# Chain Rule and Invariance Principle on Measure Chains

Christian Pötzsche

*Institut für Mathematik, Universität Augsburg, D-86135 Augsburg, Germany*

**Abstract**

In this note we prove a chain rule for mappings on abstract measure chains and apply our result to deduce an invariance principle for non-autonomous dynamic equations.

*Key words:* Time Scale, Measure Chain, Chain Rule, Invariance Principle
*1991 MSC:* 26A24, 34D20, 39A12

In the qualitative theory of dynamical systems there are plenty of applications of the chain rule, ranging from Lyapunov's Direct Method and LaSalle's Invariance Principle to invariance equations for integral manifolds. Beyond the case of ordinary differential equations, the so-called "Calculus on Measure Chains" (cf. Hilger [2]) provides additionally a useful insight into the transition between the continuous and the discrete case. In this note we prove a chain rule for mappings defined on measure chains and state LaSalle's Invariance Principle for non-autonomous dynamic equations as an application.

Our chain rule already appeared in the thesis Keller [3, p. 6, Folgerung 1.2.9], but with a proof only valid on time scales, i.e. closed subsets of $\mathbb{R}$. The chain rule stated in Lakshmikantham, Sivasundaram & Kaymakçalan [5, pp. 17–18, Theorem 1.2.3(iv)] is true solely in right-dense points, with the consequence that Sections 3.4, 4.5 and 4.9 of this monograph become questionable. Finally the scope of our result is different from the chain rule recently given in Ahlbrandt, Bohner & Ridenhour [1, Theorem 2.7]. We essentially consider Banach space-valued mappings and Fréchet-derivatives, while Ahlbrandt et al. examine time scale-valued mappings using their new concept of so-called "alpha derivatives."

Throughout this note let $(\mathbb{T}, \preceq, \mu)$ be a measure chain with forward jump

*Email address:* christian.poetzsche@math.uni-augsburg.de (Christian Pötzsche).

operator $\sigma$, graininess $\mu^*$, and $\mathcal{X}, \mathcal{Y}$ denote arbitrary Banach spaces. $\mathcal{L}(\mathcal{X};\mathcal{Y})$ is the space of linear and continuous operators between $\mathcal{X}$ and $\mathcal{Y}$. Since no confusion should arise, we always write $\|\cdot\|$ for the norms on this spaces. To introduce partial derivatives of a mapping $f : \mathbb{T}\times\mathcal{X} \to \mathcal{Y}$, for $x_0 \in \mathcal{X}$ fixed, we denote the delta derivative of $t \mapsto f(t, x_0)$ by $\Delta_1 f(\cdot, x_0)$, and for fixed $t_0 \in \mathbb{T}$, we denote the Fréchet-derivative of $x \mapsto f(t_0, x)$ by $D_2 f(t_0, \cdot)$, provided the derivatives exist.

**Theorem 1 (Chain Rule):** *For some fixed $t_0 \in \mathbb{T}^\kappa$, let $g : \mathbb{T} \to \mathcal{X}$, $f : \mathbb{T} \times \mathcal{X} \to \mathcal{Y}$ be mappings such that $g$, $f(\cdot, g(t_0))$ are differentiable in $t_0$, and let $U \subseteq \mathbb{T}$ be a neighborhood of $t_0$ such that $f(t,\cdot)$ is differentiable for $t \in U \cup \{\sigma(t_0)\}$, $D_2 f(\sigma(t_0),\cdot)$ is continuous on the line segment $\left\{g(t_0) + h\mu^*(t_0)g^\Delta(t_0) \in \mathcal{X} : \, h \in [0,1]\right\}$ and $D_2 f$ is continuous in $(t_0, g(t_0))$. Then also the composition mapping $F : \mathbb{T} \to \mathcal{Y}$, $F(t) := f(t, g(t))$ is differentiable in $t_0$ with derivative*

$$F^\Delta(t_0) = \Delta_1 f(t_0, g(t_0)) + \left[\int_0^1 D_2 f\left(\sigma(t_0), g(t_0) + h\mu^*(t_0)g^\Delta(t_0)\right) dh\right] g^\Delta(t_0).$$

**Remark 2:** (1) In case of a right-dense point $t_0$ we have $\mu^*(t_0) = 0$ and the Chain Rule possesses the expected form $F^\Delta(t_0) = \Delta_1 f(t_0, g(t_0)) + D_2 f(t_0, g(t_0))g^\Delta(t_0)$. This is not true in right-scattered points. To show this, consider the time scale $\mathbb{T} = \mathbb{Z}$, the Banach spaces $\mathcal{X} = \mathcal{Y} = \mathbb{R}$ and the functions $g(t) := t$, $f(x) := x^2$. Here we have $(f \circ g)^\Delta(t_0) = 2t_0 + 1$, but $Df(g(t_0))g^\Delta(t_0) = 2t_0$ for any $t_0 \in \mathbb{T}$.

(2) The Chain Rule stated in Theorem 1 remains true, if the domain of $f(\sigma(t_0),\cdot)$ is an — or the closure of an — open set in $\mathcal{X}$, which contains the line segment $\left\{g(t_0) + h\mu^*(t_0)g^\Delta(t_0) \in \mathcal{X} : \, h \in [0,1]\right\}$.

*Proof.* We arrange the proof in three steps and begin with some elementary preparations:

(I) First of all one can choose a neighborhood $U_0 \subseteq U$ of $t_0$ such that

$$\mu^*(t_0) \le |\mu(t, \sigma(t_0))| \quad \text{for } t \in U_0. \tag{1}$$

This is trivial (with $U_0 = U$) in a right-dense point $t_0$, but it also holds (with $U_0 = \{t \in U : \, t \prec \sigma(t_0)\}$) in a right-scattered $t_0$ by the properties of the growth calibration $\mu$ (cf. Hilger [2, Axiom 3]).

(II) We abbreviate $\Phi(t,h) := D_2 f(t, g(t_0) + h\,[g(t) - g(t_0)])$ and show the existence of a real constant $C = C(t_0) > 0$ with

$$\|\Phi(\sigma(t_0), h) - \Phi(t_0, h)\| \le C\, |\mu(t, \sigma(t_0))| \quad \text{for } t \in U_0, \ h \in [0,1]\,. \tag{2}$$

Again only the case of a right-scattered $t_0$ needs further argumentation: For that purpose, $\Phi(t_0,\cdot)$ has the constant value $D_2 f(t_0, g(t_0)) \in \mathcal{L}(\mathcal{X};\mathcal{Y})$ and the mapping $\Phi(\sigma(t_0),\cdot) : [0,1] \to \mathcal{L}(\mathcal{X};\mathcal{Y})$ is bounded, since by assumption it is a continuous function on the compact domain $[0,1]$. This immediately yields the estimate

$$\|\Phi(\sigma(t_0),h) - \Phi(t_0,h)\| \leq C\mu^*(t_0) \overset{(1)}{\leq} C\,|\mu(t,\sigma(t_0))| \quad \text{for } t \in U_0,\ h \in [0,1]$$

with $C := \frac{1}{\mu^*(t_0)}\left(\sup_{h\in[0,1]} \|\Phi(\sigma(t_0),h)\| + \|D_2 f(t_0, g(t_0))\|\right)$.

(III) During this main step of the proof, remember the well-known identity $g(\sigma(t_0)) - g(t_0) = \mu^*(t_0) g^\Delta(t_0)$ (cf. Hilger [2, Theorem 2.5(v)]). Given $\varepsilon > 0$ arbitrarily, we choose $\varepsilon_1, \varepsilon_2 > 0$ so small that

$$\text{(3)} \qquad \varepsilon_1\left(1 + C + \left\|\int_0^1 \Phi(\sigma(t_0),h)dh\right\|\right) + \varepsilon_2\left(\varepsilon_1 + 2\left\|g^\Delta(t_0)\right\|\right) \leq \varepsilon.$$

Since we assumed the differentiability of $g$ and $f(\cdot, g(t_0))$ in $t_0$, there exists a neighborhood $U_1 \subseteq U_0$ of $t_0$ with

$$\text{(4)} \qquad \|g(t) - g(t_0)\| \leq \varepsilon_1,$$

$$\text{(5)} \qquad \left\|g(t) - g(\sigma(t_0)) - \mu(t,\sigma(t_0)) g^\Delta(t_0)\right\| \leq \varepsilon_1\,|\mu(t,\sigma(t_0))|,$$

$$\text{(6)} \quad \|f(t,g(t_0)) - f(\sigma(t_0),g(t_0)) - \mu(t,\sigma(t_0))\Delta_1 f(t_0,g(t_0))\| \leq \varepsilon_1\,|\mu(t,\sigma(t_0))|$$

for $t \in U_1$, where the first inequality (4) holds, because $g$ is a continuous function in $t_0$ (cf. Hilger [2, Theorem 2.5(iii)]). Consequently by the triangle inequality it is

$$\begin{aligned}
\text{(7)} \quad \|g(t) - g(t_0)\| &\leq \left\|g(t) - g(\sigma(t_0)) - \mu(t,\sigma(t_0)) g^\Delta(t_0)\right\| \\
&\quad + \left\|g^\Delta(t_0)\right\| |\mu(t,\sigma(t_0))| + \|g(\sigma(t_0)) - g(t_0)\| \\
&\overset{(5)}{\leq} \left(\varepsilon_1 + \left\|g^\Delta(t_0)\right\|\right) |\mu(t,\sigma(t_0))| + \left\|g^\Delta(t_0)\right\| \mu^*(t_0) \\
&\overset{(1)}{\leq} \left(\varepsilon_1 + 2\left\|g^\Delta(t_0)\right\|\right) |\mu(t,\sigma(t_0))| \quad \text{for } t \in U_1.
\end{aligned}$$

On the other hand the mappings $g$ and $D_2 f : \mathbb{T}\times\mathcal{X} \to \mathcal{L}(\mathcal{X};\mathcal{Y})$ are continuous in $t_0$ and $(t_0, g(t_0))$, respectively, and hence there exists another neighborhood $U_2 \subseteq U$ of $t_0$ with

$$\text{(8)} \qquad \|\Phi(t,h) - \Phi(t_0,h)\| \leq \varepsilon_2 \quad \text{for } t \in U_2,\ h \in [0,1].$$

After the considerations above, we can deduce the estimate

$$
\begin{aligned}
&\left\| F(t) - F(\sigma(t_0)) - \mu(t,\sigma(t_0)) \left[ \Delta_1 f(t_0, g(t_0)) + \int_0^1 \Phi(\sigma(t_0), h) dh \, g^\Delta(t_0) \right] \right\| \\
&\leq \| f(t, g(t_0)) - f(\sigma(t_0), g(t_0)) - \mu(t,\sigma(t_0)) \Delta_1 f(t_0, g(t_0)) \| \\
&\quad + \left\| \int_0^1 \Phi(\sigma(t_0), h) dh \left[ g(t) - g(t_0) - \mu(t, t_0) g^\Delta(t_0) \right] \right\| \\
&\quad + \Big\| f(t, g(t)) - f(t, g(t_0)) - [f(\sigma(t_0), g(\sigma(t_0))) - f(\sigma(t_0), g(t_0))] \\
&\quad - \int_0^1 \Phi(\sigma(t_0), h) dh \, [g(t) - g(t_0)] \Big\|
\end{aligned}
$$

and the Mean Value Theorem (cf. Lang [6, p. 341, Theorem 4.2]) leads to

$$
\begin{aligned}
&\left\| F(t) - F(\sigma(t_0)) - \mu(t,\sigma(t_0)) \left[ \Delta_1 f(t_0, g(t_0)) + \int_0^1 \Phi(\sigma(t_0), h) dh \, g^\Delta(t_0) \right] \right\| \\
&\leq \| f(t, g(t_0)) - f(\sigma(t_0), g(t_0)) - \mu(t,\sigma(t_0)) \Delta_1 f(t_0, g(t_0)) \| \\
&\quad + \left\| \int_0^1 \Phi(\sigma(t_0), h) dh \right\| \left\| g(t) - g(\sigma(t_0)) - \mu(t, \sigma(t_0)) g^\Delta(t_0) \right\| \\
&\quad + \left\| \int_0^1 \left[ \Phi(t, h) - \Phi(\sigma(t_0), h) \right] dh \, [g(t) - g(t_0)] \right\| \\
&\leq \| f(t, g(t_0)) - f(\sigma(t_0), g(t_0)) - \mu(t,\sigma(t_0)) \Delta_1 f(t_0, g(t_0)) \| \\
&\quad + \left\| \int_0^1 \Phi(\sigma(t_0), h) dh \right\| \left\| g(t) - g(\sigma(t_0)) - \mu(t, \sigma(t_0)) g^\Delta(t_0) \right\| \\
&\quad + \left\| \int_0^1 \left[ \Phi(t, h) - \Phi(t_0, h) \right] dh \right\| \| g(t) - g(t_0) \| \\
&\quad + \left\| \int_0^1 \left[ \Phi(t_0, h) - \Phi(\sigma(t_0), h) \right] dh \right\| \| g(t) - g(t_0) \| .
\end{aligned}
$$

The terms of the sum on the right-hand side of the above inequality now can be estimated using (6), (5), as well as (8), (7) and (2),(4), which gives us

$$
\begin{aligned}
&\left\| F(t) - F(\sigma(t_0)) - \mu(t,\sigma(t_0)) \left[ \Delta_1 f(t_0, g(t_0)) + \int_0^1 \Phi(\sigma(t_0), h) dh \, g^\Delta(t_0) \right] \right\| \\
&\leq \left[ \varepsilon_1 \left( 1 + C + \left\| \int_0^1 \Phi(\sigma(t_0), h) dh \right\| \right) + \varepsilon_2 \left( \varepsilon_1 + 2 \left\| g^\Delta(t_0) \right\| \right) \right] |\mu(t, \sigma(t_0))| \\
&\overset{(3)}{\leq} \varepsilon \, |\mu(t, t_0)|
\end{aligned}
$$

whenever $t \in U_1 \cap U_2$, and by the definition of $\Phi(\sigma(t_0), h)$ this establishes our chain rule. $\square$

As an application of the Chain Rule from Theorem 1 we will prove a version of LaSalle's Invariance Principle (cf. LaSalle [7,8]) for non-autonomous dynamic equations. Henceforth, let $(\mathbb{T}, \preceq, \mu)$ be unbounded above. We follow closely to the considerations in Knobloch & Kappel [4, pp. 137ff] for finite-dimensional ODEs.

First of all, a point $\xi \in \mathcal{X}$ is called an *$\omega$-limit point* of a function $\lambda : \mathbb{T} \to \mathcal{X}$, if there exists a sequence $(t_k)_{k\in\mathbb{N}}$ in $\mathbb{T}$ with the properties $\lim_{k\to\infty} \mu(t_k, t_0) = \infty$ (for one and hence every $t_0 \in \mathbb{T}$) and $\lim_{k\to\infty} \lambda(t_k) = \xi$. The set of all $\omega$-limit points is denoted as *$\omega$-limit set* $\omega(\lambda) \subseteq \mathcal{X}$. Now consider the dynamic equation

$$x^\Delta = f(t, x) \tag{9}$$

with an rd-continuous right-hand side $f : \mathbb{T} \times \mathcal{X} \to \mathcal{X}$, as defined in Hilger [2, Section 5.2]. For convenience, (9) should possess solutions existing in forward time. If $\Omega \subseteq \mathcal{X}$ is an open set, a mapping $V : \mathbb{T} \times \overline{\Omega} \to \mathbb{R}$ with continuous partial derivatives is called a *Lyapunov function* of (9), if the following holds:

($V_1$) For each closed and bounded subset $B \subseteq \mathcal{X}$, the function $V$ is bounded below on the set $\left\{(t, x) \in \mathbb{T} \times \mathcal{X} : x \in B \cap \overline{\Omega}\right\}$,

($V_2$) in each right-scattered point $t \in \mathbb{T}$ and for any $x \in \overline{\Omega}$ the line segment $\{x + h\mu^*(t)f(t, x) \in \mathcal{X} : h \in [0, 1]\}$ is contained in $\overline{\Omega}$,

($V_3$) the so-called *derivative of $V$ with respect to* (9), $V^\Delta : \mathbb{T} \times \Omega \to \mathbb{R}$,

$$V^\Delta(t, x) := \Delta_1 V(t, x) + \left[\int_0^1 D_2 V\left(\sigma(t), x + h\mu^*(t)f(t, x)\right) dh\right] f(t, x)$$

fulfills $V^\Delta(t, x) \leq -W(x)$ for a non-negative and continuous function $W : \overline{\Omega} \to \mathbb{R}$.

This definition is slightly differing from the usual one to shorten our explanations.

**Theorem 3 (Invariance Principle):** *Let $V : \mathbb{T} \times \overline{\Omega} \to \mathbb{R}$ be a Lyapunov function of* (9) *and assume that for every bounded subset $M \subseteq \Omega$ the function $f$ is bounded on $\mathbb{T} \times M$. Then an arbitrary solution $\lambda$ of* (9)*, which exists in $\Omega$ on an interval unbounded to the right, fulfills $\omega(\lambda) \subseteq W^{-1}(\{0\})$.*

*Proof.* W.l.o.g. we assume $\omega(\lambda) \neq \emptyset$ and let us consider a fixed $\omega$-limit point $\xi \in \omega(\lambda)$. We proceed indirectly and suppose $W(\xi) > 0$. Since $W$ is continuous, there exists a closed and bounded neighborhood $N \subseteq \mathcal{X}$ of $\xi$ such that

$$W(x) \geq \frac{W(\xi)}{2} > 0 \quad \text{for } x \in N \cap \overline{\Omega}. \tag{10}$$

Now choose $\varepsilon > 0$ so small that $\bar{B}_{2\varepsilon}(\xi) \subseteq N$. By assumption $f$ is bounded on $\mathbb{T} \times (\Omega \cap N)$ and there exists a real $\gamma > 0$ with $\left\|\lambda^\Delta(t)\right\| = \|f(t, \lambda(t))\| < \gamma$, if $\lambda(t) \in \bar{B}_{2\varepsilon}(\xi)$. Now choose a $t_* \in \mathbb{T}$ such that $\lambda(t_*) \in \bar{B}_\varepsilon(\xi)$ and we obtain

$$\|\lambda(t) - \lambda(t_*)\| < \varepsilon \quad \text{for } 0 \leq \mu(t, t_*) \leq \frac{\varepsilon}{\gamma}, \tag{11}$$

because otherwise there would exist a $t^* \in \mathbb{T}$, $0 \leq \mu(t^*, t_*) \leq \frac{\varepsilon}{\gamma}$ with the properties $\|\lambda(t) - \lambda(t_*)\| < \varepsilon$ for $t_* \preceq t \prec t^*$ and $\|\lambda(t^*) - \lambda(t_*)\| \geq \varepsilon$. Hence the Mean Value Theorem on measure chains (cf. Hilger [2, Corollary 3.3]) would imply the contradiction

$$\varepsilon \leq \|\lambda(t^*) - \lambda(t_*)\| \leq \sup_{t_* \preceq t \prec t^*} \left\|\lambda^\Delta(t)\right\| \mu(t^*, t_*) < \gamma\mu(t^*, t_*) \leq \varepsilon.$$

Concluding we get the estimate

$$\|\lambda(t) - \xi\| \leq \|\lambda(t) - \lambda(t_*)\| + \|\lambda(t_*) - \xi\| \overset{(11)}{\leq} 2\varepsilon \quad \text{for } 0 \leq \mu(t, t_*) \leq \frac{\varepsilon}{\gamma}.$$

Since $\xi \in \omega(\lambda)$ there exists a sequence $(t_k)_{k\in\mathbb{N}}$ in $\mathbb{T}$ with $\lim_{k\to\infty} t_k = \infty$ (in the sense above) and $\lim_{k\to\infty} \lambda(t_k) = \xi$. Now, by passing over to a subsequence, we can assume

$$\lambda(t_k) \in \bar{B}_\varepsilon(\xi), \qquad \frac{\varepsilon}{\gamma} < \mu(t_{k+1}, t_k) \quad \text{for } k \in \mathbb{N} \tag{12}$$

and this implies for the function $v(t) := V(t, \lambda(t))$ by Theorem 1 and $(V_3)$

$$v^\Delta(t) = V^\Delta(t, \lambda(t)) \leq -W(\lambda(t)) \overset{(10)}{\leq} -\frac{W(\xi)}{2} \quad \text{for } 0 \leq \mu(t, t_k) \leq \frac{\varepsilon}{\gamma},\ k \in \mathbb{N}.$$

Because of $v^\Delta(t) \leq 0$, i.e. Lyapunov functions decrease along solutions, this yields

$$v(t_{k+1}) - v(t_k) = \int_{t_k}^{t_{k+1}} v^\Delta(s)\, \Delta s \overset{(12)}{\leq} \int_{0 \leq \mu(s,t_k) \leq \frac{\varepsilon}{\gamma}} v^\Delta(s)\, \Delta s \leq -\frac{W(\xi)}{2}\frac{\varepsilon}{\gamma}$$

for $k \in \mathbb{N}$ and consequently using "telescope summation"

$$v(t_n) = v(t_1) + \sum_{k=1}^{n-1} [v(t_{k+1}) - v(t_k)] \leq v(t_1) - (n-1)\frac{W(\xi)}{2}\frac{\varepsilon}{\gamma} \tag{13}$$

for $n \in \mathbb{N}$. Thus for sufficiently large $n \in \mathbb{N}$ we have $\lambda(t_n) \in N$ and we can make $v(t_n) \in \mathbb{R}$ arbitrarily negative by (13), but since $N$ was closed and bounded, by assumption $(V_1)$ the set $\{V(t_n, \lambda(t_n))\}_{n\in\mathbb{N}}$ is bounded below; this contradiction proves Theorem 3. $\square$

## References


[1] C.D. Ahlbrandt, M. Bohner and J. Ridenhour, Hamiltonian systems on time scales, *Journal of Mathematical Analysis and Applications*, **250** (2000), 561–578.

[2] S. Hilger, Analysis on measure chains — A unified approach to continuous and discrete calculus, *Results in Mathematics*, **18** (1990), 18–56.

[3] S. Keller, *Asymptotisches Verhalten invarianter Faserbündel bei Diskretisierung und Mittelwertbildung im Rahmen der Analysis auf Zeitskalen*, Augsburger mathematisch-naturwissenschaftliche Schriften 36, (Wißner Verlag, Augsburg, 1999).

[4] H.W. Knobloch und F. Kappel, *Gewöhnliche Differentialgleichungen*, (B. G. Teubner, Stuttgart, 1974).

[5] V. Lakshmikantham, S. Sivasundaram and B. Kaymakçalan, *Dynamic Systems on Measure Chains*, Mathematics and Its Applications, Vol. 370, (Kluwer Academic Publishers, Dordrecht-Boston-London, 1996).

[6] S. Lang, *Real and Functional Analysis*, Graduate Texts in Mathematics 142, (Springer-Verlag, Berlin-Heidelberg-New York, 1993).

[7] J.P. LaSalle, Stability theory for ordinary differential equations, *Journal of Differential Equations*, **4** (1968), 57–65.

[8] J.P. LaSalle, Stability theory for difference equations, in: J.K. Hale, ed., *Studies in Ordinary Differential Equations*, Math. Assoc. Amer. Studies in Math., Vol. 14, (Englewood Cliffs, 1977) 1–31.