INVARIANT FOLIATIONS AND STABILITY IN CRITICAL CASES

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ABSTRACT. We construct invariant foliations of the extended state space for nonautonomous semi-linear dynamic equations on measure chains (time scales). These equations allow a specific parameter dependence which is the key to obtain perturbation results necessary for applications to an analytical discretization theory of ODEs. Using these invariant foliations we deduce a version of Pliss's reduction principle.

1. INTRODUCTION

We begin with a motivation for this paper having its origin in the classical theory of discrete dynamical systems. For this purpose, consider a C^1-mapping $f : U \to \mathcal{X}$ from an open neighborhood $U \subseteq \mathcal{X}$ of 0 into a Banach space $\mathcal{X}$, which leaves the origin fixed ($f(0) = 0$). It is a well-established result and can be traced back to the work of Perron in the early 1930s (to be more precise, it is due to his student Ta Li (cf. [Li34])) that the origin is an asymptotically stable solution of the autonomous difference equation

$$x_{k+1} = f(x_k), \tag{1.1}$$

if the spectrum $\Sigma(Df(0))$ is contained in the open unit circle of the complex plane. Similar results also hold for continuous dynamical systems (replace the open unit disc by the negative half-plane) or nonautonomous equations (replace the assumption on the spectrum by uniform asymptotic stability of the linearization). In a time scales setting of dynamic equations these questions are addressed in the works [GH03] (for scalar equations), [Kel99] (equations in Banach spaces) and easily follow from a localized version of Theorem 2.2(a) below. Such considerations are usually summarized under the phrase *"principle of linearized stability"*, since the stability properties of the linear part dominate the nonlinear equation locally.

Significantly more interesting is the generalized situation when $\Sigma(Df(0))$ allows a decomposition into disjoint spectral sets Σ_s, Σ_c, where Σ_s is contained in the open unit disc, but Σ_c lies on its boundary. Then nonlinear effects enter the game and the *center manifold theorem* applies (cf., e.g., [Ioo79]): There exists a locally invariant submanifold $R_0 \subseteq \mathcal{X}$ which is graph of a C^1-mapping r_0 over an open neighborhood of 0 in $\mathcal{R}(P)$, where $P \in \mathcal{L}(\mathcal{X})$ is the spectral projector associated with Σ_c. Beyond that, the stability properties of the trivial solution to (1.1) are fully determined by those of

$$p_{k+1} = Pf(p_k + r_0(p_k)). \tag{1.2}$$

The advantage we obtained from this is that (1.2) is an equation in the lower-dimensional subspace $\mathcal{R}(P) \subseteq \mathcal{X}$. This is known as the *reduction principle*. From the immense literature we only cite [Pli64] — the pathbreaking paper in the framework of finite-dimensional ODEs.

The paper at hand has two primary goals:

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(1) It can be considered as a continuation of our earlier works [Pöt06, KP06]. In [Pöt06] we studied the robustness of invariant fiber bundles under parameter variation and obtained quantitative estimates. Such results were successfully applied to study the behavior of invariant manifolds under numerical discretization using one-step schemes (cf. [KP06]). Here we prepare future results in this direction on the behavior of invariant foliations under varying parameters. As a matter of course, this gives the present paper a somehow technical appearance, at least until Section 4.
(2) We want to derive a version of the above reduction principle for nonautonomous dynamic equations on measure chains. To obtain this in a geometrically transparent fashion, invariant foliations appear to be the appropriate vehicle.

In Section 2 we establish our general set-up and present an earlier result on the existence of invariant fiber bundles, which canonically generalize stable and unstable manifolds of dynamical systems to nonautonomous equations. The actual invariant foliations are constructed in Section 3 via pseudo-stable and -unstable fibers through specific points in the extended state space. Each such fiber contains all initial values of solutions approaching the invariant fiber bundles exponentially; actually they are asymptotically equivalent to a solution on the invariant fiber bundles. This behavior can be summarized under the notion of an *asymptotic phase*. While the above global results are stated in a — from an applied point of view — very restrictive setting of semi-linear equations, the final Section 4 covers a larger class of dynamic equations. For them we deduce a reduction principle and apply this technique to a specific example.

Let us close this introductory remarks by pointing out that our Proposition 3.2 is not just a "unification" of the corresponding results obtained in, e.g., [AW99] for ODEs and [AW03] for difference equations. In fact, we had to include a particular parameter dependence allowing a perturbation theory needed to study the behavior of ODEs under numerical approximation. Beyond that, invariant foliations are the key ingredient to obtain topological linearization results for dynamic equations (cf. [Hil96]).

Throughout this paper, Banach spaces $\mathcal{X}$ are all real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$) and their norm is denoted by $\|\cdot\|$. For the open ball in $\mathcal{X}$ with center 0 and radius $r > 0$ we write B_r. $\mathcal{L}(\mathcal{X})$ is the Banach space of linear bounded endomorphisms, $I_{\mathcal{X}}$ the identity on $\mathcal{X}$, and $\mathcal{R}(T) := T\mathcal{X}$ the range of an operator $T \in \mathcal{L}(\mathcal{X})$.

If a mapping $f : \mathcal{Y} \to \mathcal{Z}$ between metric spaces $\mathcal{Y}$ and $\mathcal{Z}$ satisfies a Lipschitz condition, then its smallest Lipschitz constant is denoted by $\operatorname{Lip} f$. Frequently, $f : \mathcal{Y} \times \mathcal{P} \to \mathcal{Z}$ also depends on a parameter from some set $\mathcal{P}$, and we write

$$\operatorname{Lip}_1 f := \sup_{p \in \mathcal{P}} \operatorname{Lip} f(\cdot, p).$$

In case $\mathcal{P}$ has a metric structure, we define $\operatorname{Lip}_2 f$ accordingly, and proceed along these lines for mappings depending on more than two variables.

To keep this work self-contained, we introduce some basic terminology from the calculus on measure chains (cf. [Hil90, BP01]). In all subsequent considerations we deal with a *measure chain* $(\mathbb{T}, \preceq, \mu)$, i.e. a conditionally complete totally ordered set $(\mathbb{T}, \preceq)$ (see [Hil90, Axiom 2]) with growth calibration $\mu : \mathbb{T}^2 \to \mathbb{R}$ (see [Hil90, Axiom 3]). The most intuitive and relevant examples of measure chains are *time scales*, where $\mathbb{T}$ is a canonically ordered closed subset of the reals and μ is given by $\mu(t, s) = t - s$. Continuing, $\sigma : \mathbb{T} \to \mathbb{T}$, $\sigma(t) := \inf \{s \in \mathbb{T} : t \prec s\}$ defines the *forward jump operator* and $\mu^* : \mathbb{T} \to \mathbb{R}$, $\mu^*(t) := \mu(\sigma(t), t)$ the *graininess*. For $\tau \in \mathbb{T}$ we abbreviate $\mathbb{T}_\tau^+ := \{s \in \mathbb{T} : \tau \preceq s\}$ and $\mathbb{T}_\tau^- := \{s \in \mathbb{T} : s \preceq \tau\}$.

Since we are interested in an asymptotic theory, we impose the following standing

Hypothesis. $\mu(\mathbb{T}, \tau) \subseteq \mathbb{R}$, $\tau \in \mathbb{T}$, *is unbounded above, and* μ^* *is bounded.*

$\mathcal{C}_{rd}(\mathbb{T},\mathcal{X})$ denotes the set of rd-continuous functions from $\mathbb{T}$ to $\mathcal{X}$ (cf. [Hil90, Section 4.1]). *Growth rates* are functions $a \in \mathcal{C}_{rd}(\mathbb{T},\mathbb{R})$ with $-1 < \inf_{t\in\mathbb{T}} \mu^*(t)a(t)$, $\sup_{t\in\mathbb{T}} \mu^*(t)a(t) < \infty$. Moreover, for $a, b \in \mathcal{C}_{rd}(\mathbb{T},\mathbb{R})$ we introduce the relations $\lfloor b-a \rfloor := \inf_{t\in\mathbb{T}}(b(t)-a(t))$,

$$a \lhd b \quad :\Leftrightarrow \quad 0 < \lfloor b-a \rfloor, \qquad\qquad a \unlhd b \quad :\Leftrightarrow \quad 0 \le \lfloor b-a \rfloor$$

and the set of *positively regressive functions*

$$\mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T},\mathbb{R}) := \{a \in \mathcal{C}_{rd}(\mathbb{T},\mathbb{R}) : a \text{ is a growth rate and } 1+\mu^*(t)a(t) > 0 \text{ for } t \in \mathbb{T}\}.$$

This class is technically appropriate to describe exponential growth and for $a \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T},\mathbb{R})$ the *exponential function* on $\mathbb{T}$ is denoted by $e_a(t,s) \in \mathbb{R}$, $s, t \in \mathbb{T}$ (cf. [Hil90, Theorem 7.3]).

Measure chain integrals of mappings $\phi : \mathbb{T} \to \mathcal{X}$ are always understood in Lebesgue's sense and denoted by $\int_\tau^t \phi(s)\,\Delta s$ for $\tau, t \in \mathbb{T}$, provided they exist (cf. [Nei01]).

We finally introduce the so-called *quasiboundedness* which is a convenient notion due to Bernd Aulbach describing exponentially growing functions.

Definition 1.1. For $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T},\mathbb{R})$ and $\tau \in \mathbb{T}$ we say that $\phi \in \mathcal{C}_{rd}(\mathbb{T},\mathcal{X})$ is

(a) c^+-*quasibounded*, if $\|\phi\|^+_{\tau,c} := \sup_{t\in\mathbb{T}^+_\tau} \|\phi(t)\|\, e_c(\tau,t) < \infty$,
(b) c^--*quasibounded*, if $\|\phi\|^-_{\tau,c} := \sup_{t\in\mathbb{T}^-_\tau} \|\phi(t)\|\, e_c(\tau,t) < \infty$,
(c) $c^\pm$-*quasibounded*, if $\sup_{t\in\mathbb{T}} \|\phi(t)\|\, e_c(\tau,t) < \infty$.

$\mathcal{X}^+_{\tau,c}$ and $\mathcal{X}^-_{\tau,c}$ denote the sets of c^+- and c^--quasibounded functions on $\mathbb{T}^+_\tau$ and $\mathbb{T}^-_\tau$, resp.

Remark 1.1. (1) In order to provide some intuition for these abstract notions, in case $c \lhd 0$ a c^+-quasibounded function is exponentially decaying as $t \to \infty$. Accordingly, for $0 \lhd c$ a c^--quasibounded function decays exponentially as $t \to -\infty$ (supposed $\mathbb{T}$ is unbounded below). Classical boundedness corresponds to the situation of 0^+- (or 0^--) quasiboundedness.

(2) Obviously $\mathcal{X}^+_{\tau,c}$ and $\mathcal{X}^-_{\tau,c}$ are nonempty and by [Hil90, Theorem 4.1(iii)], it is immediate that for any $c \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T},\mathbb{R})$, $\tau \in \mathbb{T}$, the sets $\mathcal{X}^+_{\tau,c}$ and $\mathcal{X}^-_{\tau,c}$ are Banach spaces with the norms $\|\cdot\|^+_{\tau,c}$ and $\|\cdot\|^-_{\tau,c}$, respectively.

2. Preliminaries on Semi-linear Equations

Given $A \in \mathcal{C}_{rd}(\mathbb{T},\mathcal{L}(\mathcal{X}))$, a *linear dynamic equation* is of the form

$$x^\Delta = A(t)x; \tag{2.1}$$

here the *transition operator* $\Phi_A(t,s) \in \mathcal{L}(\mathcal{X})$, $s \preceq t$, is the solution of the operator-valued initial value problem $X^\Delta = A(t)X$, $X(s) = I_\mathcal{X}$ in $\mathcal{L}(\mathcal{X})$.

A projection-valued mapping $P : \mathbb{T} \to \mathcal{L}(\mathcal{X})$ is called an *invariant projector* of (2.1), if

$$P(t)\Phi_A(t,s) = \Phi_A(t,s)P(s) \quad \text{for all } s,t \in \mathbb{T},\ s \preceq t \tag{2.2}$$

holds, and finally an invariant projector P is denoted as *regular*, if

$$I_\mathcal{X} + \mu^*(t)A(t)|_{\mathcal{R}(P(t))} : \mathcal{R}(P(t)) \to \mathcal{R}\big(P(\sigma(t))\big) \text{ is bijective for all } t \in \mathbb{T}.$$

Then the restriction $\bar\Phi_A(t,s) := \Phi_A(t,s)|_{\mathcal{R}(P(s))} : \mathcal{R}(P(s)) \to \mathcal{R}(P(t))$, $s \preceq t$, is a well-defined isomorphism, and we write $\bar\Phi_A(s,t)$ for its inverse (cf. [Pöt02, p. 85, Lemma 2.1.8]). These preparations allow to include non-invertible systems (2.3) into our investigation.

For the mentioned applications in discretization theory it is crucial to deal with equations admitting a certain dependence on parameters $\theta \in \mathbb{F}$ (see [KP06]). More precisely, we consider nonlinear perturbations of (2.1) given by

$$x^\Delta = A(t)x + F_1(t,x) + \theta F_2(t,x) \tag{2.3}$$

with mappings $F_i : \mathbb{T} \times \mathcal{X} \to \mathcal{X}$, such that F_i is rd-continuous (see [Hil90, Section 5.1]) for $i = 1, 2$. Further assumptions on F_1, F_2 can be found below. A *solution* of (2.3) is a function ν satisfying the identity $\nu^\Delta(t) \equiv A(t)\nu(t) + F_1(t, \nu(t)) + \theta F_2(t, \nu(t))$ on a $\mathbb{T}$-interval. Provided it exists, φ denotes the *general solution* of (2.3), i.e., $\varphi(\cdot; \tau, x_0; \theta)$ solves (2.3) on $\mathbb{T}_\tau^+$ and satisfies the initial condition $\varphi(\tau; \tau, x_0; \theta) = x_0$ for $\tau \in \mathbb{T}$, $x_0 \in \mathcal{X}$. It fulfills the *cocycle property*

$$\varphi(t; s, \varphi(s; \tau, x_0; \theta); \theta) = \varphi(t; \tau, x_0; \theta) \quad \text{for all } \tau, s, t \in \mathbb{T},\ \tau \preceq s \preceq t,\ x_0 \in \mathcal{X}. \tag{2.4}$$

We define the dynamic equation (2.3) to be *regressive* on a set $\Theta \subseteq \mathbb{F}$, if

$$I_{\mathcal{X}} + \mu^*(t)\left[A(t) + F_1(t, \cdot) + \theta F_2(t, \cdot)\right] : \mathcal{X} \to \mathcal{X} \quad \text{for all } \theta \in \Theta$$

is a homeomorphism. Then the general solution $\varphi(t; \tau, x_0; \theta)$ exists for all $t, \tau \in \mathbb{T}$ and the cocycle property (2.4) holds for arbitrary $t, s, \tau \in \mathbb{T}$.

From now on we assume:

Hypothesis 2.1. *Let $K_1, K_2 \geq 1$ be reals and $a, b \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T}, \mathbb{R})$ growth rates with $a \lhd b$.*

(i) *Exponential dichotomy: There exists a regular invariant projector $P : \mathbb{T} \to \mathcal{L}(\mathcal{X})$ of (2.1) such that the estimates*

$$\|\Phi_A(t, s)Q(s)\| \leq K_1 e_a(t, s), \qquad \left\|\bar{\Phi}_A(s, t)P(t)\right\| \leq K_2 e_b(s, t) \quad \text{for all } t \preceq s \tag{2.5}$$

are satisfied, with the complementary projector $Q(t) := I_{\mathcal{X}} - P(t)$.

(ii) *Lipschitz perturbation: We abbreviate $H_\theta := F_1 + \theta F_2$, for $i = 1, 2$ the identities $F_i(t, 0) \equiv 0$ on $\mathbb{T}$ hold and the mappings F_i satisfy the global Lipschitz estimates*

$$L_i := \sup_{t \in \mathbb{T}} \operatorname{Lip} F_i(t, \cdot) < \infty.$$

Moreover, we require

$$L_1 < \frac{\lfloor b - a \rfloor}{4(K_1 + K_2)}, \tag{2.6}$$

choose a fixed $\delta \in \left(2(K_1 + K_2)L_1, \frac{\lfloor b-a \rfloor}{2}\right)$ and abbreviate $\Theta := \{\theta \in \mathbb{F} : L_2 |\theta| \leq L_1\}$,

$$\Gamma := \left\{c \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T}, \mathbb{R}) : a + \delta \lhd c \lhd b - \delta\right\}, \quad \bar{\Gamma} := \left\{c \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T}, \mathbb{R}) : a + \delta \unlhd c \unlhd b - \delta\right\}.$$

Remark 2.1. (1) The existence of suitable values for δ yields from (2.6): Since we have $\delta < \frac{\lfloor b-a \rfloor}{2}$, there exist functions $c \in \Gamma$ and, in addition, $a + \delta, b - \delta$ are positively regressive. Furthermore, for later use we have the inequality

$$L(\theta) := \frac{K_1 + K_2}{\delta}\left(L_1 + |\theta|\, L_2\right) < 1 \quad \text{for all } \theta \in \Theta$$

and define the constant $\ell(\theta) := \frac{K_1 K_2}{K_1 + K_2} \frac{L(\theta)}{1 - L(\theta)}$.

(2) Under Hypothesis 2.1 the solutions $\varphi(\cdot; \tau, x_0; \theta)$ exist and are unique on $\mathbb{T}_\tau^+$ for arbitrary $\tau \in \mathbb{T}$, $x_0 \in \mathcal{X}$, $\theta \in \mathbb{F}$ (cf. [Pöt02, p. 38, Satz 1.2.17(a)]) and depend continuously on the data (t, τ, x_0, θ).

The next notion is helpful to understand the geometrical behavior of solutions for (2.3): Any (nonempty) subset $S(\theta)$ of the extended state space $\mathbb{T} \times \mathcal{X}$ is called a *nonautonomous set* with *τ-fibers*

$$S(\theta)_\tau := \{x \in \mathcal{X} : (\tau, x) \in S(\theta)\} \quad \text{for all } \tau \in \mathbb{T}.$$

We denote $S(\theta)$ as *forward invariant*, if for any pair $(\tau, x_0) \in S(\theta)$ one has the inclusion $(t, \varphi(t; \tau, x_0; \theta)) \in S(\theta)$ for all $t \in \mathbb{T}_\tau^+$. Presumed each fiber $S(\theta)_\tau$ is a submanifold of $\mathcal{X}$, we speak of a *fiber bundle*. Our invariant fiber bundles generalize invariant manifolds to nonautonomous equations, and consist of all initial value pairs leading to exponentially decaying solutions; admittedly in the generalized sense of quasiboundedness.

Theorem 2.2 (invariant fiber bundles)**.** *Assume that Hypothesis 2.1 is fulfilled. Then for all $\theta \in \Theta$ the following statements are true:*

(a) *The* pseudo-stable fiber bundle *of* (2.3)*, given by*

$$S(\theta) := \left\{(\tau, x_0) \in \mathbb{T} \times \mathcal{X} : \varphi(\cdot; \tau, x_0; \theta) \in \mathcal{X}^+_{\tau,c} \text{ for all } c \in \Gamma\right\}$$

is an invariant fiber bundle of (2.3) *possessing the representation*

$$S(\theta) = \{(\tau, x_0 + s(\tau, x_0; \theta)) \in \mathbb{T} \times \mathcal{X} : \tau \in \mathbb{T}, x_0 \in \mathcal{R}(Q(\tau))\}$$

with a continuous mapping $s : \mathbb{T} \times \mathcal{X} \times \Theta \to \mathcal{X}$ satisfying

$$s(\tau, x_0; \theta) = s(\tau, Q(\tau)x_0; \theta) \in \mathcal{R}(P(\tau)) \quad \text{for all } \tau \in \mathbb{T},\, x_0 \in \mathcal{X}$$

and the invariance equation

$$P(t)\varphi(t; \tau, x_0; \theta) = s(t, Q(t)\varphi(t; \tau, x_0; \theta); \theta) \quad \text{for all } (\tau, x_0) \in S(\theta),\, \tau \preceq t.$$

Furthermore, for all $\tau \in \mathbb{T}$, $x_0 \in \mathcal{X}$ it holds:
(a_1) $s(\tau, 0; \theta) \equiv 0$,
(a_2) *$s : \mathbb{T} \times \mathcal{X} \times \Theta \to \mathcal{X}$ satisfies the Lipschitz estimates*

$$\operatorname{Lip} s(\tau, \cdot; \theta) \le \ell(\theta), \qquad \operatorname{Lip} s(\tau, x_0; \cdot) \le \frac{\delta K_1 K_2 (K_1 + K_2) L_2}{[\delta - 2(K_1 + K_2)L_1]^2} \|x_0\|.$$

(b) *For $\mathbb{T}$ unbounded below, the* pseudo-unstable fiber bundle *of* (2.3)*, given by*

$$R(\theta) := \left\{(\tau, x_0) \in \mathbb{T} \times \mathcal{X} : \begin{array}{l} \text{there exists a solution } \nu : \mathbb{T} \to \mathcal{X} \text{ of } (2.3) \\ \text{with } \nu(\tau) = x_0 \text{ and } \nu \in \mathcal{X}^-_{\tau,c} \text{ for all } c \in \Gamma \end{array}\right\}$$

is an invariant fiber bundle of (2.3) *possessing the representation*

$$R(\theta) = \{(\tau, y_0 + r(\tau, y_0; \theta)) \in \mathcal{X} : \tau \in \mathbb{T}, y_0 \in \mathcal{R}(P(\tau))\}$$

with a continuous mapping $r : \mathbb{T} \times \mathcal{X} \times \Theta \to \mathcal{X}$ satisfying

$$r(\tau, x_0; \theta) = r(\tau, P(\tau)x_0; \theta) \in \mathcal{R}(Q(\tau)) \quad \text{for all } \tau \in \mathbb{T},\, x_0 \in \mathcal{X} \tag{2.7}$$

and the invariance equation

$$Q(t)\varphi(t; \tau, x_0; \theta) = r(t, P(t)\varphi(t; \tau, x_0; \theta); \theta) \quad \text{for all } (\tau, x_0) \in R(\theta),\, \tau \preceq t. \tag{2.8}$$

Furthermore, for all $\tau \in \mathbb{T}$, $x_0 \in \mathcal{X}$ it holds:
(b_1) $r(\tau, 0; \theta) \equiv 0$,
(b_2) *$r : \mathbb{T} \times \mathcal{X} \times \Theta \to \mathcal{X}$ satisfies the Lipschitz estimates*

$$\operatorname{Lip} r(\tau, \cdot; \theta) \le \ell(\theta), \qquad \operatorname{Lip} r(\tau, x_0; \cdot) \le \frac{\delta K_1 K_2 (K_1 + K_2) L_2}{[\delta - 2(K_1 + K_2)L_1]^2} \|x_0\|. \tag{2.9}$$

(c) *For $\mathbb{T}$ unbounded below, and if $L_1 < \frac{\delta}{2(K_1+K_2+\max\{K_1,K_2\})}$, then only the zero solution of* (2.3) *is contained in $S(\theta)$ and $R(\theta)$, i.e.*

$$S(\theta) \cap R(\theta) = \mathbb{T} \times \{0\}$$

and the zero solution is the only $c^{\pm}$-quasibounded solution of (2.3) *for any $c \in \Gamma$.*

Proof (of Theorem 2.2): See [Pöt06, Theorem 3.3]. □

3. Invariant Foliations

In the previous Section 2 and Theorem 2.2 we were able to characterize the set of solutions (or trajectories) for (2.3) which approach the zero solution at an exponential rate. Now we drop the restriction to the trivial solution and investigate attractivity properties of arbitrary solutions. For that purpose, we begin with an abstract lemma carrying most of the technical load for the following proofs. Due to the fact that the general solution φ of (2.3) exists uniquely in forward time, the mapping $G_\theta : \{(t, x, \tau, x_0) \in \mathbb{T} \times \mathcal{X} \times \mathbb{T} \times \mathcal{X} : \tau \in \mathbb{T}, t \in \mathbb{T}_\tau^+, x, x_0 \in \mathcal{X}\} \to \mathcal{X}$,

$$G_\theta(t, x; \tau, x_0) := H_\theta(t, x + \varphi(t; \tau, x_0; \theta)) - H_\theta(t, \varphi(t; \tau, x_0; \theta))$$

is well-defined under Hypothesis 2.1. Moreover, by Remark 2.1(2), G_θ is continuous in (τ, x_0), $G_\theta(t, 0; \tau, x_0) \equiv 0$ and $\operatorname{Lip}_2 G_\theta \le L_1 + |\theta| L_2$.

Lemma 3.1. *Assume that Hypothesis 2.1 is fulfilled and choose $\tau \in \mathbb{T}$ fixed. Then for growth rates $c \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T}, \mathbb{R})$, $a \lhd c \lhd b$, the operator $\mathcal{S}_\tau : \mathcal{X}_{\tau,c}^+ \times \mathcal{R}(Q(\tau)) \times \mathcal{X} \times \Theta \to \mathcal{X}_{\tau,c}^+$,*

$$\begin{aligned} \mathcal{S}_\tau(\psi; y_0, x_0, \theta) := \Phi_A(\cdot, \tau)\,[y_0 - Q(\tau)x_0] + \int_\tau^\cdot \Phi_A(\cdot, \sigma(s))Q(\sigma(s))G_\theta(s, \psi(s); \tau, x_0)\,\Delta s \\ - \int_\cdot^\infty \bar{\Phi}_A(\cdot, \sigma(s))P(\sigma(s))G_\theta(s, \psi(s); \tau, x_0)\,\Delta s \end{aligned} \tag{3.1}$$

is well-defined and has, for fixed $y_0 \in \mathcal{R}(Q(\tau))$, $x_0 \in \mathcal{X}$, $\theta \in \Theta$ the following properties:

(a) *There exists a $z_0 \in \mathcal{X}$ such that $\psi := \varphi(\cdot; \tau, z_0; \theta) - \varphi(\cdot; \tau, x_0; \theta) \in \mathcal{X}_{\tau,c}^+$ and satisfies*

$$Q(\tau)\psi(\tau) = y_0 - Q(\tau)x_0, \tag{3.2}$$

if and only if $\psi \in \mathcal{X}_{\tau,c}^+$ solves the fixed point problem

$$\psi = \mathcal{S}_\tau(\psi; y_0, x_0, \theta). \tag{3.3}$$

Moreover, in case $c \in \overline{\Gamma}$ we have:

(b) *$\mathcal{S}_\tau(\cdot; y_0, x_0, \theta) : \mathcal{X}_{\tau,c}^+ \to \mathcal{X}_{\tau,c}^+$ is a uniform contraction with Lipschitz constant*

$$\operatorname{Lip} \mathcal{S}_\tau(\cdot; y_0, x_0, \theta) \le L(\theta) < 1, \tag{3.4}$$

(c) *the unique fixed point $\psi_\tau^*(y_0, x_0, \theta) \in \mathcal{X}_{\tau,c}^+$ of $\mathcal{S}_\tau(\cdot; y_0, x_0, \theta)$ does not depend on the growth rate $c \in \overline{\Gamma}$ and we have the estimates*

$$\|P(\tau)\psi_\tau^*(y_0, x_0, \theta)(\tau)\| \le \ell(\theta)\,\|y_0 - x_0\|, \tag{3.5}$$

$$\operatorname{Lip} P(\tau)\psi_\tau^*(\cdot, x_0, \theta)(\tau) \le \ell(\theta), \tag{3.6}$$

(d) *for $c \in \Gamma$ the mapping $\psi_\tau^* : \mathcal{R}(Q(\tau)) \times \mathcal{X} \times \Theta \to \mathcal{X}_{\tau,c}^+$ is continuous.*

Proof. Let $\tau \in \mathbb{T}$ be fixed, and choose a growth rate $c \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T}, \mathbb{R})$ with $a \lhd c \lhd b$. We show the well-definedness of the operator $\mathcal{S}_\tau$. Thereto, pick $x_0 \in \mathcal{X}$, $y_0 \in \mathcal{R}(Q(\tau))$, $\theta \in \Theta$ arbitrarily. For $\psi, \bar{\psi} \in \mathcal{X}_{\tau,c}^+$ we obtain just as in the proof of [Pöt06, Lemma 3.2],

$$\begin{aligned} &\left\|\mathcal{S}_\tau(\psi; y_0, x_0, \theta)(t) - \mathcal{S}_\tau(\bar{\psi}; y_0, x_0, \theta)(t)\right\| e_c(\tau, t) \\ &\le \left(\frac{K_1}{\lfloor c - a \rfloor} + \frac{K_2}{\lfloor b - c \rfloor}\right) \frac{\delta L(\theta)}{K_1 + K_2} \left\|\psi - \bar{\psi}\right\|_{\tau,c}^+ \quad \text{for all } t \in \mathbb{T}_\tau^+. \end{aligned} \tag{3.7}$$

Thus, to show that $\mathcal{S}_\tau$ is well-defined, we observe $\mathcal{S}_\theta(0; y_0, x_0, \theta) = \Phi_A(\cdot, \tau)\,[y_0 - Q(\tau)x_0]$ from (3.1), whence

$$\|\mathcal{S}_\tau(\psi; y_0, x_0, \theta)(t)\|\, e_c(\tau, t)$$

$$\le \|\Phi_A(t,\tau)\,[y_0 - Q(\tau)x_0]\|\, e_c(\tau,t) + \|\mathcal{S}_\tau(\psi; y_0, x_0, \theta) - \mathcal{S}_\tau(0; y_0, x_0, \theta)\|^+_{\tau,c}$$
$$\stackrel{(2.5)}{\le} K_1 \|y_0 - x_0\| + \left(\frac{K_1}{\lfloor c-a\rfloor} + \frac{K_2}{\lfloor b-c\rfloor}\right)\frac{\delta L(\theta)}{K_1+K_2}\|\psi\|^+_{\tau,c} \quad \text{for all } t \in \mathbb{T}^+_\tau$$

and taking the supremum over $t \in \mathbb{T}^+_\tau$ implies $\mathcal{S}_\tau(\psi; y_0, x_0, \theta) \in \mathcal{X}^+_{\tau,c}$.

(*a*) Let $x_0 \in \mathcal{X}$, $\theta \in \Theta$ be arbitrary. We suppress the dependence on θ.

($\Rightarrow$) Let $y_0 \in \mathcal{R}(Q(\tau))$ and assume there exists a $z_0 \in \mathcal{X}$ such that $\psi = \varphi(\cdot;\tau,z_0) - \varphi(\cdot;\tau,x_0)$ is c^+-quasibounded and $Q(\tau)\psi(\tau) = y_0 - Q(\tau)x_0$. Then ψ is a c^+-quasibounded solution of the linear inhomogeneous equation $x^\Delta = A(t)x + G_\theta(t,\psi(t);\tau,x_0)$ and [Pöt02, p. 103, Satz 2.2.4(a)] implies that ψ is a fixed point of $\mathcal{S}_\tau(\cdot; y_0, x_0)$.

($\Leftarrow$) Conversely, assume $\psi \in \mathcal{X}^+_{\tau,c}$ satisfies (3.3) for some $y_0 \in \mathcal{R}(Q(\tau))$, $x_0 \in \mathcal{X}$. Then define $z_0 := P(\tau)\,[x_0 + \psi(\tau)] + y_0$ and set $\nu := \psi + \varphi(\cdot;\tau,x_0)$. Hence,

$$\begin{aligned} \nu(\tau) &= \psi(\tau) + x_0 \stackrel{(3.3)}{=} P(\tau)\psi(\tau) + Q(\tau)\mathcal{S}_\tau(\psi; y_0, x_0)(\tau) + x_0 \\ &\stackrel{(3.1)}{=} P(\tau)\psi(\tau) + y_0 - Q(\tau)x_0 + x_0 = P(\tau)\,[\psi(\tau)+x_0] + y_0 = z_0 \end{aligned} \tag{3.8}$$

and the difference ν also solves (2.3). Due to the uniqueness of forward solutions, this gives us $\nu = \varphi(\cdot;\tau,z_0)$, i.e., $\psi = \varphi(\cdot;\tau,z_0) - \varphi(\cdot;\tau,x_0)$. Finally, one has

$$Q(\tau)\psi(\tau) \stackrel{(3.8)}{=} Q(\tau)\,[z_0 - x_0] = Q(\tau)\,[y_0 - x_0] = y_0 - Q(\tau)x_0$$

and the equivalence in assertion (a) is established.

From now on, let $c \in \overline{\Gamma}$.

(*b*) Passing over to the least upper bound for $t \in \mathbb{T}^+_\tau$ in (3.7) yields the estimate (3.4) and our choice of δ in Hypothesis 2.1(ii) guarantees $L(\theta) < 1$ for $\theta \in \Theta$. Therefore, the contraction mapping principle implies a unique fixed point $\psi^*_\tau(y_0, x_0, \theta) \in \mathcal{X}^+_{\tau,c}$ of $\mathcal{S}_\tau(\cdot; y_0, x_0, \theta)$, which moreover satisfies

$$\|\psi^*_\tau(y_0,x_0,\theta)\|^+_{\tau,c} \le \frac{K_1}{1-L(\theta)}\|y_0 - x_0\| \tag{3.9}$$

(*c*) One proceeds as in [Pöt06, Lemma 3.2(c)] to show that $\psi^*_\tau(y_0,x_0,\theta) \in \mathcal{X}^+_{\tau,c}$ is independent of $c \in \overline{\Gamma}$. To prove the Lipschitz estimate (3.6), we suppress the dependence on the fixed parameters $x_0 \in \mathcal{X}$, $\theta \in \Theta$. To this end, consider $y_0, \bar{y}_0 \in \mathcal{R}(Q(\tau))$ and corresponding fixed points $\psi^*_\tau(y_0), \psi^*_\tau(\bar{y}_0) \in \mathcal{X}^+_{\tau,c}$ of $\mathcal{S}_\tau(\cdot; y_0)$ and $\mathcal{S}_\tau(\cdot; \bar{y}_0)$, respectively. We have

$$\begin{aligned} \|\psi^*_\tau(y_0) - \psi^*_\tau(\bar{y}_0)\|^+_{\tau,c} &\stackrel{(3.3)}{\le} \|\mathcal{S}_\tau(\psi^*_\tau(y_0); y_0) - \mathcal{S}_\tau(\psi^*_\tau(\bar{y}_0); y_0)\|^+_{\tau,c} \\ &\quad + \|\mathcal{S}_\tau(\psi^*_\tau(\bar{y}_0); y_0) - \mathcal{S}_\tau(\psi^*_\tau(\bar{y}_0); \bar{y}_0)\|^+_{\tau,c} \\ &\stackrel{(3.4)}{\le} L(\theta)\,\|\psi^*_\tau(y_0) - \psi^*_\tau(\bar{y}_0)\|^+_{\tau,c} + \|\mathcal{S}_\tau(\psi^*_\tau(\bar{y}_0); y_0) - \mathcal{S}_\tau(\psi^*_\tau(\bar{y}_0); \bar{y}_0)\|^+_{\tau,c}, \end{aligned}$$

and thus,

$$\begin{aligned} \|\psi^*_\tau(y_0) - \psi^*_\tau(\bar{y}_0)\|^+_{\tau,c} &\le \frac{1}{1-L(\theta)}\|\mathcal{S}_\tau(\psi^*_\tau(\bar{y}_0); y_0) - \mathcal{S}_\tau(\psi^*_\tau(\bar{y}_0); \bar{y}_0)\|^+_{\tau,c} \\ &\stackrel{(3.1)}{=} \frac{1}{1-L(\theta)}\sup_{t\in\mathbb{T}^+_\tau}\|\Phi_A(t,\tau)Q(\tau)\,[y_0 - \bar{y}_0]\|\, e_c(\tau,t) \stackrel{(2.5)}{\le} \frac{K_1}{1-L(\theta)}\|y_0 - \bar{y}_0\|. \end{aligned} \tag{3.10}$$

Moreover, directly from (3.1) and (3.3) we get the identity

$$P(\cdot)\psi^*_\tau(y_0) \stackrel{(2.2)}{=} -\int_\cdot^\infty \bar{\Phi}_A(\cdot,\sigma(s))P(\sigma(s))G_\theta(s,\psi^*_\tau(y_0)(s);\tau,x_0)\,\Delta s$$

and similarly to the proof of (b) this yields

$$\|P(\cdot)\left[\psi^*_\tau(y_0)-\psi^*_\tau(\bar y_0)\right]\|^+_{\tau,c} \le \frac{K_2}{\lfloor b-c\rfloor}\frac{\delta L(\theta)}{K_1+K_2}\|\psi^*_\tau(y_0)-\psi^*_\tau(\bar y_0)\|^+_{\tau,c},$$

with (3.10) this implies (3.6). The same arguments give (note $G_\theta(t,0;\tau,x_0)\equiv 0$)

$$\|P(\cdot)\psi^*_\tau(y_0)\|^+_{\tau,c} \le \frac{K_2}{\lfloor b-c\rfloor}\frac{\delta L(\theta)}{K_1+K_2}\|\psi^*_\tau(y_0)\|^+_{\tau,c},$$

and together with (3.9) we get (3.5). Therefore we have established the assertion (c).

(d) This can be shown as in [Pöt06, Lemma 3.2(d)]. □

Proposition 3.2 (invariant fibers). *Assume that Hypothesis 2.1 is fulfilled. Then for all $\tau\in\mathbb{T}$, $x_0\in\mathcal{X}$, $\theta\in\Theta$ the following holds:*

(a) *The* pseudo-stable fiber through (τ,x_0), *given by*

$$S^+(x_0,\theta)_\tau := \left\{z_0\in\mathcal{X} :\ \varphi(\cdot;\tau,z_0;\theta)-\varphi(\cdot;\tau,x_0;\theta)\in\mathcal{X}^+_{\tau,c} \text{ for all } c\in\Gamma\right\}$$

is forward invariant w.r.t. (2.3), *i.e.,*

$$\varphi(t;\tau,S^+(x_0,\theta)_\tau;\theta)\subseteq S^+(\varphi(t;\tau,x_0;\theta),\theta)_\tau \quad \text{for all } t\in\mathbb{T}^+_\tau \tag{3.11}$$

and possesses the representation

$$S^+(x_0,\theta)=\left\{(\tau,y_0+s^+(\tau,y_0,x_0;\theta)) :\ y_0\in\mathcal{R}(Q(\tau))\right\} \tag{3.12}$$

as graph of a continuous mapping $s^+:\mathbb{T}\times\mathcal{X}\times\mathcal{X}\times\Theta\to\mathcal{X}$ satisfying

$$s^+(\tau,y_0,x_0;\theta)=s^+(\tau,Q(\tau)y_0,x_0;\theta)\in\mathcal{R}(P(\tau)) \quad \text{for all } y_0\in\mathcal{X}.$$

Furthermore, for all $c\in\overline{\Gamma}$ it holds:

(a_1) *$s^+:\mathbb{T}\times\mathcal{X}\times\mathcal{X}\times\Theta\to\mathcal{X}$ is linearly bounded*

$$\left\|s^+(\tau,y_0,x_0;\theta)\right\|\le\|P(\tau)x_0\|+\ell(\theta)\|y_0-x_0\| \quad \text{for all } y_0\in\mathcal{X}, \tag{3.13}$$

(a_2) *$s^+(\tau,\cdot,x_0;\theta)$ is globally Lipschitzian with*

$$\operatorname{Lip}_2 s^+(\cdot,\theta)\le K_1\ell(\theta). \tag{3.14}$$

(b) *For $\mathbb{T}$ unbounded below and if* (2.3) *is regressive on Θ, then the* pseudo-unstable fiber through (τ,x_0), *given by*

$$R^-(x_0,\theta)_\tau := \left\{z_0\in\mathcal{X} :\ \varphi(\cdot;\tau,z_0;\theta)-\varphi(\cdot;\tau,x_0;\theta)\in\mathcal{X}^-_{\tau,c} \text{ for all } c\in\Gamma\right\}$$

is invariant w.r.t. (2.3), *i.e.,*

$$\varphi(t;\tau,R^-(x_0,\theta)_\tau;\theta)=R^-(\varphi(t;\tau,x_0;\theta),\theta)_\tau \quad \text{for all } t\in\mathbb{T}$$

and possesses the representation

$$R^-(x_0,\theta)=\left\{(\tau,y_0+r^-(\tau,y_0,x_0;\theta)) :\ y_0\in\mathcal{R}(P(\tau))\right\}$$

as graph of a continuous mapping $r^-:\mathbb{T}\times\mathcal{X}\times\mathcal{X}\times\Theta\to\mathcal{X}$ satisfying

$$r^+(\tau,y_0,x_0;\theta)=r^+(\tau,P(\tau)y_0,x_0;\theta)\in\mathcal{R}(Q(\tau)) \quad \text{for all } y_0\in\mathcal{X}.$$

Furthermore, for all $c\in\overline{\Gamma}$ it holds:

(b_1) *$r^-:\mathbb{T}\times\mathcal{X}\times\mathcal{X}\times\Theta\to\mathcal{X}$ is linearly bounded*

$$\left\|r^-(\tau,y_0,x_0;\theta)\right\|\le\|Q(\tau)x_0\|+\ell(\theta)\|y_0-x_0\| \quad \text{for all } y_0\in\mathcal{X},$$

(b_2) *$r^-(\tau,\cdot,x_0;\theta)$ is globally Lipschitzian with*

$$\operatorname{Lip}_2 r^-(\cdot,\theta)\le K_2\ell(\theta).$$

Remark 3.1. It is not difficult to see that the pseudo-stable fibers $S^+(x_0,\theta)_\tau$ are the leaves of a (forward) invariant foliation over each fiber $R(\theta)_\tau$, i.e., for each $\tau \in \mathbb{T}$ we have

$$\mathcal{X} = \bigcup_{x_0 \in R(\theta)_\tau} S^+(x_0,\theta), \qquad S^+(x_1,\theta) \cap S^+(x_2,\theta) = \emptyset \quad \text{for all } x_1, x_2 \in R(\theta)_\tau,\ x_1 \neq x_2.$$

Similarly, the fibers $R^-(x_0,\theta)$ form a foliation over $S(\theta)_\tau$.

Proof. Keep $\theta \in \Theta$ fixed and note that we suppress the dependence on θ to a large extend.

(a) Let $x_0, y_0 \in \mathcal{X}$ and $c \in \overline{\Gamma}$. We aim to show the invariance assertion (3.11) for $S^+(x_0)_\tau$. Let $\hat{x}_0 \in \varphi(t;\tau,S^+(x_0)_\tau)$ for some $t \in \mathbb{T}_\tau^+$, and by definition this is equivalent to the existence of a $z_0 \in \mathcal{X}$ such that $\hat{x}_0 = \varphi(t;\tau,z_0)$ and $\varphi(\cdot;\tau,z_0) - \varphi(\cdot;\tau,x_0) \in \mathcal{X}_{\tau,c}^+$. Therefore,

$$\varphi(\cdot;t,\hat{x}_0) - \varphi(\cdot;t,\varphi(t;\tau,x_0)) = \varphi(\cdot;t,\varphi(t;\tau,z_0)) - \varphi(\cdot;t,\varphi(t;\tau,x_0)) \overset{(2.4)}{=} \varphi(\cdot;\tau,z_0) - \varphi(\cdot;\tau,x_0),$$

i.e., $\hat{x}_0 \in S^+(\varphi(t;\tau,x_0))_\tau$ for all $t \in \mathbb{T}_\tau^+$.

The above Lemma 3.1 implies that $\mathcal{S}_\tau(\cdot;y_0,x_0) : \mathcal{X}_{\tau,c}^+ \to \mathcal{X}_{\tau,c}^+$ possesses a unique fixed point $\psi_\tau^*(y_0,x_0) \in \mathcal{X}_{\tau,c}^+$. Furthermore, this fixed point is of the form $\psi_\tau^*(y_0,x_0) = \varphi(\cdot;\tau,z_0) - \varphi(\cdot;\tau,x_0)$ with some $z_0 \in \mathcal{X}$ (cf. Lemma 3.1(a)). We define

$$s^+(\tau,y_0,x_0;\theta) := P(\tau)\left[x_0 + \psi_\tau^*(Q(\tau)y_0,x_0;\theta)(\tau)\right] \tag{3.15}$$

and evidently have $s^+(\tau,y_0,x_0) \in \mathcal{R}(P(\tau))$. Let us verify the representation (3.12).

($\subseteq$) Let $z_0 \in S^+(x_0)_\tau$, i.e., $\psi = \varphi(\cdot;\tau,z_0) - \varphi(\cdot;\tau,x_0) \in \mathcal{X}_{\tau,c}^+$. Then Lemma 3.1 implies

$$z_0 = \psi(\tau) + x_0 \overset{(3.2)}{=} P(\tau)\psi(\tau) + y_0 - Q(\tau)x_0 + x_0 = P(\tau)\psi(\tau) + y_0 + P(\tau)x_0,$$

hence $Q(\tau)z_0 = y_0$, and $z_0 = Q(\tau)z_0 + P(\tau)\left[x_0 + \psi_\tau^*(y_0,x_0)(\tau)\right]$. Thus, z_0 is contained in the graph of $s^+(\tau,\cdot,x_0)$ over $\mathcal{R}(Q(\tau))$.

($\supseteq$) On the other hand, let $z_0 \in \mathcal{X}$ be of the form $z_0 = y_0 + s^+(\tau,y_0,x_0)$ with $y_0 \in \mathcal{R}(Q(\tau))$. Then (3.1) and (3.3) imply $Q(\tau)\psi_\tau^*(y_0,x_0)(\tau) = y_0 - Q(\tau)x_0$, which yields $z_0 = y_0 + P(\tau)\left[x_0 + \psi_\tau^*(y_0,x_0)(\tau)\right] = x_0 + \psi_\tau^*(y_0,x_0)(\tau)$, and consequently $\varphi(\cdot;\tau,z_0) - \varphi(\cdot;\tau,x_0) \in \mathcal{X}_{\tau,c}^+$, i.e., $z_0 \in S^+(x_0)_\tau$. We postpone the continuity proof for s^+ to the end (a_2) below.

(a_1) Referring to (3.15), the inequality (3.13) is an immediate consequence of (3.5).

(a_2) The estimate (3.14) is a consequence of (3.6) and (3.15). Addressing the continuity of s^+, we know from Lemma 3.1(d) that $\psi_\tau^* : \mathcal{R}(Q(\tau)) \times \mathcal{X} \times \Theta \to \mathcal{X}_{\tau,c}^+$ is continuous, and by definition in (3.15) we get the continuity of $s^+(\tau,\cdot)$. Finally, the strategy to show that $s^+ : \mathbb{T} \times \mathcal{X} \times \mathcal{X} \times \Theta \to \mathcal{X}$ is continuous can be adapted from [Pöt06, Proof of Theorem 3.3].

(b) Since (2.3) is assumed to be regressive, its general solution $\varphi(t;\tau,x_0;\theta)$ exists for all $t,\tau \in \mathbb{T}$, as well as the mapping $G_\theta(t,x;\tau,x_0)$. Analogously to Lemma 3.1 we can show that the operator $\bar{\mathcal{S}}_\tau : \mathcal{X}_{\tau,c}^- \times \mathcal{R}(P(\tau)) \times \mathcal{X} \times \Theta \to \mathcal{X}_{\tau,c}^-$,

$$\begin{aligned}\bar{\mathcal{S}}_\tau(\psi;y_0,x_0,\theta) := \bar{\Phi}_A(\cdot,\tau)\left[y_0 - P(\tau)x_0\right] - \int_\tau^\cdot \bar{\Phi}_A(\cdot,\sigma(s))P(\sigma(s))G_\theta(s,\psi(s);\tau,x_0)\,\Delta s \\ + \int_{-\infty}^\cdot \Phi_A(\cdot,\sigma(s))Q(\sigma(s))G_\theta(s,\psi(s);\tau,x_0)\,\Delta s\end{aligned}$$

possesses a unique fixed point $\psi_\tau^*(y_0,x_0,\theta) \in \mathcal{X}_{\tau,c}^-$. We define $r^-(\tau,y_0,x_0;\theta) := Q(\tau)[x_0 + \psi_\tau^*(P(\tau)y_0,x_0;\theta)(\tau)]$ and proceed as in (a). □

In a more geometrically descriptive way, the subsequent result states that the invariant fiber bundles from Theorem 2.2 are exponentially attractive in a generalized sense of quasiboundedness. In fact, this convergence is actually "in phase" with solutions on the fiber bundles, and for that reason we speak of an *asymptotic phase*: For each solution ν of (2.3) there exists a solution ν_0 in the fiber bundles from Theorem 2.2 such that the difference $\nu - \nu_0$ is quasibounded.

Theorem 3.3 (asymptotic phase). *Assume $\mathbb{T}$ is unbounded below, that Hypothesis 2.1 is fulfilled with*

$$L_1 < \frac{\lfloor b-a \rfloor}{4(K_1 + K_2 + K_1 K_2 \max\{K_1, K_2\})}, \tag{3.16}$$

and choose a fixed $\delta \in \left(2(K_1 + K_2 + K_1K_2 \max\{K_1, K_2\})L_1, \frac{\lfloor b-a \rfloor}{2}\right)$ and $c \in \overline{\Gamma}$. Then for all $\tau \in \mathbb{T}$, $x_0 \in \mathcal{X}$, $\theta \in \Theta$ the following holds:

(a) *The pseudo-unstable fiber bundle $R(\theta)$ from Theorem 2.2(b) possesses an* asymptotic (forward) phase*, i.e., there exists a retraction $\pi^+(\tau, \cdot; \theta) : \mathcal{X} \to R(\theta)_\tau$ onto $R(\theta)_\tau$ with the property:*

$$\left\|\varphi(t; \tau, x_0; \theta) - \varphi(t; \tau, \pi^+(\tau, x_0; \theta); \theta)\right\| \leq \frac{K_1}{1 - L(\theta)} \frac{1 + (K_2 - 1)\ell(\theta)}{1 - \ell(\theta)} \|x_0\| \, e_c(t, \tau) \tag{3.17}$$

for all $t \in \mathbb{T}_\tau^+$. Geometrically, $\pi^+(\tau, x_0, \theta)$ is the unique intersection

$$R(\theta)_\tau \cap S^+(x_0, \theta)_\tau = \left\{\pi^+(\tau, x_0; \theta)\right\} \quad \textit{for all } x_0 \in \mathcal{X} \tag{3.18}$$

and one has:

(a_1) *$\pi^+ : \mathbb{T} \times \mathcal{X} \times \Theta \to \mathcal{X}$ is continuous and linearly bounded*

$$\left\|\pi^+(\tau, x_0; \theta)\right\| \leq K_2 \frac{1 + \ell(\theta)}{1 - \ell(\theta)} \|x_0\| \quad \textit{for all } x_0 \in \mathcal{X}, \tag{3.19}$$

(a_2) *$\varphi(t; \tau, \cdot; \theta) \circ \pi^+(\tau, \cdot; \theta) = \pi^+(t, \cdot; \theta) \circ \varphi(t; \tau, \cdot; \theta)$ for $t \in \mathbb{T}_\tau^+$.*

(b) *In case* (2.3) *is regressive on Θ, the pseudo-stable fiber bundle $S(\theta)$ from Theorem 2.2(a) possesses an* asymptotic (backward) phase*, i.e., there exists a retraction $\pi^-(\tau, \cdot; \theta) : \mathcal{X} \to S(\theta)_\tau$ onto $S(\theta)_\tau$ with the property:*

$$\left\|\varphi(t; \tau, x_0; \theta) - \varphi(t; \tau, \pi^-(\tau, x_0; \theta); \theta)\right\| \leq \frac{K_2}{1 - L(\theta)} \frac{1 + (K_1 - 1)\ell(\theta)}{1 - \ell(\theta)} \|x_0\| \, e_c(t, \tau)$$

for all $t \in \mathbb{T}_\tau^-$. Geometrically, $\pi^-(\tau, x_0, \theta)$ is the unique intersection

$$S(\theta)_\tau \cap R^-(x_0, \theta)_\tau = \left\{\pi^-(\tau, x_0; \theta)\right\} \quad \textit{for all } x_0 \in \mathcal{X}$$

and one has:

(b_1) *$\pi^- : \mathbb{T} \times \mathcal{X} \times \Theta \to \mathcal{X}$ is continuous and linearly bounded*

$$\left\|\pi^-(\tau, x_0; \theta)\right\| \leq K_1 \frac{1 + \ell(\theta)}{1 - \ell(\theta)} \|x_0\| \quad \textit{for all } x_0 \in \mathcal{X},$$

(b_2) *$\varphi(t; \tau, \cdot; \theta) \circ \pi^-(\tau, \cdot; \theta) = \pi^-(t, \cdot; \theta) \circ \varphi(t; \tau, \cdot; \theta)$ for $t \in \mathbb{T}_\tau^-$.*

Remark 3.2. Note that condition (3.16) is stronger than the corresponding inequality (2.6) necessary for Theorem 2.2 and Proposition 3.2. Consequently, all the above results remain applicable. The fact that (3.16) holds, implies $\max\{K_1, K_2\}\, \ell(\theta) < 1$ and thus $\ell(\theta) < 1$ for all $\theta \in \Theta$. Using Theorem 2.2 and Proposition 3.2 this gives us

$$\begin{aligned} &\operatorname{Lip}_2 r < 1, && \operatorname{Lip}_2 s^+ < 1, \\ &\operatorname{Lip}_2 s < 1, && \operatorname{Lip}_2 r^- < 1. \end{aligned} \tag{3.20}$$

Proof. Let $\theta \in \Theta$, $c \in \overline{\Gamma}$ and fix $\tau \in \mathbb{T}$, $x_0 \in \mathcal{X}$.

(a) We derive that there exists a unique $z_0 \in R(\theta)_\tau \cap S^+(x_0,\theta)_\tau$. For that purpose, note that $z_0 \in R(\theta)_\tau \cap S^+(x_0,\theta)_\tau$ if and only if $z_0 = P(\tau)z_0 + r(\tau, P(\tau)z_0;\theta)$ and $z_0 = Q(\tau)z_0 + s^+(\tau, Q(\tau)z_0, x_0;\theta)$, which is equivalent to

$$Q(\tau)z_0 = r(\tau, P(\tau)z_0;\theta) \quad \text{and} \quad P(\tau)z_0 = s^+(\tau, Q(\tau)z_0, x_0;\theta). \tag{3.21}$$

Due to Theorem 2.2(b_2) and Proposition 3.2(a_2) we know from (3.20) that $\operatorname{Lip}_2 r \cdot \operatorname{Lip}_2 s^+ < 1$ and [GD03, p. 19, (A.13)] applies to the equations (3.21). Thus, there exist two unique functions $q_\tau : \mathcal{X} \times \Theta \to \mathcal{R}(Q(\tau))$, $p_\tau : \mathcal{X} \times \Theta \to \mathcal{R}(P(\tau))$ satisfying (3.21), i.e.,

$$q_\tau(x_0,\theta) \equiv r(\tau, p_\tau(x_0,\theta);\theta) \quad \text{and} \quad p_\tau(x_0,\theta) \equiv s^+(\tau, q_\tau(x_0,\theta), x_0;\theta) \quad \text{on } \mathcal{X} \times \Theta. \tag{3.22}$$

Therefore, $\pi^+(\tau, x_0;\theta) := p_\tau(x_0;\theta) + q_\tau(x_0;\theta)$ is the unique element in the intersection $R(\theta)_\tau \cap S^+(x_0,\theta)_\tau$. As preparation for later use we deduce two estimates. From (3.22) and Theorem 2.2(b_1) one has

$$\|q_\tau(x_0,\theta)\| \overset{(2.9)}{\leq} \ell(\theta)\,\|p_\tau(x_0,\theta)\|$$

and also

$$\begin{aligned}
\|p_\tau(x_0,\theta)\| &\overset{(3.13)}{\leq} \|P(\tau)x_0\| + \ell(\theta)\,\|q_\tau(x_0,\theta) - x_0\| \\
&\overset{(2.5)}{\leq} K_2\,(1+\ell(\theta))\,\|x_0\| + \ell(\theta)^2\,\|p_\tau(x_0,\theta)\|,
\end{aligned}$$

which implies (note Remark 3.2)

$$\|p_\tau(x_0,\theta)\| \leq \frac{K_2}{1-\ell(\theta)}\,\|x_0\|, \qquad \|q_\tau(x_0,\theta)\| \leq \frac{K_2\ell(\theta)}{1-\ell(\theta)}\,\|x_0\|. \tag{3.23}$$

Now we can show (3.17) and neglect the dependence on θ. Since by definition, $\pi^+(\tau,x_0) \in S^+(x_0)_\tau$ for $x_0 \in \mathcal{X}$, it follows from Lemma 3.1(a) that $\varphi(\cdot;\tau,x_0) - \varphi(\cdot;\tau,\pi^+(\tau,x_0)) = \psi^*_\tau(Q(\tau)\pi^+(\tau,x_0),x_0)$ and Lemma 3.1 together with (3.9) implies

$$\left\|\varphi(t;\tau,x_0) - \varphi(t;\tau,\pi^+(\tau,x_0))\right\|^+_{\tau,c} \leq \frac{K_1}{1-L(\theta)}\,\|q_\tau(x_0) - x_0\|;$$

so the triangle inequality implies (3.17). Theorem 2.2(b) and Proposition 3.2(a_1), together with the uniform contraction principle (cf. [GD03, p. 18, (A.4)]) easily yield that the functions $p_\tau(x_0,\theta)$, $q_\tau(x_0,\theta)$ are continuous in (τ,x_0,θ); thus also $\pi^+ : \mathbb{T} \times \mathcal{X} \times \Theta \to \mathcal{X}$ shares this property.

(a_1) It remains to derive the estimate (3.19), which is an easy consequence of the above inequalities for $\|p_\tau(x_0,\theta)\|$ and $\|q_\tau(x_0,\theta)\|$, respectively.

(a_2) The (forward) invariance of $R(\theta)$ and $S^+(x_0,\theta)_\tau$ implies

$$\begin{aligned}
\varphi(t;\tau,\pi^+(\tau,x_0)) &\overset{(3.18)}{\in} \varphi(t;\tau,R(\theta)_\tau \cap S^+(x_0)_\tau) \subseteq \varphi(t;\tau,R(\theta)_\tau) \cap \varphi(t;\tau,S^+(x_0)_\tau) \\
&\overset{(3.11)}{\subseteq} R(\theta)_\tau \cap S^+(\varphi(t;\tau,x_0))_\tau \overset{(3.18)}{=} \left\{\pi^+(\tau,\varphi(t;\tau,x_0))\right\} \quad \text{for all } t \in \mathbb{T}^+_\tau.
\end{aligned}$$

(b) Since equation (2.3) is supposed to be regressive on Θ, we can construct pseudo-unstable fibers $R^-(x_0,\theta)_\tau$ by virtue of Proposition 3.2(b). With an analogous argumentation one shows that the intersection of $S(\theta)_\tau$ and $R^-(x_0,\theta)_\tau$ consists of a single point $\pi^-(\tau,x_0;\theta)$ and proceeds as in the proof of assertion (a).

This finishes our proof. □

Before we continue to the more applied part of this work, let us conclude and give a geometrical interpretation of the results obtained until now (we keep $\theta \in \Theta$ fixed):

Under Hypothesis 2.1 the semi-linear dynamic equation (2.3) possesses a pseudo-unstable fiber bundle $R(\theta) \subseteq \mathbb{T} \times \mathcal{X}$. In case $a \lhd 0$ and for sufficiently small Lipschitz constant of the nonlinearity H_θ, we can choose $c \lhd 0$ and $R(\theta)$ contains all solutions to (2.3) which exist in backward time and tend away from the origin at an exponential rate (they are c^--quasibounded). The pseudo-unstable fiber bundle $R(\theta)$ is invariant, i.e., for any solution $\nu : \mathbb{T}_\tau^+ \to \mathcal{X}$ of (2.3) with $\nu(\tau) \in R(\theta)_\tau$ one has $\nu(t) \in R(\theta)_t$ for all $t \in \mathbb{T}_\tau^+$, consequently (cf. (2.8))

$$\nu(t) \equiv P(t)\nu(t) + r(t, P(t)\nu(t); \theta) \quad \text{on } \mathbb{T}_\tau^+$$

and the projection $\nu_0(t) := P(t)\nu(t)$ solves the *reduced equation*

$$p^\Delta = A(t)p + P(t)H_\theta(t, p + r(t, p; \theta)). \tag{3.24}$$

This is a dynamic equation evolving in the lower-dimensional set $\{(t, x) \in \mathbb{T} \times \mathcal{X} : t \in \mathbb{T}, x \in \mathcal{R}(P(t))\}$, i.e., any solution ν_0 of (3.24) satisfies $\nu_0(t) \in \mathcal{R}(P(t))$ for all $t \in \mathbb{T}_\tau^+$ (see (2.2)), provided $\nu_0(\tau) \in \mathcal{R}(P(\tau))$.

Conversely, the solutions of (3.24) are related to the solutions of (2.3) starting on $R(\theta)$ via the relation

$$\varphi\big(t; \tau, \nu_0(\tau) + r(\tau, \nu_0(\tau); \theta); \theta\big) \equiv \nu_0(t) + r(t, \nu_0(t); \theta) \quad \text{on } \mathbb{T}_\tau^+. \tag{3.25}$$

Then Theorem 3.3(a) states that for every solution $\nu : \mathbb{T}_\tau^+ \to \mathcal{X}$ of (2.3) there exists a solution ν_0 of (3.24) such that the difference $\nu - \varphi(\cdot; \tau, \nu_0(\tau) + r(\tau, \nu_0(\tau); \theta); \theta)$ is exponentially decaying as $t \to \infty$. The initial value for ν_0 is given by $\nu_0(\tau) = P(\tau)\pi^+(\tau, \nu(\tau); \theta)$.

Dual considerations also hold for the pseudo-stable fiber bundle $S(\theta)$ and its asymptotic (backward) phase π^-, if (2.3) is regressive.

4. Stability in Critical Cases

So far the present paper had an abstract and quite technical flavor since our main concern was to provide general existence results for invariant foliations. Nevertheless, the harvest of these considerations will be a version of Pliss's reduction principle from the introduction for a quite general class of nonautonomous dynamic equations on measure chains. Here we can restrict to the parameter-free situation and consider (2.3) for $\theta = 0$, i.e. the system

$$x^\Delta = A(t)x + F_1(t, x) \tag{4.1}$$

to deduce the subsequent center manifold theorem:

Theorem 4.1 (reduction principle)**.** *Let $K_1, K_2 \geq 1$ be reals, $a, b \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T}, \mathbb{R})$ growth rates with $a \lhd b$, $a \lhd 0$, assume $\mathbb{T}$ is unbounded below and let $U \subseteq \mathcal{X}$ be an open neighborhood of 0. Moreover, suppose*

(i) *Exponential dichotomy: There exists a regular invariant projector $P : \mathbb{T} \to \mathcal{L}(\mathcal{X})$ of (2.1) such that the estimates*

$$\|\Phi_A(t, s)Q(s)\| \leq K_1 e_a(t, s), \qquad \left\|\bar{\Phi}_A(s, t)P(t)\right\| \leq K_2 e_b(s, t) \quad \textit{for all } t \preceq s \tag{4.2}$$

are satisfied, with the complementary projector $Q(t) := I_\mathcal{X} - P(t)$.

(ii) *$o(x)$-perturbation: The identity $F_1(t, 0) \equiv 0$ on $\mathbb{T}$ holds and one has*

$$\lim_{x,y \to 0} \frac{F_1(t, x) - F_1(t, y)}{\|x - y\|} = 0 \quad \textit{uniformly in } t \in \mathbb{T}. \tag{4.3}$$

Then for all $\lambda > 0$ there exists a $\rho > 0$ and a continuous mapping $r_0 : \mathbb{T} \times B_\rho \to B_{\rho\lambda} \subseteq \mathcal{X}$ with

$$r_0(\tau, x_0) = r_0(\tau, P(\tau)x_0) \in \mathcal{R}(Q(\tau)) \quad \text{for all } \tau \in \mathbb{T},\ x_0 \in B_\rho \tag{4.4}$$

and the following properties:

(a) $r_0(\tau, 0) \equiv 0$ *on* $\mathbb{T}$ *and* $\operatorname{Lip}_2 r_0 \le \lambda$,

(b) *the graph* $R_0 := \{(\tau, y_0 + r_0(\tau, y_0)) \in \mathbb{T} \times \mathcal{X} : y \in \mathcal{R}(P(\tau)) \cap B_\rho\}$ *is locally invariant w.r.t.* (4.1)*, i.e. for all* $(\tau, x_0) \in R_0$ *one has the inclusion* $(t, \varphi(t; \tau, x_0)) \in R$ *as long as* $\varphi([\tau, t]_\mathbb{T}; \tau, x_0) \subseteq B_\rho$,

(c) *if the zero solution of the* reduced equation

$$p^\Delta = A(t)p + P(t)F_1(t, p + r_0(t, p)) \tag{4.5}$$

is stable (uniformly stable, asymptotically stable, uniformly asymptotically stable or unstable, resp.), then the zero solution of (4.1) *is stable (uniformly stable, asymptotically stable, uniformly asymptotically stable or unstable, resp.).*

Proof. Let $\lambda > 0$ be given.

Thanks to our assumption (ii) we can choose a fixed $\rho > 0$ so small that beyond $B_\rho \subseteq \mathcal{X}$ also the Lipschitz condition

$$\|F_1(t, x) - F_1(t, \bar{x})\| \le \tfrac{L_1}{2} \|x - \bar{x}\| \quad \text{for all } t \in \mathbb{T},\ x, \bar{x} \in B_\rho$$

holds with $L_1 := \min\left\{\frac{\min\{\lfloor -a \rfloor, \lfloor b-a \rfloor\}}{8(K_1+K_2+K_1K_2\max\{K_1,K_2\})}, \lambda\frac{K_1+K_2}{K_1K_2}\right\}$. On the other hand, it is well-known that the radial retraction $\chi : \mathcal{X} \to B_1$,

$$\chi(x) := \begin{cases} x & \text{for } \|x\| \le 1 \\ \frac{x}{\|x\|} & \text{for } \|x\| > 1 \end{cases}$$

is globally Lipschitz with $\operatorname{Lip}\chi \le 2$. Then the globally extended nonlinearity $\tilde{F}_1 : \mathbb{T} \times \mathcal{X} \to \mathcal{X}$, $\tilde{F}_1(t, x) := F_1\big(t, \rho\chi\big(\frac{x}{\rho}\big)\big)$ satisfies $\tilde{F}_1(t, x) \equiv F_1(t, x)$ on $\mathbb{T} \times B_\rho$ and $\operatorname{Lip}_2 \tilde{F}_1 \le L_1$. Therefore, due to our choice of L_1, Theorem 2.2(b) guarantees that the extended system

$$x^\Delta = A(t)x + \tilde{F}_1(t, x) \tag{4.6}$$

possesses a (global) pseudo-unstable fiber bundle $\tilde{R} \subseteq \mathbb{T} \times \mathcal{X}$ given as graph of a continuous mapping $\tilde{r} : \mathbb{T} \times \mathcal{X} \to \mathcal{X}$ satisfying (2.7). Furthermore, the growth rate $c \in \Gamma$ can be chosen so that $c \lhd 0$. We now define the continuous restriction $r_0 := \tilde{r}|_{\mathbb{T} \times B_\rho}$ and verify that it has all the desired properties claimed in Theorem 4.1. Above all, (4.4) is a direct consequence of (2.7).

(*a*) While $r_0(\tau, 0) \equiv 0$ on $\mathbb{T}$ follows from Theorem 2.2(b_1), we get the Lipschitz estimate from (2.9) and our choice for L_1. Combining this, we also have $\|r_0(t, x)\| \le \lambda \|x\| \le \lambda\rho$ for all $t \in \mathbb{T}$, $x \in B_\rho$.

(*b*) The invariance of $\tilde{R}$ from Theorem 2.2(b) immediately gives us local invariance of R_0.

(*c*) If the zero solution of the reduced equation (4.5) is unstable, then by invariance of R_0, also the zero solution of (4.1) is unstable (cf. (3.25)). Now, let $\varepsilon > 0$, $\tau \in \mathbb{T}$ be given, but w.l.o.g. $\varepsilon \le 2(1 + \lambda)\rho$. We suppose the zero solution of (4.5) is stable, i.e., there exists a $\delta \in (0, \rho)$ so that

$$\|\nu_0(t)\| < \tfrac{\varepsilon}{2(1+\lambda)} \quad \text{for all } t \in \mathbb{T}_\tau^+ \tag{4.7}$$

and any solution $\nu_0 : \mathbb{T}_\tau^+ \to \mathcal{X}$ of (4.5) with $\nu_0(\tau) \in \mathcal{R}(P(\tau)) \cap B_\delta$. In the following, let $\nu : \mathbb{T}_\tau^+ \to \mathcal{X}$ be an arbitrary solution of (4.1) with $\|\nu(\tau)\| < \min\big\{\delta\frac{1-\ell(0)}{K_2}, \frac{\varepsilon}{2}\frac{(1-\ell(0))(1-L(0))}{K_1(1+(K_2-1)\ell(0))}\big\}$.

From Theorem 3.3(a) we know that there exists a solution $\tilde{\nu}_0 : \mathbb{T}_\tau^+ \to \mathcal{X}$ of $p^\Delta = A(t)p + P(t)\tilde{F}_1(t, p + \tilde{r}(t,p))$ with

$$\left\|\tilde{\varphi}(t;\tau,\nu(\tau)) - \tilde{\varphi}\big(t;\tau,\tilde{\nu}_0(\tau) + \tilde{r}(\tau,\tilde{\nu}_0(\tau))\big)\right\| \overset{(3.17)}{\leq} \frac{K_1}{1-L(0)} \frac{1+(K_2-1)\ell(0)}{1-\ell(0)} \|\nu(\tau)\| \, e_c(t,\tau)$$

for all $t \in \mathbb{T}_\tau^+$, where $\tilde{\varphi}$ denotes the general solution of (4.6). We have from Theorem 3.3(a)

$$\|\tilde{\nu}_0(\tau)\| = \left\|P(\tau)\pi^+(\tau,\nu(\tau))\right\| \overset{(3.23)}{\leq} \frac{K_2}{1-\ell(0)} \|\nu(\tau)\| < \delta$$

and consequently (4.7) gives us $\|\tilde{\nu}_0(t)\| < \frac{\varepsilon}{2(1+\lambda)}$ for all $t \in \mathbb{T}_\tau^+$. But this yields (note that we have $e_c(t,\tau) \leq 1$ for $t \in \mathbb{T}_\tau^+$) with the triangle inequality

$$\begin{aligned} \|\tilde{\varphi}(t;\tau,\nu(\tau))\| &\leq \left\|\tilde{\varphi}(t;\tau,\nu(\tau)) - \tilde{\varphi}\big(t;\tau,\pi^+(\tau,\nu(\tau))\big)\right\| + \left\|\tilde{\varphi}\big(t;\tau,\pi^+(\tau,\nu(\tau))\big)\right\| \\ &\leq \frac{K_1}{1-L(0)} \frac{1+(K_2-1)\ell(0)}{1-\ell(0)} \|\nu(\tau)\| \, e_c(t,\tau) + \|\tilde{\nu}_0(t) + \tilde{r}(t,\tilde{\nu}_0(t))\| \\ &\leq \frac{K_1}{1-L(0)} \frac{1+(K_2-1)\ell(0)}{1-\ell(0)} \|\nu(\tau)\| + (1+\lambda)\|\tilde{\nu}_0(t)\| < \varepsilon \quad \text{for all } t \in \mathbb{T}_\tau^+ \end{aligned}$$

and 0 is a stable solution of (4.6). However, since the systems (4.1) and (4.6) coincide on $\mathbb{T} \times B_\rho$, and due to $\tilde{\varphi}(t;\tau,\nu(\tau)) \in B_\rho$ for all $t \in \mathbb{T}_\tau^+$, it is $\nu = \tilde{\varphi}(\cdot;\tau,\nu(\tau))$. Thus, the zero solution is also stable w.r.t. (4.1). Keeping in mind that R_0 is uniformly exponentially attracting (cf. (3.17)), a similar reasoning gives us the assertion on the remaining stability properties. □

In our concluding example we make use of the "Hilger discs" given by

$$\mathbb{H}_0 := \{z \in \mathbb{C} : \Re z < 0\}, \qquad \mathbb{H}_h := \left\{z \in \mathbb{C} : \left|z + \tfrac{1}{h}\right| < \tfrac{1}{h}\right\} \quad \text{for all } h > 0,$$

which are crucial for a stability analysis on general measure chains.

Example 4.1. In a population-dynamical framework, Rosenzweig (see [Ros71]) studied an autonomous version of the following planar ODE

$$\left\{ \begin{array}{l} \dot{x}_1 = -x_1(1-x_1) - b(t)x_2(1-e^{-x_1}) \\ \dot{x}_2 = c(t)x_2(1-e^{-x_1}) - 2x_2 \end{array} \right. , \tag{4.8}$$

whereas we allow an explicit time-dependence in form of the bounded continuous functions $b, c : \mathbb{R} \to \mathbb{R}$. In its equilibrium $(0,0)$ the above system has the linearization $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -2 \end{smallmatrix}\right)$ and by the principle of linearized stability the zero solution is asymptotically stable. Now we want to study the time scale version of (4.8) on time scales $\mathbb{T}$ satisfying $\mu^*(\mathbb{T}) \subseteq [h,H]$ for $0 \leq h \leq H$. This system can be represented in the form (4.1) with $\mathcal{X} = \mathbb{R}^2$ and

$$A(t) \equiv \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \qquad F_1(t,x) = \begin{pmatrix} x_1^2 - b(t)x_2(1-e^{-x_1}) \\ c(t)x_2(1-e^{-x_1}) \end{pmatrix}.$$

Hence, $A = A(t)$ has the spectrum $\Sigma(A) = \{-2,-1\}$ and concerning the stability properties for the trivial solution of (4.1), the following can be stated:

- For $\Sigma(A) \subseteq \mathbb{H}_H$ the zero solution is asymptotically stable.
- For $\Sigma(A) \not\subseteq \overline{\mathbb{H}_h}$ the zero solution is unstable, since (4.1) possesses an unstable fiber bundle consisting of solutions tending exponentially away from 0.

The interesting situation is given when, for instance, on the homogeneous time scale $\mathbb{T} = \mathbb{Z}$ the constant matrix A has eigenvalues on the boundary of $\mathbb{H}_1$. The principle of linearized stability does not apply, but we can use Theorem 4.1: Its assumption (i) is fulfilled with $K_1 = K_2 = 1$, constant functions $a(t) \equiv \alpha$, $b(t) \equiv \beta$ with $-1 < \alpha < \beta < 0$ and an invariant projector $P(t) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Due to $\lim_{x\to 0} D_2F_1(t,x) = 0$ uniformly in $t \in \mathbb{T}$ also assumption (ii) holds. Thus, if we choose $\lambda = 1$, there exists a function $r_0 : \mathbb{T} \times (-\rho,\rho) \to (-\rho,\rho)$ with $r_0(t,0) \equiv 0$, such that the stability of the zero solution of the planar system (4.1) is determined by the stability of the trivial solution for the scalar equation

$$x_2^\Delta = -2x_2 + c(t)x_2 \left(1 - e^{-r_0(t,x_2)}\right).$$

Nevertheless, due to our limited space the stability analysis of this equation is beyond the scope of the paper. Such methods, in particular a procedure to obtain approximations of the mapping r_0, have been developed in [PR05].

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