there exists an $n_0 \geq N_0$ such that $n > n_0$ implies

$$\left\|\varphi^n(\tau; \tau - s, u) - \varphi^0(\tau; \tau - s, u)\right\| < \varepsilon \quad \text{for all } u \in U_{\tau-s} \cap K,\ 0 < s \leq S.$$

This finally verifies Thm. 2.6(iii). □

3.3. **Forward dynamics.** Let all nonempty, closed $Z :\equiv Z_t$ be time-invariant yielding the constant domains $U :\equiv U_t$ on a discrete interval $\mathbb{I}$ unbounded above.

The forward dynamics counterpart to Cor. 3.7 has a slightly different structure:

Corollary 3.9. *The forward absorbing set $\mathcal{A} \subseteq \mathcal{U}$ from Prop. 3.1(b) has constant fibers, i.e. there exists a $R > 0$ with $R = R_\tau$ for all $\tau \in \mathbb{I}$. Moreover, in case*

$$pb_\tau \leq \rho R(1 - pa_\tau) \quad \textit{for all } \tau \in \mathbb{I} \tag{3.10}$$

the nonautonomous set $\mathcal{A}$ is positively invariant for every (I_n), $n \in \mathbb{N}_0$.

Proof. Above all, for $\tau, t \in \mathbb{I}$ we define $r_\tau(t) := p\sum_{l_1=\tau}^{t-1} b_{l_1} \prod_{l_2=l_1+1}^{t-1} pa_{l_2}$ and passing to the limit $t \to \infty$ in

$$r_\tau(t) = p \sum_{l_1=\tau+1}^{t-1} b_{l_1} \prod_{l_2=l_1+1}^{t-1} pa_{l_2} + pb_\tau \prod_{l_2=\tau+1}^{t-1} pa_{l_2} = r_{\tau+1}(t) + pb_\tau \prod_{l_2=\tau+1}^{t-1} pa_{l_2}$$

implies that $R_\tau = R_{\tau+1}$ due to (3.5). With $R := R_\tau$ and $u \in u \cap \bar{B}_{\rho R}(0)$ we obtain as in the proof of Cor. 3.7 that $\|\mathcal{F}_\tau^n(u)\| \leq pb_\tau + \rho pa_\tau R$ for all $n \in \mathbb{N}_0$ and therefore (3.10) yields the assertion. □

Theorem 3.10 (Forward limit sets for (I_n))**.** *Suppose $(H_1$–$H_4)$, the limit relations* (3.5) *and* (3.10) *hold for all $\tau \in \mathbb{I}$ yielding the nonautonomous set $\mathcal{A} \subseteq \mathcal{U}$ from Prop. 3.1(b). If each (I_n) is strongly $\mathcal{A}$-asymptotically compact with a compact $K_n \subseteq U$, $n \in \mathbb{N}_0$, then its forward limit set $\omega^+_{\mathcal{A},n} \subseteq K_n$ is nonempty, compact and forward attracts $\mathcal{A}$. If moreover $K := \bigcup_{n\in\mathbb{N}_0} K_n$ is bounded, $\mathcal{F}_t : U \to U$ maps bounded subsets of U into bounded sets uniformly in $t \in \mathbb{I}$ and*

(i) *$\omega^+_{\mathcal{A},0}$ attracts K uniformly in $\tau \in \mathbb{I}$, i.e.,*

$$\lim_{s\to\infty} \sup_{\tau\in\mathbb{I}} \operatorname{dist}\left(\varphi^0(\tau + s; \tau, K), \omega^+_{\mathcal{A},0}\right) = 0,$$

(ii) *for every sequence $((s_k, \tau_k))_{k\in\mathbb{N}}$ in $\mathbb{N} \times \mathbb{I}$ with $\tau_k \to \infty$ as $k \to \infty$ one has*

$$\lim_{k\to\infty} \operatorname{dist}\left(\varphi^n(\tau_k + s_k; \tau_k, K_n), K_n\right) = 0 \quad \textit{for all } n \in \mathbb{N}_0,$$

(iii) *(H_5) is satisfied with $\ell(r) := \sup_{t\in\mathbb{I}} \ell_t(r) < \infty$ for all $r > 0$,*

(iv) *the collocation discretizations (I_n), $n \in \mathbb{N}$, are convergent*

hold, then $\lim_{n\to\infty} \operatorname{dist}\left(\omega^+_{\mathcal{A},n}, \omega^+_{\mathcal{A},0}\right) = 0$.

Proof. We aim to apply Thm. 2.7 to (I_n), $n \in \mathbb{N}_0$. Above all, it results from the above Cor. 3.9 that $\mathcal{A}$ is forward absorbing and positively invariant for each (I_n). Then Thm. 2.7 yields the existence of forward limit sets $\omega^+_{\mathcal{A},n}$ with the claimed properties for all $n \in \mathbb{N}_0$.

It remains to show that the assumptions (iii) and (iv) of Thm. 2.7 hold.

ad (iii): Choose $r > 0$ so large that $\mathcal{A}(\tau) \cup K \subseteq \bar{B}_r(0)$ holds for all $\tau \in \mathbb{I}$, which is possible due to the boundedness of $\mathcal{A}$ and K. By assumption, it results inductively from Lemma 3.3 that

$$\|\varphi^n(\tau+s;\tau,u) - \varphi^n(\tau+s;\tau,v)\| \leq (p\ell(r))^s \|u-v\| \quad \text{for all } n \in \mathbb{N}_0,\ s \in \mathbb{N}$$

and $\tau \in \mathbb{I}$, $u, v \in \mathcal{A}(\tau) \cup K_n$. Let $\varepsilon > 0$ and $S \in \mathbb{N}$ be arbitrarily given. If we choose $\delta := \min_{0<s\leq S} \frac{\varepsilon}{\ell(r)^s} > 0$, then $\|u-v\| < \delta$ readily leads to the estimate $\|\varphi^n(\tau+s;\tau,u) - \varphi^n(\tau+s;\tau,v)\| < \varepsilon$ for $0 < s \leq S$, which implies (iii).

ad (iv): Let $\varepsilon > 0$, $S \in \mathbb{N}$ and $\tau \in \mathbb{I}$. As in the proof of Thm. 3.8 one shows that the Lipschitz estimate (2.4) holds in our present setting with the (local) Lipschitz constant $p\ell(r)$. For $u \in \mathcal{A}(\tau) \cup K$ choose a bounded nonautonomous set $\mathcal{B} \subseteq \mathcal{U}$ so large that $\varphi^0(t;\tau,u) \in \mathcal{B}(t)$ for all $\tau \leq t \leq \tau + S$; since $\mathcal{F}_t$ is assumed to be bounded uniformly in $t \in \mathbb{I}$ the nonautonomous set $\mathcal{B}$ can be chosen to be independent of $\tau \in \mathbb{I}$. Furthermore, choose $r > 0$ so large that $\mathcal{B}(t) \subseteq \bar{B}_r(0)$ for all $\tau \leq t \leq \tau + S$. For $\rho > r$, then Prop. 2.5 implies that there is an $N_0 \in \mathbb{N}$ with

$$\begin{aligned}\left\|\varphi^n(\tau+s;\tau,u) - \varphi^0(\tau+s;\tau,u)\right\| &\overset{(2.6)}{\leq} \Gamma\left(\tfrac{1}{n}\right) \sum_{l_1=\tau}^{\tau+s-1} C(\mathcal{B}(l_1)) \prod_{l_2=l_1+1}^{\tau+s-1} \ell(\rho) \\ &\leq \Gamma\left(\tfrac{1}{n}\right) C(\bar{B}_r(0)) \sum_{l_1=\tau}^{\tau+s-1} \prod_{l_2=l_1+1}^{\tau+s-1} \ell(\rho)\end{aligned}$$

for all $0 < s \leq S$ and $n > N_0$. Note that the right-hand side of this inequality does not depend on $\tau \in \mathbb{I}$. Consequently, our convergence condition $\lim_{n\to\infty} \Gamma(\frac{1}{n}) = 0$ implies that there exists an $n_0 \geq N_0$ such that $n > n_0$ implies

$$\left\|\varphi^n(\tau+s;\tau,u) - \varphi^0(\tau+s;\tau,u)\right\| < \varepsilon \quad \text{for all } u \in \mathcal{A}(\tau) \cap K,\ 0 < s \leq S$$

and therefore Thm. 2.7(iii) is verified. □

Theorem 3.11 (Persistence of forward attractors for (I_n))**.** *Let* $\mathbb{I} = \mathbb{Z}$. *Suppose* $(H_1$–$H_4)$, *the limit relations* (3.5) *and* (3.10) *hold for all* $\tau \in \mathbb{Z}$ *yielding the nonautonomous set* $\mathcal{A} \subseteq \mathcal{U}$ *from Prop. 3.1(b). If each* (I_n) *is strongly* $\mathcal{A}$*-asymptotically compact with a compact* $K_n \subseteq U$, $n \in \mathbb{N}_0$, *and*

(i) *for every sequence* $((s_k,\tau_k))_{k\in\mathbb{N}}$ *in* $\mathbb{N} \times \mathbb{Z}$ *with* $\tau_k \to \infty$ *as* $k \to \infty$ *one has*

$$\lim_{k\to\infty} \operatorname{dist}\left(\varphi^n(\tau_k+s_k;\tau_k,K_n), K_n\right) = 0 \quad \textit{for all } n \in \mathbb{N}_0,$$

(ii) (H_5) *is satisfied with* $\ell(r) := \sup_{t\in\mathbb{I}} \ell_t(r) < \infty$ *for all* $r > 0$,

(iii) *the forward limit sets* $\omega^+_{\mathcal{A},n}$ *of* (I_n) *satisfy*

$$\omega^+_{\mathcal{A},n} = \limsup_{t\to\infty} \mathcal{A}_n^*(t) \quad \textit{for all } n \in \mathbb{N}_0$$

with $\mathcal{A}_n^\star(t) := \bigcap_{0\leq s} \varphi^n(t;t-s,\mathcal{A}(t-s))$ *for all* $t \in \mathbb{Z}$

hold, then every (I_n), $n \in \mathbb{N}_0$, *has* $\mathcal{A}_n^\star \subseteq \mathcal{A}$ *as a forward attractor.*

Proof. Let $n \in \mathbb{N}_0$. We verify that (I_n) satisfies the assumptions of Thm. 2.3. First of all, Cor. 3.9 ensures that the closed set $\mathcal{A}$ is forward absorbing and positively invariant w.r.t. (I_n). Because each $\mathcal{F}_t : U \to C_d$ is completely continuous, also the discretizations $\mathcal{F}_t^n = \Pi_n \mathcal{F}_t$ have this property for all $t \in \mathbb{Z}$. Therefore, (I_n) is pullback asymptotically compact (see [33, p. 13, Cor. 1.2.22]). Our assumption (i) yields that (I_n) fulfills the assumption in Thm. 2.3(i). As in the above proof of

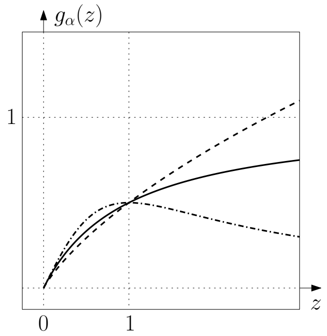

FIGURE 1. Shape of generalized Beverton-Holt functions $g_\alpha(z) := \frac{z}{1+z^\alpha}$ for $\alpha = \frac{1}{2}$ (dashed), $\alpha = 1$ (solid) and $\alpha = 2$ (dash dotted)

Thm. 3.10 one shows that also Thm. 2.3(ii) holds for (I_n). Finally, (iii) ensures that finally the assumption in Thm. 2.3(iii) is satisfied for (I_n). □

4. ILLUSTRATIONS

Let $\mathbb{I}$ be a discrete interval and $\Omega \subset \mathbb{R}^\kappa$ be a compact habitat. We are discussing IDEs defined on the (closed) cone

$$U = C_+(\Omega) := \{u \in C(\Omega) :\ 0 \leq u(x) \text{ for all } x \in \Omega\}$$

of nonnegative continuous functions.

4.1. **Pullback attractors for generalized Beverton-Holt equations.** Consider the scalar nonautonomous IDE

$$u_{t+1} = \mathscr{F}_t(u_t), \qquad \mathscr{F}_t(u) := \int_\Omega k_t(\cdot, y)\frac{\gamma_t(y)u(y)}{1 + u(y)^\alpha}\,\mathrm{d}y \tag{4.1}$$

depending on a parameter $\alpha > 0$ with continuous kernels $k_t : \Omega^2 \to \mathbb{R}_+$ and integrable growth rates $\gamma_t : \Omega \to \mathbb{R}_+$, $t \in \mathbb{I}'$.

The nonlinearity in this Hammerstein IDE is given by the family of generalized Beverton-Holt functions $g_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, $g_\alpha(z) := \frac{z}{1+z^\alpha}$, whose behavior depends on $\alpha > 0$ (see Fig. 1). All these growth functions have in common to be globally Lipschitz with constant 1. Consequently, in order to fulfill the assumptions of Lemma 3.3 we choose $\lambda_t(x, y) := k_t(x, y)\gamma_t(y)$ guaranteeing even a global Lipschitz estimate in (3.7) with

$$\ell_t := \sup_{x\in\Omega} \int_\Omega k_t(x, y)\gamma_t(y)\,\mathrm{d}y \quad \text{for all } t \in \mathbb{I}'.$$

In order to establish that the generalized Beverton-Holt equation (4.1) is dissipative using Prop. 3.1, we suppose that $(\ell_t)_{t\in\mathbb{I}'}$ is bounded and proceed as follows:

- For $\alpha \in (0, 1)$ the function g_α is unbounded and its tangent in a point $(\zeta, g_\alpha(\zeta))$ for $\zeta > 0$ reads as

$$z \mapsto \tfrac{1+(1-\alpha)\zeta^\alpha}{(1+\zeta^\alpha)^2}(z - \zeta) + \tfrac{\zeta}{1+\zeta^\alpha} = \tfrac{\alpha\zeta^{1+\alpha}}{(1+\zeta^\alpha)^2} + \tfrac{1+(1-\alpha)\zeta^\alpha}{(1+\zeta^\alpha)^2}z.$$

  Concerning the estimate (3.1) this allows us to choose

$$\alpha_t(x, y) := \tfrac{1+(1-\alpha)\zeta^\alpha}{(1+\zeta^\alpha)^2}k_t(x, y)\gamma_t(y), \qquad \beta_t(x, y) := \tfrac{\alpha\zeta^{1+\alpha}}{(1+\zeta^\alpha)^2}k_t(x, y)\gamma_t(y)$$

  and consequently

$$a_t = \tfrac{1+(1-\alpha)\zeta^\alpha}{(1+\zeta^\alpha)^2}\ell_t, \qquad b_t = \tfrac{\alpha\zeta^{1+\alpha}}{(1+\zeta^\alpha)^2}\ell_t.$$

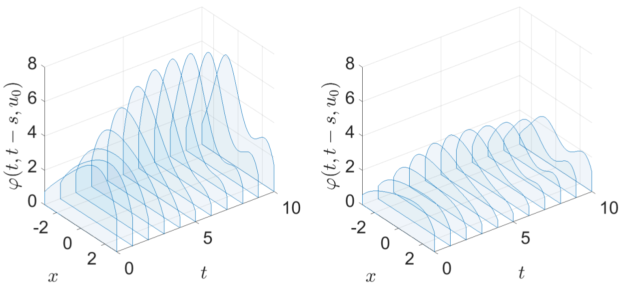

FIGURE 2. Sequences of sets containing functions in the fibers $\mathcal{A}_n^*(t)$, $0 \le t \le 10$, for $\alpha \in \left\{\frac{1}{2}, 1\right\}$, respectively from left to right.

If $\zeta > 0$ is chosen so large that

$$\tfrac{1+(1-\alpha)\zeta^\alpha}{(1+\zeta^\alpha)^2} \limsup_{t\to-\infty} \ell_t < 1 \quad \text{on } \mathbb{I} \text{ unbounded below,}$$

then (4.1) is pullback dissipative, while

$$\tfrac{1+(1-\alpha)\zeta^\alpha}{(1+\zeta^\alpha)^2} \limsup_{t\to\infty} \ell_t < 1 \quad \text{on } \mathbb{I} \text{ unbounded above}$$

yields forward dissipativity of the Beverton-Holt equation (4.1).

- For $\alpha \ge 1$, the function g_α is globally bounded with

$$\sup_{z\ge 0} g_\alpha(z) = \tfrac{1}{\alpha}(\alpha-1)^{1-\frac{1}{\alpha}}$$

and therefore in (3.1) we can choose

$$\alpha_t(x,y) :\equiv 0, \qquad \beta_t(x,y) := \tfrac{1}{\alpha}(\alpha-1)^{1-\frac{1}{\alpha}} k_t(x,y)\gamma_t(y),$$

which results in

$$a_t \equiv 0, \qquad b_t = \tfrac{1}{\alpha}(\alpha-1)^{1-\frac{1}{\alpha}} \ell_t.$$

Consequently, the IDE (4.1) is both forward and pullback dissipative.

Now let $\mathbb{I}$ be unbounded below. In any case, we conclude from Thm. 2.1 that the nonautonomous Beverton-Holt equation (4.1) has a pullback attractor $\mathcal{A}_0^*$. In particular, for parameters $\alpha \in (0,1]$ the growth function g_α is strictly increasing and therefore the right-hand sides of (4.1) are order-preserving. This means that for all $u, v \in U$ the implication

$$u(x) \le v(x) \quad \Rightarrow \quad \mathcal{F}_t^n(u)(x) \le \mathcal{F}_t(v)(x) \quad \text{for all } t \in \mathbb{I}',\ x \in \Omega$$

holds. Given a sequence $(\xi_t)_{t\in\mathbb{I}}$ in $C_+(\Omega)$ we introduce the order intervals

$$[0,\xi] := \{(t,v) \in \mathbb{I}\times C(\Omega) : 0 \le v(x) \le \xi_t(x) \text{ for all } x \in \Omega\} \subseteq \mathbb{I}\times C_+(\Omega)$$

and obtain from [34, Prop. 8] that the pullback attractor $\mathcal{A}_0^*$ of (4.1) satisfies the inclusion $\mathcal{A}_0^* \subseteq [0, \xi^0]$ with a bounded entire solution $(\xi_t^0)_{t\in\mathbb{I}}$ to (4.1). This entire solution is contained in $\mathcal{A}_0^*$ due to the characterization (2.2). In addition, ξ^0 is the pullback limit for all sequences starting from 'above' ξ^0 (cf. [34, Prop. 8]).

Based on Thm. 3.8 also stable collocation discretizations (I_n), $n \in \mathbb{N}$, of the generalized Beverton-Holt equation (4.1) possess pullback attractors $\mathcal{A}_n^*$. In order

n	$\alpha = \frac{1}{2}$	$\alpha = 1$
16	2.112614126300029	2.100856100109834
32	2.055209004601208	2.051123510423984
64	2.026777868073563	2.025681916479720
128	2.013096435137189	2.012865860858589
256	2.006536458546063	2.006433295172438
512	2.003256451919377	2.003220576642864
1024	2.001624177537549	2.001610772707442

TABLE 1. Values of the approximate convergence rates $c(n)$ for $n \in \left\{2^4, \dots, 2^{10}\right\}$ nodes and parameters $\alpha \in \left\{\frac{1}{2}, 1, 2\right\}$.

to illustrate convergence of the pullback attractors $\mathcal{A}_n^*$ to $\mathcal{A}_0^*$ as $n \to \infty$, we restrict to habitats $\Omega = [a, b]$ and piecewise linear splines from Ex. 3.4 having the nodes $x_j := a + j\frac{b-a}{n}$, $0 \le j \le n$, for all $n \in \mathbb{N}$. This results in the spatial discretizations

$$u_{t+1} = \mathcal{F}_t^n(u_t), \qquad \mathcal{F}_t^n(u) = \sum_{j=0}^{n} \int_\Omega k_t(x_j, y) \frac{\gamma_t(y)u(y)}{1 + u(y)^\alpha} \,\mathrm{d}y \beta_{j-1}^1. \tag{4.2}$$

In particular, $p = 1$ and $\overline{h}_n = \frac{b-a}{n}$ for all $n \in \mathbb{N}$ and for appropriate kernels k_t one obtains the convergence rate 2 from (3.9).

Spatial discretization using piecewise linear functions guarantees that also the right-hand sides $\mathcal{F}_t^n : U \to X_n$, $t \in \mathbb{I}'$, are order preserving for $\alpha \in (0, 1]$. Thus, [34, Prop. 8] applies to (4.2) as well, and ensures that its pullback attractors fulfill $\mathcal{A}_n^* \subseteq [0, \xi^n]$ with bounded entire solutions $(\xi_t^n)_{t \in \mathbb{I}}$ to (4.2) in $\mathcal{A}_n^*$.

On this basis, in order to quantify convergence rates of the Hausdorff distances $\operatorname{dist}(\mathcal{A}_n^*(t), \mathcal{A}_0^*(t))$ between the fibers of the pullback attractors to (4.2) and $\mathcal{A}_0^*$ guaranteed by Thm. 3.8, we make use the entire solutions ξ^n and approximate the rates of their convergence to ξ^0 as $n \to \infty$. For this purpose, choose the habitat $\Omega = [-3, 3]$, the Laplace kernel $k_t(x, y) := \frac{\delta_t}{2} e^{-\delta_t |x-y|}$ yielding $C^2[-3, 3]$-smoothness in the right-hand side of (4.1) for the almost periodic dispersal rates $\delta_t := 2 + \sin(\frac{t}{3})$, the growth rates $\gamma_t(x) := 3 - \sin(\frac{tx}{5})$ and $n = 4096$. In order to evaluate the remaining integrals in (4.2) we apply the trapezoidal rule and the functions ξ_t^n are approximated as pullback limits $\varphi^n(t; t-s, \xi)$, $t \in \mathbb{I}$, with $s := 15$ and upper solutions ξ. Then the sequence $c(n) := \log_2 \frac{\|\xi^n - \xi^{2n}\|}{\|\xi^{2n} - \xi^{4n}\|}$ approximates the desired convergence rates for ξ^n to ξ^0 given large values of n. This results in the values listed in Tab. 1 which where illustrated in Fig. 3, respectively. Clearly, the rate 2 (cf. Ex. 3.4) is preserved.

4.2. **Forward limit sets for asymptotically autonomous Ricker equations.** Consider the scalar nonautonomous IDE

$$u_{t+1} = \mathcal{F}_t(u_t), \qquad \mathcal{F}_t(u) := \gamma_t \int_\Omega k(\cdot, y) u_t(y) e^{-u_t(y)} \,\mathrm{d}y + b \tag{4.3}$$

with a continuous kernel $k : \Omega \times \Omega \to \mathbb{R}_+$, an inhomogeneity $b \in C_+(\Omega)$ and a bounded sequence of growth rates $(\gamma_t)_{t \in \mathbb{I}}$ in $\mathbb{R}_+$. For the Ricker growth function $g : \mathbb{R}_+ \to \mathbb{R}_+$, $g(z) := ze^{-z}$ it is elementary to show

$$|g(z)| \le \tfrac{1}{e}, \qquad |g(z) - g(\bar{z})| \le \tfrac{1}{e^2} |z - \bar{z}| \quad \text{for all } z, \bar{z} \in \mathbb{R}_+.$$

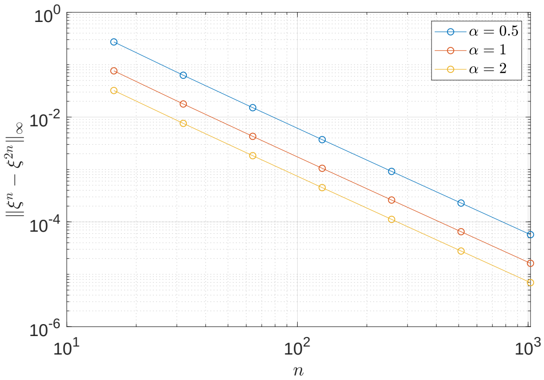

FIGURE 3. Development of the averaged error $\|\xi^n - \xi^{2n}\|$ for $n \in \{2^4, \dots, 2^{10}\}$ nodes and parameters $\alpha \in \{\frac{1}{2}, 1, 2\}$.

Given this, on intervals $\mathbb{I}$ unbounded below it is not difficult to establish that (4.3) has a pullback attractor. However, we are interested in the forward dynamics of (4.3) under discretization. For this purpose, let us suppose that $\mathbb{I}$ is bounded below, but unbounded above. Moreover, it is crucial to assume that (4.3) is asymptotically autonomous in the sense that the coefficient sequence $(\gamma_t)_{t\in\mathbb{I}}$ converges exponentially. In order to be more precise, let us assume that there exist reals $\gamma > 0$ and $\tilde{K}_0, \tilde{K}_1 \geq 1$ such that the following holds with $k_0 := \sup_{x\in\Omega} \int_\Omega k(x, y)\,\mathrm{d}y$:

- $\tilde{K}_0 := \sup_{s\leq t} \prod_{l=s}^{t-1} \frac{\gamma_l}{\gamma} < \infty$ and $\gamma k_0 < 1$,
- $|\gamma_t - \gamma| \leq \tilde{K}_1 (k_0\gamma)^t$, which implies $\lim_{t\to\infty} \gamma_t = \gamma$,
- $k_0 \sup_{t\in\mathbb{I}} \gamma_t < \frac{e^2}{1+e^2} \frac{1-k_0\gamma}{\tilde{K}_0}$.

As a result, it is shown in [18, Exam. 5.6] that the autonomous limit equation

$$u_{t+1} = \mathcal{F}(u_t), \qquad \mathcal{F}(u) := \gamma \int_\Omega k(\cdot, y) u_t(y) e^{-u_t(y)}\,\mathrm{d}y + b \tag{4.4}$$

has a singleton global attractor $A^* = \{u^*\}$ and for any bounded set $A \subseteq C_+(\Omega)$ absorbing for (4.4) the forward limit set of (4.3) arising from $\mathcal{A} := \mathbb{I} \times A$ is just $\omega^+_{\mathcal{A},0} = \{u^*\}$.

In addition, it can be shown using [18, Thm. 4.14] that for every bounded subset $B \subseteq C_+(\Omega)$ one has the limit relation

$$\lim_{s\to\infty} \sup_{b\in B} \left\|\varphi^0(\tau + s; \tau, b) - \mathcal{F}^s(b)\right\| = 0 \quad \text{uniformly in } \tau \in \mathbb{I};$$

for this $\mathbb{I}$ needs to be bounded below. Consequently, passing to the least upper bound over $b \in B$ in the inequality

$$\inf_{a\in B} \left\|\varphi^0(\tau + s; \tau, b) - \mathcal{F}^s(a)\right\| \leq \left\|\varphi^0(\tau + s; \tau, b) - \mathcal{F}^s(b)\right\| \quad \text{for all } b \in B$$

leads to the limit relation

$$0 \leq \operatorname{dist}\left(\varphi^0(\tau + s; \tau, B), \mathcal{F}^s(B)\right) \leq \sup_{b\in B} \left\|\varphi^0(\tau + s; \tau, b) - \mathcal{F}^s(b)\right\| \xrightarrow[s\to\infty]{} 0$$

uniformly in the initial time $\tau \in \mathbb{I}$. Since the global attractor A^* of (4.4) attracts bounded subsets $B \subseteq X$, one therefore arrives at

$$\operatorname{dist}\left(\varphi^0(\tau + s; \tau, B), \omega^+_{\mathcal{A},0}\right)$$

$$\begin{aligned}&\overset{(1.1)}{\leq} \operatorname{dist}\left(\varphi^0(\tau+s;\tau,B),\mathcal{F}^s(B)\right)+\operatorname{dist}\left(\mathcal{F}^s(B),\omega^+_{\mathcal{A},0}\right)\\ &=\operatorname{dist}\left(\varphi^0(\tau+s;\tau,B),\mathcal{F}^s(B)\right)+\operatorname{dist}\left(\mathcal{F}^s(B),A^*\right)\xrightarrow[s\to\infty]{}0 \qquad (4.5)\end{aligned}$$

uniformly in $\tau\in\mathbb{I}$.

Along with the asymptotically autonomous Ricker equation (4.3) we now turn to their convergent collocation discretizations (I_n), $n\in\mathbb{N}$, satisfying $(H_3$–$H_4)$. With the abbreviations

$$R:=\tfrac{pk_0}{e}\sup_{t\in\mathbb{I}}\gamma_t, \qquad b_n:=\pi_n b\in X_n$$

one can show that the nonautonomous set

$$\mathcal{A}:=\Big\{(t,x)\in\mathcal{U}:\ u\in\overline{\bigcup_{n\in\mathbb{N}_0}\mathcal{F}^n_{t^*}(U\cap B_R(b_n))}\Big\},\quad t^*:=\begin{cases}t-1, & t>\min\mathbb{I},\\ t, & t=\min\mathbb{I}\end{cases}$$

is bounded, compact, positively invariant and forward absorbing w.r.t. every (I_n), $n\in\mathbb{N}_0$, with absorption time 2. In particular, each fiber $\mathcal{A}(t)$, $t\in\mathbb{I}$, is compact, since every $\mathcal{F}^n_{t^*}(U\cap B_R(b_n))$ is relatively compact and (I_n), $n\in\mathbb{N}$, is convergent.

The above exponential convergence assumption for $(\gamma_t)_{t\in\mathbb{I}}$ implies that the union $\bigcup_{t\in\mathbb{I}}\mathcal{F}^n_t(U\cap B_R(b_n))$ is relatively compact. Thanks to the Lipschitz estimate

$$\|\mathcal{F}^n_t(u)-\mathcal{F}^n_t(v)\|\leq\tfrac{pk_0}{e^2}\sup_{t\in\mathbb{I}}\gamma_t\,\|u-v\| \quad \text{for all } n\in\mathbb{N}_0,\ t\in\mathbb{I},\ u,v\in U,$$

we can apply [18, Thm. 5.5] in order to obtain that the forward limit sets $\omega^+_{\mathcal{A},n}$ of (I_n), $n\in\mathbb{N}$, are asymptotically negatively invariant (cf. the proof of Thm. 2.7).

Because $K:=\bigcup_{n\in\mathbb{N}_0}K_n$ with $K_n:=\overline{\bigcup_{t\in\mathbb{I}}\mathcal{F}^n_t(U\cap B_R(b_n))}$ is bounded, we obtain from the uniform limit relation (4.5) established above that the assumption in Thm. 3.10(i) is satisfied. Combining this with the asymptotic negative invariance of each $\omega^+_{\mathcal{A},n}$, $n\in\mathbb{N}_0$, we conclude as in Thm. 3.10 that the forward limit sets of the collocation discretizations (I_n), $n\in\mathbb{N}$, fulfill

$$\lim_{n\to\infty}\operatorname{dist}\left(\omega^+_{\mathcal{A},n},\omega^+_{\mathcal{A},0}\right)=\lim_{n\to\infty}\sup_{a\in\omega^+_{\mathcal{A},n}}\|a-u^*\|=0.$$

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