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ON PERIODIC DYNAMIC EQUATIONS ON MEASURE CHAINS

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ABSTRACT. We prove some basic results on the existence of periodic solutions for linear and non-linear dynamic equations on measure chains or time scales.
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1. INTRODUCTION AND PRELIMINARIES

The calculus on measure chains (or time scales) has originally been developed in [Hil90] to unify the theory of ordinary differential equations (ODEs)

$$\dot{x} = f(t,x) \qquad \text{with } t \in \mathbb{R}$$

and of difference equations (OΔEs)

$$\Delta x = f(t,x) \qquad \text{with } t \in \mathbb{N}_0 \text{ or } t \in \mathbb{Z}$$

within the framework of so-called *dynamic equations* on time scales

$$x^\Delta = f(t,x) \qquad \text{with } t \in \mathbb{T}, \tag{1.1}$$

where $\mathbb{T}$ is allowed to be any closed subset of the reals, or more general, an abstract measure chain, and the derivative $x^\Delta(t)$ reduces to $\dot{x}(t) = \frac{dx}{dt}(t)$ for $\mathbb{T} = \mathbb{R}$, or to the forward difference $\Delta x(t) = x(t+1) - x(t)$ for $\mathbb{T} = \mathbb{N}_0$, $\mathbb{T} = \mathbb{Z}$. Moreover, this flexible set-up makes it possible to consider dynamic equations on time scales beyond $\mathbb{R}$ or $\mathbb{N}_0$, $\mathbb{Z}$, like, e.g., $\mathbb{T} = \bigcup_{k\in\mathbb{N}_0} [k, k+h]$ with $h \geq 0$. Here one is able to model systems possessing continuous, as well as discrete growth features, simultaneously. A possible application comes from biology, to describe the behavior of populations with hibernation periods, or of insect populations laying eggs before dying out over the winter period and hatching in spring (cf. Example 1.5).

Despite these interesting perspectives, periodic dynamic equations are rarely studied until now. So the first goal of this paper is to introduce the notion of periodicity on general measure chains. To overcome the lacking algebraic structure on such sets, one needs so-called translations as a key tool. Having this concept available, we are in the position to present some existence results for periodic solutions of periodic dynamic equations on (periodic) measure chains, which turn out to be very similar to the corresponding well-known results for ODEs (cf., e.g., [Ama95, Far94]).

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Our terminology is as follows. We write $\mathbb{Z}$ for the integers, $\mathbb{N}_0$ for the nonnegative integers and $\mathbb{R}$ for the real field. Throughout this paper, Banach spaces $\mathcal{X}$ are real or complex and their norm is denoted by $\|\cdot\|$. $\mathcal{L}(\mathcal{X})$ is the Banach space of endomorphisms T on $\mathcal{X}$ with $\|T\| := \sup_{\|x\|=1} \|Tx\| < \infty$, $\mathcal{GL}(\mathcal{X})$ the group of toplinear isomorphisms on $\mathcal{X}$, $I_{\mathcal{X}}$ the identity map on $\mathcal{X}$ and $\mathcal{X}'$ the dual space of $\mathcal{X}$, i.e. the linear space of linear bounded functionals on $\mathcal{X}$; $\langle x, x'\rangle$ stands for the duality pairing of $x \in \mathcal{X}$, $x' \in \mathcal{X}'$ and the dual operator $T' \in \mathcal{L}(\mathcal{X}')$ of $T \in \mathcal{L}(\mathcal{X})$ is given by $\langle x, T'x'\rangle = \langle Tx, x'\rangle$ for $x \in \mathcal{X}$, $x' \in \mathcal{X}'$. After all, we write $\bar{B}_r := \{x \in \mathcal{X} : \|x\| \leq r\}$ for the closed ball around $0 \in \mathcal{X}$ with radius $r > 0$; Ω° denotes the interior and $\partial\Omega$ the boundary of a set $\Omega \subseteq \mathcal{X}$.

We also introduce some typical notations from the calculus on measure chains (cf. [Hil90, BP01]). In all the subsequent considerations we deal with a *measure chain* $(\mathbb{T}, \preceq, \mu)$, i.e. a conditionally complete totally ordered set $(\mathbb{T}, \preceq)$ (see [Hil90, Axiom 2]) with the *growth calibration* $\mu : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ (see [Hil90, Axiom 3]), such that $\mu(\mathbb{T}, \tau) \subseteq \mathbb{R}$, $\tau \in \mathbb{T}$, is unbounded above. For readers not familiar with these abstract concepts, the most intuitive and relevant examples of measure chains are *time scales*, where $\mathbb{T}$ is a canonically ordered closed subset of the reals $\mathbb{R}$ and μ is given by $\mu(t, s) = t - s$. Continuing, $\sigma : \mathbb{T} \to \mathbb{T}$, $\sigma(t) := \inf \{s \in \mathbb{T} : t \prec s\}$ defines the *forward jump operator* and the *graininess* is $\mu^* : \mathbb{T} \to \mathbb{R}$, $\mu^*(t) := \mu(\sigma(t), t)$. For $\tau, t \in \mathbb{T}$ we abbreviate $\mathbb{T}_\tau^+ := \{s \in \mathbb{T} : \tau \preceq s\}$, $[\tau, t]_{\mathbb{T}} := \{s \in \mathbb{T} : \tau \preceq s \preceq t\}$, and (half-) open intervals are defined analogously.

$\mathcal{C}_{rd}(\mathbb{T}, \mathcal{X})$ denotes the linear space of rd-continuous functions from $\mathbb{T}$ to $\mathcal{X}$ and $\mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T}, \mathbb{R}) := \{c \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}) : 1 + \mu^*(t)c(t) > 0 \text{ for } t \in \mathbb{T}\}$ forms the so-called *positively regressive group* (cf. [Hil90, Section 7]). *Growth rates* are functions $a \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T}, \mathbb{R})$ such that $\inf_{t\in\mathbb{T}} \mu^*(t)a(t) > -1$ and $\sup_{t\in\mathbb{T}} \mu^*(t)a(t) < \infty$ holds. Furthermore, we define the relation

$$a \lhd b \quad :\Leftrightarrow \quad 0 < \inf_{t\in\mathbb{T}} (b(t) - a(t)) .$$

A mapping $\phi : \mathbb{T} \to \mathcal{X}$ is said to be *differentiable* (in a point $t_0 \in \mathbb{T}$), if there exists a unique *derivative* $\phi^\Delta(t_0) \in \mathcal{X}$, such that for any $\epsilon > 0$ the estimate

$$\left\|\phi(\sigma(t_0)) - \phi(t) - \mu(\sigma(t_0), t)\phi^\Delta(t_0)\right\| \leq \epsilon \left|\mu(\sigma(t_0), t)\right| \quad \text{for } t \in U$$

holds in a $\mathbb{T}$-neighborhood U of t_0 (see [Hil90, Section 2.4]). The Cauchy integral of ϕ is denoted as $\int_\tau^t \phi(s)\,\Delta s$ for $\tau, t \in \mathbb{T}$, provided it exists (cf. [Hil90, Section 4.3]).

Let $T \in \mathbb{R}$ be given. A function $\sigma_T : \mathbb{T} \to \mathbb{T}$ is called *translation*, if the identity $\mu(\sigma_T(t), t) \equiv T$ is valid on $\mathbb{T}$. In case, for given $T > 0$ there exists a translation σ_T, then the measure chain $(\mathbb{T}, \preceq, \mu)$ is denoted as *T-periodic* and this will be a standard assumption from now on throughout the present paper. Moreover, we say a mapping $\phi : \mathbb{T} \to \mathcal{X}$ is *T-periodic*, if $\phi(\sigma_T(t)) \equiv \phi(t)$ holds on $\mathbb{T}$. Thus the graininess μ^* is T-periodic in our setting (cf. [Pöt02, p. 3, Korollar 1.1.11(e)]). Using the differentiation concept for measure chain-valued mappings introduced in [Pöt02, p. 6, Definition 1.1.14(a)], one can show that $\sigma_T : \mathbb{T} \to \mathbb{T}$ is differentiable with $\sigma_T^\Delta(t) \equiv 1$ on $\mathbb{T}$ (cf. [Pöt02, p. 6, Korollar 1.1.16]).

Example 1.1. *On time scales, translations are of the form $\sigma_T(t) = t+T$. The time scale $\mathbb{R}$ is T-periodic for any real $T > 0$, and $\mathbb{N}_0$ or $\mathbb{Z}$ are T-periodic for arbitrary integers $T > 0$. Less trivial examples of 1-periodic time scales are sets of the form $\mathbb{T} = \bigcup_{k\in\mathbb{N}_0} \{k + s \in \mathbb{R} : s \in S\}$, where $S \subseteq [0, 1]$ is closed, nonempty.*

Let Ω be a non-empty, open and connected subset of $\mathcal{X}$. For a non-autonomous dynamic equation (1.1) with a *right-hand side* $f : \mathbb{T}\times\Omega \to \mathcal{X}$, we say that a function $\nu : I \to \Omega$ is a *solution* of (1.1), if $\nu^\Delta(t) \in \mathcal{X}$ exists and the identity $\nu^\Delta(t) \equiv f(t, \nu(t))$ holds on a subset $I \subseteq \mathbb{T}$. For right-hand sides f guaranteeing existence and uniqueness of solutions in forward time (cf., e.g., [Pöt02, p. 38, Satz 1.2.17(a)]), as well as their continuous dependence on the initial conditions, let $\varphi(t; \tau, \xi)$ denote the *general solution* of (1.1), i.e. $\varphi(\cdot; \tau, \xi)$ solves (1.1) on $\mathbb{T}^+_\tau$ and satisfies the initial condition $\varphi(\tau; \tau, \xi) = \xi$ for $\tau \in \mathbb{T}$, $\xi \in \Omega$. Finally, we say (1.1), or its right-hand side, is T*-periodic*, if $f(\sigma_T(t), x) \equiv f(t, x)$ is satisfied on $\mathbb{T} \times \Omega$. In this situation, it is easy to see by direct computation using the chain rule (cf. [Pöt02, p. 13, Satz 1.1.25]) that, if ν is a solution of equation (1.1), then for any integer $k \in \mathbb{N}_0$, $\nu \circ \sigma_{kT}$ is also a solution. This observation, in conjunction with the uniqueness of solutions of initial value problems, implies the identities

$$\varphi(\sigma_T(t); \sigma_T(\tau), \xi) = \varphi(t; \tau, \xi), \qquad \varphi(\sigma_T(t); \tau, \xi) = \varphi(t; \tau, \varphi(\sigma_T(\tau); \tau, \xi)) \tag{1.2}$$

for $\tau, t \in \mathbb{T}$, $\tau \preceq t$, $\xi \in \Omega$. Before proceeding, we present an elementary result which is valid for general periodic systems.

Lemma 1.2. *Let $\tau \in \mathbb{T}$ and assume equation* (1.1) *is T-periodic. Then a solution $\nu : \mathbb{T}^+_\tau \to \Omega$ of* (1.1) *is T-periodic, if and only if a $t_0 \in \mathbb{T}^+_\tau$ exists for which*

$$\nu(\sigma_T(t_0)) = \nu(t_0). \tag{1.3}$$

Proof. If ν is a T-periodic solution, then (1.3) obviously holds true. Conversely, the function $\nu \circ \sigma_T : \mathbb{T}^+_\tau \to \Omega$ is also a solution of (1.1), because the periodicity of f implies

$$(\nu \circ \sigma_T)^\Delta(t) \equiv \nu^\Delta(\sigma_T(t)) \overset{(1.1)}{\equiv} f\big(\sigma_T(t), \nu(\sigma_T(t))\big) \equiv f\big(t, \nu(\sigma_T(t))\big) \quad \text{on } \mathbb{T}^+_\tau$$

by the chain rule (cf. [Pöt02, p. 13, Satz 1.1.25]). Due to (1.3), we have $\nu\circ\sigma_T(t_0) = \nu(t_0)$, and the uniqueness of solutions yields $\nu \circ \sigma_T = \nu$, i.e. ν is T-periodic. □

From now on keep a reference instant $\tau_0 \in \mathbb{T}$ arbitrarily fixed throughout the paper and assume that $\varphi(\cdot; \tau_0, \xi)$ exists (at least) on $[\tau_0, \sigma_T(\tau_0)]_\mathbb{T}$ for any $\xi \in \Omega$. Then we denote $\phi_T : \Omega \to \Omega$,

$$\phi_T(\xi) := \varphi(\sigma_T(\tau_0); \tau_0, \xi)$$

as *time-T-map* of (1.1). Because of our assumptions, ϕ_T is continuous.

Lemma 1.3 (time-T-map)**.** *Assume that equation* (1.1) *is T-periodic. Then* (1.1) *possesses a T-periodic solution, if and only if ϕ_T has a fixed point.*

Proof. ($\Rightarrow$) Let ν be a T-periodic solution of (1.1). In case ν is defined only on $\mathbb{T}^+_\tau$ for some $\tau \in \mathbb{T}$, we extend ν T-periodically on $\mathbb{T}$, and due to the periodicity of f, this extension is a solution of (1.1) on $\mathbb{T}$. Now we pick a $t_0 \in \mathbb{T}$, $\tau_0 \preceq t_0$, choose $k \in \mathbb{N}_0$ so large that $t_0 \preceq \sigma_{kT}(\tau_0)$ and obtain for $\xi_0 := \varphi(\sigma_{kT}(\tau_0); t_0, \nu(t_0))$ the fixed point relation

$$\begin{aligned}
\phi_T(\xi_0) &= \phi_T\big(\varphi(\sigma_{kT}(\tau_0); t_0, \nu(t_0))\big) \\
&= \varphi\big(\sigma_{(k+1)T}(\tau_0); \sigma_{kT}(\tau_0), \varphi(\sigma_{kT}(\tau_0); t_0, \nu(t_0))\big) \\
&\overset{(1.2)}{=} \varphi(\sigma_{(k+1)T}(\tau_0); t_0, \nu(t_0)) = \varphi(\sigma_{kT}(\tau_0); t_0, \nu(t_0)) = \xi_0.
\end{aligned}$$

($\Leftarrow$) If $\xi \in \Omega$ is a fixed point of ϕ_T, then the relation (1.3) is satisfied with $t_0 = \tau_0$, $\nu = \varphi(\cdot; \tau_0, \xi)$ and Lemma 1.2 implies the assertion. □

Theorem 1.4. *Let $\mathcal{X}$ be finite-dimensional and equation* (1.1) *be T-periodic. Assume that a non-empty, convex, compact subset $\Omega_0 \subseteq \Omega$ exists, for which $\xi \in \Omega_0$ implies that $\varphi(\cdot;\tau_0,\xi)$ is defined on $[\tau_0,\sigma_T(\tau_0)]_{\mathbb{T}}$ at least and $\varphi(\sigma_T(\tau_0);\tau_0,\xi) \in \Omega_0$. Then* (1.1) *possesses a T-periodic solution.*

Proof. The assumptions imply that the time-T-map ϕ_T of (1.1) is continuous and maps Ω_0 into itself. Since Ω_0 and ϕ_T satisfy the conditions of Brouwer's Fixed Point Theorem (cf., e.g., [Zei93, p. 51, Proposition 2.6]), ϕ_T has a fixed point in Ω_0 and the assertion follows from Lemma 1.3. □

We illustrate Theorem 1.4 using the following example, which is inspired by the one-dimensional ODE discussed in [Far94, pp. 154–156, Example 4.1.1].

Example 1.5. *Consider an interaction between $d \geq 1$, e.g., insect populations. The life span of each population is given by the interval $[k, k+h]$, $k \in \mathbb{N}_0$ and $h \in (0,1)$, which can be interpreted as summer period. Suppose that just before the populations die out, eggs are laid at time $t = k+h$ and hatched after the winter period $(k+h, k+1)$ at time $t = k+1$. Hence, we are interested in the time scale*

$$\mathbb{T} = \bigcup_{k\in\mathbb{N}_0} [k, k+h]\,.$$

During the summer, we describe the growth behavior by a generalized logistic model, while in winter, no interaction between species occurs and the number of eggs remains constant. Precisely, we arrive at a dynamic equation (1.1) *with right-hand side $f = (f_1,\ldots,f_d) : \mathbb{T}\times(0,\infty)^d \to \mathbb{R}^d$,*

$$f_i(t,x) := \begin{cases} r_i(t)x_i\frac{K_i(t)-x_i}{K_i(t)+\sum_{j=1}^d c_{ij}(t)x_j} & \text{for } t\in[k,k+h) \\ 0 & \text{for } t = k+h \end{cases}, \tag{1.4}$$

with continuous real-valued mappings r_i, K_i, c_{ij} satisfying $0 < r_i(t)$, $0 \leq c_{ij}(t)$ for all $t\in[0,h)$, $i,j\in\{1,\ldots,d\}$, and

$$0 < \inf_{t\in[0,h)} K_i(t) < \sup_{t\in[0,h)} K_i(t) \quad \text{for } i\in\{1,\ldots,d\}\,. \tag{1.5}$$

To describe seasonal changes, say, we moreover assume that the intrinsic growth rates *r_i, the* carrying capacities *K_i and the coupling parameters c_{ij}, describing the interaction between species, are* 1*-periodic. We are going to show that under these assumptions,* (1.1) *has a non-constant* 1*-periodic solution in the positive orthant of $\mathbb{R}^d$. Thereto, we abbreviate $K_i^- := \inf_{t\in[0,h)} K_i(t)$, $K_i^+ := \sup_{t\in[0,h)} K_i(t)$, define the box $\Omega_0 := \prod_{i=1}^d \left[K_i^-, K_i^+\right]$ and apply the induction principle (cf.* [Hil90, Theorem 1.4(c)]*) to prove that the statement*

$$\mathcal{A}(t):\ \varphi(s;0,\xi)\in\Omega_0 \quad \text{for } s\in[0,t]_{\mathbb{T}}$$

holds true for arbitrary $t\in\mathbb{T}$ and $\xi\in\Omega_0$. We write $\nu(t) := \varphi(t;0,\xi)$:
(I) Evidently, $\mathcal{A}(0)$ is valid.
(II) Let $t\in\mathbb{T}$ be right-scattered, i.e. $t = k+h$ for some $k\in\mathbb{N}_0$. Then (1.4) *immediately implies $\nu(\sigma(t)) = \nu(t)\in\Omega_0$ due to the induction hypothesis.*
(III) Let $t\in\mathbb{T}$ be right-dense, i.e. $t\in[k,k+h)$ for some $k\in\mathbb{N}_0$, and assume that $\mathcal{A}(t)$ holds. In case $\nu(t)\in\Omega_0^\circ$, the continuity of ν implies that there exists a $\mathbb{T}$-neighborhood U of t with $\nu(\tau)\in\Omega_0$ for all $\tau\in U$. In case $\nu(t)\in\partial\Omega_0$, there exists an $i\in\{1,\ldots,d\}$ with $\nu_i(t)\in\left\{K_i^-,K_i^+\right\}$. Now $\nu_i(t) = K_i^+$ implies $\nu_i(t)\geq K_i(t)$, then (1.4) *yields $\nu_i^\Delta(t)\leq 0$ and we have $\nu_i(\tau)\leq K_i^+$ for τ in a*

$\mathbb{T}$*-neighborhood* U *of* t*. Analogously, one obtains* $\nu_i(\tau) \geq K_i^-$ *from* $\nu_i(t) = K_i^-$ *and we get* $\varphi(\tau; 0, \xi) \in \Omega_0$ *for* $\tau \in U$*.*
(IV) Let $t \in \mathbb{T}$ *be left-dense, i.e.* $t \in (k, k+h]$ *for some* $k \in \mathbb{N}_0$*, and assume that* $\mathcal{A}(\tau)$ *holds for all* $\tau \in [0, t)_{\mathbb{T}}$*. Then the continuity of* ν *and the closedness of* Ω_0 *imply* $\nu(t) \in \Omega_0$*, i.e.* $\mathcal{A}(t)$ *holds.*

Since Ω_0 *is convex and compact, our Theorem 1.4 implies that the dynamic equation* (1.1) *(with right-hand side given by* (1.4)*) has a* 1*-periodic solution* ν^* *such that* $K_i^- \leq \nu_i^*(t) \leq K_i^+$ *for all* $t \in \mathbb{T}$*,* $i \in \{1, \ldots, d\}$*. Here the relation* (1.5) *guarantees that* ν^* *is non-constant.*

2. LINEAR DYNAMIC EQUATIONS

Now we are interested in linear homogeneous dynamic equations of the form

$$x^{\Delta} = A(t)x, \tag{2.1}$$

where the mapping $A \in \mathcal{C}_{rd}(\mathbb{T}, \mathcal{L}(\mathcal{X}))$ is assumed to be T-periodic. We denote its *transition operator* by $\Phi_A(t, \tau) \in \mathcal{L}(\mathcal{X})$, $\tau \preceq t$, i.e. $\Phi_A(\cdot, \tau)$ is the solution of the initial value problem $X^{\Delta} = A(t)X$, $X(\tau) = I_{\mathcal{X}}$ in $\mathcal{L}(\mathcal{X})$. From (1.2) we readily get

$$\Phi_A(\sigma_T(t), \sigma_T(\tau)) = \Phi_A(t, \tau), \qquad \Phi(\sigma_T(t), \tau) = \Phi_A(t, \tau)\Phi(\sigma_T(\tau), \tau) \tag{2.2}$$

for $\tau, t \in \mathbb{T}$, $\tau \preceq t$. In the present situation, the time-T-map of (2.1) is linear and given by $\Phi_T := \Phi_A(\sigma_T(\tau_0), \tau_0) \in \mathcal{L}(\mathcal{X})$. This operator is called the *monodromy operator* of (2.1). For a *regressive* equation (2.1), i.e. if there exists a $\tau \in \mathbb{T}$ with $I_{\mathcal{X}} + \mu^*(t)A(t) \in \mathcal{GL}(\mathcal{X})$ for $t \in [\tau, \sigma_T(\tau)]_{\mathbb{T}}$, one has $\Phi_A(t, \tau) \in \mathcal{GL}(\mathcal{X})$ and (2.2) is valid for all $\tau, t \in \mathbb{T}$. Additionally, in this situation [Hil90, Theorem 6.2(ii)] yields

$$\Phi_A(\sigma_T(t), t) = \Phi_A(\sigma_T(t), \sigma_T(\tau_0))\Phi_T\Phi_A(\tau_0, t) \stackrel{(2.2)}{=} \Phi_A(t, \tau_0)\Phi_T\Phi_A(\tau_0, t)$$

and therefore the spectrum of Φ_T does only depend on the system (2.1) and not on the particular choice of $\tau_0 \in \mathbb{T}$. We call the eigenvalues (the elements of the point spectrum) of the monodromy operator its *characteristic multipliers*.

Theorem 2.1 (characteristic multipliers)**.** *Consider the linear system* (2.1)*.*

(a) *If* λ *is a characteristic multiplier, then there exists a non-trivial solution* $\nu : \mathbb{T}_{\tau_0}^+ \to \mathcal{X}$ *of* (2.1) *satisfying* $\nu(\sigma_T(t)) = \lambda\nu(t)$ *for* $\tau_0 \preceq t$*.*
(b) *Conversely, if for a non-trivial solution* ν *of* (2.1) *one has the relation*

$$\nu(\sigma_T(\tau_0)) = \lambda\nu(\tau_0), \tag{2.3}$$

then λ *is a characteristic multiplier with corresponding eigenvector* $\nu(\tau_0) \in \mathcal{X}$*.*

Proof. (a) Let λ be a characteristic multiplier. Then there exists a $x_0 \neq 0$ such that $\Phi_T x_0 = \lambda x_0$ and the solution $\nu : \mathbb{T}_{\tau_0}^+ \to \mathcal{X}$, $\nu(t) := \Phi_A(t, \tau_0)x_0$ of (2.1) satisfies

$$\begin{aligned} \nu(\sigma_T(t)) &= \Phi_A(\sigma_T(t), \tau_0)x_0 = \Phi_A(\sigma_T(t), \sigma_T(\tau_0))\Phi_T x_0 \\ &\stackrel{(2.2)}{=} \lambda\Phi_A(t, \tau_0)x_0 = \lambda\nu(t) \quad \text{for } \tau_0 \preceq t. \end{aligned}$$

(b) Assume the relation (2.3) holds for some non-zero solution ν of (2.1). Since $\nu(\sigma_T(\tau_0)) = \Phi_T\nu(\tau_0)$, this means $[\Phi_T - \lambda I_{\mathcal{X}}]\,\nu(\tau_0) = 0$ and consequently the point spectrum of the monodromy operator contains λ. □

Now we consider linear inhomogeneous dynamic equations of the form

$$x^{\Delta} = A(t)x + h(t), \tag{2.4}$$

and the forcing term $h : \mathbb{T} \to \mathcal{X}$ is assumed to be T-periodic and piecewise rd-continuous (cf. [Pöt02, p. 16]). For ODEs the next result has its origins in [Mas50].

Theorem 2.2 (linear equations)**.** *Consider the linear inhomogeneous system* (2.4)*.*

(a) *If it has a T-periodic solution, then there exists a bounded solution of* (2.4)*.*
(b) *Conversely, if* (2.4) *has a bounded solution and if Φ_T is a compact operator, then there exists a T-periodic solution of* (2.4)*.*

Proof. Since the assertion (a) is trivial, we only have to verify (b). From Lemma 1.3 we know that (2.4) has a T-periodic solution if and only if there exists a $\xi \in \mathcal{X}$ such that

$$\xi = \phi_T(\xi) = \Phi_T\xi + \eta \tag{2.5}$$

with $\eta := \int_{\tau_0}^{\sigma_T(\tau_0)} \Phi_A(\sigma_T(\tau_0), \sigma(s))h(s)\,\Delta s$, where we have used the variation of constants formula (cf. [Pöt02, p. 56, Satz 1.3.11]). Hence it suffices to show that any solution of (2.4) is unbounded, if one cannot solve (2.5). From a theorem of Fredholm (cf., e.g., [Zei93, pp. 372–373, Section 8.5]), we obtain that (2.5) is not solvable, if and only if there exists a $x' \in \mathcal{X}'$ such that

$$x' = \Phi_T' x' \qquad \text{and} \qquad \langle \eta, x'\rangle \neq 0. \tag{2.6}$$

For an arbitrary solution ν of (2.4) satisfying $\nu(\tau_0) = \xi$ we have

$$\nu(t) = \Phi_A(t, \tau_0)\xi + \int_{\tau_0}^{t} \Phi_A(t, \sigma(s))h(s)\,\Delta s \quad \text{for } \tau_0 \preceq t$$

and therefore $\nu(\sigma_T(\tau_0)) = \Phi_T\xi + \eta$, as well as

$$\nu^{\Delta}(\sigma_{kT}(t)) \overset{(2.4)}{=} A(\sigma_{kT}(t))\nu(\sigma_{kT}(t))+h(\sigma_{kT}(t)) = A(t)\nu(\sigma_{kT}(t))+h(t) \quad \text{for } k \in \mathbb{N}_0.$$

Due to the uniqueness of solutions we obtain that $\nu_k(t) := \nu(\sigma_{kT}(t))$ is a solution of the initial value problem (2.4), $x(\tau_0) = \nu(\sigma_{kT}(\tau_0))$. Consequently, (2.5) gives us $\nu_k(\sigma_T(\tau_0)) = \nu(\sigma_{(k+1)T}(\tau_0)) = \Phi_T\nu(\sigma_{kT}(\tau_0)) + \eta$ and this, in turn, yields $\nu_k(\sigma_T(\tau_0)) = \Phi_T^k\xi + \sum_{j=0}^{k-1} \Phi_T^j\eta$ by mathematical induction. Using (2.6), we get

$$\langle \nu_k(\sigma_T(\tau_0)), x'\rangle = \left\langle \xi, [\Phi_T']^k x'\right\rangle + \sum_{j=0}^{k-1} \left\langle \eta, [\Phi_T']^j x'\right\rangle \quad \text{for } k \in \mathbb{N}_0$$

and because of $\langle \eta, x'\rangle \neq 0$ we have

$$\lim_{k\to\infty} \langle \nu(\sigma_{kT}(\tau_0)), x'\rangle = \lim_{k\to\infty} \langle \nu_k(\sigma_T(\tau_0)), x'\rangle = \infty.$$

Thus ν is unbounded. □

We close this section with a result about exponentially dichotomous linear dynamic equations. We say (2.1) possesses an *exponential dichotomy*, if there exists an invariant projector $P : \mathbb{T} \to \mathcal{L}(\mathcal{X})$ (cf. [Pöt01, Proposition 2.1]) such that the dichotomy estimates

$$\|\Phi_A(t, s)P(s)\| \leq K_1 e_a(t, s), \quad \left\|\bar{\Phi}_A(s, t)\,[I_{\mathcal{X}} - P(t)]\right\| \leq K_2 e_b(s, t) \quad \text{for } s \preceq t$$

hold, with reals $K_1, K_2 \geq 1$ and growth rates $a, b \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T}, \mathbb{R})$, $a \lhd 0 \lhd b$. Here $e_a(t,s)$, $s,t \in \mathbb{T}$, denotes the *real exponential function* on $\mathbb{T}$ (cf. [Hil90, Section 7]) and $\bar{\Phi}_A(t,s)$ stands for the *extended transition operator* of (2.1) (cf. [Pöt01]).

Corollary 2.3. *Assume that* (2.1) *possesses an exponential dichotomy and that* Φ_T *is compact. Then* (2.4) *has a* T*-periodic solution, which is unique on measure chains unbounded above and below.*

Proof. Since (2.1) has an exponential dichotomy, we obtain from [Pöt02, p. 103, Satz 2.2.4(a)] that there exists a bounded solution of (2.4). Due to Theorem 2.2(b) this solution is T-periodic. Moreover, [Pöt02, p. 106, Satz 2.2.7] guarantees the uniqueness on measure chains unbounded above and below. □

3. NON-LINEAR DYNAMIC EQUATIONS

In this section we turn our interest to non-linear dynamic equations of the form

$$x^\Delta = A(t)x + g(t,x) \tag{3.1}$$

with rd-continuous and T-periodic mappings $A : \mathbb{T} \to \mathcal{L}(\mathcal{X})$, $g : \mathbb{T} \times \mathcal{X} \to \mathcal{X}$. Assume that the forward solutions of (3.1) are uniquely determined by the initial conditions and let φ denote the general solution of (3.1).

Theorem 3.1 (semi-linear equations)**.** *Suppose that* $I_\mathcal{X} + \mu^*(t)A(t) \neq 0$ *for all* $t \in [\tau_0, \sigma_T(\tau_0)]_\mathbb{T}$*, define* $a(t) := \lim_{h \searrow \mu^*(t)} \frac{\|I_\mathcal{X} + hA(t)\| - 1}{h}$ *and we assume that:*

(i) *The spectrum of* Φ_T *does not contain* 1*,*
(ii) *there exists an* $L \geq 0$ *such that*

$$\|g(t,x) - g(t,\bar{x})\| \leq L\,\|x - \bar{x}\| \quad \text{for } t \in [\tau_0, \sigma_T(\tau_0)]_\mathbb{T},\ x, \bar{x} \in \mathcal{X}$$

$$\text{and } \left\|[I_\mathcal{X} - \Phi_T]^{-1}\right\| [e_{a+L}(\sigma_T(\tau_0), \tau_0) - e_a(\sigma_T(\tau_0), \tau_0)] < 1.$$

Then there exists exactly one T*-periodic solution of* (3.1)*.*

Proof. Because of the variation of constants formula (cf. [Pöt02, p. 56, Satz 1.3.11]), the time-T-map of (3.1) is given by $\phi_T(\xi) = \Phi_T\xi + F(\xi)$ with $F : \mathcal{X} \to \mathcal{X}$,

$$F(\xi) := \int_{\tau_0}^{\sigma_T(\tau_0)} \Phi_A(\sigma_T(\tau_0), \sigma(s)) g(s, \varphi(s; \tau_0, \xi))\, \Delta s.$$

Using the estimates $\|\Phi_A(t,s)\| \leq e_a(t,s)$ for $s \preceq t$ (cf. [Pöt02, p. 69, Satz 1.2.36(a)]), hypothesis (ii) and [Pöt02, p. 138, Lemma 3.2.5(b)], one obtains

$$\left\|F(\xi) - F(\bar{\xi})\right\| \leq [e_{a+L}(\sigma_T(\tau_0), \tau_0) - e_a(\sigma_T(\tau_0), \tau_0)]\left\|\xi - \bar{\xi}\right\| \quad \text{for } \xi, \bar{\xi} \in \mathcal{X}.$$

Thus our assumptions imply that the operator $[I_\mathcal{X} - \Phi_T]^{-1} F$ is a contraction and consequently has a unique fixed point $\xi^* \in \mathcal{X}$ due to Banach's Fixed Point Theorem. Since ξ^* is also a fixed point of ϕ_T, the assertion follows from Lemma 1.3. □

For ODEs the next result can be found in [Ama95, pp. 327–328, Theorem (22.1)].

Theorem 3.2 (asymptotically linear equations)**.** *Let* $\mathcal{X}$ *be finite-dimensional and we assume that:*

(i) 1 *is not a characteristic multiplier,*
(ii) *for all* $\epsilon > 0$ *there exists an* $R > 0$ *satisfying*

$$\|g(t,x)\| \leq \epsilon\,\|x\| \quad \text{for } t \in [\tau_0, \sigma_T(\tau_0)]_\mathbb{T},\ \|x\| \geq R. \tag{3.2}$$

Then there exists a T*-periodic solution of* (3.1)*.*

Proof. We subdivide the proof in two steps:

(I) Due to the periodicity of A we know $\alpha := \sup_{\tau_0 \preceq s \preceq t \preceq \sigma_T(\tau_0)} \|\Phi_A(t,s)\| < \infty$ and for any $\epsilon > 0$ one obtains from (3.2) the existence of a $C(\epsilon) > 0$ such that

$$\|g(t,x)\| \leq C(\epsilon) + \epsilon \|x\| \quad \text{for } t \in [\tau_0, \sigma_T(\tau_0)]_{\mathbb{T}}\,,\ x \in \mathcal{X}.$$

Let $\xi \in \mathcal{X}$ be arbitrary. The variation of constants formula (cf. [Pöt02, p. 56, Satz 1.3.11]) implies that the solution $\varphi(\cdot;\tau_0,\xi)$ of (3.1) satisfies

$$\varphi(t;\tau_0,\xi) = \Phi_A(t,\tau_0)\xi + \int_{\tau_0}^t \Phi_A(t,\sigma(s))g(s,\varphi(s;\tau_0,\xi))\,\Delta s \quad \text{for } \tau_0 \preceq t, \tag{3.3}$$

together with the above estimates this yields

$$\|\varphi(t;\tau_0,\xi)\| \leq \alpha\|\xi\| + \alpha C(\epsilon)T + \alpha\epsilon \int_{\tau_0}^t \|\varphi(s;\tau_0,\xi)\|\ \Delta s \quad \text{for } t \in [\tau_0, \sigma_T(\tau_0)]_{\mathbb{T}}$$

and Gronwall's Lemma (cf., e.g., [Pöt02, p. 66, Korollar 1.3.31]) gives us

$$\|\varphi(t;\tau_0,\xi)\| \leq \bar{C}(\epsilon) + \alpha e_{\alpha\epsilon}(\sigma_T(\tau_0),\tau_0)\|\xi\|$$

with $\bar{C}(\epsilon) := \alpha e_{\alpha\epsilon}(\sigma_T(\tau_0),\tau_0)TC(\epsilon)$. Additionally, we get

$$\begin{aligned}\|\varphi(t;\tau_0,\xi) - \Phi_A(t,\tau_0)\xi\| &\overset{(3.3)}{\leq} \alpha \int_{\tau_0}^t (C(\epsilon) + \epsilon\|\varphi(s;\tau_0,\xi)\|)\ \Delta s \\ &\leq \alpha TC(\epsilon) + \alpha\epsilon T\bar{C}(\epsilon) + \alpha\epsilon Te_{\alpha\epsilon}(\sigma_T(\tau_0),\tau_0)\|\xi\|\end{aligned}$$

and therefore

$$\|\phi_T(\xi) - \Phi_A(t,\tau_0)\xi\| \leq \alpha TC(\epsilon) + \alpha\epsilon T\bar{C}(\epsilon) + \alpha\epsilon Te_{\alpha\epsilon}(\sigma_T(\tau_0),\tau_0)\|\xi\|\,. \tag{3.4}$$

(II) Since the space $\mathcal{X}$ is finite-dimensional, the spectrum of Φ_T consists of characteristic multipliers. Thus $I_{\mathcal{X}} - \Phi_T \in \mathcal{GL}(\mathcal{X})$ and $F : \mathcal{X} \to \mathcal{X}$,

$$F(\xi) := [I_{\mathcal{X}} - \Phi_T]^{-1}(\phi_T(\xi) - \Phi_T\xi)$$

defines a continuous mapping. From the inequality (3.4) we see that there exists a $\rho > 0$ with $\|F(\xi)\| \leq \rho + \frac{1}{2}\|\xi\|$ for $\xi \in \mathcal{X}$, and consequently f maps the ball $\bar{B}_{2\rho}$ into itself. Now Brouwer's Fixed Point Theorem (cf., e.g., [Zei93, p. 51, Proposition 2.6]) implies the existence of a point $\xi^* \in \bar{B}_{2\rho}$ satisfying $\xi^* = F(\xi^*)$. This relation is equivalent to $\xi^* = \phi_T(\xi^*)$ and Lemma 1.3 yields the assertion. □

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