Reducibility of Linear Dynamic Equations on Measure Chains

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Abstract

The concepts of reducibility and kinematic similarity are of major significance in the theory of stability of linear differential and difference equations. In this paper we generalize some fundamental results on reducibility from the finite-dimensional differential equations context to dynamic equations on measure chains in arbitrary Hilbert spaces. In fact, we derive sufficient conditions for dynamic equations to be kinematically similar to an equation with zero right-hand side or to an equation in Hermitian or block diagonal form.

Key words: Time scale, Measure chain, Linear dynamic equation, Lyapunov transformation, Reducibility, Kinematic similarity
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1 Introduction

The transition operators of linear ordinary differential equations (ODEs) or difference equations (OΔEs) play an important role in the qualitative and quantitative theory of such equations. However, aside from certain examples it is generally difficult to determine transition operators explicitly or to gain some insight into their asymptotic behavior. On the other hand, if a linear system is autonomous or in block diagonal form then the asymptotic behavior of the corresponding transition operator can be determined by means of the spectrum of the coefficient operator or with the aid of equations of lower dimension, respectively. For this reason it is important to know under which conditions a given linear system can be simplified by means of a linear transformation which preserves the qualitative properties of this system.

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In this paper we generalize some results of this kind from the differential equations context to dynamic equations on measure chains. A rough summary of our main results reads as follows:

- only restrictively stable systems can be reduced to a system with zero right-hand side,
- systems possessing a so-called *RS*-decomposition (see Section 4), in particular ODEs, can be transformed into Hermitian form,
- dichotomous systems are reducible to systems in block diagonal form.

For finite-dimensional ordinary differential equations results of this kind have a long tradition which can be traced back to Lyapunov. For differential equations nowadays primary references are Coppel [8] as well as Harris & Miles [11]. In Daleckiĭ & Kreĭn [9] also equations in Hilbert spaces are examined. For results on difference equations we refer to Agarwal [1] and Gohberg, Kaashoek & Kos [10].

The role of kinematic similarity in the theory of structurally stable linear systems has been investigated in Palmer [18,19] (for ODEs) and Kurzweil & Papaschinopoulos [14,15] as well as Aulbach, Van Minh & Zabreiko [5] (for OΔEs).

Studying dynamic equations on measure chains is important from a theoretical point of view (unification of discrete and continuous dynamics), but also for applications like e.g. in discretization theory with varying step-sizes (cf. Keller [13]). As an introduction we recommend the articles Hilger [12], Aulbach & Hilger [3] as well as the monograph Lakshmikantham, Sivasundaram & Kaymakçalan [16].

2 Preliminaries

Throughout this paper we consider an arbitrary measure chain $(\mathbb{T}, \preceq, \mu)$ with graininess μ^* and a complex Hilbert space $\mathcal{X}$ with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$. Even though some of our considerations also make sense in Banach spaces, our main results are valid in the Hilbert space setting only. $\mathcal{L}(\mathcal{X})$ denotes the linear space of continuous endomorphisms of $\mathcal{X}$ equipped with the norm $\|T\| := \sup_{\|x\|=1} \|Tx\|$. The symbol $\mathcal{GL}(\mathcal{X})$ stands for the multiplicative group of bijective mappings in $\mathcal{L}(\mathcal{X})$, its neutral element is denoted by $I_{\mathcal{X}}$. An operator $T^* \in \mathcal{L}(\mathcal{X})$ is called the *adjoint* of $T \in \mathcal{L}(\mathcal{X})$ if the identity $\langle Tx, y \rangle = \langle x, T^*y \rangle$ holds for all $x, y \in \mathcal{X}$. The operators belonging to the additive group $\mathcal{S}(\mathcal{X}) := \{T \in \mathcal{L}(\mathcal{X}) : T = T^*\}$ are called *Hermitian*, and in case we have $\langle x, Tx \rangle > 0$ for all $x \in \mathcal{X} \setminus \{0\}$ the operator $T \in \mathcal{S}(\mathcal{X})$ is called *positive*.

We also introduce some notions which are specific to the calculus on measure chains. Above all, $\mathbb{T}_\tau^+$ is the interval $\{t \in \mathbb{T} : \tau \preceq t\}$ and σ : $\mathbb{T} \to \mathbb{T}$ denotes the forward jump operator. $C_{rd}(\mathbb{T}^\kappa, \mathcal{L}(\mathcal{X}))$ denotes the rd-continuous, $C_{rd}R(\mathbb{T}^\kappa, \mathcal{L}(\mathcal{X}))$ the rd-continuous, regressive and $C_{rd}^1(\mathbb{T}^\kappa, \mathcal{L}(\mathcal{X}))$ the rd-continuously differentiable mappings from $\mathbb{T}^\kappa$ to $\mathcal{L}(\mathcal{X})$. Recall that $C_{rd}R(\mathbb{T}^\kappa, \mathcal{L}(\mathcal{X}))$ forms a group with respect to the addition $(A \oplus B)(t) := A(t) + B(t) + \mu^*(t)A(t)B(t)$, the so-called *regressive group*. The neutral element of this group is 0, the zero-mapping, and the inverse element of A is $(\ominus A)(t) = -A(t)\left[I_\mathcal{X} + \mu^*(t)A(t)\right]^{-1}$ (cf. Hilger [12] or Aulbach & Hilger [3]). The regressive group can be extended to a *regressive module* by introducing the product

$$(k \odot A)(t) := \lim_{h \searrow \mu^*(t)} \frac{1}{h}\left[(I_\mathcal{X} + hA(t))^k - I_\mathcal{X}\right] \quad \text{for all } t \in \mathbb{T}^\kappa \text{ and } k \in \mathbb{Z}.$$

For the regressive module we easily get the following:

- In any right dense point $t \in \mathbb{T}^\kappa$ the limit is well defined and we obtain $(k \odot A)(t) = kA(t)$,
- the product $\odot$ is *consistent* with the addition $\oplus$ on $C_{rd}R(\mathbb{T}^\kappa, \mathcal{L}(\mathcal{X}))$, i.e. for $k \in \mathbb{Z}$ we have

$$0 \odot A = 0, \quad (-1) \odot A = \ominus A, \quad (k+1) \odot A = k \odot A \oplus A,$$

- the product $\odot$ makes $(C_{rd}R(\mathbb{T}^\kappa, \mathcal{L}(\mathcal{X})), \oplus)$ to a left module (generally non-abelian) over the integers $\mathbb{Z}$.

Finally $e_\alpha(t, s)$ denotes the real exponential function on $\mathbb{T}$ (cf. Hilger [12, Section 7]) for any $\alpha \in \mathbb{R}$ which is *positively regressive*, i.e. $1 + \alpha\mu^*(t) > 0$.

3 Kinematic Similarity

We consider a linear dynamic equation

$$x^\Delta = A(t)x \tag{1}$$

with coefficient operator $A \in C_{rd}R(\mathbb{T}^\kappa, \mathcal{L}(\mathcal{X}))$. As known from Hilger [12, Theorem 5.7] or Aulbach & Hilger [3, Theorem 8] all solutions of such an equation exist on the whole measure chain $\mathbb{T}$. We denote the transition operator of (1) by $\Phi_A(t, \tau) \in \mathcal{GL}(\mathcal{X})$, and an arbitrary fundamental operator by $\Psi_A(t)$, i.e. an operator solution of (1) with $\Psi_A(t) = \Phi_A(t, \tau)C$ for some $C \in \mathcal{GL}(\mathcal{X})$. Another linear dynamic equation

$$x^\Delta = B(t)x \tag{2}$$

with (not necessarily regressive) $B \in C_{rd}(\mathbb{T}^\kappa, \mathcal{L}(\mathcal{X}))$ is said to be *kinematically similar* to (1) on an interval $J \subseteq \mathbb{T}$ if there exists a function $\Lambda \in C^1_{rd}(J, \mathcal{GL}(\mathcal{X}))$ with the following properties:

(H_1) $\Lambda(\cdot)$ and $\Lambda(\cdot)^{-1}$ are bounded as functions from J to $\mathcal{L}(\mathcal{X})$,
(H_2) the identity $\Lambda^\Delta(t) \equiv A(t)\Lambda(t) - \Lambda(\sigma(t))B(t)$ holds on J^κ.

A function $\Lambda : J \to \mathcal{GL}(\mathcal{X})$ with these properties is called a *Lyapunov transformation*. It is known (cf. Hilger [12, Theorem 6.4(i)]) that the corresponding linear change of variables $x \mapsto \Lambda(t)^{-1}x$ transforms (2) into (1).

Remarks 3.1 (1) Kinematic similarity defines an equivalence relation on the set of all linear homogeneous dynamic equations in $\mathcal{X}$.

(2) For ODEs in $\mathbb{C}^N$ Söderlind & Mattheij [24, Theorem 6] have shown that every system (1) is kinematically similar to a totally decoupled linear system (diagonal coefficient matrix) if one does not require $\Lambda(\cdot)^{-1}$ to be bounded. The boundedness assumption on $\Lambda(\cdot)^{-1}$, however, is essential since otherwise stability properties may not carry over from (1) to (2).

(3) For difference equations and, more generally, for dynamic equation on discrete measure chains (all points are right and left scattered) the boundedness of the coefficient mappings is preserved under kinematic similarity. That this is not true in the case of measure chains with right dense points can be seen by considering the measure chain $\mathbb{R}$, the Hilbert space $\mathcal{X} = \mathbb{C}^2$ and the Lyapunov transformation $\Lambda(t) := \left(\begin{smallmatrix} \sin(t^2) & \cos(t^2) \\ -\cos(t^2) & \sin(t^2) \end{smallmatrix} \right)$ showing that the two differential equations

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x \,, \qquad \dot{x} = \begin{pmatrix} 1 & 2t \\ -2t & 1 \end{pmatrix} x$$

are kinematically similar. That regressivity, on the other hand, is preserved under kinematic similarity on any measure chain is the content of our first lemma.

Lemma 3.2 *If equations (1) and (2) are kinematically similar on J then the regressivity on J carries over from equation (1) to (2).*

Proof. Using hypothesis (H_2) we see that in every right scattered point $t \in J^\kappa$ we have the identity $I_\mathcal{X} + \mu^*(t)B(t) = \Lambda(\sigma(t))^{-1} \left[I_\mathcal{X} + \mu^*(t)A(t) \right] \Lambda(t)$. Therefore also the linear operator $I_\mathcal{X} + \mu^*(t)B(t) \in \mathcal{L}(\mathcal{X})$ is a bijection on the space $\mathcal{X}$. ∎

Variants of the following lemma can already be found in Hilger [12, Theorem 6.2(iv)].

Lemma 3.3 *Equations (1) and (2) are kinematically similar on J if and only if there exists a function $\Lambda \in C^1_{rd}(J, \mathcal{GL}(\mathcal{X}))$ such that in addition to (H_1) one of the following conditions holds:*

(a) *hypothesis (H_2) is true,*
(b) *for every solution $\nu_1 : J \to \mathcal{X}$ of (1) the function $\nu_2(t) := \Lambda(t)^{-1}\nu_1(t)$ is a solution of (2), and for every solution $\nu_2 : J \to \mathcal{X}$ of (2) the function $\nu_1(t) := \Lambda(t)\nu_2(t)$ is a solution (1),*
(c) $\Phi_A(t,s)\Lambda(s) = \Lambda(t)\Phi_B(t,s)$ *for all $s,t \in J$,*
(d) *there exist fundamental operators $\Psi_A(t)$ of equation (1) and $\Psi_B(t)$ of equation (2) such that $\Psi_A(t) = \Lambda(t)\Psi_B(t)$ for all $t \in J$.*

Proof. The identities for solutions which have to be verified are easily shown to be valid by using the product and quotient rule from Hilger [12, Theorem 2.6]. ∎

While hypothesis (H_2) essentially states that kinematically similar systems can be transferred into each other by a linear transformation, assumption (H_1) guarantees that certain stability properties are preserved. In order to demonstrate this for a quite general type of conditional stability we choose an arbitrary interval $J \subseteq \mathbb{T}$ and introduce two functions

$$a : \{(t,s) \in J^2 : s \preceq t\} \to (0,\infty), \quad b : \{(t,s) \in J^2 : t \preceq s\} \to (0,\infty)$$

which satisfy the identities $a(t,t) \equiv b(t,t) \equiv 1$ and are rd-continuous in their second argument. We say that equation (1) possesses a *dichotomy* (with a, b, K_1, K_2 and P) on J if there exist functions a, b as above, real constants $K_1, K_2 \geq 1$ and a projection $P \in \mathcal{L}(\mathcal{X})$ such that for some fundamental operator $\Psi_A(t)$ of (1) we have

$$\|\Psi_A(t)P\Psi_A(s)^{-1}\| \leq K_1 a(t,s) \quad \text{for all } s \preceq t,\ s,t \in J,$$
$$\|\Psi_A(t)\left[I_\mathcal{X} - P\right]\Psi_A(s)^{-1}\| \leq K_2 b(t,s) \quad \text{for all } t \preceq s,\ s,t \in J.$$

Even though this very general definition is appropriate for our purposes it leads to relevant and applicable results only in the following special cases:

- Uniform stability ($P = I_\mathcal{X}$, $a(t,s) :\equiv 1$) and uniform asymptotic stability ($P = I_\mathcal{X}$, $a(t,s) := e_\alpha(t,s)$ with $\alpha < 0$) as discussed in Coppel [8, pp. 1–2] (ODEs) and in Agarwal [1, pp. 245-246, Theorem 5.5.1] (OΔEs).
- Ordinary dichotomies ($a(t,s) :\equiv b(t,s) :\equiv 1$) as considered in Coppel [8, p. 10] (ODEs), in Agarwal [1, p. 265] (OΔEs) and in Bohner & Lutz [6, Theorem 3.1] for dynamic equations on time scales.
- Exponential dichotomies ($a(t,s) := e_\alpha(t,s)$, $b(t,s) := e_\beta(t,s)$ with $\alpha < 0 < \beta$) as treated in Coppel [8, p. 10] (ODEs) and in Agarwal [1, p. 264] (OΔEs).

- (h, k)-Dichotomies $(a(t, s) = \frac{h(t)}{h(s)}, b(t, s) = \frac{k(t)}{k(s)}$ with positive real functions $h, k)$ which have been introduced in Pinto [21] (ODEs) and in Pinto [20] (OΔEs).

Theorem 3.4 *If equation (1) possesses a dichotomy with a, b, K_1, K_2, P and if it is kinematically similar to (2) on an interval $J \subseteq \mathbb{T}$, then also system (2) possesses a dichotomy on J with a, b,*

$$L_1 := K_1 \sup_{t \in J} \left(\|\Lambda(t)\| \, \|\Lambda(t)^{-1}\| \right), \quad L_2 := K_2 \sup_{t \in J} \left(\|\Lambda(t)\| \, \|\Lambda(t)^{-1}\| \right)$$

and a projection $Q \in \mathcal{L}(\mathcal{X})$ which is similar to $P \in \mathcal{L}(\mathcal{X})$.

Proof. Since equations (1) and (2) are kinematically similar, using Lemma 3.3(d) we get a relation of the form $\Psi_B(t) = \Lambda(t)^{-1} \Psi_A(t) C$ where C is an element of $\mathcal{GL}(\mathcal{X})$ and $\Psi_A(t)$ is the fundamental operator of equation (1) describing the dichotomy of this equation. Consequently, for the projection $Q := C^{-1} P C$ we obtain the estimate

$$\|\Psi_B(t) Q \Psi_B(s)^{-1}\| \leq \|\Lambda(t)^{-1} \Psi_A(t) C Q C^{-1} \Psi_A(s)^{-1} \Lambda(s)\| \leq L_1 a(t, s).$$

for all $s \preceq t$. The second dichotomy inequality follows accordingly. ∎

Our next result roughly states that the notion of kinematic similarity is robust in the sense that in any neighborhood of each dichotomous system there exists at least one more equation which is kinematically similar to the given one.

Theorem 3.5 *Let equation (1) possess a dichotomy with a, b, K_1, K_2 on an interval J. Furthermore consider a mapping $B \in C_{rd}(J, \mathcal{L}(\mathcal{X}))$ such that*

$$\gamma(a, b) := K_1 \sup_{t \in J} \int_{\inf J}^{t} a(t, \sigma(s)) \, \|B(s) - A(s)\| \, \Delta s +$$

$$+ K_2 \sup_{t \in J} \int_{t}^{\sup J} b(t, \sigma(s)) \, \|B(s) - A(s)\| \, \Delta s < 1. \tag{3}$$

Then there exists a mapping $C \in C_{rd}^1(J, \mathcal{L}(\mathcal{X}))$ with the following properties:

(a) C is globally bounded, more precisely, $\|C(t)\| \leq \frac{\gamma(a,b)}{1 - \gamma(a,b)}$ for all $t \in J$,
(b) equation (1) is kinematically similar on J to the equation $x^\Delta = [I_{\mathcal{X}} + C(t)] B(t) x$.

Remark 3.6 If equation (1) possesses an ordinary dichotomy on $\mathbb{T}$ then one can choose $\gamma(a, b) := \max\{K_1, K_2\} \int_J \|B(s) - A(s)\| \, \Delta s$ in condition (3),

i.e. A and B have to be L^1-close. For an exponentially dichotomous system (1) (i.e. $a(t,s) = e_\alpha(t,s), b(t,s) = e_\beta(t,s)$ with positively regressive reals $\alpha < 0 < \beta$) we can take $\gamma(\alpha, \beta) := \left(\frac{K_2}{\beta} - \frac{K_1}{\alpha} \right) \sup_{t \in J} \|B(t) - A(t)\|$ for $J = \mathbb{T} = \mathbb{R}$, or

$$\gamma(\alpha, \beta) := \frac{1}{h} \left(\frac{K_1}{(1 + \alpha h)^{\frac{1}{h}}} + \frac{K_2}{(1 + \beta h)^{\frac{1}{h}}} \right) \sup_{t \in J} \|B(t) - A(t)\|$$

for $J = \mathbb{T} = h\mathbb{Z}$, $h > 0$, and require the estimate (3) to be fulfilled, i.e. A and B have to be L^∞-close.

Proof. First of all we consider the operator-valued function $S : J \to \mathcal{L}(\mathcal{X})$,

$$S(t) := \int_{\inf J}^{t} \Psi_A(t) P \Psi_A(\sigma(s))^{-1} \left[B(s) - A(s) \right] \Delta s -$$

$$- \int_{t}^{\sup J} \Psi_A(t) \left[I_{\mathcal{X}} - P \right] \Psi_A(\sigma(s))^{-1} \left[B(s) - A(s) \right] \Delta s,$$

where $P \in \mathcal{L}(\mathcal{X})$ is the projection corresponding to the dichotomy of (1). Then the assumption (3) immediately yields

$$\|S(t)\| \leq \gamma(a, b) < 1 \quad \text{for all } t \in J. \tag{4}$$

Consequently S is well-defined and S is in fact a solution of the dynamic operator equation $X^\Delta = A(t)X + B(t) - A(t)$. This can be seen along the lines of Bohner & Lutz [6, Lemma 3.3]. Hence the function $\Lambda : J \to \mathcal{L}(\mathcal{X})$ defined as $\Lambda(t) := I_{\mathcal{X}} - S(t)$ is rd-continuously differentiable, it is a bounded solution of $X^\Delta = A(t)X - B(t)$ and by the Neumann series (cf. Lang [17, p. 74, Theorem 2.1]) together with (4) it follows that $\Lambda(t)$ belongs to $\mathcal{GL}(\mathcal{X})$ for each $t \in J$. This proves the kinematic similarity of equations (1) and $x^\Delta = \Lambda(\sigma(t))^{-1} B(t) x$. Furthermore, the inverse operator of $\Lambda(\sigma(t))$ is given by the Neumann series

$$\Lambda(\sigma(t))^{-1} = [I_{\mathcal{X}} - S(\sigma(t))]^{-1} = \sum_{k=0}^{\infty} S(\sigma(t))^k \quad \text{for all } t \in J.$$

Now defining $C : J \to \mathcal{L}(\mathcal{X})$ by $C(t) := \sum_{k=1}^{\infty} S(\sigma(t))^k$ the assertions (a) and (b) follow easily. ∎

4   Reducibility

Since kinematic similarity is an equivalence relation we aim at a classification or at least at a description of those equivalence classes which have a particularly "simple" representative. In this context the *reducible* equations play a prominent role where the term *reducibility* generalizes the corresponding notion introduced by Lyapunov who called a linear differential equation reducible if it is kinematically similar to an autonomous system.

We start with a preparatory result.

Lemma 4.1 *The transition operators $\Phi_A(t, \tau)$ of equation (1) and $\Phi_A^*(t, \tau)$ of the* adjoint equation

$$x^\Delta = (\ominus A)(t)^* x \tag{5}$$

are related by the identity $\Phi_A^*(t, \tau) = \Phi_A(\tau, t)^*$ which holds true for all $t, \tau \in \mathbb{T}$.

Proof. See Hilger [12, Theorem 6.2(ix)]. ∎

Our first result on reducibility generalizes the corresponding result for ODEs due to Coppel [7] and the one for OΔEs due to Agarwal [1, p. 249, Theorem 5.5.3].

Theorem 4.2 *For each fixed $\tau \in \mathbb{T}$ the following statements are equivalent:*

(a) equation (1) is stable together with its adjoint equation (5),
(b) there exists a real constant $K \geq 1$ such that

$$\|\Phi_A(t, \tau)\| \leq K \quad and \quad \|\Phi_A(\tau, t)\| \leq K \quad for \; all \; t \in \mathbb{T}_\tau^+,$$

(c) equation (1) is kinematically similar to $x^\Delta = 0$ on $\mathbb{T}_\tau^+$.

Remark 4.3 Linear systems of the form (1) having property (a) are called *restrictively stable* (cf. Agarwal [1, p. 248, Definition 5.5.1]). They are obviously identical with the *strongly stable* linear equations (cf. Agarwal [1, pp. 245–246, Theorem 5.5.1(iii)]) which are characterized by statement (b).

Proof. Let some time $\tau \in \mathbb{T}$ be fixed.

$(a) \Rightarrow (b)$ Since equation (1) is stable its trivial solution is stable and thus there exists a $\delta > 0$ such that $\|\Phi_A(t, \tau)\xi\| \leq 1$ for all $t \in \mathbb{T}_\tau^+$ and all $\xi \in \mathcal{X}$ with $\|\xi\| \leq \frac{\delta}{2}$. This immediately implies $\|\Phi_A(t, \tau)\| \leq \frac{2}{\delta}$ for all $t \in \mathbb{T}_\tau^+$. Because the adjoint equation (5) is stable as well, we obtain the existence of a $\delta^* > 0$ with $\|\Phi_A^*(t, \tau)\| \leq \frac{2}{\delta^*}$ for all $t \in \mathbb{T}_\tau^+$. Applying Lemma 4.1 we then get $\|\Phi_A(t, \tau)\| =$

$\|\Phi_A(\tau,t)^*\| = \|\Phi_A(t,\tau)\| \leq \frac{2}{\delta^*}$ for all $t \in \mathbb{T}_\tau^+$ and putting $K := \max\left\{\frac{2}{\delta}, \frac{2}{\delta^*}\right\}$ leads to assertion (b).

$(b) \Rightarrow (c)$ The mapping $\Lambda : \mathbb{T}_\tau^+ \to \mathcal{GL}(\mathcal{X})$, $\Lambda(t) := \Phi_A(t,\tau)$ is rd-continuously differentiable and satisfies condition (H_1) by hypothesis (b). Due to (H_2) the coefficient mapping of the particular equation which is kinematically similar to (1) by virtue of Λ has the form

$$B(t) = \Lambda(\sigma(t))^{-1}\left[A(t)\Lambda(t) - \Lambda^\Delta(t)\right] = 0 \quad \text{for all } t \in (\mathbb{T}_\tau^+)^\kappa.$$

$(c) \Rightarrow (b)$ If equation (1) is kinematically similar on $\mathbb{T}_\tau^+$ to the trivial equation $x^\Delta = 0$ then there exists a mapping $\Lambda \in C_{rd}^1(\mathbb{T}_\tau^+, \mathcal{GL}(\mathcal{X}))$ with the properties (H_1) and (due to (H_2)) $\Lambda^\Delta(t) \equiv A(t)\Lambda(t)$ on $(\mathbb{T}_\tau^+)^\kappa$. Consequently the Lyapunov transformation Λ is a fundamental operator of (1) which is bounded on $\mathbb{T}_\tau^+$ together with its inverse $\Lambda(\cdot)^{-1}$. Because of the relation $\Phi_A(t,\tau) = \Lambda(t)\Lambda(\tau)^{-1}$ statement (b) follows.

$(b) \Rightarrow (a)$ This implication easily follows from Lemma 4.1. ∎

Lemma 4.4 *Suppose the coefficient mapping $A(t) \in \mathcal{L}(\mathcal{X})$ is generalized skew-Hermitian, i.e. $A(t)^* = (\ominus A)(t)$ for all $t \in \mathbb{T}^\kappa$. Then the transition operator $\Phi_A(t,s)$ of (1) is unitary for all $s, t \in \mathbb{T}$.*

Proof. Since by assumption equation (1) coincides with its adjoint, from Lemma 4.1 we get

$$\Phi_A(t,s)\Phi_A(t,s)^* = \Phi_A(t,s)\Phi_A^*(s,t) = \Phi_A(t,s)\Phi_A(s,t) = I_\mathcal{X} =$$
$$= \Phi_A(s,t)\Phi_A(t,s) = \Phi_A^*(s,t)\Phi_A(t,s) = \Phi_A(t,s)^*\Phi_A(t,s)$$

for all $s, t \in \mathbb{T}$. This had to be proved. ∎

Equation (1) is said to possess an *RS-decomposition* if the representation $A = R \oplus S$ holds true with mappings $R, S \in C_{rd}R(\mathbb{T}^\kappa, \mathcal{L}(\mathcal{X}))$ where $R(t)$ is Hermitian and $S(t)$ is generalized skew-Hermitian on $\mathbb{T}^\kappa$. It is easy to verify that the validity of the two equations

$$A \oplus A^* = 2 \odot R, \qquad \ominus A^* \oplus A = 2 \odot S \tag{6}$$

is necessary for A to be *RS*-decomposable, and that in turn the relations (6) immediately yield the relations $2 \odot A = 2 \odot (R \oplus S)$ and $(-2) \odot A^* = 2 \odot (S \ominus R)$.

Remark 4.5 If the measure chain $(\mathbb{T}, \preceq, \mu)$ contains only right dense points then every mapping $A \in C_{rd}R(\mathbb{T}^\kappa, \mathcal{L}(\mathcal{X}))$ possesses an *RS*-decomposition. In fact, in this case one can choose $R(t) = \frac{1}{2}(A(t) + A(t)^*)$ and $S(t) =$

$\frac{1}{2}\left(A(t) - A(t)^*\right)$. In the right scattered case $\mu^*(t) > 0$, on the other hand, one has to solve the two non-linear operator equations (6) point-wise.

For ordinary differential equations the next result can be found in Daleckiĭ & Kreĭn [9, p. 160, Lemma 2.2].

Theorem 4.6 *If equation (1) possesses an RS-decomposition on an interval $J \subseteq \mathbb{T}$ then it is kinematically similar on J to a system of the form*

$$x^\Delta = U(\sigma(t))^* R(t) U(\sigma(t)) x \tag{7}$$

where the coefficient mapping is Hermitian and $U(t) \in \mathcal{L}(\mathcal{X})$ is a unitary fundamental operator of $x^\Delta = S(t)x$.

Proof. Let some time $\tau \in J$ be fixed. Then the fundamental operator $U(t) := \Phi_S(t, \tau)$ of $x^\Delta = S(t)x$ is unitary by Lemma 4.4, hence U and $U(\cdot)^{-1}$ are norm-wise bounded above by 1. The operator $U : J \to \mathcal{GL}(\mathcal{X})$ satisfies hypothesis (H_1) and using U as a Lyapunov transformation applied to (1) we obtain

$$U(\sigma(t))^{-1}\left[A(t)U(t) - U^\Delta(t)\right] \equiv U(\sigma(t))^{-1}\left[(R \oplus S)(t) - S(t)\right]U(t) \equiv$$
$$\equiv U(\sigma(t))^{-1}R(t)\left[I_\mathcal{X} + \mu^*(t)S(t)\right]U(t) \equiv$$
$$\equiv U(\sigma(t))^* R(t) U(\sigma(t)) \quad \text{on } J^\kappa.$$

In the last identity the relation $U(t)^{-1} = U(t)^*$ has been used which also implies that the right hand side of (7) is Hermitian. ∎

For the following abstract lemma we give an ad-hoc proof which does not use any sophisticated result from operator theory. An alternative proof using tools from spectral theory and contour integrals is suggested in Daleckiĭ & Kreĭn [9, p. 63, Exercise 27].

Lemma 4.7 *Let $J \subseteq \mathbb{T}$ be an arbitrary interval and let $\Gamma \in C^1_{rd}(J, \mathcal{S}(\mathcal{X}))$ be a mapping with the property that for every $t \in J$ there exists a real $\gamma(t) > 0$ such that*

$$\langle \Gamma(t)x, x \rangle \geq \gamma(t) \|x\|^2 \quad \text{for all } x \in \mathcal{X}.$$

Then there exists a unique function $\Theta \in C^1_{rd}(J, \mathcal{S}(\mathcal{X}))$ with the following properties:

(a) $\Theta(t)^2 \equiv \Gamma(t)$ on J,
(b) $\Theta(t)$ is positive for all $t \in J$.

Proof. First of all we fix an arbitrary $t_0 \in J$ and define a non-linear mapping $s : \mathcal{L}(\mathcal{X}) \rightarrow \mathcal{L}(\mathcal{X})$ by $s(T) := T^2$. This mapping is differentiable and its derivative $Ds : \mathcal{L}(\mathcal{X}) \rightarrow \mathcal{L}(\mathcal{L}(\mathcal{X}))$,

$$(Ds)(T)X = XT + TX \tag{8}$$

is continuous. Now, using Lang [17, p. 446, Theorem 4.3], there exists a unique square root of the operator $\Gamma(t_0) \in \mathcal{S}(\mathcal{X})$. To be more precise, there exists a positive operator $\Theta(t_0) \in \mathcal{S}(\mathcal{X})$ such that $s(\Theta(t_0)) = \Gamma(t_0)$. Because of the positivity of $\Theta(t_0)$ the point 0 is not contained in the spectrum of $\Theta(t_0)$ and hence the Sylvester equation $X\Theta(t_0) + \Theta(t_0)X = Y$ has exactly one solution $X \in \mathcal{L}(\mathcal{X})$ for each $Y \in \mathcal{L}(\mathcal{X})$ (cf. Daleckiĭ & Kreĭn [9, p. 23, Theorem 3.2]). Using relation (8) we obtain the inclusion $(Ds)(\Theta(t_0)) \in \mathcal{GL}(\mathcal{L}(\mathcal{X}))$. Because of the inverse function theorem (see Lang [17, p. 361, Theorem 1.2]) the mapping s is a local C^1-diffeomorphism near $\Theta(t_0) \in \mathcal{S}(\mathcal{X})$ and thus we get $s^{-1}(\Gamma(t_0)) = \Theta(t_0)$; in particular s^{-1} is defined on a ball $B_\rho(\Gamma(t_0)) \subseteq \mathcal{X}$ for some $\rho = \rho(t_0) > 0$.

It remains to be shown that the mapping $\Theta : J \rightarrow \mathcal{L}(\mathcal{X})$ is rd-continuously differentiable. In right dense points $t_0 \in J^\kappa$ there exists a neighborhood U of t_0 such that $\|\Gamma(t) - \Gamma(t_0)\| < \frac{\rho}{2}$ and $\mu^*(t) \left\|\Gamma^\Delta(t)\right\| < \frac{\rho}{2}$ for $t \in U$, since μ^*, Γ, Γ^Δ are continuous in t_0. This yields $\Gamma(t) + h\mu^*(t)\Gamma^\Delta(t) \in B_\rho(\Gamma(t_0))$ for $h \in [0, 1]$, $t \in U$ and by the chain rule (cf. Pötzsche [23, Theorem 1]) one obtains

$$\Theta^\Delta(t) = \int_0^1 (Ds^{-1})(\Gamma(t) + h\mu^*(t)\Gamma^\Delta(t))\, dh\Gamma^\Delta(t) \quad \text{for all } t \in U, \tag{9}$$

since $B_\rho(\Gamma(t_0))$ is convex. Now $\Theta^\Delta(t)$ is the product of rd-continuous functions and therefore continuous in t_0, with the aid of Hilger [12, Theorem 4.1(ii)]. The arguments in the case of a left dense, right scattered $t_0 \in J$ are similar. Here one has to work with a one-sided neighborhood $U \subseteq \{t \in \mathbb{T} : t \prec t_0\}$ of t_0 and equation (9) in order to prove the existence of $\lim_{t \nearrow t_0} \Theta^\Delta(t)$. ∎

For finite-dimensional spaces the next lemma can be found in Coppel [8, p. 39, Lemma 1] or in Harris & Miles [11, p. 215, Lemma A2.1]. Our version in Hilbert spaces is based on Daleckiĭ & Kreĭn [9, p. 154, Theorem 1.2].

Lemma 4.8 *Consider an interval $J \subseteq \mathbb{T}$, mappings $\Psi \in C^1_{rd}(J, \mathcal{GL}(\mathcal{X}))$, $T \in \mathcal{GL}(\mathcal{X})$ and a projection $P \in \mathcal{L}(\mathcal{X})$ with the property $(TPT^{-1})^* = TPT^{-1}$. Then there exists a function $\Lambda \in C^1_{rd}(J, \mathcal{GL}(\mathcal{X}))$ with the following properties:*

(a) $\Lambda(t)P\Lambda(t)^{-1} \equiv \Psi(t)P\Psi(t)^{-1}$ on J,
(b) $\|\Lambda(t)\| \leq \sqrt{2}\,\|T\|$ for all $t \in J$,

(c) $\|\Lambda(t)^{-1}\| \leq \sqrt{\|\Psi(t)P\Psi(t)^{-1}\|^2 + \|\Psi(t)[I_\mathcal{X} - P]\Psi(t)^{-1}\|^2} \|T^{-1}\|$ *for all* $t \in J$.

Remarks 4.9 (1) By choosing an appropriate inner product on $\mathcal{X}$ which is equivalent to the given one we can always assume that the projection P is orthogonal and that consequently $T = I_\mathcal{X}$ (cf. Daleckiĭ & Kreĭn [9, pp. 45ff]).

(2) In finite-dimensional Hilbert spaces a proof of the estimate

$$\max\left\{\|T\|, \|T^{-1}\|\right\} \leq 1 + \|P\|$$

can be found in Gohberg, Kaashoek & Kos [10, Lemma 2.2].

Proof. The proof is divided in two parts.

(I) Referring to Daleckiĭ & Kreĭn [9, p. 154, Theorem 1.2] for details we only sketch the first part of the proof where we suppose to have $P = P^*$ and $T = I_\mathcal{X}$. In this case we define the mapping $\Gamma : J \to \mathcal{L}(\mathcal{X})$,

$$\Gamma(t) := P\Psi(t)^*\Psi(t)P + [I_\mathcal{X} - P]\Psi(t)^*\Psi(t)[I_\mathcal{X} - P] \tag{10}$$

which is Hermitian and uniformly positive since we have

$$\langle \Gamma(t)x, x \rangle \geq \frac{1}{\|\Psi(t)^{-1}\|^2}\|x\|^2 \quad \text{for all } x \in \mathcal{X}.$$

This is due to the Theorem of Pythagoras (cf. Lang [17, p. 98]) and the fact that P is orthogonal. Because of Lemma 4.7 there exists a uniquely determined positive operator $\Theta(t) \in \mathcal{S}(\mathcal{X})$ for each $t \in J$ with $\Theta(t)^2 \equiv \Gamma(t)$ and $\Theta \in C^1_{rd}(J, \mathcal{L}(\mathcal{X}))$. Thus the function $\tilde{\Lambda} : J \to \mathcal{L}(\mathcal{X})$, $\tilde{\Lambda}(t) := \Psi(t)\Theta(t)^{-1}$ possesses the claimed properties.

(II) For arbitrary projections $P \in \mathcal{L}(\mathcal{X})$ we obtain the assertions of Lemma 4.8 by applying the above arguments to the function $\Psi(t)T^{-1}$ and the orthogonal projection TPT^{-1}. Then one can choose $\Lambda(t) := \tilde{\Lambda}(t)T$ as Lyapunov transformation. ∎

Theorem 4.10 *Suppose we are given a mapping $T \in \mathcal{GL}(\mathcal{X})$ and a projection $P \in \mathcal{L}(\mathcal{X})$ such that $(TPT^{-1})^* = TPT^{-1}$. Then if there exists an interval $J \subseteq \mathbb{T}$, real constants $K_1, K_2 \geq 1$ and a fundamental operator $\Psi_A(t)$ of (1) with*

$$\|\Psi_A(t)P\Psi_A(t)^{-1}\| \leq K_1, \qquad \|\Psi_A(t)[I_\mathcal{X} - P]\Psi_A(t)^{-1}\| \leq K_2$$

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for all $t \in J$, then equation (1) is kinematically similar on J to a linear dynamic equation (2) with the following properties:

(a) $B(t)P \equiv PB(t)$ on J^κ,
(b) for the corresponding Lyapunov transformation $\Lambda \in C_{rd}^1(J, \mathcal{GL}(\mathcal{X}))$ we have

$$\|\Lambda(t)\| \le \sqrt{2} \, \|T\|\,, \quad \|\Lambda(t)^{-1}\| \le \sqrt{K_1^2 + K_2^2} \, \|T^{-1}\| \quad \text{for all } t \in J,$$

(c) $\|((T^*B^* (T^*)^{-1}) \oplus (TBT^{-1}))(t)\| \le \|(A^* \oplus A)(t)\| \quad \text{for all } t \in J^\kappa.$

Remark 4.11 If equation (1) is autonomous then the kinematically similar system (2) provided by Theorem 4.10 does not have to be autonomous. Neither does periodicity of A automatically lead to a periodic Lyapunov transformation Λ or a periodic coefficient mapping B.

Proof. We arrange the proof in three steps:

(I) We first apply Lemma 4.8 to the fundamental operator $\Psi_A(t) \in \mathcal{GL}(\mathcal{X})$ and obtain a mapping $\Lambda \in C_{rd}^1(J, \mathcal{GL}(\mathcal{X}))$ with the following properties:

$$\Lambda(t)P\Lambda(t)^{-1} \equiv \Psi_A(t)P\Psi_A(t)^{-1} \quad \text{on } J,$$
$$\|\Lambda(t)\| \le \sqrt{2} \, \|T\| \quad \text{for all } t \in J,$$
$$\|\Lambda(t)^{-1}\| \le \sqrt{\|\Psi_A(t)P\Psi_A(t)^{-1}\|^2 + \|\Psi_A(t)[I_\mathcal{X} - P]\Psi_A(t)^{-1}\|^2} \, \|T^{-1}\|$$
$$\le \sqrt{K_1^2 + K_2^2} \, \|T^{-1}\| \quad \text{for all } t \in J.$$

Hence the assertion (b) is fulfilled.

(II) Until further notice let P be orthogonal and hence $T = I_\mathcal{X}$. Using the notation from the proof of Lemma 4.8, differentiating the identity $\Psi_A(t) \equiv \tilde{\Lambda}(t)\Theta(t)$ and applying the product rule (cf. Hilger [12, Theorem 2.6(ii)]) we get the identity $A(t)\Psi_A(t) \equiv \Psi_A^\Delta(t) \equiv \tilde{\Lambda}(\sigma(t))\Theta^\Delta(t) + \tilde{\Lambda}^\Delta(t)\Theta(t)$ on J^κ. Thus the coefficient mapping of the particular equation which is kinematically similar to (1) by means of $\tilde{\Lambda}$ has the form

$$B(t) = \tilde{\Lambda}(\sigma(t))^{-1}[A(t)\tilde{\Lambda}(t) - \tilde{\Lambda}^\Delta(t)] =$$
$$= \tilde{\Lambda}(\sigma(t))^{-1}[A(t)\tilde{\Lambda}(t) - A(t)\Psi_A(t)\Theta(t)^{-1} + \tilde{\Lambda}(\sigma(t))\Theta^\Delta(t)\Theta(t)^{-1}] =$$
$$= \Theta^\Delta(t)\Theta(t)^{-1} \quad \text{for all } J^\kappa.$$

Hence $\Theta(t) \in \mathcal{L}(\mathcal{X})$ is a fundamental operator of (2). ¿From relation (10) we conclude that $P\Gamma(t) \equiv \Gamma(t)P$, and consequently the two operators $\Theta(t)$ and $\Theta(t)^{-1}$ commute with P. Differentiating the identity $P\Theta(t) \equiv \Theta(t)P$ we

obtain

$$PB(t) \equiv P\Theta^\Delta(t)\Theta(t)^{-1} \equiv \Theta^\Delta(t)P\Theta(t)^{-1} \equiv \Theta^\Delta(t)\Theta(t)^{-1}P \equiv B(t)P$$

on J^κ. Thus (a) is verified and only (c) remains to be proved. To this end we derive from (10) the identity

$$\Theta(t)^2 \equiv \sum_{k=1}^{2} P_k \Psi_A(t)^* \Psi_A(t) P_k \quad \text{on } J,$$

where we use the abbreviations $P_1 := P$ and $P_2 := I_{\mathcal{X}} - P$. Differentiation of this identity and application of the product rule (cf. Hilger [12, Theorem 2.6(ii)]) leads to

$$\Theta(\sigma(t))\Theta^\Delta(t) + \Theta^\Delta(t)\Theta(t) \equiv$$
$$\equiv \sum_{k=1}^{2} P_k \left[\Psi_A(\sigma(t))^* \Psi_A^\Delta(t) + \Psi_A^\Delta(t)^* \Psi_A(t) \right] P_k \equiv$$
$$\equiv \sum_{k=1}^{2} P_k \Psi_A(t)^* \left[(\Psi_A(t)^*)^{-1} \Psi_A(\sigma(t))^* A(t) + A(t)^* \right] \Psi_A(t) P_k \equiv$$
$$\equiv \sum_{k=1}^{2} P_k \Psi_A(t)^* \left(A^* \oplus A \right)(t) \Psi_A(t) P_k \quad \text{on } J^\kappa.$$

Denoting the greatest lower bound and the least upper bound of the Hermitian operator $(A^* \oplus A)(t) \in \mathcal{L}(\mathcal{X})$ by $\alpha(t)$ and $\beta(t)$, respectively, we get

$$\alpha(t)\|x\|^2 \leq \langle (A^* \oplus A)(t)x, x \rangle \leq \beta(t)\|x\|^2 \quad \text{for all } t \in J^\kappa, \ x \in \mathcal{X}.$$

Hence the relation

$$\langle (\Theta(\sigma(t))\Theta^\Delta(t) + \Theta^\Delta(t)\Theta(t))x, x \rangle \equiv$$
$$\equiv \sum_{k=1}^{2} \langle P_k \Psi_A(t)^* \left(A^* \oplus A \right)(t) \Psi_A(t) P_k x, x \rangle \equiv$$
$$\equiv \sum_{k=1}^{2} \langle (A^* \oplus A)(t) \Psi_A(t) P_k x, \Psi_A(t) P_k x \rangle \quad \text{for all } t \in J^\kappa, \ x \in \mathcal{X}$$

implies the estimate

$$\alpha(t) \sum_{k=1}^{2} \|\Psi_A(t)P_k x\|^2 \leq \langle (\Theta(\sigma(t))\Theta^\Delta(t) + \Theta^\Delta(t)\Theta(t))x, x \rangle \leq$$

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$$\leq \beta(t) \sum_{k=1}^{2} \|\Psi_A(t) P_k x\|^2 \quad \text{for all } t \in J^\kappa,\ x \in \mathcal{X}.$$

This estimate in turn can be written in the form

$$\alpha(t)\langle \Theta(t)^2 x, x\rangle \leq \langle (\Theta(\sigma(t))\Theta^\Delta(t) + \Theta^\Delta(t)\Theta(t)) x, x\rangle \leq$$
$$\leq \beta(t)\,\langle \Theta(t)^2 x, x\rangle \quad \text{for all } t \in J^\kappa,\ x \in \mathcal{X}.$$

Setting $x := \Theta(t)^{-1} y$ and using $\Theta(t) \in \mathcal{S}(\mathcal{X})$ we get

$$\langle (\Theta(\sigma(t))\Theta^\Delta(t) + \Theta^\Delta(t)\Theta(t)) x, x\rangle =$$
$$= \langle \Theta(t)^{-1}\Theta(\sigma(t))\Theta^\Delta(t)\Theta(t)^{-1} y + \Theta(t)^{-1}\Theta^\Delta(t) y, y\rangle =$$
$$= \langle \Theta(t)^{-1}\Theta(\sigma(t)) B(t) y + \Theta(t)^{-1}\Theta^\Delta(t)^* y, y\rangle =$$
$$= \langle \Theta(t)^{-1}\Theta(\sigma(t)) B(t) y + B(t)^* y, y\rangle =$$
$$= \langle (B^* \oplus B)(t) y, y\rangle \quad \text{for all } t \in J^\kappa.$$

Altogether we obtain the inequality $\alpha(t)\,\|y\|^2 \leq \langle (B^* \oplus B)(t) y, y\rangle \leq \beta(t)\,\|y\|^2$ for all $t \in J^\kappa$ and $y \in \mathcal{X}$. This finally yields the estimate

$$\|(B^* \oplus B)(t)\| \leq \max\{|\alpha(t)|, |\beta(t)|\} = \|(A^* \oplus A)(t)\| \quad \text{for all } t \in J^\kappa$$

which is nothing but assertion (c).

(III) In case of an arbitrary projection $P \in \mathcal{L}(\mathcal{X})$ the assertions (a) and (c) follow from an application of the second step of the proof to the fundamental operator $\Psi_A(t) T^{-1}$ and the orthogonal projection TPT^{-1}. The choice $\Lambda(t) := \tilde{\Lambda}(t) T$ for the Lyapunov transformation then completes the proof. ∎

Our final corollary is concerned with the problem of decoupling of finite-dimensional dynamic equations.

Corollary 4.12 *Consider the Hilbert space $\mathcal{X} = \mathbb{C}^N$ ($N \geq 2$) and let equation (1) possess a dichotomy with a, b, K_1, K_2 and projection P with rank $M \leq N$ on an interval $J \subseteq \mathbb{T}$. Then if $T \in \mathcal{GL}(\mathbb{C}^N)$ is a transformation with $TPT^{-1} = \begin{pmatrix} I & \\ & 0 \end{pmatrix}$ system (1) is kinematically similar on J to the block diagonal system*

$$x^\Delta = \begin{pmatrix} B_1(t) & \\ & B_2(t) \end{pmatrix} x \tag{11}$$

which has the following properties:

(a) $B_1(t) \in \mathbb{C}^{M \times M}$ and $B_2(t) \in \mathbb{C}^{(N-M) \times (N-M)}$ for all $t \in J^\kappa$,

(b) for the transition operators of the two subsystems of (11) we get

$$\|\Phi_{B_1}(t,s)\| \leq \sqrt{2} K_1 \|T\|^2 \|T^{-1}\|^2 \sqrt{K_1^2 + K_2^2}\, a(t,s) \quad \text{for all } s \preceq t,$$

$$\|\Phi_{B_2}(t,s)\| \leq \sqrt{2} K_2 \|T\|^2 \|T^{-1}\|^2 \sqrt{K_1^2 + K_2^2}\, b(t,s) \quad \text{for all } t \preceq s,$$

and $s, t \in J$.

Remark 4.13 The existence of the matrix $T \in \mathcal{GL}(\mathbb{C}^N)$ diagonalizing the projection P is shown in Gohberg, Kaashoek & Kos [10, Lemma 2.2].

Proof. Equation (1) is kinematically similar on J to equation

$$x^{\Delta} = T A(t) T^{-1} x \tag{12}$$

which possesses a dichotomy with $a, b, K_1 \|T\| \|T^{-1}\|, K_2 \|T\| \|T^{-1}\|$ and projection $Q = TPT^{-1}$ by Theorem 3.4. Because of Theorem 4.10 system (12) in turn is kinematically similar to system (2) whose coefficient matrix $B(t)$ commutes with $Q = \left(\begin{smallmatrix} I & \\ & 0 \end{smallmatrix}\right)$. Hence system (2) is in block diagonal form. Consequently also $\Phi_B(t,s)$ is in block diagonal form and we get

$$\|\Phi_{B_1}(t,s)\| = \|\Psi_B(t) Q \Psi_B(s)^{-1}\| \leq$$
$$\leq \sqrt{2} K_1 \|T\| \|T^{-1}\| \sqrt{K_1^2 \|T\|^2 \|T^{-1}\|^2 + K_2^2 \|T\|^2 \|T^{-1}\|^2}\, a(t,s) =$$
$$= \sqrt{2} K_1 \|T\|^2 \|T^{-1}\|^2 \sqrt{K_1^2 + K_2^2}\, a(t,s) \quad \text{for all } s \preceq t,\ s,t \in J.$$

The corresponding estimate for $\Phi_{B_2}(t,s)$ follows along the same lines. ∎

We close this paper with a few perspectives to possible applications and generalizations:

- The assumption of regressivity or invertibility of the right-hand side of a dynamic or difference equation, respectively, is frequently too restrictive, particularly in an infinite-dimensional setting. Therefore the question arises whether this assumption may be dropped in the context of reducibility. A closer look at our proofs — in particular the one of Lemma 4.8 — demonstrates that this is not a simple task. Yet reduction to block diagonal form can be done in the case of non-invertible difference equations in $\mathbb{R}^N$ with an exponential dichotomy or trichotomy (see Pötzsche [22]).
- If one assumes in Corollary 4.12 that the right-hand side of (1) possesses a trichotomy or a suitable splitting of the extended phase space into more than three invariant families of subspaces, then a repeated application of the above results provides reducibility into more than three diagonal blocks.
- An application of Corollary 4.12 to semi-linear equations $x^{\Delta} = A(t)x + F(t,x)$ allows to subsequently use the existence theorems on invariant fiber

bundles from Aulbach & Wanner [4] (ODEs), from Aulbach [2] (OΔEs) or from Keller [13] (dynamic equations).

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