Pseudo-stable and pseudo-unstable Fiber Bundles for Dynamic Equations on Measure Chains

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Abstract

Invariant fiber bundles are the generalization of invariant manifolds from classical discrete or continuous dynamical systems to non-autonomous dynamic equations on measure chains. In this paper we present a self-contained proof of their existence and smoothness. Our main result generalizes the so-called "Hadamard-Perron-Theorem" for hyperbolic finite-dimensional diffeomorphisms to pseudo-hyperbolic time-dependent non-regressive dynamic equations in Banach spaces. The proof of their smoothness uses a fixed point theorem of Vanderbauwhede-Van Gils.

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1 Introduction

Since the days of Poincaré, Hadamard, Lyapunov and Perron invariant manifolds have played an eminent role in the theory of dynamical systems. A particularly prominent example in this subject is the "Stable-Manifold-Theorem", which is also known as the Theorem of Hadamard-Perron. It states that the domain of attraction of a hyperbolic rest point can be locally represented as a graph over an algebraic eigenspace of the corresponding linearized system. Meanwhile this basic result has been generalized in various directions and because the present paper is not a survey article, we do not even try to summarize all the related literature and refer simply to the references given in [2, 4, 5, 9, 11, 15].

The primary objective of this paper is to unify the two results Aulbach & Wanner [4, Theorem 4.1] (i.e. its special case of ODEs) and Aulbach [2, Theorem 4.1] within the so-called "Calculus on Measure Chains" developed e.g. in Hilger [8]. Both [4] and [2] contain a very general version of the "Stable-Manifold-Theorem" for non-autonomous infinite-dimensional equations with a pseudo-hyperbolic linear part. The first article deals with Carathéodory type differential equations, in particular ordinary differential equations (ODEs), while the latter one treats non-invertible difference equations (OΔEs) in Banach spaces. The basic notion hereby is the concept

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of an invariant fiber bundle which is an appropriate pendant to the invariant manifolds in a non-autonomous setting. However, our construction of the invariant fiber bundles is different from [4, 2], being rather a direct adaptation from Siegmund [15] and Aulbach, Pötzsche & Siegmund [5], which allows an accessible approach to prove their smoothness using a theorem by Vanderbauwhede & Van Gils [16]. Again the literature on the smoothness of invariant manifolds is vast and therefore we refer to the references given in [5, 9, 11, 15].

Our main result (Theorem 4.9) applies to non-autonomous, non-regressive dynamic equations in Banach spaces on nearly arbitrary measure chains, with a pseudo-hyperbolic linear part where the growth rates are not assumed to be constant. It guarantees the existence and C^1-smoothness of so-called “pseudo-stable” and “pseudo-unstable fiber bundles” and additionally a higher order smoothness in a weakened hyperbolic sense. So far there are only three other contributions to the theory of invariant manifolds for dynamic equations on measure chains or time scales (closed subsets of the real line), respectively. Hilger [10, Theorem 4.1] shows the existence of a “center fiber bundle” (in our terminology) for non-autonomous systems on measure chains. A rigorous proof of the smoothness of generalized center manifolds for autonomous dynamic equations on homogeneous time scales can be found in Hilger [9]. Finally the thesis Keller [11] deals with classical stable, unstable and center invariant fiber bundles and their smoothness for dynamic equations on arbitrary time scales. As an introduction to dynamic equations on measure chains we recommend Hilger [8] or Bohner & Peterson [6, Section 8.1].

2 Preliminaries

First, $\mathbb{N}$ is the set of positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}$ the real field. Throughout this paper Banach spaces $\mathcal{X}, \mathcal{Y}$ are all real or complex and their norms are denoted by $\|\cdot\|_{\mathcal{X}}$, $\|\cdot\|_{\mathcal{Y}}$ or simply by $\|\cdot\|$. $\mathcal{L}_n(\mathcal{X};\mathcal{Y})$ is the Banach space of n-linear continuous operators from $\mathcal{X}^n$ to $\mathcal{Y}$ for $n \in \mathbb{N}$, $\mathcal{L}_0(\mathcal{X};\mathcal{Y}) := \mathcal{Y}$, $\mathcal{L}_n(\mathcal{X}) := \mathcal{L}_n(\mathcal{X};\mathcal{X})$, $\mathcal{L}(\mathcal{X};\mathcal{Y}) := \mathcal{L}_1(\mathcal{X};\mathcal{Y})$, $\mathcal{L}(\mathcal{X}) := \mathcal{L}_1(\mathcal{X})$ and $I_{\mathcal{X}}$ the identity map on $\mathcal{X}$. On the cartesian product $\mathcal{X} \times \mathcal{Y}$ we always use the norm

$$\|(x,y)\|_{\mathcal{X}\times\mathcal{Y}} := \max\left\{\|x\|_{\mathcal{X}}, \|y\|_{\mathcal{Y}}\right\} \tag{2.1}$$

and write $\Pi_1 : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ and $\Pi_2 : \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ for the projections on the first and second component, respectively. We say that a linear subspace $\mathcal{X}_1 \subseteq \mathcal{X}$ is continuously embedded into $\mathcal{X}$ if the embedding operator $J : \mathcal{X}_1 \to \mathcal{X}$, $Jx := x$ is continuous and in this case we use the notation $\mathcal{X}_1 \overset{J}{\hookrightarrow} \mathcal{X}$. The ball in $\mathcal{X}$ with center $x \in \mathcal{X}$ and radius $\varepsilon > 0$ is denoted by $B_\varepsilon(x)$. We write Df for the Fréchet derivative of a mapping f and if f depends differentiable on e.g. two variables, then the partial derivatives are denoted by $D_1 f$ and $D_2 f$, respectively.

We also introduce some notions which are specific to the calculus on measure chains. In all the subsequent considerations we deal with a *measure chain* $(\mathbb{T}, \preceq, \mu)$ unbounded above and below, i.e. a conditionally complete totally ordered set $(\mathbb{T}, \preceq)$ (see Hilger [8, Axiom 2]) with the growth calibration $\mu : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ (see [8, Axiom 3]), such that the set $\mu(\mathbb{T}, \tau) \subseteq \mathbb{R}$, $\tau \in \mathbb{T}$, is unbounded above and below. In addition $\rho_+ : \mathbb{T} \to \mathbb{T}$, $\rho_+(t) := \inf\{s \in \mathbb{T} : t \prec s\}$ defines the *forward jump operator* and the *graininess* $\mu^* : \mathbb{T} \to \mathbb{R}$, $\mu^*(t) := \mu(\rho_+(t), t)$ is assumed to be bounded from now on. For $\tau, t \in \mathbb{T}$ we define

$$[\tau, t]_{\mathbb{T}} := \{s \in \mathbb{T} : \tau \preceq s \preceq t\}, \qquad \mathbb{T}_\tau^+ := \{s \in \mathbb{T} : \tau \preceq s\}, \qquad \mathbb{T}_\tau^- := \{s \in \mathbb{T} : s \preceq \tau\},$$

where (half-) open $\mathbb{T}$-intervals are given analogously. $\mathcal{C}_{rd}(I, \mathcal{L}(\mathcal{X}))$ denotes the rd-continuous and $\mathcal{C}_{rd}\mathcal{R}(I, \mathcal{L}(\mathcal{X}))$ the rd-continuous, regressive functions from a $\mathbb{T}$-interval I to $\mathcal{L}(\mathcal{X})$. Recall that $\mathcal{C}_{rd}^{+}\mathcal{R}(\mathbb{T},\mathbb{R}) := \{c \in \mathcal{C}_{rd}\mathcal{R}(\mathbb{T},\mathbb{R}) : 1 + \mu^*(t)a(t) > 0 \text{ for } t \in \mathbb{T}\}$ forms the so-called *regressive module* with respect to the algebraic operations

$$(a \oplus b)(t) := a(t) + b(t) + \mu^*(t)a(t)b(t), \qquad (n \odot a)(t) := \lim_{h \searrow \mu^*(t)} \frac{(1 + ha(t))^n - 1}{h}$$

for $t \in \mathbb{T}$, integers n and $a, b \in \mathcal{C}_{rd}^{+}\mathcal{R}(\mathbb{T},\mathbb{R})$. *Growth rates* are functions $a \in \mathcal{C}_{rd}^{+}\mathcal{R}(\mathbb{T},\mathbb{R})$ such that $1 + \inf_{t\in\mathbb{T}} \mu^*(t)a(t) > 0$ and $\sup_{t\in\mathbb{T}} \mu^*(t)a(t) < \infty$ holds. Moreover we define the relations

$$a \lhd b \quad :\Leftrightarrow \quad 0 < \lfloor b - a \rfloor := \inf_{t\in\mathbb{T}}(b(t) - a(t)), \qquad a \unlhd b \quad :\Leftrightarrow \quad 0 \leq \lfloor b - a \rfloor,$$

and $e_a(t, s) \in \mathbb{R}$ stands for the real exponential function on $\mathbb{T}$ (see [8, Section 7]). A mapping $\phi : \mathbb{T} \to \mathcal{X}$ is said to be *differentiable* (at $t_0 \in \mathbb{T}$), if there exists a unique *derivative* $\phi^{\Delta}(t_0) \in \mathcal{X}$, such that for any $\varepsilon > 0$ the estimate

$$\left\| \phi(\rho_+(t_0)) - \phi(t) - \mu(\rho_+(t_0), t)\phi^{\Delta}(t_0) \right\| \leq \varepsilon \left| \mu(\rho_+(t_0), t) \right| \quad \text{for } t \in U$$

holds in a neighborhood $U \subseteq \mathbb{T}$ of t_0 (see [8, Section 2.4]). We write $\Delta_1 s : \mathbb{T} \times \mathcal{X} \to \mathcal{Y}$ for the partial derivative w.r.t. the first variable of a mapping $s : \mathbb{T} \times \mathcal{X} \to \mathcal{Y}$, provided it exists. If $\phi : \mathbb{T} \to \mathcal{X}$ possesses an *antiderivative*, i.e. a differentiable mapping $\Phi : \mathbb{T} \to \mathcal{X}$ such that $\Phi^{\Delta}(t) \equiv \phi(t)$ on $\mathbb{T}$, then the *Cauchy integral* of ϕ is defined to be

$$\int_{\tau}^{t} \phi(s)\, \Delta s := \Phi(t) - \Phi(\tau) \quad \text{for } \tau, t \in \mathbb{T}.$$

Any $\phi \in \mathcal{C}_{rd}(\mathbb{T}, \mathcal{X})$ has an antiderivative (see [8, Theorem 4.4]). For a dynamic equation

$$\boxed{x^{\Delta} = f(t, x)} \tag{2.2}$$

with a right-hand side $f : \mathbb{T} \times \mathcal{X} \to \mathcal{X}$ guaranteeing existence and uniqueness of solutions in forward time (e.g. Pötzsche [14, p. 38, Satz 1.2.7(a)]), let $\lambda(t; \tau, \xi)$ denote the *general solution*, i.e. $\lambda(\cdot; \tau, \xi)$ solves (2.2) on $\mathbb{T}_{\tau}^{+} \cap I$ and satisfies the initial condition $\lambda(\tau; \tau, \xi) = \xi$ for $\tau \in I$, $\xi \in \mathcal{X}$. It fulfills the *cocycle property*

$$\lambda(t; s, \lambda(s; \tau, \xi)) = \lambda(t; \tau, \xi) \quad \text{for } \tau, s, t \in I,\ \tau \preceq s \preceq t \tag{2.3}$$

and $\xi \in \mathcal{X}$. Given $A \in \mathcal{C}_{rd}(\mathbb{T}, \mathcal{L}(\mathcal{X}))$, the *transition operator* $\Phi_A(t, \tau) \in \mathcal{L}(\mathcal{X})$, $\tau \preceq t$, of a linear dynamic equation $x^{\Delta} = A(t)x$ is the solution of the operator-valued initial value problem $X^{\Delta} = A(t)X$, $X(\tau) = I_{\mathcal{X}}$ in $\mathcal{L}(\mathcal{X})$. If A is regressive then $\Phi_A(t, \tau)$ is defined for all $\tau, t \in \mathbb{T}$.

3 Quasibounded functions

In this section we introduce the so-called quasiboundedness which is a convenient notion describing exponential growth of functions.

Definition 3.1: *For a function $c \in \mathcal{C}_{rd}^{+}\mathcal{R}(\mathbb{T},\mathbb{R})$, a fixed time $\tau_0 \in \mathbb{T}$, a Banach space $\mathcal{X}$, a $\mathbb{T}$-interval I and a rd-continuous function $\lambda : I \to \mathcal{X}$ we say that*

*(a) λ is c^+-*quasibounded *if $I = \mathbb{T}_{\tau_0}^{+}$ and if $\|\lambda\|_{\tau,c}^{+} := \sup_{t\in\mathbb{T}_{\tau}^{+}} \|\lambda(t)\|\, e_{\ominus c}(t, \tau) < \infty$ for $\tau \in \mathbb{T}_{\tau_0}^{+}$,*

(b) λ *is* c^--quasibounded *if* $I = \mathbb{T}_{\tau_0}^-$ *and if* $\|\lambda\|_{\tau,c}^- := \sup_{t\in\mathbb{T}_\tau^-} \|\lambda(t)\| e_{\ominus c}(t,\tau) < \infty$ *for* $\tau \in \mathbb{T}_{\tau_0}^-$,

(c) λ *is* $c^\pm$-quasibounded *if* $I = \mathbb{T}$ *and if* $\|\lambda\|_{\tau,c}^\pm := \sup_{t\in\mathbb{T}} \|\lambda(t)\| e_{\ominus c}(t,\tau) < \infty$ *for* $\tau \in \mathbb{T}$.

By $\mathcal{B}_{\tau,c}^+(\mathcal{X}), \mathcal{B}_{\tau,c}^-(\mathcal{X})$ *and* $\mathcal{B}_c^\pm(\mathcal{X})$ *we denote the sets of all* c^+*-,* c^-*- and* $c^\pm$*-quasibounded functions* $\lambda : I \to \mathcal{X}$*, respectively.*

Obviously $\mathcal{B}_{\tau,c}^+(\mathcal{X}), \mathcal{B}_{\tau,c}^-(\mathcal{X})$ and $\mathcal{B}_c^\pm(\mathcal{X})$ are non-empty and using Hilger [8, Theorem 4.1(iii)], the following result is immediate:

Lemma 3.2: *For any* $c \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T},\mathbb{R})$*,* $\tau \in \mathbb{T}$*, the sets* $\mathcal{B}_{\tau,c}^+(\mathcal{X}), \mathcal{B}_{\tau,c}^-(\mathcal{X})$ *and* $\mathcal{B}_c^\pm(\mathcal{X})$ *are Banach spaces with the norms* $\|\cdot\|_{\tau,c}^+$*,* $\|\cdot\|_{\tau,c}^-$ *and* $\|\cdot\|_{\tau,c}^\pm$*, respectively.*

We state the next lemma only in forward time. It will simplify our differential calculus.

Lemma 3.3: *For functions* $c, d \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T},\mathbb{R})$ *with* $c \trianglelefteq d$*,* $n \in \mathbb{N}$*,* $\tau \in \mathbb{T}$ *and Banach spaces* $\mathcal{X}, \mathcal{Y}$ *the following statements are valid:*

(a) The Banach spaces $\mathcal{B}_{\tau,c}^+(\mathcal{X}) \times \mathcal{B}_{\tau,c}^+(\mathcal{Y})$ *and* $\mathcal{B}_{\tau,c}^+(\mathcal{X}\times\mathcal{Y})$ *are isometrically isomorphic, and thus they can be identified,*

(b) we have $\mathcal{B}_{\tau,c}^+(\mathcal{X}) \overset{J_c^d}{\hookrightarrow} \mathcal{B}_{\tau,d}^+(\mathcal{X})$*, and the embedding operator* $J_c^d : \mathcal{B}_{\tau,c}^+(\mathcal{X}) \to \mathcal{B}_{\tau,d}^+(\mathcal{X})$ *satisfies*

$$\left\|J_c^d\right\|_{\mathcal{L}(\mathcal{B}_{\tau,c}^+(\mathcal{X});\mathcal{B}_{\tau,d}^+(\mathcal{X}))} \le 1, \tag{3.1}$$

(c) the spaces $\mathcal{B}_{\tau,d\ominus n\odot c}^+(\mathcal{L}_n(\mathcal{X}))$ *and* $\mathcal{L}_n(\mathcal{B}_{\tau,c}^+(\mathcal{X});\mathcal{B}_{\tau,d}^+(\mathcal{X}))$ *are isometrically isomorphic by means of the isomorphism* $J_n : \mathcal{B}_{\tau,d\ominus n\odot c}^+(\mathcal{L}_n(\mathcal{X})) \to \mathcal{L}_n(\mathcal{B}_{\tau,c}^+(\mathcal{X});\mathcal{B}_{\tau,d}^+(\mathcal{X}))$*,*

$$\big((J_n\Lambda)(\lambda_1,\ldots,\lambda_n)\big)(t) := \Lambda(t)\lambda_1(t)\cdots\lambda_n(t) \quad \textit{for } t \in \mathbb{T}_\tau^+$$

for any $\Lambda \in \mathcal{B}_{\tau,d\ominus n\odot c}^+(\mathcal{L}_n(\mathcal{X}))$ *and* $\lambda_1,\ldots,\lambda_n \in \mathcal{B}_{\tau,c}^+(\mathcal{X})$*.*

Proof. We show only the assertion (c). At first J_n is a homomorphism. With arbitrary functions $\Lambda \in \mathcal{B}_{\tau,d\ominus n\odot c}^+(\mathcal{L}_n(\mathcal{X}))$ and $\lambda_1,\ldots,\lambda_n \in \mathcal{B}_{\tau,c}^+(\mathcal{X})$ we obtain the estimate

$$\begin{aligned}\|\Lambda(t)\lambda_1(t)\cdot\ldots\cdot\lambda_n(t)\| e_{\ominus d}(t,\tau) &\le \|\Lambda(t)\|_{\mathcal{L}_n(\mathcal{X})} e_{\ominus d\oplus n\odot c}(t,\tau)\prod_{k=1}^n \|\lambda_k(t)\| e_{\ominus c}(t,\tau) \le \\ &\le \|\Lambda\|_{\tau,d\ominus n\odot c}^+ \prod_{k=1}^n \|\lambda_k\|_{\tau,c}^+ \quad \text{for } t \in \mathbb{T}_\tau^+.\end{aligned}$$

Thus the continuity of J_n follows from

$$\|J_n\Lambda\|_{\mathcal{L}_n(\mathcal{B}_{\tau,c}^+(\mathcal{X});\mathcal{B}_{\tau,d}^+(\mathcal{X}))} = \sup_{\substack{\|\lambda_l\|_{\tau,c}^+\le 1,\\ l\in\{1,\ldots,n\}}} \|(J_n\Lambda)\lambda_1\cdots\lambda_n\|_{\tau,d}^+ \le \|\Lambda\|_{\tau,d\ominus n\odot c}^+. \tag{3.2}$$

Otherwise, the inverse $J_n^{-1} : \mathcal{L}_n(\mathcal{B}_{\tau,c}^+(\mathcal{X});\mathcal{B}_{\tau,d}^+(\mathcal{X})) \to \mathcal{B}_{\tau,d\ominus n\odot c}^+(\mathcal{L}_n(\mathcal{X}))$ of J_n is given by

$$\big(J_n^{-1}\bar\Lambda\big)(t)\lambda_1(t)\cdots\lambda_n(t) := \big(\bar\Lambda\lambda_1\cdots\lambda_n\big)(t) \quad \text{for } t \in \mathbb{T}_\tau^+$$

for any $\bar{\Lambda} \in \mathcal{L}_n(\mathcal{B}^+_{\tau,c}(\mathcal{X}); \mathcal{B}^+_{\tau,d}(\mathcal{X}))$ and $\lambda_1, \dots, \lambda_n \in \mathcal{B}^+_{\tau,c}(\mathcal{X})$. By the open mapping theorem (e.g. LANG [12, p. 388, Corollary 1.4]) J_n^{-1} is continuous and it remains to show that it is non-expanding. To this purpose we choose n points $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$ arbitrarily with $\|x_l\| \leq 1$, $l \in \{1, \dots, n\}$ and define functions $\lambda_l(t) := e_c(t, \tau) x_l$. Obviously $\|\lambda_l\|^+_{\tau,c} \leq 1$ and hence

$$\begin{aligned} \left\| \left(J_n^{-1}\bar{\Lambda}\right)(t) x_1 \cdots x_n \right\| e_{\ominus d \oplus n \odot c}(t, \tau) &= \left\| \left(\bar{\Lambda}\lambda_1 \cdots \lambda_n\right)(t) \right\| e_{\ominus d}(t, \tau) \leq \left\| \bar{\Lambda}\lambda_1 \cdots \lambda_n \right\|^+_{\tau,d} \leq \\ &\leq \left\| \bar{\Lambda} \right\|_{\mathcal{L}_n(\mathcal{B}^+_{\tau,c}(\mathcal{X}); \mathcal{B}^+_{\tau,d}(\mathcal{X}))} \quad \text{for } t \in \mathbb{T}^+_\tau. \end{aligned}$$

Now this implies $\left\| \left(J_n^{-1}\bar{\Lambda}\right)(t) \right\|_{\mathcal{L}_n(\mathcal{X})} e_{\ominus d \oplus n \odot c}(t, \tau) \leq \left\| \bar{\Lambda} \right\|_{\mathcal{L}_n(\mathcal{B}^+_{\tau,c}(\mathcal{X}); \mathcal{B}^+_{\tau,d}(\mathcal{X}))}$ for all $t \in \mathbb{T}^+_\tau$ and hence $\left\| J_n^{-1}\bar{\Lambda} \right\|^+_{\tau, d \ominus n \odot c} \leq \left\| \bar{\Lambda} \right\|_{\mathcal{L}_n(\mathcal{B}^+_{\tau,c}(\mathcal{X}); \mathcal{B}^+_{\tau,d}(\mathcal{X}))}$. Therefore J_n is an isometry. $\square$

Lemma 3.4: *Consider functions $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$, a time $\tau \in \mathbb{T}$, Banach spaces $\mathcal{X}, \mathcal{Y}$ and a mapping $f \in C^m(\mathcal{X}, \mathcal{B}^+_{\tau,c}(\mathcal{Y}))$ for some $m \in \mathbb{N}_0$. Then $\big(f(\cdot)\big)(t) \in C^m(\mathcal{X}, \mathcal{Y})$ for every $t \in \mathbb{T}^+_\tau$.*

Proof. Let $t \in \mathbb{T}^+_\tau$ be fixed. Then the evaluation map $\mathrm{ev}_t : \mathcal{B}^+_{\tau,c}(\mathcal{Y}) \to \mathcal{Y}$, $\mathrm{ev}_t(\lambda) := \lambda(t)$ is a continuous homomorphism and hence of class C^∞. It follows from the usual chain rule for Fréchet derivatives that the composition $\big(f(\cdot)\big)(t) = \mathrm{ev}_t \circ f$ has the same smoothness as f. $\square$

4 Construction of invariant fiber bundles

We begin this section by stating our frequently used main assumptions.

Hypothesis 4.1: Suppose the $\mathbb{T}$-interval I is unbounded above and $\mathcal{X}$, $\mathcal{Y}$ are Banach spaces. Let us consider the system of non-autonomous dynamic equations

$$\boxed{\begin{cases} x^\Delta = A(t)x + F(t, x, y) \\ y^\Delta = B(t)y + G(t, x, y) \end{cases}}, \tag{4.1}$$

where $A \in \mathcal{C}_{rd}(I, \mathcal{L}(\mathcal{X}))$, $B \in \mathcal{C}_{rd}\mathcal{R}(I, \mathcal{L}(\mathcal{Y}))$ and the rd-continuous mappings $F : I \times \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$, $G : I \times \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ are m-times, $m \in \mathbb{N}$, continuously differentiable with respect to (x, y). Moreover we assume:

(i) Hypothesis on the linear part: The evolution operators Φ_A and Φ_B satisfy for all $s, t \in I$ the dichotomy estimates

$$\|\Phi_A(t, s)\| \leq K_1 e_a(t, s) \quad \text{for } s \preceq t, \qquad \|\Phi_B(t, s)\| \leq K_2 e_b(t, s) \quad \text{for } t \preceq s, \tag{4.2}$$

with real constants $K_1, K_2 \geq 1$ and $a, b \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$ with $a \lhd b$.

(ii) Hypothesis on the perturbation: The identities

$$F(t, 0, 0) \equiv 0, \qquad\qquad G(t, 0, 0) \equiv 0 \quad \text{on } I, \tag{4.3}$$

hold and the partial derivatives of F and G are globally bounded, i.e. for all $n \in \{1, \dots, m\}$ we have

$$\begin{aligned} |F|_n &:= \sup_{(t,x,y) \in I \times \mathcal{X} \times \mathcal{Y}} \left\| D^n_{(2,3)} F(t, x, y) \right\|_{\mathcal{L}_n(\mathcal{X} \times \mathcal{Y}; \mathcal{X})} < \infty, \\ |G|_n &:= \sup_{(t,x,y) \in I \times \mathcal{X} \times \mathcal{Y}} \left\| D^n_{(2,3)} G(t, x, y) \right\|_{\mathcal{L}_n(\mathcal{X} \times \mathcal{Y}; \mathcal{Y})} < \infty. \end{aligned} \tag{4.4}$$

(iii) Hypothesis on higher order smoothness (if $m \geq 2$): The partial derivatives of F and G are uniformly continuous: For any $\varepsilon > 0$ there exists an $\delta > 0$ such that for all $t \in I$ and $(x_0, y_0) \in \mathcal{X} \times \mathcal{Y}$ we have

$$\left\| D^m_{(2,3)}(F,G)(t,x,y) - D^m_{(2,3)}(F,G)(t,x_0,y_0) \right\|_{\mathcal{L}_m(\mathcal{X}\times\mathcal{Y})} < \varepsilon \quad \text{for } (x,y) \in B_\delta(x_0,y_0).$$

Remark 4.2: (1) It is an immediate consequence of the mean value theorem (see e.g. LANG [12, p. 341, Theorem 4.2]) and Hypothesis 4.1(ii) that the partial derivatives $D^{n-1}_{(2,3)}F$, $D^{n-1}_{(2,3)}G$ are globally Lipschitz continuous (with constants $|F|_n$, $|G|_n$, respectively) for $n \in \{1,\dots,m\}$, and hence Hypothesis 4.1(iii) also holds for $D^n_{(2,3)}F$, $D^n_{(2,3)}G$, with $n \in \{0,\dots,m\}$. Nevertheless, Hypothesis 4.1(iii) is of technical nature and is only needed for $m \geq 2$.

(2) In a Hilbert space $\mathcal{Z}$ and for a mapping $R : I \times \mathcal{Z} \to \mathcal{Z}$ with globally bounded derivatives $D^n_2 R$, $n \in \{1,\dots,m\}$, any system of the form $z^\Delta = C(t)z + R(t,z)$ can be transformed into the "decoupled" equation (4.1) if $C \in \mathcal{C}_{rd}\mathcal{R}(I, \mathcal{L}(\mathcal{X}))$ possesses an exponential dichotomy. This can be shown using methods from AULBACH & PÖTZSCHE [3] via a Lyapunov transformation.

In the sequel we define two linear and two non-linear operators on spaces of quasibounded functions and derive their basic properties concerning continuity and differentiability. This allows to characterize the quasibounded solutions of (4.1) as fixed points of an equation based on these operators.

Lemma 4.3 (the operator $\boldsymbol{\mathcal{S}_\tau}$)**:** *We assume $I = \mathbb{T}^+_{\tau_0}$, $\tau_0 \in \mathbb{T}$, in Hypothesis 4.1 and choose any $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T},\mathbb{R})$, $a \trianglelefteq c$. Then for every $\tau \in \mathbb{T}^+_{\tau_0}$ the operator $\mathcal{S}_\tau : \mathcal{X} \to \mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$,*

$$\big(\mathcal{S}_\tau\, \xi\big)(t) := \big(\Phi_A(t,\tau)\xi, 0\big) \quad \textit{for } t \in \mathbb{T}^+_\tau,$$

is linear and continuous with

$$\|\mathcal{S}_\tau\|_{\mathcal{L}(\mathcal{X};\mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y}))} \leq K_1. \tag{4.5}$$

Proof. The linearity of $\mathcal{S}_\tau$ is evident. Furthermore, by HILGER [8, Theorem 7.4] for any $\xi \in \mathcal{X}$ we get the estimate

$$\big\|\big(\mathcal{S}_\tau\xi\big)(t)\big\|\, e_{\ominus c}(t,\tau) \leq \|\Phi_A(t,\tau)\|\,\|\xi\|\, e_{\ominus c}(t,\tau) \overset{(4.2)}{\leq} K_1\,\|\xi\| \quad \text{for } t \in \mathbb{T}^+_\tau,$$

hence $\|\mathcal{S}_\tau\xi\|^+_{\tau,c} \leq K_1\,\|\xi\|$, and $\mathcal{S}_\tau\xi$ is well-defined, continuous and (4.5) holds. □

Lemma 4.4 (the operator $\boldsymbol{\mathcal{K}_\tau}$)**:** *We assume $I = \mathbb{T}^+_{\tau_0}$, $\tau_0 \in \mathbb{T}$, in Hypothesis 4.1(i) and choose an arbitrary growth rate $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T},\mathbb{R})$, $a \lhd c \lhd b$. Then for any $\tau \in \mathbb{T}^+_{\tau_0}$ the operator $\mathcal{K}_\tau : \mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y}) \to \mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$,*

$$\big(\mathcal{K}_\tau(\nu,\upsilon)\big)(t) := \left(\int_\tau^t \Phi_A(t,\rho^+(s))\nu(s)\,\Delta s,\; -\int_t^\infty \Phi_B(t,\rho^+(s))\upsilon(s)\,\Delta s\right) \quad \textit{for } t \in \mathbb{T}^+_\tau,$$

is linear and continuous with

$$\|\mathcal{K}_\tau\|_{\mathcal{L}(\mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y}))} \leq \max\left\{\frac{K_1}{\lfloor c-a\rfloor}, \frac{K_2}{\lfloor b-c\rfloor}\right\}. \tag{4.6}$$

In particular we have

$$\|\Pi_2\mathcal{K}_\tau(\nu,\upsilon)\|^+_{\tau,c} \leq \frac{K_2}{\lfloor b-c\rfloor}\,\|\upsilon\|^+_{\tau,c} \quad \textit{for } (\nu,\upsilon) \in \mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y}). \tag{4.7}$$

Proof. Obviously $\mathcal{K}_\tau$ is linear. Now choose any pair $(\nu, \upsilon) \in \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y})$. Then, by the variation of constants formula, which can be shown to be valid in forward time without assuming regressivity (see PÖTZSCHE [14, p. 56, Satz 1.3.11]), the inhomogeneous dynamic equation $x^\Delta = A(t)x + \nu(t)$ has the solution $\tilde{\nu} := \Pi_1 \mathcal{K}_\tau(\nu, \upsilon) : \mathbb{T}^+_\tau \to \mathcal{X}$, satisfying the initial condition $x(\tau) = 0$. Due to PÖTZSCHE [13, Theorem 2(a)] the function $\tilde{\nu}$ is c^+-quasibounded and we get $\|\Pi_1 \mathcal{K}_\tau(\nu, \upsilon)\|^+_{\tau,c} = \|\tilde{\nu}\|^+_{\tau,c} \le \frac{K_1}{\lfloor c-a \rfloor} \|\nu\|^+_{\tau,c}$. Similarly PÖTZSCHE [13, Theorem 4(b)] implies that the linear system $y^\Delta = B(t)y + \upsilon(t)$ has exactly one c^+-quasibounded solution, namely $\tilde{\upsilon} := \Pi_2 \mathcal{K}_\tau(\nu, \upsilon) : \mathbb{T}^+_\tau \to \mathcal{Y}$, and this solution satisfies the estimate $\|\Pi_2 \mathcal{K}_\tau(\nu, \upsilon)\|^+_{\tau,c} = \|\tilde{\upsilon}\|^+_{\tau,c} \le \frac{K_2}{\lfloor b-c \rfloor} \|\upsilon\|^+_{\tau,c}$, which is identical with inequality (4.7). Finally we get

$$\begin{aligned}
\left\| \big(\mathcal{K}_\tau(\nu, \upsilon)\big)(t) \right\| e_{\ominus c}(t, \tau) &\overset{(2.1)}{=} \max\left\{ \|\tilde{\nu}(t)\| \, e_{\ominus c}(t, \tau), \|\tilde{\upsilon}(t)\| \, e_{\ominus c}(t, \tau) \right\} \le \\
&\le \max\left\{ \frac{K_1}{\lfloor c-a \rfloor} \|\nu\|^+_{\tau,c}, \frac{K_2}{\lfloor b-c \rfloor} \|\upsilon\|^+_{\tau,c} \right\} \le \\
&\overset{(2.1)}{\le} \max\left\{ \frac{K_1}{\lfloor c-a \rfloor}, \frac{K_2}{\lfloor b-c \rfloor} \right\} \|(\nu, \upsilon)\|^+_{\tau,c} \quad \text{for } t \in \mathbb{T}^+_\tau.
\end{aligned}$$

Passing to the least upper bound over $t \in \mathbb{T}^+_\tau$ we finally obtain the estimate $\|\mathcal{K}_\tau(\nu, \upsilon)\|^+_{\tau,c} \le \max\left\{ \frac{K_1}{\lfloor c-a \rfloor}, \frac{K_2}{\lfloor b-c \rfloor} \right\} \|(\nu, \upsilon)\|^+_{\tau,c}$ and hence the continuity of $\mathcal{K}_\tau$, as well as the estimate (4.6). □

The two operators $\mathcal{S}_\tau$ and $\mathcal{K}_\tau$ given in Lemmas 4.3 and 4.4 are continuous homomorphisms and hence continuously differentiable. The non-linear mapping $\mathcal{G}$, which we define next, does not have this property in general. This operator describes the composition of the non-linearities F and G with quasibounded functions and is a special case of a so-called *substitution operator*.

Lemma 4.5 (the operator $\mathcal{G}$)**:** *We assume $I = \mathbb{T}^+_{\tau_0}$, $\tau_0 \in \mathbb{T}$ in Hypothesis 4.1(ii), choose $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$ and $\tau \in \mathbb{T}^+_{\tau_0}$. Then the non-linear operator $\mathcal{G} : \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y}) \to \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y})$,*

$$\big(\mathcal{G}(\nu, \upsilon)\big)(t) := \big(F(t, \nu(t), \upsilon(t)), G(t, \nu(t), \upsilon(t))\big) \quad \textit{for } t \in \mathbb{T}^+_\tau,$$

has the following properties:

(a) $\mathcal{G}(0,0) = (0,0) \in \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y})$,

(b) $\mathcal{G}$ *is globally Lipschitzian with*

$$\|\mathcal{G}(\nu, \upsilon) - \mathcal{G}(\bar{\nu}, \bar{\upsilon})\|^+_{\tau,c} \le \max\{|F|_1, |G|_1\} \|(\nu, \upsilon) - (\bar{\nu}, \bar{\upsilon})\|^+_{\tau,c} \tag{4.8}$$

for all $(\nu, \upsilon), (\bar{\nu}, \bar{\upsilon}) \in \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y})$.

Proof. (a) As a result of (4.3) in Hypothesis 4.1(ii) we obtain statement (a).

(b) For arbitrary $(\nu, \upsilon), (\bar{\nu}, \bar{\upsilon}) \in \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y})$ it follows from the mean value theorem that

$$\begin{aligned}
&\left\| \big(\mathcal{G}(\nu, \upsilon)\big)(t) - \big(\mathcal{G}(\bar{\nu}, \bar{\upsilon})\big)(t) \right\| e_{\ominus c}(t, \tau) \le \\
&\overset{(4.4)}{\le} \max\left\{ |F|_1 \|(\nu, \upsilon)(t) - (\bar{\nu}, \bar{\upsilon})(t)\| \, e_{\ominus c}(t, \tau), |G|_1 \|(\nu, \upsilon)(t) - (\bar{\nu}, \bar{\upsilon})(t)\| \, e_{\ominus c}(t, \tau) \right\} \le \\
&\le \max\{|F|_1, |G|_1\} \|(\nu, \upsilon) - (\bar{\nu}, \bar{\upsilon})\|^+_{\tau,c} \quad \text{for } t \in \mathbb{T}^+_\tau.
\end{aligned}$$

The Lipschitz condition (4.8) is obtained by taking the least upper bound over $t \in \mathbb{T}^+_\tau$ in the above estimate. Setting $(\bar{\nu}, \bar{\upsilon}) := (0, 0)$ together with statement (a) implies that the operator $\mathcal{G}$ is well-defined, i.e. $\mathcal{G}(\nu, \upsilon) \in \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y})$. □

Lemma 4.6 (the operator $\mathcal{G}^{(n)}$)**:** *We assume $I = \mathbb{T}^+_{\tau_0}$, $\tau_0 \in \mathbb{T}$ in Hypothesis 4.1, choose integers $n \in \{1, \dots, m\}$, $\tau \in \mathbb{T}^+_{\tau_0}$, functions $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$ and growth rates $d \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$ such that $c \trianglelefteq d$ and $n \odot c \trianglelefteq d$. Then the operator $\mathcal{G}^{(n)} : \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y}) \to \mathcal{B}^+_{\tau,d\ominus n\odot c}(\mathcal{L}_n(\mathcal{X} \times \mathcal{Y}))$,*

$$\big(\mathcal{G}^{(n)}(\nu, \upsilon)\big)(t) := D^n_{(2,3)}(F, G)(t, \nu(t), \upsilon(t)) \quad \text{for } t \in \mathbb{T}^+_\tau,$$

has the following properties:

(a) It is well-defined and globally bounded with

$$\left\|\mathcal{G}^{(n)}(\nu, \upsilon)\right\|^+_{\tau,d\ominus n\odot c} \leq \max\{|F|_n, |G|_n\} \quad \text{for } (\nu, \upsilon) \in \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y}), \tag{4.9}$$

(b) for $c \triangleleft d$ and $n = 1$ the operator $\mathcal{G}^{(1)}$ is continuous,

(c) for $c \trianglelefteq 0$ and $m \geq 2$ the operator $\mathcal{G}^{(n)}$ is continuous as well.

Proof. (a) For arbitrary functions $(\nu, \upsilon) \in \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y})$ we get (cf. Hilger [8, Theorem 7.4])

$$\begin{aligned} \left\|\big(\mathcal{G}^{(n)}(\nu, \upsilon)\big)(t)\right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y})} e_{\ominus d\oplus n\odot c}(t, \tau) &\leq \left\|D^n_{(2,3)}(F, G)(t, \nu(t), \upsilon(t))\right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y})} \leq \\ &\overset{(4.4)}{\leq} \max\{|F|_n, |G|_n\} \quad \text{for } t \in \mathbb{T}^+_\tau, \end{aligned}$$

since $0 \trianglelefteq d \ominus n \odot c$. Therefore we have $\mathcal{G}^{(n)}(\nu, \upsilon) \in \mathcal{B}^+_{\tau,d\ominus n\odot c}(\mathcal{L}_n(\mathcal{X} \times \mathcal{Y}))$ and (4.9) holds.

(b) To prove the continuity of $\mathcal{G}^{(1)}$ under the assumption $c \triangleleft d$, $n = 1$ we choose $\varepsilon > 0$ and $(\nu_0, \upsilon_0) \in \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y})$ arbitrarily, but fixed. Since $c \triangleleft d$, there exists a $T \in (\tau, \infty)_\mathbb{T}$ with $2\max\{|F|_1, |G|_1\} e_{c\ominus d}(T, \tau) < \frac{\varepsilon}{2}$. Using the triangle inequality we get

$$\begin{aligned} &\left\|\big(\mathcal{G}^{(1)}(\nu, \upsilon)\big)(t) - \big(\mathcal{G}^{(1)}(\nu_0, \upsilon_0)\big)(t)\right\| e_{\ominus d\oplus c}(t, \tau) \leq \\ \leq\ & \big(\left\|D_{(2,3)}(F, G)(t, \nu(t), \upsilon(t))\right\| + \left\|D_{(2,3)}(F, G)(t, \nu_0(t), \upsilon_0(t))\right\|\big) e_{\ominus d\oplus c}(t, \tau) \leq \\ \overset{(4.4)}{\leq}\ & 2\max\{|F|_1, |G|_1\} e_{\ominus d\oplus c}(T, \tau) < \frac{\varepsilon}{2} \quad \text{for } t \in (T, \infty)_\mathbb{T} \end{aligned}$$

and for all $(\nu, \upsilon) \in \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y})$. Since the partial derivative $D_{(2,3)}(F, G)$ is continuous, there exists a constant $\delta_1 = \delta_1(\varepsilon) > 0$ such that for $(\nu, \upsilon) \in \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y})$ the estimate

$$\|(\nu, \upsilon)(t) - (\nu_0, \upsilon_0)(t)\| < \delta_1 \quad \text{for } t \in [\tau, T]_\mathbb{T}$$

implies

$$\left\|\big(\mathcal{G}^{(1)}(\nu, \upsilon)\big)(t) - \big(\mathcal{G}^{(1)}(\nu_0, \upsilon_0)\big)(t)\right\| < \frac{\varepsilon}{2} \quad \text{for } t \in [\tau, T]_\mathbb{T}.$$

Besides, using Hilger [8, Theorem 7.4] again, one gets

$$\begin{aligned} &\|(\nu, \upsilon)(t) - (\nu_0, \upsilon_0)(t)\| \leq e_c(t, \tau) \|(\nu, \upsilon) - (\nu_0, \upsilon_0)\|^+_{\tau,c} \leq \\ \leq\ & \max\{1, e_c(T, \tau)\} \|(\nu, \upsilon) - (\nu_0, \upsilon_0)\|^+_{\tau,c} < \delta_1 \quad \text{for } t \in [\tau, T]_\mathbb{T}, \end{aligned}$$

for every $(\nu, \upsilon) \in \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y})$ with $\|(\nu, \upsilon) - (\nu_0, \upsilon_0)\|^+_{\tau,c} < \delta_2 := \frac{\delta_1}{\max\{1, e_c(T,\tau)\}}$. For such pairs of c^+-quasibounded functions $(\nu, \upsilon) \in B_{\delta_2}(\nu_0, \upsilon_0)$ one has

$$\left\|\big(\mathcal{G}^{(1)}(\nu, \upsilon)\big) - \big(\mathcal{G}^{(1)}(\nu_0, \upsilon_0)\big)\right\|^+_{\tau,d\ominus c} =$$

$$= \sup_{t\in\mathbb{T}_\tau^+} \left\| \left(\mathcal{G}^{(1)}(\nu,\upsilon)\right)(t) - \left(\mathcal{G}^{(1)}(\nu_0,\upsilon_0)\right)(t) \right\| e_{\ominus d\oplus c}(t,\tau) \le$$

$$\le \max\left\{ \sup_{t\in[\tau,T]_\mathbb{T}} \left\| \left(\mathcal{G}^{(1)}(\nu,\upsilon)\right)(t) - \left(\mathcal{G}^{(1)}(\nu_0,\upsilon_0)\right)(t) \right\|, \right.$$

$$\left. \sup_{T\prec t} \left\| \left(\mathcal{G}^{(1)}(\nu,\upsilon)\right)(t) - \left(\mathcal{G}^{(1)}(\nu_0,\upsilon_0)\right)(t) \right\| e_{\ominus d\oplus c}(t,\tau) \right\} \le \frac{\varepsilon}{2} < \varepsilon.$$

This proves the continuity of the operator $\mathcal{G}^{(1)}$.

(c) Choose $\varepsilon > 0$ and $(\nu_0, \upsilon_0) \in \mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$ arbitrarily to prove the continuity of $\mathcal{G}^{(n)}$ in case $c \trianglelefteq 0$ and $m \ge 2$. By means of Hypothesis 4.1(iii) and the definition of the operator $\mathcal{G}^{(n)}$ there exists an $\delta = \delta(\varepsilon) > 0$ such that the estimate $\|(\nu,\upsilon)(t) - (\nu_0,\upsilon_0)(t)\| < \delta$ for all $t \in \mathbb{T}^+_\tau$ implies

$$\left\| \left(\mathcal{G}^{(n)}(\nu,\upsilon)\right)(t) - \left(\mathcal{G}^{(n)}(\nu_0,\upsilon_0)\right)(t) \right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y})} e_{\ominus d\oplus n\odot c}(t,\tau) \le$$

$$\le \left\| \left(\mathcal{G}^{(n)}(\nu,\upsilon)\right)(t) - \left(\mathcal{G}^{(n)}(\nu_0,\upsilon_0)\right)(t) \right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y})} < \frac{\varepsilon}{2} \quad \text{for } t \in \mathbb{T}^+_\tau$$

for arbitrary $(\nu,\upsilon) \in \mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$, since $0 \trianglelefteq d \ominus n \odot c$. Moreover we get

$$\begin{aligned} \|(\nu,\upsilon)(t) - (\nu_0,\upsilon_0)(t)\| &\le e_c(t,\tau)\, \|(\nu,\upsilon) - (\nu_0,\upsilon_0)\|^+_{\tau,c} \le \\ &\le \|(\nu,\upsilon) - (\nu_0,\upsilon_0)\|^+_{\tau,c} \quad \text{for } t \in \mathbb{T}^+_\tau. \end{aligned}$$

Taking $(\nu,\upsilon) \in B_\delta(\nu_0,\upsilon_0) \subseteq \mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$ our assertion follows because we have

$$\left\| \mathcal{G}^{(n)}(\nu,\upsilon) - \mathcal{G}^{(n)}(\nu_0,\upsilon_0) \right\|^+_{\tau,d\ominus n\odot c} =$$

$$= \sup_{t\in\mathbb{T}^+_\tau} \left\| \left(\mathcal{G}^{(n)}(\nu,\upsilon)\right)(t) - \left(\mathcal{G}^{(n)}(\nu_0,\upsilon_0)\right)(t) \right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y})} e_{\ominus d\oplus n\odot c}(t,\tau) \le \frac{\varepsilon}{2} < \varepsilon.$$

Therefore Lemma 4.6 is proved. □

Now the question arises whether the assumption $c \triangleleft d$ in statement (b) of Lemma 4.6 is of a purely technical nature. In fact, one cannot get rid of it because c^+-quasibounded functions can be unbounded. An example which demonstrates that the non-linear operator $\mathcal{G}^{(1)}$ may not be continuous for $0 \triangleleft c = d$ can be found in Aulbach, Pötzsche & Siegmund [5, Example 4.7] for difference equations.

Next we investigate the differentiability of $\mathcal{G}$. It will turn out that not only the smoothness of the mappings F and G is essential but also the particular choice of the spaces of quasibounded functions as domain and range of $\mathcal{G}$. The mappings $\mathcal{G}^{(n)}$ seem to be a good choice as candidates for the derivatives of the substitution operator $\mathcal{G}$ since they are defined with the aid of the derivatives of the mappings (F, G).

Lemma 4.7 (continuous differentiability of $\boldsymbol{\mathcal{G}}$)**:** *We assume $I = \mathbb{T}^+_{\tau_0}$, $\tau_0 \in \mathbb{T}$ in Hypothesis 4.1, choose functions $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T},\mathbb{R})$ and growth rates $d \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T},\mathbb{R})$ with $c \trianglelefteq d$ and a time $\tau \in \mathbb{T}^+_{\tau_0}$. Then the operator $\mathcal{G}^{(0)} := J^d_c \mathcal{G} : \mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y}) \to \mathcal{B}^+_{\tau,d}(\mathcal{X}\times\mathcal{Y})$ has the following properties:*

(a) For $c \triangleleft d$ it is continuously differentiable,

(b) for $c \trianglelefteq 0$ and $m \ge 2$ it is m-times continuously differentiable.

In any case and for any $n \in \{1, \dots, m\}$ the derivatives are given by

$$D^n \mathcal{G}^{(0)} = J_n \mathcal{G}^{(n)} : \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y}) \to \mathcal{L}_n(\mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y}); \mathcal{B}^+_{\tau,d}(\mathcal{X} \times \mathcal{Y})), \tag{4.10}$$

and they are globally bounded with

$$\left\| \left(D^n \mathcal{G}^{(0)}\right)(\nu, \upsilon) \right\|_{\mathcal{L}_n(\mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y}); \mathcal{B}^+_{\tau,d}(\mathcal{X}\times\mathcal{Y}))} \le \max\{|F|_n, |G|_n\} \quad \textit{for } (\nu, \upsilon) \in \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y}).$$

Proof. We start with some preparations. To this end consider two arbitrary functions $(\nu, \upsilon), (\nu_0, \upsilon_0) \in \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y})$. Now we keep the pair (ν_0, υ_0) fixed and define the functions $r_n : \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y}) \to [0, \infty)$ for $n \in \{1, \dots, m-1\}$ by

$$r_n(\nu, \upsilon) := \sup_{h \in [0,1]} \left\| \mathcal{G}^{(n)}(\nu_0 + h\nu, \upsilon_0 + h\upsilon) - \mathcal{G}^{(n)}(\nu_0, \upsilon_0) \right\|^+_{\tau, d \ominus n \odot c}.$$

By Lemma 3.3(c), $\mathcal{G}^{(n)}(\nu_0, \upsilon_0)$ can be considered as a mapping in $\mathcal{B}^+_{\tau,d\ominus n\odot c}(\mathcal{L}_n(\mathcal{X} \times \mathcal{Y}))$ as well as a n-linear mapping in $\mathcal{L}_n(\mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y}); \mathcal{B}^+_{\tau,d}(\mathcal{X} \times \mathcal{Y}))$. The mean value theorem implies for $n \in \{0, \dots, m-1\}$ the estimate

$$\begin{aligned}
&\left\| \mathcal{G}^{(n)}(\nu_0 + \nu, \upsilon_0 + \upsilon) - \mathcal{G}^{(n)}(\nu_0, \upsilon_0) - \mathcal{G}^{(n+1)}(\nu_0, \upsilon_0) \begin{pmatrix} \nu \\ \upsilon \end{pmatrix} \right\|^+_{\tau, d\ominus n\odot c} = \\
&= \sup_{t \in \mathbb{T}^+_\tau} \Bigg\| D^n_{(2,3)}(F,G)(t, \nu_0(t) + \nu(t), \upsilon_0(t) + \upsilon(t)) - D^n_{(2,3)}(F,G)(t, \nu_0(t), \upsilon_0(t)) - \\
&\quad - D^{n+1}_{(2,3)}(F,G)(t, \nu_0(t), \upsilon_0(t)) \begin{pmatrix} \nu(t) \\ \upsilon(t) \end{pmatrix} \Bigg\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y})} e_{\ominus d \oplus n\odot c}(t, \tau) = \\
&= \sup_{t \in \mathbb{T}^+_\tau} \Bigg\| \left(\int_0^1 D^{n+1}_{(2,3)}(F,G)(t, \nu_0(t) + h\nu(t), \upsilon_0(t) + h\upsilon(t))\, dh \right) \begin{pmatrix} \nu(t) \\ \upsilon(t) \end{pmatrix} - \\
&\quad - D^{n+1}_{(2,3)}(t, \nu_0(t), \upsilon_0(t)) \begin{pmatrix} \nu(t) \\ \upsilon(t) \end{pmatrix} \Bigg\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y})} e_{\ominus d\oplus n\odot c}(t, \tau) \le \\
&\le \sup_{t \in \mathbb{T}^+_\tau} \int_0^1 \Big\| D^{n+1}_{(2,3)}(F,G)(t, \nu_0(t) + h\nu(t), \upsilon_0(t) + h\upsilon(t)) - \\
&\quad - D^{n+1}_{(2,3)}(F,G)(t, \nu_0(t), \upsilon_0(t)) \Big\|_{\mathcal{L}_{n+1}(\mathcal{X}\times\mathcal{Y})} dh e_{\ominus d\oplus(n+1)\odot c}(t, \tau) \left\|(\nu(t), \upsilon(t))\right\| e_{\ominus c}(t, \tau).
\end{aligned}$$

Estimating the integral we get

$$\begin{aligned}
&\left\| \mathcal{G}^{(n)}(\nu_0 + \nu, \upsilon_0 + \upsilon) - \mathcal{G}^{(n)}(\nu_0, \upsilon_0) - \mathcal{G}^{(n+1)}(\nu_0, \upsilon_0) \begin{pmatrix} \nu \\ \upsilon \end{pmatrix} \right\|^+_{\tau, d\ominus n\odot c} \le \\
&\le \sup_{t \in \mathbb{T}^+_\tau} \sup_{h \in [0,1]} \Big\| D^{n+1}_{(2,3)}(F,G)(t, \nu_0(t) + h\nu(t), \upsilon_0(t) + h\upsilon(t)) - \\
&\quad - D^{n+1}_{(2,3)}(F,G)(t, \nu_0(t), \upsilon_0(t)) \Big\|_{\mathcal{L}_{n+1}(\mathcal{X}\times\mathcal{Y})} e_{\ominus d\oplus(n+1)\odot c}(t, \tau) \left\|(\nu(t), \upsilon(t))\right\| e_{\ominus c}(t, \tau) \le \\
&\le \sup_{h \in [0,1]} \left\| \mathcal{G}^{(n+1)}(\nu_0 + h\nu, \upsilon_0 + h\upsilon) - \mathcal{G}^{(n+1)}(\nu_0, \upsilon_0) \right\|^+_{\tau, d\ominus(n+1)\odot c} \left\|(\nu, \upsilon)\right\|^+_{\tau,c} = \\
&= r_{n+1}(\nu, \upsilon) \left\|(\nu, \upsilon)\right\|^+_{\tau,c}.
\end{aligned}$$

(a) In case $c \lhd d$ the operator $\mathcal{G}^{(1)} : \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y}) \to \mathcal{L}(\mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y}); \mathcal{B}^+_{\tau,d}(\mathcal{X} \times \mathcal{Y}))$ is continuous by Lemma 4.6(b), hence the function r_1 is well-defined and we get $\lim_{(\nu,\upsilon)\to(0,0)} r_1(\nu,\upsilon) = 0$. Now the above estimate for $n = 0$ shows the differentiability of $\mathcal{G}^{(0)} = J_c^d \mathcal{G}$ in any point $(\nu_0, \upsilon_0) \in \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y})$ with continuous derivative $\mathcal{G}^{(1)}$.

(b) For functions $c \unlhd 0$ and integers $m \geq 2$, $n \in \{0, \dots, m-1\}$ obviously $(n+1) \odot c \unlhd d$ holds. Therefore the operators $\mathcal{G}^{(n+1)} : \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y}) \to \mathcal{L}_{n+1}(\mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y}); \mathcal{B}^+_{\tau,d}(\mathcal{X} \times \mathcal{Y}))$ are continuous by Lemma 4.6(c), the functions r_{n+1} are well-defined and fulfill $\lim_{(\nu,\upsilon)\to(0,0)} r_{n+1}(\nu,\upsilon) = 0$. Furthermore the above estimate shows again that each mapping $\mathcal{G}^{(n)} : \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y}) \to \mathcal{L}_n(\mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y}); \mathcal{B}^+_{\tau,d}(\mathcal{X} \times \mathcal{Y}))$ is differentiable and has the continuous derivative $\mathcal{G}^{(n+1)}$. Now the assertion follows by mathematical induction. Finally we get the estimate

$$
\begin{aligned}
& \left\| \left(D^n \mathcal{G}^{(0)}\right)(\nu,\upsilon) \right\|_{\mathcal{L}_n(\mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y}); \mathcal{B}^+_{\tau,d}(\mathcal{X}\times\mathcal{Y}))} \leq \\
\stackrel{(4.10)}{\leq}\ & \|J_n\|_{\mathcal{L}(\mathcal{B}^+_{\tau,d\ominus n\odot c}(\mathcal{L}_n(\mathcal{X}\times\mathcal{Y})); \mathcal{L}_n(\mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y}); \mathcal{B}^+_{\tau,d}(\mathcal{X}\times\mathcal{Y})))} \left\| \mathcal{G}^{(n)}(\nu,\upsilon) \right\|^+_{\tau,d\ominus n\odot c} \leq \\
\stackrel{(3.2)}{\leq}\ & \left\| \mathcal{G}^{(n)}(\nu,\upsilon) \right\|^+_{\tau,d\ominus n\odot c} \stackrel{(4.9)}{\leq} \max\left\{|F|_n, |G|_n\right\} \quad \text{for } n \in \{1, \dots, m\},
\end{aligned}
$$

and the proof of Lemma 4.7 is complete. □

We already pointed out above that for $0 \lhd c = d$ the continuity of the operator $\mathcal{G}^{(1)}$ may be lost. In fact, one cannot even expect the differentiability of the operator $\mathcal{G} = \mathcal{G}^{(0)}$. Nonlinearities demonstrating this can be found in Siegmund [15, pp. 35–38] (ODEs) or Aulbach, Pötzsche & Siegmund [5] (OΔEs).

In the previous last lemmas we investigated the linear operators $\mathcal{S}_\tau$, $\mathcal{K}_\tau$ and the more subtle substitution operator $\mathcal{G}$. With the help of these mappings we can now characterize the quasibounded solutions of the dynamic equation (4.1) quite easily as fixed points.

Lemma 4.8 (solutions in $\mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y})$ as fixed points)**:** *Let us assume $I = \mathbb{T}^+_{\tau_0}$, $\tau_0 \in \mathbb{T}$ in Hypothesis 4.1, choose growth rates $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$, $a \lhd c \lhd b$, $\tau \in \mathbb{T}^+_{\tau_0}$ and a vector $\xi \in \mathcal{X}$. Then for the operator $\mathcal{T}_\tau : \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y}) \times \mathcal{X} \to \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y})$,*

$$\mathcal{T}_\tau(\nu, \upsilon; \xi) := \mathcal{S}_\tau \xi + \mathcal{K}_\tau \mathcal{G}(\nu, \upsilon), \tag{4.11}$$

the following two statements are equivalent:

- *(a) $(\nu^*, \upsilon^*) \in \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y})$ is a solution of the dynamic equation (4.1) with $\nu^*(\tau) = \xi$,*
- *(b) $(\nu^*, \upsilon^*) \in \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y})$ is a solution of the equation*

$$(\nu^*, \upsilon^*) = \mathcal{T}_\tau(\nu^*, \upsilon^*; \xi). \tag{4.12}$$

Proof. Lemmas 4.3, 4.4 and 4.5 imply that $\mathcal{T}_\tau$ is well-defined and may be written explicitly as

$$
\begin{aligned}
\big(\mathcal{T}_\tau(\nu^*, \upsilon^*; \xi)\big)(t) \ = \ & \Big(\Phi_A(t,\tau)\xi + \int_\tau^t \Phi_A(t, \rho^+(s)) F(s, \nu^*(s), \upsilon^*(s)) \, \Delta s, \\
& - \int_t^\infty \Phi_B(t, \rho^+(s)) G(s, \nu^*(s), \upsilon^*(s)) \Big).
\end{aligned}
$$

(a) $\Rightarrow$ (b) The function ν^* is also a solution of the linear inhomogeneous equation

$$x^\Delta = A(t)x + F(t, \nu^*(t), \upsilon^*(t)) \tag{4.13}$$

with the initial condition $x(\tau) = \xi$. By the variation of constant formula it is given by $\Pi_1 \mathcal{T}_\tau(\nu^*, \upsilon^*; \xi)$. Additionally using the mean value theorem we get

$$\begin{aligned} \|G(t, \nu^*(t), \upsilon^*(t))\| \, e_{\ominus c}(t,\tau) &\overset{(4.3)}{=} \|G(t, \nu^*(t), \upsilon^*(t)) - G(t,0,0)\| \, e_{\ominus c}(t,\tau) \leq \\ &\overset{(4.4)}{\leq} |G|_1 \, \|(\nu^*, \upsilon^*)\|^+_{\tau,c} \quad \text{for } t \in \mathbb{T}^+_\tau, \end{aligned}$$

and hence the inhomogeneous part of equation

$$y^\Delta = B(t)y + G(t, \nu^*(t), \upsilon^*(t)) \tag{4.14}$$

is c^+-quasibounded. With the aid of Pötzsche [13, Theorem 4(b)] one can show that the function υ^* is the only c^+-quasibounded solution of (4.14) and has the claimed form.

(b) $\Rightarrow$ (a) If $(\nu^*, \upsilon^*) \in \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y})$ is a solution of the fixed point problem (4.12), the function ν^* has to be the (unique) solution of the dynamic equation (4.13) with $\nu^*(\tau) = \xi$ by the variation of constant formula. Furthermore Pötzsche [13, Theorem 4(b)] again implies that the function υ^* is a solution of the linear system (4.14) in $\mathcal{B}^+_{\tau,c}(\mathcal{Y})$. □

Having all preparatory results at hand, we may now head for our main theorem. As mentioned in the introduction, invariant fiber bundles are generalizations of invariant manifolds to non-autonomous equations. In order to be more precise we call a subset S of the extended state space $I \times \mathcal{X} \times \mathcal{Y}$ an *invariant fiber bundle* if for any triple $(\tau, \xi, \eta) \in S$ one has $(t, \lambda(t; \tau, \xi, \eta)) \in S$ for all $t \in I$, $\tau \preceq t$, where λ denotes the general solution of (4.1).

Theorem 4.9 (pseudo-stable and -unstable fiber bundles)**:** *We assume Hypothesis 4.1 with some $m \in \mathbb{N}$, and let the global Lipschitz constants $|F|_1$ and $|G|_1$ satisfy the estimate*

$$0 \leq \max\{|F|_1, |G|_1\} < \frac{\lfloor b - a \rfloor}{2 \max\{K_1, K_2\}}. \tag{4.15}$$

In addition we choose a fixed real number $\sigma \in \left(\max\{K_1, K_2\} \max\{|F|_1, |G|_1\}, \frac{\lfloor b-a \rfloor}{2}\right]$ and let λ denote the general solution of (4.1). Then the following statements are true:

(a) In case $I = \mathbb{T}^+_{\tau_0}$, $\tau_0 \in \mathbb{T}$, there exists a uniquely determined continuous mapping $s : I \times \mathcal{X} \to \mathcal{Y}$ whose graph $S := \{(\tau, \xi, s(\tau, \xi)) : \tau \in I, \xi \in \mathcal{X}\}$ can be characterized dynamically for any growth rate $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$, $a + \sigma \trianglelefteq c \trianglelefteq b - \sigma$ as

$$S = \left\{(\tau, \xi, \eta) \in I \times \mathcal{X} \times \mathcal{Y} : \lambda(\cdot; \tau, \xi, \eta) \in \mathcal{B}^+_{\tau,c}(\mathcal{X} \times \mathcal{Y})\right\}. \tag{4.16}$$

Furthermore we have

(a_1) $s(\tau, 0) \equiv 0$ on I,

(a_2) $s : I \times \mathcal{X} \to \mathcal{Y}$ is continuously differentiable in its second argument with globally bounded derivative

$$\|D_2 s(\tau, \xi)\| \leq \frac{K_1 K_2 \max\{|F|_1, |G|_1\}}{\sigma - \max\{K_1, K_2\} \max\{|F|_1, |G|_1\}} \quad \text{for } (\tau, \xi) \in I \times \mathcal{X} \tag{4.17}$$

and $s(\cdot, \xi) : \mathcal{X} \to \mathcal{Y}$ is rd-continuously differentiable for any $\xi \in \mathcal{X}$,

(a_3) *provided that* $a + \sigma \trianglelefteq 0$, *then* $D_2^m s : I \times \mathcal{X} \to \mathcal{L}_m(\mathcal{X};\mathcal{Y})$ *exists, is continuous and globally bounded,*

(a_4) *the graph* $S \subseteq I \times \mathcal{X} \times \mathcal{Y}$ *is an invariant fiber bundle of* (4.1) *and* $s : I \times \mathcal{X} \to \mathcal{Y}$ *satisfies the* invariance equation

$$\begin{aligned}\Delta_1 s(\tau,\xi) &= B(\tau)s(\tau,\xi) + G(\tau,\xi,s(\tau,\xi)) - \\ &- \int_0^1 D_2 s\big(\rho^+(\tau),\xi + h\mu^*(\tau)\,[A(\tau)\xi + F(\tau,\xi,s(\tau,\xi))]\big)\,dh\,[A(\tau)\xi + F(\tau,\xi,s(\tau,\xi))]\end{aligned} \tag{4.18}$$

for all $\tau \in I$, $\xi \in \mathcal{X}$.

S is called the pseudo-stable fiber bundle *of* (4.1).

(*b*) *In case* $I = \mathbb{T}$ *there exists a uniquely determined continuous mapping* $r : I \times \mathcal{Y} \to \mathcal{X}$ *whose graph* $R := \{(\tau, r(\tau,\eta),\eta) : \tau \in I, \eta \in \mathcal{Y}\}$ *can be characterized dynamically for any growth rate* $c \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T},\mathbb{R})$, $a + \sigma \trianglelefteq c \trianglelefteq b - \sigma$ *as*

$$R = \left\{(\tau,\xi,\eta) \in I \times \mathcal{X} \times \mathcal{Y} : \lambda(\cdot;\tau,\xi,\eta) \in \mathcal{B}_{\tau,c}^-(\mathcal{X} \times \mathcal{Y})\right\}. \tag{4.19}$$

Furthermore we have

(b_1) $r(\tau,0) \equiv 0$ *on* I,

(b_2) $r : I \times \mathcal{Y} \to \mathcal{X}$ *is continuously differentiable in its second argument with globally bounded derivative*

$$\|D_2 r(\tau,\eta)\| \le \frac{K_1 K_2 \max\{|F|_1, |G|_1\}}{\sigma - \max\{K_1,K_2\}\max\{|F|_1,|G|_1\}} \quad \text{for } (\tau,\eta) \in I \times \mathcal{Y}$$

and $r(\cdot,\eta) : I \to \mathcal{X}$ *is rd-continuously differentiable for any* $\eta \in \mathcal{Y}$,

(b_3) *provided that* $0 \trianglelefteq b - \sigma$, *then* $D_2^m r : I \times \mathcal{Y} \to \mathcal{L}_m(\mathcal{Y};\mathcal{X})$ *exists, is continuous and globally bounded,*

(b_4) *the graph* $R \subseteq I \times \mathcal{X} \times \mathcal{Y}$ *is an invariant fiber bundle of* (4.1) *and* $r : I \times \mathcal{Y} \to \mathcal{X}$ *satisfies the* invariance equation

$$\begin{aligned}\Delta_1 r(\tau,\eta) &= A(\tau)r(\tau,\eta) + F(\tau,r(\tau,\eta),\eta) - \\ &- \int_0^1 D_2 r\big(\rho^+(\tau),\eta + h\mu^*(\tau)\,[B(\tau)\eta + G(\tau,r(\tau,\eta),\eta)]\big)\,dh\,[B(\tau)\eta + G(\tau,r(\tau,\eta),\eta)]\end{aligned}$$

for all $\tau \in I$, $\eta \in \mathcal{Y}$.

R is called the pseudo-unstable fiber bundle *of* (4.1).

(*c*) *In case* $I = \mathbb{T}$ *only the zero solution of equation* (4.1) *is contained both in S and R, i.e.* $S \cap R = \mathbb{T} \times \{0\} \times \{0\}$, *and hence the zero solution is the only* $c^{\pm}$*-quasibounded solution of* (4.1) *for any growth rate* $c \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T},\mathbb{R})$, $a + \sigma \trianglelefteq c \trianglelefteq b - \sigma$.

Remark 4.10: (1) It is easy to see that the existence of suitable values for σ follows from assumption (4.15). Since we have $0 < \sigma \le \frac{\lfloor b-a \rfloor}{2}$ there exist functions $c \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T},\mathbb{R})$ such that $a + \sigma \trianglelefteq c \trianglelefteq b - \sigma$ and in addition $a + \sigma, b - \sigma$ are positively regressive.

(2) Since we have not assumed the regressivity of the dynamic equation (4.1) one has to interpret the dynamical characterization (4.19) of the pseudo-unstable fiber bundle R as follows: A point

$(\tau, \xi, \eta) \in I \times \mathcal{X} \times \mathcal{Y}$ is contained in R if and only if there exists a c^--quasibounded solution $\lambda(\cdot; \tau, \xi, \eta) : I \to \mathcal{X} \times \mathcal{Y}$ of (4.1) satisfying the initial condition $x(\tau) = \xi$, $y(\tau) = \eta$. In this case the solution $\lambda(\cdot; \tau, \xi, \eta)$ is uniquely determined.

(3) The assumption of global boundedness of the derivatives in (4.4) can be replaced by the global Lipschitz continuity of F and G. If this is done the mappings s and r defining the invariant fiber bundles S and R are also only globally Lipschitz continuous with respect to $\xi \in \mathcal{X}$ and $\eta \in \mathcal{Y}$ (uniformly in $\tau \in I$), respectively. This result can be easily derived by slight modifications in the subsequent proof of Theorem 4.9.

(4) By means of cut-off-functions we can deduce a theorem on locally invariant C^1-fiber bundles for equation (4.1) from the above Theorem 4.9. The essential fact here is that one can replace the strong assumption of the existence of $|F|_1, |G|_1 < \infty$ and (4.15) by

$$\lim_{(x,y)\to(0,0)} D_{(2,3)}(F, G)(t, x, y) = 0 \quad \text{uniformly in } t \in I.$$

The detailed construction can be found in many references (cf. e.g. Vanderbauwhede & Van Gils [16]) for autonomous equations and it is easily lifted to our non-autonomous setting. Nonetheless, it is worth mentioning that although C^∞-cut-off-functions always exist in Hilbert spaces, in general infinite-dimensional Banach spaces even C^1-cut-off-functions may fail to exist (cf. Abraham, Marsden & Ratiu [1, p. 273, Lemma 4.2.13]).

(5) If the mappings F and G are m-times continuously differentiable in their variables $(x, y) \in \mathcal{X} \times \mathcal{Y}$ with globally bounded derivatives $D^n_{(2,3)}F$ and $D^n_{(2,3)}G$ for $n \in \{1, \dots, m\}$ then one would expect the same degree of smoothness for the invariant fiber bundles S and R. For this to be true the growth rates $a \lhd b$ have to satisfy a so-called *Gap-Condition* $m \odot a \lhd b$ or $a \lhd m \odot b$, respectively. Such a generalization of Theorem 4.9 can be found for ordinary differential equations in Siegmund [15, p. 73, Satz 8.1]. For difference equations and dynamic equations it will be shown in forthcoming papers. Our Theorem 4.9 at least provides higher order smoothness under the following conditions:

- For $a \lhd b \unlhd 0$ the pseudo-stable fiber bundle S is of class C^m.
- For $0 \unlhd a \lhd b$ the pseudo-unstable fiber bundle R is of class C^m.
- In the hyperbolic case $a \lhd 0 \lhd b$ and under the additional assumption

$$0 \le \max\{|F|_1, |G|_1\} < \frac{\min\{\lfloor -a \rfloor, \lfloor b \rfloor\}}{\max\{K_1, K_2\}}$$

  one can always choose a real number $\sigma \in \left(\max\{K_1, K_2\} \max\{|F|_1, |G|_1\}, \frac{\lfloor b-a \rfloor}{2}\right]$ such that $a + \sigma \unlhd 0 \unlhd b - \sigma$. In this case S and R are as smooth as the right-hand side of the dynamic equation (4.1) and they are called the *stable fiber bundle* and the *unstable fiber bundle* of (4.1), respectively.

However the paper Aulbach, Pötzsche & Siegmund [5, Example 4.13] contains an example of an autonomous difference equations where the pseudo-unstable fiber bundle R is not C^2, even though the nonlinearities F and G are C^∞-mappings.

Proof (of Theorem 4.9): (a) By $\lambda = (\lambda_1, \lambda_2)$ we denote the general solution of the dynamic equation (4.1). We show first that for any pair $(\tau, \xi) \in I \times \mathcal{X}$ there exists exactly one $s(\tau, \xi) \in \mathcal{Y}$

such that $\lambda(\cdot;\tau,\xi,s(\tau,\xi)) \in \mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$ for every $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T},\mathbb{R})$, $a+\sigma \trianglelefteq c \trianglelefteq b-\sigma$. Then the mapping $s : I \times \mathcal{X} \to \mathcal{Y}$ defines the invariant fiber bundle S. Now for $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T},\mathbb{R})$, $a+\sigma \trianglelefteq c \trianglelefteq b-\sigma$ we obtain the estimate

$$\max\left\{\frac{K_1}{\lfloor c-a\rfloor}, \frac{K_2}{\lfloor b-c\rfloor}\right\}\max\left\{|F|_1, |G|_1\right\} \le \frac{\max\left\{K_1,K_2\right\}}{\sigma}\max\left\{|F|_1, |G|_1\right\} =: L. \tag{4.20}$$

Due to assumption (4.15) we immediately get $L \in [0,1)$ and hence the operator $\mathcal{T}_\tau : \mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})\times\mathcal{X} \to \mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$ defined in Lemma 4.8 is a uniform contraction in $\xi \in \mathcal{X}$, since for any $(\nu,\upsilon),(\bar\nu,\bar\upsilon) \in \mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$ and $\xi \in \mathcal{X}$ we have the estimate

$$\begin{aligned}
&\left\|\mathcal{T}_\tau(\nu,\upsilon;\xi) - \mathcal{T}_\tau(\bar\nu,\bar\upsilon;\xi)\right\|^+_{\tau,c} \overset{(4.11)}{=} \left\|\mathcal{K}_\tau\big(\mathcal{G}(\nu,\upsilon)-\mathcal{G}(\bar\nu,\bar\upsilon)\big)\right\|^+_{\tau,c} \le \\
\overset{(4.6)}{\le}\; &\max\left\{\frac{K_1}{\lfloor c-a\rfloor}, \frac{K_2}{\lfloor b-c\rfloor}\right\}\left\|\mathcal{G}(\nu,\upsilon)-\mathcal{G}(\bar\nu,\bar\upsilon)\right\|^+_{\tau,c} \le \\
\overset{(4.8)}{\le}\; &\max\left\{\frac{K_1}{\lfloor c-a\rfloor}, \frac{K_2}{\lfloor b-c\rfloor}\right\}\max\left\{|F|_1,|G|_1\right\}\left\|(\nu,\upsilon)-(\bar\nu,\bar\upsilon)\right\|^+_{\tau,c} \le \\
\overset{(4.20)}{\le}\; &L\left\|(\nu,\upsilon)-(\bar\nu,\bar\upsilon)\right\|^+_{\tau,c}.
\end{aligned} \tag{4.21}$$

Consequently Banach's fixed point theorem (e.g Lang [12, p. 360, Lemma 1.1]) guarantees the unique existence of a fixed point $(\nu^*_\tau(\xi),\upsilon^*_\tau(\xi)) \in \mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$ of $\mathcal{T}_\tau(\cdot;\xi)$. This fixed point is independent of the growth rate $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T},\mathbb{R})$, $a+\sigma \trianglelefteq c \trianglelefteq b-\sigma$ because with Lemma 3.3(b) we have $\mathcal{B}^+_{\tau,a+\sigma}(\mathcal{X}\times\mathcal{Y}) \subseteq \mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$ and every operator $\mathcal{T}_\tau(\cdot;\xi) : \mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y}) \to \mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$ has the same fixed point as the restriction $\mathcal{T}_\tau(\cdot;\xi)\big|_{\mathcal{B}^+_{\tau,a+\sigma}(\mathcal{X}\times\mathcal{Y})}$. Formally we can write

$$J^c_{a+\sigma}(\nu^*_\tau(\xi),\upsilon^*_\tau(\xi)) \equiv (\nu^*_\tau(\xi),\upsilon^*_\tau(\xi)) \equiv \mathcal{T}_\tau(\nu^*_\tau(\xi),\upsilon^*_\tau(\xi);\xi) \quad \text{on } \mathcal{X}. \tag{4.22}$$

By means of Lemma 4.8 the fixed point $(\nu^*_\tau(\xi),\upsilon^*_\tau(\xi))$ is a solution of equation (4.1) with $\big(\nu^*_\tau(\xi)\big)(\tau) = \xi$. Now we define $s(\tau,\xi) := \big(\upsilon^*_\tau(\xi)\big)(\tau)$ and have to prove (4.16).
($\subseteq$) Due to of the uniqueness of solutions (cf. Pötzsche [14, p. 38, Satz 1.2.7(a)]) we obtain the identity $\lambda(\cdot;\tau,\xi,s(\tau,\xi)) = (\nu^*_\tau(\xi),\upsilon^*_\tau(\xi))$ and therefore $\lambda(\cdot;\tau,\xi,s(\tau,\xi)) \in \mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$.
($\supseteq$) On the other hand the function $\lambda(\cdot;\tau,\xi,\eta)$ is a c^+-quasibounded solution of the dynamic equation (4.1) with $\lambda_1(\tau;\tau,\xi,\eta) = \xi$, and with Lemma 4.8 it is the unique solution of the fixed point problem (4.12). So we obtain $(\lambda_1,\lambda_2)(\cdot;\tau,\xi,\eta) = (\nu^*_\tau(\xi),\upsilon^*_\tau(\xi))$ and, hence, $\eta = \big(\upsilon^*_\tau(\xi)\big)(\tau) = s(\tau,\xi)$.

(a_1) Using Hypothesis 4.1(ii) we have $\lambda(t;\tau,0,0) \equiv (0,0)$ on $\mathbb{T}^+_\tau$ and since this zero solution is obviously c^+-quasibounded, the identity

$$s(\tau,0) \equiv \big(\upsilon^*_\tau(0)\big)(\tau) \equiv \lambda_2(\tau;\tau,0,0) \overset{(4.3)}{\equiv} 0 \quad \text{on } I$$

follows from the uniqueness statement proved before.

(a_2) In this step we initially examine the continuous differentiability of the mapping $s(\tau,\cdot) : \mathcal{X} \to \mathcal{Y}$ defining the invariant fiber bundle S. The primary tool in this endeavor is Theorem 5.1 from the appendix whose assumptions we check now. To obtain the notation from Theorem 5.1 we declare for any $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T},\mathbb{R})$, $a+\sigma \triangleleft c \trianglelefteq b-\sigma$, the Banach spaces $\mathcal{X}_0 := \mathcal{B}^+_{\tau,a+\sigma}(\mathcal{X}\times\mathcal{Y})$, $\mathcal{X}_1 := \mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$, $\mathcal{A} := \mathcal{X}$ and consider the operator $\mathcal{T}_\tau$. Due to Lemma 3.3(b) we have the continuous embedding $\mathcal{B}^+_{\tau,a+\sigma}(\mathcal{X}\times\mathcal{Y}) \overset{J^c_{a+\sigma}}{\hookrightarrow} \mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$. By means of of relation (4.21) the operator $\mathcal{T}_\tau : \mathcal{B}^+_{\tau,a+\sigma}(\mathcal{X}\times\mathcal{Y})\times\mathcal{X} \to \mathcal{B}^+_{\tau,a+\sigma}(\mathcal{X}\times\mathcal{Y})$ satisfies a uniform Lipschitz condition

$$\|\mathcal{T}_\tau(\nu,\upsilon;\xi)-\mathcal{T}_\tau(\bar\nu,\bar\upsilon;\xi)\|^+_{\tau,a+\sigma} \le L\,\|(\nu,\upsilon)-(\bar\nu,\bar\upsilon)\|^+_{\tau,a+\sigma}$$

for arbitrary pairs $(\nu,\upsilon),(\bar\nu,\bar\upsilon)\in\mathcal{B}^+_{\tau,a+\sigma}(\mathcal{X}\times\mathcal{Y})$ and $\xi\in\mathcal{X}$. We define the substitution operator $\mathcal{G}^{(1)}_1 : \mathcal{B}^+_{\tau,a+\sigma}(\mathcal{X}\times\mathcal{Y}) \to \mathcal{B}^+_{\tau,0}(\mathcal{L}(\mathcal{X}\times\mathcal{Y}))$ by $\mathcal{G}^{(1)}_1(\nu,\upsilon):=\mathcal{G}^{(1)}(\nu,\upsilon)$; one should keep in mind the range of $\mathcal{G}^{(1)}$ in Lemma 4.6. Now the embedded mapping $J^c_{a+\sigma}\mathcal{T}_\tau$ is continuously differentiable with respect to (ν,υ). This follows from the identity

$$\begin{aligned} D_{(1,2)}(J^c_{a+\sigma}\mathcal{T}_\tau)(\nu,\upsilon;\xi) &\overset{(4.11)}{\equiv} D\big(J^c_{a+\sigma}\mathcal{K}_\tau\mathcal{G}\big)(\nu,\upsilon) \equiv D\big(\mathcal{K}_\tau J^c_{a+\sigma}\mathcal{G}\big)(\nu,\upsilon) \equiv \\ &\equiv \mathcal{K}_\tau D\big(J^c_{a+\sigma}\mathcal{G}\big)(\nu,\upsilon) \equiv \\ &\overset{(4.10)}{\equiv} \mathcal{K}_\tau J_1\mathcal{G}^{(1)}(\nu,\upsilon) \in \mathcal{L}(\mathcal{B}^+_{\tau,a+\sigma}(\mathcal{X}\times\mathcal{Y});\mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})) \end{aligned}$$

and Lemma 4.7. It is obvious that the two linear operators $\mathcal{K}_\tau\in\mathcal{L}(\mathcal{B}^+_{\tau,a+\sigma}(\mathcal{X}\times\mathcal{Y}))$ and $\mathcal{K}_\tau\in\mathcal{L}(\mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y}))$, respectively, and $J^c_{a+\sigma}$ commute; the continuous homomorphism $\mathcal{K}_\tau$ and the differential operator D commute because of Lang [12, p. 339, Corollary 3.2]. Furthermore, we have

$$D_{(1,2)}(J^c_{a+\sigma}\mathcal{T}_\tau)(\nu,\upsilon;\xi) \equiv J^c_{a+\sigma}\mathcal{K}_\tau J_1\mathcal{G}^{(1)}_1(\nu,\upsilon) \equiv \mathcal{K}_\tau J_1\mathcal{G}^{(1)}(\nu,\upsilon)J^c_{a+\sigma},$$

and hence relation (5.1) is verified for the (not necessarily continuous) operators

$$\begin{aligned} \mathcal{T}^{(1)}_1 &:= \mathcal{K}_\tau J_1\mathcal{G}^{(1)}_1 : \mathcal{B}^+_{\tau,a+\sigma}(\mathcal{X}\times\mathcal{Y}) \to \mathcal{L}(\mathcal{B}^+_{\tau,a+\sigma}(\mathcal{X}\times\mathcal{Y})), \\ \mathcal{T}^{(1)} &:= \mathcal{K}_\tau J_1\mathcal{G}^{(1)} : \mathcal{B}^+_{\tau,a+\sigma}(\mathcal{X}\times\mathcal{Y}) \to \mathcal{L}(\mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})). \end{aligned}$$

Since $\mathcal{T}_\tau$ is linear in ξ, it is differentiable and the derivative is given by

$$D_3\mathcal{T}_\tau(\nu,\upsilon;\xi) \overset{(4.11)}{\equiv} \mathcal{S}_\tau \in \mathcal{L}(\mathcal{X};\mathcal{B}^+_{\tau,a+\sigma}(\mathcal{X}\times\mathcal{Y}));$$

obviously $D_3\mathcal{T}_\tau$ is continuous and hence $\mathcal{T}_\tau$ is continuously differentiable with respect to the parameter $\xi\in\mathcal{X}$. After all, for any pair $(\nu,\upsilon)\in\mathcal{B}^+_{\tau,a+\sigma}(\mathcal{X}\times\mathcal{Y})$ we have the estimate

$$\begin{aligned} \left\|\mathcal{T}^{(1)}_1(\nu,\upsilon)\right\|_{\mathcal{L}(\mathcal{B}^+_{\tau,a+\sigma}(\mathcal{X}\times\mathcal{Y}))} &\overset{(4.6)}{\le} \max\left\{\frac{K_1}{\sigma},\frac{K_2}{\lfloor b-a\rfloor-\sigma}\right\}\left\|J_1\mathcal{G}^{(1)}_1(\nu,\upsilon)\right\|_{\mathcal{L}(\mathcal{B}^+_{\tau,a+\sigma}(\mathcal{X}\times\mathcal{Y}))} \le \\ &\overset{(3.2)}{\le} \max\left\{\frac{K_1}{\sigma},\frac{K_2}{\lfloor b-a\rfloor-\sigma}\right\}\left\|\mathcal{G}^{(1)}_1(\nu,\upsilon)\right\|^+_{\tau,1} \le \\ &\overset{(4.9)}{\le} \max\left\{\frac{K_1}{\sigma},\frac{K_2}{\lfloor b-a\rfloor-\sigma}\right\}\max\{|F|_1,|G|_1\} \overset{(4.20)}{\le} L \end{aligned}$$

as well as $\left\|\mathcal{T}^{(1)}(\nu,\upsilon)\right\|_{\mathcal{L}(\mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y}))} \le L$. From Theorem 5.1 the mapping $J^c_{a+\sigma}\upsilon^*_\tau = \upsilon^*_\tau : \mathcal{X}\to\mathcal{B}^+_{\tau,c}(\mathcal{Y})$ has to be of class C^1. Using Lemma 3.4 we also know that $s(\tau,\cdot)=\big(\upsilon^*_\tau(\cdot)\big)(\tau):\mathcal{X}\to\mathcal{Y}$ is, for any fixed $\tau\in I$, a continuously differentiable mapping. Next we prove the estimate (4.17). To this end we consider $\xi,\bar\xi\in\mathcal{X}$ and the corresponding fixed points $(\nu^*_\tau(\xi),\upsilon^*_\tau(\xi)),(\nu^*_\tau(\bar\xi),\upsilon^*_\tau(\bar\xi))\in\mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$ of $\mathcal{T}_\tau(\cdot;\xi)$ and $\mathcal{T}_\tau(\cdot;\bar\xi)$. For growth rates $c\in\mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T},\mathbb{R})$, $a+\sigma\trianglelefteq c\trianglelefteq b-\sigma$ we have

$$\left\|(\nu^*_\tau(\xi),\upsilon^*_\tau(\xi))-(\nu^*_\tau(\bar\xi),\upsilon^*_\tau(\bar\xi))\right\|^+_{\tau,c} \le$$

$$
\begin{aligned}
&\overset{(4.22)}{\leq} \left\|\mathcal{T}_\tau(\nu_\tau^*(\xi), \upsilon_\tau^*(\xi); \xi) - \mathcal{T}_\tau(\nu_\tau^*(\bar{\xi}), \upsilon_\tau^*(\bar{\xi}); \xi)\right\|_{\tau,c}^+ + \\
&\quad + \left\|\mathcal{T}_\tau(\nu_\tau^*(\bar{\xi}), \upsilon_\tau^*(\bar{\xi}); \xi) - \mathcal{T}_\tau(\nu_\tau^*(\bar{\xi}), \upsilon_\tau^*(\bar{\xi}); \bar{\xi})\right\|_{\tau,c}^+ \leq \\
&\overset{(4.21)}{\leq} \frac{\max\{K_1, K_2\}}{\sigma} \max\{|F|_1, |G|_1\} \left\|(\nu_\tau^*(\xi), \upsilon_\tau^*(\xi)) - (\nu_\tau^*(\bar{\xi}), \upsilon_\tau^*(\bar{\xi}))\right\|_{\tau,c}^+ + \\
&\quad + \left\|\mathcal{T}_\tau(\nu_\tau^*(\bar{\xi}), \upsilon_\tau^*(\bar{\xi}); \xi) - \mathcal{T}_\tau(\nu_\tau^*(\bar{\xi}), \upsilon_\tau^*(\bar{\xi}); \bar{\xi})\right\|_{\tau,c}^+ ,
\end{aligned}
$$

consequently,

$$
\begin{aligned}
&\left\|(\nu_\tau^*(\xi), \upsilon_\tau^*(\xi)) - (\nu_\tau^*(\bar{\xi}), \upsilon_\tau^*(\bar{\xi}))\right\|_{\tau,c}^+ \leq \\
\leq\ & \frac{\sigma}{\sigma - \max\{K_1, K_2\}\max\{|F|_1, |G|_1\}} \cdot \\
&\quad \cdot \left\|\mathcal{T}_\tau(\nu_\tau^*(\bar{\xi}), \upsilon_\tau^*(\bar{\xi}); \xi) - \mathcal{T}_\tau(\nu_\tau^*(\bar{\xi}), \upsilon_\tau^*(\bar{\xi}); \bar{\xi})\right\|_{\tau,c}^+ = \\
\overset{(4.11)}{=}\ & \frac{\sigma}{\sigma - \max\{K_1, K_2\}\max\{|F|_1, |G|_1\}} \left\|\mathcal{S}_\tau(\xi - \bar{\xi})\right\|_{\tau,c}^+ \leq \\
\overset{(4.5)}{\leq}\ & \frac{\sigma K_1}{\sigma - \max\{K_1, K_2\}\max\{|F|_1, |G|_1\}} \left\|\xi - \bar{\xi}\right\| .
\end{aligned}
\tag{4.23}
$$

Finally, the mapping $s(\tau, \cdot) : \mathcal{X} \to \mathcal{Y}$ is globally Lipschitzian uniformly in $\tau \in I$ because

$$
\begin{aligned}
\left\|s(\tau, \xi) - s(\tau, \bar{\xi})\right\| &= \left\|\big(\upsilon_\tau^*(\xi)\big)(\tau) - \big(\upsilon_\tau^*(\bar{\xi})\big)(\tau)\right\| \leq \left\|\upsilon_\tau^*(\xi) - \upsilon_\tau^*(\bar{\xi})\right\|_{\tau,c}^+ = \\
&= \left\|\Pi_2\left(\mathcal{T}_\tau(\nu_\tau^*(\xi), \upsilon_\tau^*(\xi); \xi) - \mathcal{T}_\tau(\nu_\tau^*(\bar{\xi}), \upsilon_\tau^*(\bar{\xi}); \bar{\xi})\right)\right\|_{\tau,c}^+ = \\
&\overset{(4.11)}{=} \left\|\Pi_2\left(\mathcal{K}_\tau\big(\mathcal{G}(\nu_\tau^*(\xi), \upsilon_\tau^*(\xi)) - \mathcal{G}(\nu_\tau^*(\bar{\xi}), \upsilon_\tau^*(\bar{\xi}))\big)\right)\right\|_{\tau,c}^+ \leq \\
&\overset{(4.7)}{\leq} \frac{K_2}{\lfloor b - c \rfloor} \left\|\mathcal{G}(\nu_\tau^*(\xi), \upsilon_\tau^*(\xi)) - \mathcal{G}(\nu_\tau^*(\bar{\xi}), \upsilon_\tau^*(\bar{\xi}))\right\|_{\tau,c}^+ \leq \\
&\overset{(4.8)}{\leq} \frac{K_2 \max\{|F|_1, |G|_1\}}{\sigma} \left\|(\nu_\tau^*(\xi), \upsilon_\tau^*(\xi)) - (\nu_\tau^*(\bar{\xi}), \upsilon_\tau^*(\bar{\xi}))\right\|_{\tau,c}^+ \leq \\
&\overset{(4.23)}{\leq} \frac{K_1 K_2 \max\{|F|_1, |G|_1\}}{\sigma - \max\{K_1, K_2\}\max\{|F|_1, |G|_1\}} \left\|\xi - \bar{\xi}\right\| .
\end{aligned}
$$

Since differentiable and globally Lipschitz continuous mappings (here with Lipschitz constant $L_0 := \frac{K_1 K_2 \max\{|F|_1, |G|_1\}}{\sigma - \max\{K_1, K_2\}\max\{|F|_1, |G|_1\}}$) have a derivative which is bounded by L_0, the estimate (4.17) follows. Finally the continuity of $s : I \times \mathcal{Y} \to \mathcal{Y}$ and of the derivative $D_2 s : I \times \mathcal{X} \to \mathcal{Y}$ result from Pötzsche [14, p. 139, Lemma 3.1.3(a)].

(a_3) We are going to show now that $s(\tau, \cdot) : \mathcal{X} \to \mathcal{Y}$ is m-times continuously differentiable under the assumption $a + \sigma \trianglelefteq 0$. Thereby we do not have to use the whole embedding procedure, but rather may use the well-known uniform contraction principle (see e.g. Chow & Hale [7, p. 25, Theorem 2.2]), applied to the uniform contraction $\mathcal{T}_\tau : \mathcal{B}_{\tau,c}^+(\mathcal{X} \times \mathcal{Y}) \times \mathcal{X} \to \mathcal{B}_{\tau,c}^+(\mathcal{X} \times \mathcal{Y})$. We may choose $c \trianglelefteq 0$ and $m \geq 2$ here. Using the chain rule and by setting $c = d$ in Lemma 4.6(c), we see that $\mathcal{T}_\tau(\cdot; \xi) : \mathcal{B}_{\tau,c}^+(\mathcal{X} \times \mathcal{Y}) \to \mathcal{B}_{\tau,c}^+(\mathcal{X} \times \mathcal{Y})$, $\xi \in \mathcal{X}$, is m-times continuously differentiable with the derivative $D_{(1,2)}^m \mathcal{T}_\tau(\nu, \upsilon; \xi) = \mathcal{K}_\tau J_m \mathcal{G}^{(m)}(\nu, \upsilon)$. On the other hand, $\mathcal{T}_\tau(\nu, \upsilon; \cdot) : \mathcal{X} \to \mathcal{B}_{\tau,c}^+(\mathcal{X} \times \mathcal{Y})$, $(\nu, \upsilon) \in \mathcal{B}_{\tau,c}^+(\mathcal{X} \times \mathcal{Y})$ is a linear continuous mapping and consequently C^∞ with identically vanishing derivatives of order $m \geq 2$. For this reason $\mathcal{T}_\tau$ is m-times continuously differentiable and with the aid of the uniform contraction principle the fixed-point mapping $(\nu_\tau^*, \upsilon_\tau^*) : \mathcal{X} \to$

$\mathcal{B}^+_{\tau,c}(\mathcal{X}\times\mathcal{Y})$ is of the class C^m as well. Using Lemma 3.4 again, for arbitrarily fixed times $\tau\in I$, the mapping $s(\tau,\cdot)=\big(\upsilon^*_\tau(\cdot)\big)(\tau):\mathcal{X}\to\mathcal{Y}$ is m-times continuously differentiable and PÖTZSCHE [14, p. 139, Lemma 3.1.3(a)] implies the continuity of $D_2^m s: I\times\mathcal{X}\to\mathcal{L}_m(\mathcal{X};\mathcal{Y})$. To show the global boundedness of the derivatives we differentiate the identity

$$(\nu^*_\tau(\xi),\upsilon^*_\tau(\xi))=\mathcal{S}_\tau\xi+\mathcal{K}_\tau\mathcal{G}(\nu^*_\tau(\xi),\upsilon^*_\tau(\xi))\quad\text{on }\mathcal{X}$$

with respect to $\xi\in\mathcal{X}$ by using the higher order chain rule (see ABRAHAM, MARSDEN & RATIU [1, pp. 96–97]). Then mathematical induction and Lemma 4.6 lead to the assertion, since the derivatives $D^n\mathcal{G}$, $n\in\{1,\dots,m\}$, are globally bounded and $\mathcal{K}_\tau$ is also continuous.

(a_4) So far the proof has established the fact that the function $\lambda(\cdot;\tau,\xi,\eta)$ is c^+-quasibounded for arbitrary pairs of initial values $(\tau,\xi,\eta)\in S$. The cocycle property (2.3) now implies for any time $t_0\in\mathbb{T}^+_\tau$ that

$$\lambda(t;t_0,\lambda(t_0;\tau,\xi,\eta))\overset{(2.3)}{\equiv}\lambda(t;\tau,\xi,\eta)\quad\text{on }\mathbb{T}^+_{t_0}.$$

Hence also $\lambda(\cdot;t_0,\lambda(t_0;\tau,\xi,\eta))$ is a c^+-quasibounded function and additionally this yields $(t_0,\lambda(t_0;\tau,\xi,\eta))\in S$ for any $t_0\in\mathbb{T}^+_\tau$. The invariance equation (4.18) is a consequence of PÖTZSCHE [14, p. 139, Lemma 3.1.3(b)].

(b) Since part (b) of the theorem can be proved along the same lines as part (a) we present only a sketch of the proof. Analogously to Lemma 4.8, for initial values $\eta\in\mathcal{Y}$, the c^--quasibounded solutions of (4.1) may be characterized as the fixed points of a mapping $\bar{\mathcal{T}}_\tau:\mathcal{B}^-_{\tau,c}(\mathcal{X}\times\mathcal{Y})\times\mathcal{Y}\to\mathcal{B}^-_{\tau,c}(\mathcal{X}\times\mathcal{Y})$,

$$\begin{aligned}\big(\bar{\mathcal{T}}_\tau(\nu^*,\upsilon^*;\eta)\big)(t) &:= \Bigg(\int_{-\infty}^t\Phi_A(t,\rho^+(s))F(s,\nu^*(s),\upsilon^*(s))\,\Delta s,\\ &\qquad \Phi_B(t,\tau)\eta+\int_\tau^t\Phi_B(t,\rho^+(s))G(s,\nu^*(s),\upsilon^*(s))\,\Delta s\Bigg).\end{aligned}$$

Furthermore, $\bar{\mathcal{T}}_\tau$ can be decomposed into two linear operators and a substitution operator, as we have done for $\mathcal{T}_\tau$ in relation (4.11). Now counterparts to our preparatory Lemmas 4.3, 4.4, 4.5, 4.6 and 4.7 hold true in the Banach spaces $\mathcal{B}^-_{\tau,c}(\mathcal{X}\times\mathcal{Y})$. Note that in order to prove the counterpart of Lemma 4.4 (on the linear operator $\mathcal{K}_\tau$) one has to use PÖTZSCHE [13, Theorem 2(b), Theorem 4(a)] for c^--quasibounded perturbations and solutions. It follows from the assumption (4.15) that $\bar{\mathcal{T}}_\tau$ is a contraction on $\mathcal{B}^-_{\tau,c}(\mathcal{X}\times\mathcal{Y})$ and if $(\nu^*_\tau(\eta),\upsilon^*_\tau(\eta))\in\mathcal{B}^-_{\tau,c}(\mathcal{X}\times\mathcal{Y})$ denotes its unique fixed point we define the mapping $r:I\times\mathcal{Y}\to\mathcal{X}$ by $r(\tau,\eta):=\big(\nu^*_\tau(\eta)\big)(\tau)$. The claimed properties of r can be proved using the same arguments as in step (a).

(c) Let $(\nu,\upsilon):\mathbb{T}\to\mathcal{X}\times\mathcal{Y}$ be an arbitrary solution of (4.1) in $\mathcal{B}^\pm_c(\mathcal{X}\times\mathcal{Y})$. By means of Hypothesis 4.1(ii), the mapping $G(\cdot,\nu(\cdot),\upsilon(\cdot))$ is $c^\pm$-quasibounded and as a consequence of PÖTZSCHE [13, Theorem 2(c)] ν is the unique $c^\pm$-quasibounded solution of $x^\Delta=A(t)x+F(t,\nu(t),\upsilon(t))$, which additionally satisfies the estimate

$$\|\nu\|^\pm_{\tau,c}\le\frac{K_1}{\lfloor c-a\rfloor}\|F(\cdot,\nu(\cdot),\upsilon(\cdot))\|^\pm_{\tau,c}\overset{(4.4)}{\le}\frac{K_1\,|F|_1}{\lfloor c-a\rfloor}\|(\nu,\upsilon)\|^\pm_{\tau,c}\overset{(4.20)}{\le}L\,\|(\nu,\upsilon)\|^\pm_{\tau,c}.\tag{4.24}$$

A dual argument using PÖTZSCHE [13, Theorem 4(c)] applied to the linear dynamic equation $y^\Delta=B(t)y+G(t,\nu(t),\upsilon(t))$ leads to $\|\upsilon\|^\pm_{\tau,c}\le L\,\|(\nu,\upsilon)\|^\pm_{\tau,c}$ and together with (4.24) this yields

$$\|(\nu,\upsilon)\|^\pm_{\tau,c}\overset{(2.1)}{\le}L\,\|(\nu,\upsilon)\|^\pm_{\tau,c}.$$

We therefore obtain $(\nu,\upsilon)=(0,0)$ because of $L<1$ and the proof of Theorem 4.9 is complete. □

5 Appendix

Since the substitution operators under consideration (see Lemma 4.5) become differentiable only after composition with certain embeddings, we present here an appropriate fixed point theorem to keep the paper self-contained. It goes back to VANDERBAUWHEDE & VAN GILS [16].

Theorem 5.1 (contractions between embedded Banach spaces)**:** *Consider three Banach spaces $\mathcal{X}_0, \mathcal{X}_1$ and $\mathcal{A}$ with a continuous embedding*

$$\mathcal{X}_0 \overset{J}{\hookrightarrow} \mathcal{X}_1.$$

Furthermore the operator $\mathcal{T} : \mathcal{X}_0 \times \mathcal{A} \to \mathcal{X}_0$ is assumed to satisfy the following assumptions:

(i) There exists a constant $L \in [0, 1)$ with

$$\|\mathcal{T}(x; \alpha) - \mathcal{T}(\bar{x}; \alpha)\| \leq L \|x - \bar{x}\| \quad \text{for } x, \bar{x} \in \mathcal{X}_0,\ \alpha \in \mathcal{A},$$

(ii) $J\mathcal{T} : \mathcal{X}_0 \times \mathcal{A} \to \mathcal{X}_1$ has a continuous partial derivative $D_1(J\mathcal{T}) : \mathcal{X}_0 \times \mathcal{A} \to \mathcal{L}(\mathcal{X}_0; \mathcal{X}_1)$, where

$$D_1(J\mathcal{T})(x; \alpha) \equiv J\mathcal{T}_1^{(1)}(x; \alpha) \equiv \mathcal{T}^{(1)}(x; \alpha)J \quad \text{on } \mathcal{X}_0 \times \mathcal{A} \tag{5.1}$$

for certain operators $\mathcal{T}_1^{(1)} : \mathcal{X}_0 \times \mathcal{A} \to \mathcal{L}(\mathcal{X}_0)$ and $\mathcal{T}^{(1)} : \mathcal{X}_0 \times \mathcal{A} \to \mathcal{L}(\mathcal{X}_1)$,

(iii) $\mathcal{T}$ has a continuous partial derivative $D_2\mathcal{T} : \mathcal{X}_0 \times \mathcal{A} \to \mathcal{L}(\mathcal{A}; \mathcal{X}_0)$,

(iv) $\mathcal{T}_1^{(1)}$ and $\mathcal{T}^{(1)}$ are bounded by the constant L defined above, i.e.

$$\left\|\mathcal{T}_1^{(1)}(x; \alpha)\right\|_{\mathcal{L}(\mathcal{X}_0)} \leq L, \qquad \left\|\mathcal{T}^{(1)}(x; \alpha)\right\|_{\mathcal{L}(\mathcal{X}_1)} \leq L \quad \text{for } (x, \alpha) \in \mathcal{X}_0 \times \mathcal{A}.$$

Then for arbitrary parameter values $\alpha \in \mathcal{A}$ the operator $\mathcal{T}(\cdot; \alpha)$ has exactly one fixed point $x^*(\alpha) \in \mathcal{X}_0$, i.e. there exists a mapping $x^* : \mathcal{A} \to \mathcal{X}_0$ with the property $\mathcal{T}(x^*(\alpha); \alpha) \equiv x^*(\alpha)$ on $\mathcal{A}$. Additionally x^* is Lipschitz continuous and $Jx^* : \mathcal{A} \to \mathcal{X}_1$ is continuously differentiable with derivative $D\big(Jx^*\big)(\alpha) = JT(\alpha)$, where $T(\alpha) \in \mathcal{L}(\mathcal{A}; \mathcal{X}_0)$ is the unique fixed point of the linear operator equation $T = \mathcal{T}_1^{(1)}(x^*(\alpha); \alpha)T + D_2\mathcal{T}(x^*(\alpha); \alpha)$.

Proof. Theorem 5.1 follows from VANDERBAUWHEDE & VAN GILS [16, Theorem 3] as well as from HILGER [9, Theorem 6.1]. □

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