Exponential Dichotomies for Dynamic Equations on Measure Chains

Christian Pötzsche*

Institut für Mathematik, Universität Augsburg,
D-86135 Augsburg, Germany

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Abstract

In this paper we introduce the notion of an exponential dichotomy for not necessarily invertible linear dynamic equations in Banach spaces within the framework of the "Calculus on Measure Chains." Particularly this unifies the corresponding theories for difference and differential equations. We apply our approach to obtain results on perturbed systems.

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1 Introduction and Preliminaries

Basically an exponential dichotomy is a generalization of the concept of hyperbolicity from autonomous to nonautonomous linear equations, where the stability properties of the solutions in the nontrivial invariant sets, or precisely in the invariant vector bundles, are uniform. Thorough introductions into the theory of exponentially dichotomic ordinary differential equations (ODEs) can be found in e.g. the books Daleckiĭ & Kreĭn [7] or Coppel [6]. For difference equations (OΔEs) the literature is slightly sparser, but Coffman & Schäffer [5] and Henry [9, Section 7.6] pioneered here and meanwhile dichotomies are widely used. They are so important in the theory of nonautonomous dynamical systems, since they are a very useful tool to solve nonlinear problems as perturbations of linear ones, like in the persistence of integral manifolds (cf. e.g. Sakamoto [18]). Moreover the applications range from stability theory, because a dichotomy is a type of conditional stability, to modern chaos theory (see Palmer [17], which is also a good introduction into discrete dichotomies).

In the present paper we introduce the notion of an exponential dichotomy for nonregressive linear dynamic equations in Banach spaces and prove some of its central properties. This allows to consider difference, differential and equations on inhomogeneous time scales, i.e. closed subsets of $\mathbb{R}$, simultaneously. As an application we also deduce some results about inhomogeneously and semi-linearly perturbed systems. Here we have to point out that our exponential growth rates are not assumed to be constant. To quote a good reference about dynamic equations on measure

*Fax: 49 821 598 2200, `email: christian.poetzsche@math.uni-augsburg.de`

chains we still strongly recommend Hilger [11] and with a focus on the linear theory Aulbach & Hilger [1]. Nevertheless there exists the monograph Lakshmikantham, Sivasundaram & Kaymakçalan [13].

Now suppose for the following that $(\mathbb{T}, \preceq, \mu)$ is an arbitrary measure chain with bounded graininess μ^* and $\mathcal{X}$ is a real or complex Banach space with the norm $\|\cdot\|$. $\mathcal{L}(\mathcal{X}_1; \mathcal{X}_2)$ stands for the linear space of continuous homomorphisms with the norm $\|T\| := \sup_{\|x\|=1} \|Tx\|$ for any $T \in \mathcal{L}(\mathcal{X}_1; \mathcal{X}_2)$, and $\mathcal{GL}(\mathcal{X}_1; \mathcal{X}_2)$ for the set of toplinear isomorphisms between two linear subspaces $\mathcal{X}_1, \mathcal{X}_2$ of $\mathcal{X}$; $I_{\mathcal{X}_1}$ is the identity mapping on $\mathcal{X}_1$. Additionally we write $\mathcal{L}(\mathcal{X}) := \mathcal{L}(\mathcal{X}; \mathcal{X})$ and $\mathcal{N}(T) := T^{-1}(\{0\})$ is the nullspace and $\mathcal{R}(T) := T\mathcal{X}$ the range of $T \in \mathcal{L}(\mathcal{X})$.

We also shortly introduce some notions, which are specific for the calculus on measure chains. Above all, $\mathbb{T}_\tau^+$ and $\mathbb{T}_\tau^-$ are the $\mathbb{T}$-intervals $\{t \in \mathbb{T} : \tau \preceq t\}$ and $\{t \in \mathbb{T} : t \preceq \tau\}$, respectively, for any $\tau \in \mathbb{T}$; differing from the usual standard, $\rho^+ : \mathbb{T} \to \mathbb{T}$ is the forward jump operator. A subset $J \subseteq \mathbb{T}$ is said to be unbounded above (resp. below), if the set $\{\mu(t, \tau) \in \mathbb{R} : t \in J\}$ is unbounded above (resp. below) for one and hence (by the properties of the growth calibration μ) every $\tau \in \mathbb{T}$. The partial derivative of a mapping $\Phi : \mathbb{T} \times \mathbb{T} \to \mathcal{X}$ with respect to the first variable is denoted by $\Delta_1 \Phi$. $\mathcal{C}_{rd}(\mathbb{T}^\kappa, \mathcal{X})$ are the rd-continuous mappings from $\mathbb{T}^\kappa$ into $\mathcal{X}$ and $\mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T}^\kappa, \mathbb{R}) := \{a \in \mathcal{C}_{rd}(\mathbb{T}^\kappa, \mathbb{R}) : 1 + \mu^*(t)a(t) > 0 \text{ for } t \in \mathbb{T}^\kappa\}$ is the linear space of positively regressive functions with the algebraic operations

$$(a \oplus b)(t) := a(t) + b(t) + \mu^*(t)a(t)b(t),$$
$$(\alpha \odot a)(t) := \lim_{h \searrow \mu^*(t)} \frac{(1 + ha(t))^\alpha - 1}{h} \quad \text{for } t \in \mathbb{T}^\kappa$$

for $a, b \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T}^\kappa, \mathbb{R})$ and reals $\alpha \in \mathbb{R}$. With fixed $\tau \in \mathbb{T}$ and $c, d \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T}^\kappa, \mathbb{R})$ we define the two linear spaces

$$\mathcal{B}_{\tau,c}^+(\mathcal{X}) := \left\{\lambda \in \mathcal{C}_{rd}(\mathbb{T}_\tau^+, \mathcal{X}) : \sup_{\tau \preceq t} \|\lambda(t)\| \, e_{\ominus c}(t, \tau) < \infty\right\},$$
$$\mathcal{B}_{\tau,d}^-(\mathcal{X}) := \left\{\lambda \in \mathcal{C}_{rd}(\mathbb{T}_\tau^-, \mathcal{X}) : \sup_{t \preceq \tau} \|\lambda(t)\| \, e_{\ominus d}(t, \tau) < \infty\right\}$$

of so-called c^+-*quasibounded* and d^--*quasibounded mappings*, which are immediately seen to be Banach spaces with regard to the norms

$$\|\lambda\|_{\tau,c}^+ := \sup_{\tau \preceq t} \|\lambda(t)\| \, e_{\ominus c}(t, \tau), \qquad \|\lambda\|_{\tau,d}^- := \sup_{t \preceq \tau} \|\lambda(t)\| \, e_{\ominus d}(t, \tau),$$

respectively.

2 Exponential Dichotomies

We consider a linear dynamic equation

$$\boxed{x^\Delta = A(t)x,} \tag{2.1}$$

with coefficient operator $A \in \mathcal{C}_{rd}(\mathbb{T}^\kappa, \mathcal{L}(\mathcal{X}))$ and the transition operator $\Phi_A(t, \tau) \in \mathcal{L}(\mathcal{X})$, i.e. the solution of the corresponding operator-valued initial value problem $X^\Delta = A(t)X$, $X(\tau) = I_\mathcal{X}$ in $\mathcal{L}(\mathcal{X})$ for $\tau, t \in \mathbb{T}$, $\tau \preceq t$. Since we do not assume $I_\mathcal{X} + \mu^*(t)A(t) \in \mathcal{GL}(\mathcal{X}; \mathcal{X})$, in other words, (2.1) needs not to be regressive, $\Phi_A(t,\tau)$ in general is not invertible and exists only for $\tau \preceq t$, which can be shown like in the standard existence and uniqueness theorem for dynamic equations from HILGER [11, Theorem 5.7].

For all the subsequent, let $J \subseteq \mathbb{T}^\kappa$ be a bounded or unbounded $\mathbb{T}$-interval. A nonempty set $\mathcal{W} \subseteq J \times \mathcal{X}$ is called an *invariant vector bundle* of (2.1), if the following holds:

(i) $\mathcal{W}$ is positively invariant with respect to (2.1), i.e.

$$(\tau,\xi) \in \mathcal{W} \quad \Rightarrow \quad (t, \Phi_A(t,\tau)\xi) \in \mathcal{W} \quad \text{for } \tau \preceq t,\ \tau, t \in J,$$

(ii) for each $\tau \in J$ the *fiber* $\mathcal{W}(\tau) := \{\xi \in \mathcal{X} : (\tau,\xi) \in \mathcal{W}\}$ is a closed subspace of $\mathcal{X}$;

see also SIEGMUND [19]. In case of a regressive equation (2.1) two fibers $\mathcal{W}(t)$ and $\mathcal{W}(s)$ are homeomorphic by virtue of the toplinear homeomorphism $\Phi_A(t,s)$ $(s,t \in J)$. Otherwise only the inclusion $\Phi_A(t,s)\mathcal{W}(s) \subseteq \mathcal{W}(t)$ for $s \preceq t$ holds. Trivially the *zero bundle* $J \times \{0\}$ and the whole *extended state space* $J \times \mathcal{X}$ are invariant vector bundles, but also the following holds.

Proposition 2.1 (invariant projector)**:** *If $P : J \to \mathcal{L}(\mathcal{X})$ is an* invariant projector *of equation* (2.1)*, i.e. a projection-valued function $(P(t)^2 \equiv P(t))$ such that*

$$P(t)\Phi_A(t,s) = \Phi_A(t,s)P(s) \quad \textit{for } s \preceq t,\ s,t \in J, \tag{2.2}$$

then the two sets

$$\mathcal{S} := \{(\tau,\eta) \in J \times \mathcal{X} : \eta \in \mathcal{R}(P(\tau))\}, \qquad \mathcal{U} := \{(\tau,\xi) \in J \times \mathcal{X} : \xi \in \mathcal{N}(P(\tau))\}$$

are invariant vector bundles of (2.1) *and it is $\mathcal{S}(\tau) \oplus \mathcal{U}(\tau) = \mathcal{X}$ for all $\tau \in J$.*

Remark 2.2: (1) On a discrete measure chain, i.e. if there exists a real $\gamma > 0$ such that $\gamma \leq \mu^*(t)$ for $t \in J$, a function of projections $P : J \to \mathcal{L}(\mathcal{X})$ is an invariant projector of (2.1), if and only if $P(\rho^+(t))\,[I_{\mathcal{X}} + \mu^*(t)A(t)] = [I_{\mathcal{X}} + \mu^*(t)A(t)]\,P(t)$ for $t \in J^\kappa$ holds. This is a consequence of HILGER [11, Theorem 6.2] and mathematical induction.

(2) In case of a regressive equation (2.1), we get $P(t) = \Phi_A(t,s)P(s)\Phi_A(s,t)$ for $s,t \in J$ and all projections $P(t) \in \mathcal{L}(\mathcal{X})$ are similar. In addition to this, each $P(t)$ has the same rank, if $\mathcal{S}(s)$ is finite-dimensional for one and consequently every $s \in J$.

Proof. The proof is a straight-forward verification of the definition of an invariant vector bundle. □

Because we do not restrict our considerations on regressive equations, in particular solutions in the vector bundle $\mathcal{U}$ might not exist in backward time. To overcome this problem, we postulate invertibility only between the fibers of $\mathcal{U}$.

Proposition 2.3 (regularity condition)**:** *Let $P : J \to \mathcal{L}(\mathcal{X})$ be an invariant projector of equation* (2.1) *such that the* regularity condition

$$[I_{\mathcal{X}} + \mu^*(t)A(t)]\,|_{\mathcal{U}(t)} : \mathcal{U}(t) \to \mathcal{U}(\rho^+(t)) \ \textit{is bijective for right-scattered } t \in J^\kappa \tag{2.3}$$

is fulfilled. Then we have $\Phi_A(t,s)|_{\mathcal{U}(s)} \in \mathcal{GL}(\mathcal{U}(s);\mathcal{U}(t))$ for any $s \preceq t,\ s,t \in J$.

Remark 2.4: (1) For regressive equations (2.1) and especially ODEs, the regularity condition is always fulfilled.

(2) Under the regularity condition invariant projectors $P : J \to \mathcal{L}(\mathcal{X})$ are rd-continuously differentiable with the derivative $P^\Delta(t) = A(t)P(t) - P(\rho^+(t))A(t)$ for $t \in J^\kappa$.

Proof. First of all, the mapping (2.3) is well-defined by Proposition 2.1. For fixed $s \in J$ we use the induction principle from Hilger [11, Theorem 1.4(c)] to prove the statement

$$\mathcal{A}(t) : \ \Phi_A(t,s)|_{\mathcal{U}(s)} : \mathcal{U}(s) \to \mathcal{U}(t) \text{ is continuous and bijective}$$

for each $t \in J$, $s \preceq t$. Therefore we have to proceed in four steps:
(I): Because of $\Phi_A(s,s)|_{\mathcal{U}(s)} = I_{\mathcal{U}(s)}$ obviously $\mathcal{A}(s)$ is fulfilled.
(II): Now let $t \in J$, $s \preceq t$ be right-scattered. Since $\mathcal{A}(t)$ holds by assumption, the mapping $\Phi_A(\rho^+(t),s)|_{\mathcal{U}(s)} = [I_{\mathcal{X}} + \mu^*(t)A(t)]\,|_{\mathcal{U}(t)}\Phi_A(t,s)|_{\mathcal{U}(s)}$ is a composition of two continuous bijective homomorphisms, and this yields $\mathcal{A}(\rho^+(t))$.
(III): Let $t \in J$, $s \preceq t$ be right-dense and let $\mathcal{A}(t)$ be true. Thus there exists a $\mathbb{T}$-neighborhood $U \subseteq \mathbb{T}$ of t such that $\sup_{t\in U} \mu^*(t)\,\|A(t)\| < 1$ and (2.1) is regressive on U. Consequently $\Phi_A(r,t)$ is a toplinear isomorphism on $\mathcal{X}$ for $r \in U$, $t \preceq r$ and with the aid of the identity $\Phi_A(r,s)|_{\mathcal{U}(s)} = \Phi_A(r,t)|_{\mathcal{U}(t)}\Phi_A(t,s)|_{\mathcal{U}(s)}$ we obtain $\mathcal{A}(r)$ for $r \in U$, $t \preceq r$.
(IV): Finally let $t \in J$, $s \preceq t$ be left-dense and $\mathcal{A}(r)$ be true for each $r \in J$, $s \preceq r \prec t$. Then also $\mathcal{A}(t)$ follows similarly to step (III).
Now the proof is finished, since the continuous isomorphism $\Phi_A(t,s)|_{\mathcal{U}(s)} : \mathcal{U}(s) \to \mathcal{U}(t)$ has a continuous inverse by Lang [14, p. 388, Corollary 1.4]). □

At this point the Proposition 2.3 allows us to define the *extended transition operator* $\bar{\Phi}_A(t,s) : \mathcal{U}(s) \to \mathcal{U}(t)$,

$$\bar{\Phi}_A(t,s) := \begin{cases} \left[\Phi_A(s,t)|_{\mathcal{U}(t)}\right]^{-1} & \text{if } t \prec s \\ \Phi(t,s)|_{\mathcal{U}(s)} & \text{if } s \preceq t \end{cases}$$

for any pairs $(t,s) \in J \times J$. It is easy to show that $\bar{\Phi}_A(t,s) \in \mathcal{GL}(\mathcal{U}(s);\mathcal{U}(t))$ inherits certain distinctive features of the usual transition operator, namely the *cocycle property*

$$\bar{\Phi}_A(t,\tau) = \bar{\Phi}_A(t,s)\bar{\Phi}_A(s,\tau) \quad \text{for } \tau, s, t \in J,$$

or the two identities

$$[I_{\mathcal{X}} - P(t)]\,\bar{\Phi}_A(t,s) = \bar{\Phi}_A(t,s)\,[I_{\mathcal{X}} - P(s)]\,, \qquad \bar{\Phi}_A(t,s)^{-1} = \bar{\Phi}_A(s,t) \quad \text{for } s,t \in J. \tag{2.4}$$

Finally one can show the differentiability of $\bar{\Phi}_A(\cdot,\tau) : J^\kappa \to \mathcal{L}(\mathcal{U}(\tau);\mathcal{X})$ with the derivative

$$(\Delta_1\bar{\Phi}_A)(t,\tau) = A(t)\bar{\Phi}_A(t,\tau) \quad \text{for } \tau, t \in J^\kappa.$$

To bring our preparations to an end, in the definition of an exponential dichotomy we use the abbreviation $\lfloor b - a \rfloor := \inf_{t\in\mathbb{T}^\kappa}(b(t) - a(t))$ and introduce the notations

$$a \lhd b \quad :\Leftrightarrow \quad 0 < \lfloor b - a \rfloor\,, \qquad\qquad a \unlhd b \quad :\Leftrightarrow \quad 0 \le \lfloor b - a \rfloor\,,$$

where two positively regressive functions $a, b \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T}^\kappa,\mathbb{R})$ are denoted as *growth rates*, if $\sup_{t\in\mathbb{T}^\kappa} \mu^*(t)a(t) < \infty$ and $\sup_{t\in\mathbb{T}^\kappa} \mu^*(t)b(t) < \infty$, respectively. Then we obtain the limits

$$\lim_{t\to\infty} e_{a\ominus b}(t,\tau) = 0, \qquad\qquad \lim_{t\to-\infty} e_{b\ominus a}(t,\tau) = 0 \tag{2.5}$$

for growth rates $a \lhd b$ and on a measure chain, which is unbounded above resp. below. The relations (2.5) can be shown using similar arguments as in Hilger [10, Satz 9.2].

Definition 2.5 (exponential dichotomy)**:** *Let $P : J \to \mathcal{L}(\mathcal{X})$ be an invariant projector of* (2.1) *such that the regularity condition* (2.3) *is fulfilled. Then equation* (2.1) *is said to possess an* exponential dichotomy*, if the estimates*

$$\|\Phi_A(t,s)P(s)\| \leq K_1 e_a(t,s) \quad \textit{for } s \preceq t,\ s,t \in J, \tag{2.6}$$
$$\left\|\bar{\Phi}_A(t,s)\left[I_{\mathcal{X}} - P(s)\right]\right\| \leq K_2 e_b(t,s) \quad \textit{for } t \preceq s,\ s,t \in J \tag{2.7}$$

hold for real constants $K_1, K_2 \geq 1$ and growth rates $a, b \in \mathcal{C}^+_{rd}\mathcal{R}(J,\mathbb{R})$, $a \lhd b$.

Remark 2.6: (1) The growth rates a, b do not have to be constant functions. For ODEs this goes back to Muldowney [16]. A second feature of our definition is that we do not insist on a hyperbolicity condition like $a \lhd 0 \lhd b$. Thus one can speak of a *pseudo-hyperbolic dichotomy*. Although this makes Definition 2.5 more flexible, it is not a real generalization, because one can transform each pseudo-hyperbolic into a hyperbolic system (see Corollary 2.7). Eventually we point out again that equation (2.1) does not have to be regressive. For OΔEs this can be traced back to Henry [9, p. 229, Definition 7.6.4] and with a different, but equivalent definition to Kalkbrenner [12]. A dichotomy notion for OΔEs without any regularity condition is contained in Aulbach & Kalkbrenner [2].

(2) The equation (2.1) is said to have an *ordinary dichotomy* if the estimates (2.6) and (2.7) hold with $a = b = 0$. Having the current concept available, one can generalize results from Bohner & Lutz [3] to nonregressive equations in finite-dimensional Banach spaces, like it has been done in Elaydi, Papaschinopoulos & Schinas [8] for OΔEs.

(3) Setting $s = t$, the inequalities (2.6) and (2.7) imply the boundedness of the invariant projectors P and $I_{\mathcal{X}} - P$, respectively.

The following result can be seen as a first step to introduce a spectral notion for linear dynamic equations (see Siegmund [19]).

Corollary 2.7 (shifted system)**:** *If equation* (2.1) *possesses an exponential dichotomy with a, b, K_1, K_2 and P on a $\mathbb{T}$-interval J, then for arbitrary growth rates $c \in \mathcal{C}^+_{rd}\mathcal{R}(J,\mathbb{R})$ also the* shifted system

$$\boxed{x^\Delta = (A \ominus cI_{\mathcal{X}})(t)x} \tag{2.8}$$

has an exponential dichotomy with $a \ominus c, b \ominus c$, K_1, K_2 and P on J.

Remark 2.8: Specifically for $c := \frac{1}{2} \odot (a \oplus b)$ the shifted system (2.8) has a hyperbolic exponential dichotomy with the growth rates $\frac{1}{2} \odot (a \ominus b), \frac{1}{2} \odot (b \ominus a)$.

Proof. For $\tau \preceq t$ define the mapping $\Phi(t,\tau) := e_c(\tau,t)\Phi_A(t,\tau)$. Then the product rule (cf. Hilger [11, Theorem 2.6(ii)]) leads to

$$\begin{aligned}
(\Delta_1\Phi)(t,\tau) &= e_c(\tau,\rho^+(t))A(t)\Phi_A(t,\tau) - c(t)e_c(\tau,\rho^+(t))\Phi_A(t,\tau) = \\
&= e_c(t,\rho^+(t))\left[A(t) - c(t)I_{\mathcal{X}}\right]e_c(\tau,t)\Phi_A(t,\tau) = \\
&= \left[1 + \mu^*(t)c(t)\right]^{-1}\left[A(t) - c(t)I_{\mathcal{X}}\right]\Phi(t,\tau) \quad \text{for } \tau \preceq t,\ t \in J^\kappa
\end{aligned}$$

and consequently $\Phi(t,\tau)$ is the transition operator of the shifted system (2.8) since we already have $\Phi(\tau,\tau) = I_{\mathcal{X}}$. Now it is easy to see that P is also an invariant projector of (2.8) satisfying the regularity condition (2.3), and that the dichotomy estimates (2.6) and (2.7) for $\Phi(t,\tau)$ hold true. $\square$

Eventually, for systems possessing an exponential dichotomy, we can characterize the invariant vector bundles $\mathcal{S}$ and $\mathcal{U}$ dynamically.

Theorem 2.9 (dynamical characterization of $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{U}}$)**:** *Let equation* (2.1) *possess an exponential dichotomy with a, b and P on J. For a fixed $\tau_0 \in \mathbb{T}^\kappa$ the following holds:*

(a) If $J = \mathbb{T}^+_{\tau_0}$ is unbounded above and $c \in \mathcal{C}^+_{rd}\mathcal{R}(J, \mathbb{R})$ with $a \trianglelefteq c \lhd b$, then

$$\mathcal{S} = \left\{(\tau, \eta) \in J \times \mathcal{X} : \Phi_A(\cdot, \tau)\eta \in \mathcal{B}^+_{\tau,c}(\mathcal{X})\right\},$$

(b) if $J = \mathbb{T}^-_{\tau_0}$ is unbounded below and $d \in \mathcal{C}^+_{rd}\mathcal{R}(J, \mathbb{R})$ with $a \lhd d \trianglelefteq b$, then

$$\mathcal{U} = \left\{(\tau, \xi) \in J \times \mathcal{X} : \begin{array}{l} \textit{there exists a solution } \nu : \mathbb{T}^-_\tau \to \mathcal{X} \textit{ of} \\ (2.1) \textit{ with } \nu(\tau) = \xi \textit{ and } \nu \in \mathcal{B}^-_{\tau,d}(\mathcal{X}) \end{array}\right\}.$$

Hence we call $\mathcal{S}$ and $\mathcal{U}$ the pseudo-stable *and the* pseudo-unstable vector bundle *of* (2.1), *respectively.*

Remark 2.10: In particular the sets $\mathcal{S}$ and $\mathcal{U}$ are independent of the growth rates c and d, respectively, since they are defined in Proposition 2.1 only using invariant projectors.

Proof. We have to show two inclusions each.

(a)($\subseteq$) For $(\tau, \eta) \in \mathcal{S}$, i.e. $\eta = P(\tau)\eta$ it follows

$$\|\Phi_A(t, \tau)\eta\| \, e_{\ominus c}(t, \tau) \overset{(2.6)}{\leq} K_1 e_{a \ominus c}(t, \tau) \, \|\eta\| \leq K_1 \, \|\eta\| \quad \text{for } \tau \preceq t$$

and we obtain $\Phi_A(\cdot, \tau)\eta \in \mathcal{B}^+_{\tau,c}(\mathcal{X})$ by passing over to the least upper bound over $t \in \mathbb{T}^+_\tau$ in the last estimate.

($\supseteq$) Because of $\Phi_A(\cdot, \tau)\eta \in \mathcal{B}^+_{\tau,c}(\mathcal{X})$ there exists a real $C_\tau \geq 0$ such that $\|\Phi_A(t, \tau)\eta\| \leq C_\tau e_c(t, \tau) \, \|\eta\|$ for $\tau \preceq t$ and the identity $[I_\mathcal{X} - P(\tau)]\, \eta = \bar{\Phi}_A(\tau, t) \, [I_\mathcal{X} - P(t)] \, \Phi_A(t, \tau)\eta$ yields

$$\|[I_\mathcal{X} - P(\tau)]\, \eta\| \leq \left\|\bar{\Phi}_A(\tau, t) \, [I_\mathcal{X} - P(t)]\right\| \, \|\Phi_A(t, \tau)\eta\| \overset{(2.7)}{\leq} C_\tau K_2 e_{c \ominus b}(t, \tau) \, \|\eta\| \quad \text{for } \tau \preceq t.$$

By taking the limit $t \to \infty$ in this estimate and considering (2.5), it follows $\eta = P(\tau)\eta$ and consequently $(\tau, \eta) \in \mathcal{S}$.

(b)($\subseteq$) This inclusion results similarly to the first inclusion in (a), if one defines the function $\nu(t) := \bar{\Phi}_A(t, \tau)\xi$ for $t \preceq \tau$.

($\subseteq$) Vice versa let $\nu \in \mathcal{B}^-_{\tau,d}(\mathcal{X})$ be a solution of (2.1) with $\nu(\tau) = \xi$. Then we have $\Phi_A(\tau, t)\nu(t) = \xi$ for $t \preceq \tau$ and one gets

$$\|P(\tau)\xi\| \overset{(2.2)}{=} \|\Phi_A(\tau, t)P(t)\nu(t)\| \overset{(2.6)}{\leq} K_1 e_{d \ominus a}(t, \tau) \, \|\nu\|^-_{\tau,d} \quad \text{for } t \preceq \tau.$$

Now passing over to the limit $t \to -\infty$ in this estimate and by considering (2.5) we obtain $P(\tau)\xi = 0$ and $(\tau, \xi) \in \mathcal{U}$. □

Without proof we state an immediate consequence of Theorem 2.9 about the possible choices of the invariant projectors. Its partial converse can be found in Siegmund [19, Lemma 1.1].

Corollary 2.11: *Let equation* (2.1) *possess an exponential dichotomy with* a, b *and the invariant projectors* P *and* Q *on* J*. For fixed* $\tau_0 \in \mathbb{T}^\kappa$ *and* $c, d \in \mathcal{C}^+_{rd}\mathcal{R}(J, \mathbb{R})$ *the following holds:*

(a) If $J = \mathbb{T}^+_{\tau_0}$ *is unbounded above,* $\tau \in J$*, then* $\mathcal{R}(P(\tau)) = \mathcal{R}(Q(\tau))$ *and for* $c \lhd b$ *the equation* (2.1) *has no nontrivial* c^+*-quasibounded solution in* $\mathcal{U}$ *on* $\mathbb{T}^+_\tau$*,*

(b) if $J = \mathbb{T}^-_{\tau_0}$ *is unbounded below,* $\tau \in J$*, then* $\mathcal{N}(P(\tau)) = \mathcal{N}(Q(\tau))$ *and for* $a \lhd d$ *the equation* (2.1) *has no nontrivial* d^-*-quasibounded solution in* $\mathcal{S}$ *on* $\mathbb{T}^-_\tau$*,*

(c) if $J = \mathbb{T}$ *is unbounded above and below, then* $P = Q$ *and for* $a \lhd c \lhd b$*,* $a \lhd d \lhd b$ *the equation* (2.1) *has no nontrivial* c^+*- and* d^-*-quasibounded solution on* $\mathbb{T}$*.*

3 Perturbation Results

To prepare the proofs of our perturbation theorems, we have to derive an elementary lemma. Yet its importance should not be underestimated, since it is the key to consider nonconstant growth rates, too.

Lemma 3.1: *For* $\tau, t, t_1, t_2 \in \mathbb{T}^\kappa$*,* $t_1 \preceq t_2$ *and* $a, b \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T}^\kappa, \mathbb{R})$ *we obtain*

$$\int_{t_1}^{t_2} e_a(t, \rho^+(s)) e_b(s, \tau)\, \Delta s \leq \begin{cases} \frac{e_a(t,\tau)}{\lfloor b-a \rfloor}\left[e_{b\ominus a}(t_2, \tau) - e_{b\ominus a}(t_1, \tau)\right] & \text{if } a \lhd b \\ \frac{e_a(t,\tau)}{\lfloor a-b \rfloor}\left[e_{b\ominus a}(t_1, \tau) - e_{b\ominus a}(t_2, \tau)\right] & \text{if } b \lhd a \end{cases}. \tag{3.1}$$

Proof. The desired estimates follow from the identity

$$\int_{t_1}^{t_2} e_a(t, \rho^+(s)) e_b(s, \tau)\, \Delta s = e_a(t, \tau) \int_{t_1}^{t_2} \frac{(\Delta_1 e_{b\ominus a})(s, \tau)}{b(s) - a(s)}\, \Delta s,$$

by the properties of the Cauchy-Integral (cf. Hilger [11, Theorem 4.3]). □

Or primary aim is to show the existence of some quasibounded solutions of perturbed dynamic equations. Here we follow closely to Palmer [17, Lemma 2.7, Proposition 2.8], who considered finite-dimensional invertible OΔEs. One can also apply the Theorems 3.2 and 3.4 in the situation of a "trivial dichotomy," where $P(t) \equiv I_{\mathcal{X}}$ or $P(t) \equiv 0$. In this case they state that certain exponential growth properties of the solutions of (2.1) are preserved under linear-inhomogeneous and semi-linear perturbations.

On the other hand, both subsequent results can be seen as a generalization of Corollary 2.11, while the following theorem is also related to the problem of admissibility (cf. Massera & Schäffer [15, pp. 165ff]), because it gives sufficient conditions, under which the pairs $(\mathcal{B}^+_{\tau,c}(\mathcal{X}), \mathcal{B}^+_{\tau,c}(\mathcal{X}))$ and $(\mathcal{B}^-_{\tau,d}(\mathcal{X}), \mathcal{B}^-_{\tau,d}(\mathcal{X}))$ are admissible for the linear dynamic equation (2.1).

Theorem 3.2 (inhomogeneous perturbations)**:** *Let equation* (2.1) *possess an exponential dichotomy with* a, b*,* K_1, K_2 *and* P *on* J*. For the linear-inhomogeneous equation*

$$\boxed{x^\Delta = A(t)x + r(t)} \tag{3.2}$$

and fixed $\tau_0 \in \mathbb{T}^\kappa$*,* $c, d \in \mathcal{C}^+_{rd}\mathcal{R}(J, \mathbb{R})$*,* $a \lhd c \lhd b$*,* $a \lhd d \lhd b$ *the following holds:*

(a) *If $J = \mathbb{T}_{\tau_0}^+$ is unbounded above, then for each $\tau \in J$, $x_0 \in \mathcal{X}$ and $r \in \mathcal{B}_{\tau,c}^+(\mathcal{X})$ there exists exactly one solution $\lambda_* \in \mathcal{B}_{\tau,c}^+(\mathcal{X})$ of (3.2) with $(\tau, \lambda_*(\tau) - x_0) \in \mathcal{U}$; additionally it is*

$$\|\lambda_*\|_{\tau,c}^+ \leq K_1 \|x_0\| + C(c) \|r\|_{\tau,c}^+ , \tag{3.3}$$

(b) *if $J = \mathbb{T}_{\tau_0}^-$ is unbounded below, then for each $\tau \in J$, $x_0 \in \mathcal{X}$ and $r \in \mathcal{B}_{\tau,d}^-(\mathcal{X})$ there exists exactly one solution $\lambda_* \in \mathcal{B}_{\tau,d}^-(\mathcal{X})$ of (3.2) with $(\tau, \lambda_*(\tau) - x_0) \in \mathcal{S}$; additionally it is*

$$\|\lambda_*\|_{\tau,d}^- \leq K_2 \|x_0\| + C(d) \|r\|_{\tau,d}^- ,$$

where we have used the abbreviation $C(c) := \frac{K_1}{\lfloor c-a \rfloor} + \frac{K_2}{\lfloor b-c \rfloor}$.

Remark 3.3: The inclusion $(\tau, \lambda_*(\tau) - x_0) \in \mathcal{U}$ (resp. $(\tau, \lambda_*(\tau) - x_0) \in \mathcal{S}$) means that the initial value $\lambda_*(\tau)$ and the point x_0 have the same projection onto the fiber $\mathcal{S}(\tau)$ (resp. $\mathcal{U}(\tau)$).

Proof. (a) Keep the points $\tau \in J$, $x_0 \in \mathcal{X}$ and the inhomogeneity $r \in \mathcal{B}_{\tau,c}^+(\mathcal{X})$ fixed. Above all the function $\lambda_* : \mathbb{T}_\tau^+ \to \mathcal{X}$,

$$\begin{aligned} \lambda_*(t) \;:=\; & \Phi_A(t,\tau)P(\tau)x_0 + \int_\tau^t \Phi_A(t,\rho^+(s))P(\rho^+(s))r(s)\,\Delta s - \\ & - \int_t^\infty \bar{\Phi}_A(t,\rho^+(s))\left[I_{\mathcal{X}} - P(\rho^+(s))\right] r(s)\,\Delta s \end{aligned} \tag{3.4}$$

is well-defined, since the integrands are rd-continuous in the variable s (hence they have an anti-derivative by Hilger [11, Theorem 4.4]) and using Lemma 3.1 we obtain the estimate

$$\begin{aligned} & \|\lambda_*(t)\| \leq \\ \stackrel{(3.4)}{\leq} \; & K_1 e_a(t,\tau)\|x_0\| + K_1 \int_\tau^t e_a(t,\rho^+(s))\|r(s)\|\,\Delta s + K_2 \int_t^\infty e_b(t,\rho^+(s))\|r(s)\|\,\Delta s \leq \\ \leq \; & K_1 e_a(t,\tau)\|x_0\| + \\ & + \left[K_1 \int_\tau^t e_a(t,\rho^+(s))e_c(s,\tau)\,\Delta s + K_2 \int_t^\infty e_b(t,\rho^+(s))e_c(s,\tau)\,\Delta s\right] \|r\|_{\tau,c}^+ \leq \\ \stackrel{(3.1)}{\leq} \; & K_1 e_a(t,\tau)\|x_0\| + \left[\frac{K_1}{\lfloor c-a \rfloor}\left(e_c(t,\tau) - e_a(t,\tau)\right) + \frac{K_2}{\lfloor b-c \rfloor} e_c(t,\tau)\right] \|r\|_{\tau,c}^+ \quad \text{for } \tau \preceq t, \end{aligned}$$

which on the other side implies the c^+-quasiboundedness of λ_* as well as the inequality (3.3). Now the Lemma 4.1 applies to the first integral in (3.4) and Lemma 4.2 can be applied to the indefinite integral in (3.4), which leads to

$$\begin{aligned} \lambda_*^\Delta(t) \stackrel{(3.4)}{=} \; & A(t)\Phi_A(t,\tau)P(\tau)x_0 + P(\rho^+(t))r(t) + \int_\tau^t A(t)\Phi_A(t,\rho^+(s))P(\rho^+(s))r(s)\,\Delta s + \\ & + \left[I_{\mathcal{X}} - P(\rho^+(t))\right] r(t) - \int_t^\infty A(t)\bar{\Phi}_A(t,\rho^+(s))\left[I_{\mathcal{X}} - P(\rho^+(s))\right] r(s)\,\Delta s = \\ \stackrel{(3.4)}{=} \; & A(t)\lambda_*(t) + r(t) \quad \text{for } \tau \preceq t; \end{aligned}$$

whence λ_* is a solution of (3.2). It remains to show $(\tau, \lambda_*(\tau) - x_0) \in \mathcal{U}$, which however follows from

$$P(\tau)\lambda_*(\tau) \stackrel{(3.4)}{=} P(\tau)x_0 - P(\tau)\int_\tau^\infty \bar{\Phi}_A(\tau,\rho^+(s))\left[I_{\mathcal{X}} - P(\rho^+(s))\right] r(s)\,\Delta s \stackrel{(2.4)}{=} P(\tau)x_0.$$

Eventually λ_* is uniquely defined, because if ν_* would be another solution of (3.2) with $(\tau, \nu_*(\tau) - x_0) \in \mathcal{U}$, then the difference $\lambda_* - \nu_* \in \mathcal{B}^+_{\tau,c}(\mathcal{X})$ would be a solution of the homogeneous equation (2.1) and $P(\tau)\,[\lambda_*(\tau) - \nu_*(\tau)] = P(\tau)x_0 - P(\tau)x_0 = 0$. Consequently $\lambda_* - \nu_*$ is a c^+-quasibounded solution of (2.1) in $\mathcal{U}$ which has to vanish identically on $\mathbb{T}^+_\tau$ by Corollary 2.11(a).

(b) Completely analogously to (a), the function $\lambda_* : \mathbb{T}^-_\tau \to \mathcal{X}$,

$$\begin{aligned}\lambda_*(t) \;:=\; & \bar{\Phi}_A(t,\tau)\,[I_{\mathcal{X}} - P(\tau)]\,x_0 + \int_{-\infty}^t \Phi_A(t,\rho^+(s))P(\rho^+(s))r(s)\,\Delta s - \\ & - \int_t^\tau \bar{\Phi}_A(t,\rho^+(s))\,\left[I_{\mathcal{X}} - P(\rho^+(s))\right] r(s)\,\Delta s\end{aligned}$$

is a d^--quasibounded solution of equation (3.2) with $[I_{\mathcal{X}} - P(\tau)]\,\lambda_*(\tau) = [I_{\mathcal{X}} - P(\tau)]\,x_0$, i.e. $(\tau, \lambda_*(\tau) - x_0) \in \mathcal{S}$. The uniqueness in the present case follows from Corollary 2.11(b). □

Theorem 3.4 (semi-linear perturbations)**:** *Let equation* (2.1) *possess an exponential dichotomy with* a, b, K_1, K_2 *and* P *on* J. *For the semi-linear equation*

$$x^\Delta = A(t)x + g(t,x) + h(t),^\Delta \tag{3.5}$$

fixed $\tau_0 \in \mathbb{T}^\kappa$, $c, d \in \mathcal{C}^+_{rd}\mathcal{R}(J,\mathbb{R})$, $a \lhd c \lhd b$, $a \lhd d \lhd b$ *and under the assumptions*

(i) $g : J \times \mathcal{X} \to \mathcal{X}$ *is rd-continuous and fulfills*

$$\|g(t,x) - g(t,\bar{x})\| \le L\,\|x - \bar{x}\| \quad \text{for } t \in J,\ x, \bar{x} \in \mathcal{X}, \tag{3.6}$$

for a real constant $L \ge 0$,

(ii) $L \max\{C(c), C(d)\} < 1$,

the following holds:

(a) *If* $J = \mathbb{T}^+_{\tau_0}$ *is unbounded above and* $g(\cdot,0) \in \mathcal{B}^+_{\tau,c}(\mathcal{X})$, *then for each* $\tau \in J$, $x_0 \in \mathcal{X}$ *and* $h \in \mathcal{B}^+_{\tau,c}(\mathcal{X})$ *there exists exactly one solution* $\lambda_* \in \mathcal{B}^+_{\tau,c}(\mathcal{X})$ *of* (3.5) *with* $(\tau, \lambda_*(\tau) - x_0) \in \mathcal{U}$; *additionally it is*

$$\|\lambda_*\|^+_{\tau,c} \le \frac{1}{1 - LC(c)}\left[K_1\,\|x_0\| + C(c)\,\|g(\cdot,0) + h\|^+_{\tau,c}\right], \tag{3.7}$$

(b) *if* $J = \mathbb{T}^-_{\tau_0}$ *is unbounded below and* $g(\cdot,0) \in \mathcal{B}^-_{\tau,d}(\mathcal{X})$, *then for each* $\tau \in J$, $x_0 \in \mathcal{X}$ *and* $h \in \mathcal{B}^-_{\tau,d}(\mathcal{X})$ *there exists exactly one solution* $\lambda_* \in \mathcal{B}^-_{\tau,d}(\mathcal{X})$ *of* (3.5) *with* $(\tau, \lambda_*(\tau) - x_0) \in \mathcal{S}$; *additionally it is*

$$\|\lambda_*\|^-_{\tau,d} \le \frac{1}{1 - LC(d)}\left[K_2\,\|x_0\| + C(d)\,\|g(\cdot,0) + h\|^-_{\tau,d}\right],$$

where the constant $C(c) > 0$ *is given in Theorem 3.2.*

Remark 3.5: Under the additional assumptions $g(t,0) \equiv 0$, $h(t) \equiv 0$ on J, the semi-linear equation (3.5) has no nontrivial c^+-quasibounded (resp. d^--quasibounded) solution with $(\tau, \lambda_*(\tau)) \in \mathcal{U}$ on $\mathbb{T}^+_\tau$ (resp. $(\tau, \lambda_*(\tau)) \in \mathcal{S}$ on $\mathbb{T}^-_\tau$).

Proof. (a) First of all, the solutions of equation (3.5) are unique and they exist on $\mathbb{T}_\tau^+$, which follows from the proof of HILGER [11, Theorem 5.7]. Keep $\tau \in J$, $x_0 \in \mathcal{X}$ fixed and consider any functions $\lambda, \bar{\lambda} \in \mathcal{B}_{\tau,c}^+(\mathcal{X})$ for the moment. Then the mapping $r_\lambda : J \to \mathcal{X}$, $r_\lambda(t) := g(t, \lambda(t)) + h(t)$ has the property

$$\|r_\lambda(t)\| \leq \|g(t, \lambda(t)) - g(t, 0)\| + \|g(t, 0) + h(t)\| \overset{(3.6)}{\leq} L\,\|\lambda(t)\| + \|g(t, 0) + h(t)\| \quad \text{for } \tau \preceq t$$

and accordingly it is c^+-quasibounded with

$$\|r_\lambda\|_{\tau,c}^+ \leq L\,\|\lambda\|_{\tau,c}^+ + \|g(\cdot, 0) + h\|_{\tau,c}^+ . \tag{3.8}$$

Therefore Theorem 3.2(a) implies that

$$x^\Delta = A(t)x + r_\lambda(t) \tag{3.9}$$

has a unique c^+-quasibounded solution $\mathcal{T}_{x_0}\lambda : \mathbb{T}_\tau^+ \to \mathcal{X}$ with $(\tau, (\mathcal{T}_{x_0}\lambda)(\tau) - x_0) \in \mathcal{U}$, which is given by the expression (3.4), namely

$$\begin{aligned}(\mathcal{T}_{x_0}\lambda)(t) \;:=\; & \Phi_A(t, \tau)P(\tau)x_0 + \int_\tau^t \Phi_A(t, \rho^+(s))P(\rho^+(s))r_\lambda(s)\,\Delta s - \\ & - \int_t^\infty \bar{\Phi}_A(t, \rho^+(s))\left[I_{\mathcal{X}} - P(\rho^+(s))\right] r_\lambda(s)\,\Delta s.\end{aligned}$$

In particular the operator $\mathcal{T}_{x_0} : \mathcal{B}_{\tau,c}^+(\mathcal{X}) \to \mathcal{B}_{\tau,c}^+(\mathcal{X})$ is well-defined and the difference $\mathcal{T}_{x_0}\lambda - \mathcal{T}_{x_0}\bar{\lambda} \in \mathcal{B}_{\tau,c}^+(\mathcal{X})$ solves the equation

$$x^\Delta = A(t)x + g(t, \lambda(t)) - g(t, \bar{\lambda}(t)) \tag{3.10}$$

with $(\tau, (\mathcal{T}_{x_0}\lambda)(\tau) - (\mathcal{T}_{x_0}\bar{\lambda})(\tau)) \in \mathcal{U}$, where the inhomogeneity fulfills $\left\|g(\cdot, \lambda(\cdot)) - g(\cdot, \bar{\lambda}(\cdot))\right\|_{\tau,c}^+ \leq L\left\|\lambda - \bar{\lambda}\right\|_{\tau,c}^+$ by (3.6). Now Theorem 3.2(a) applied to the equation (3.10) has the consequence

$$\left\|\mathcal{T}_{x_0}\lambda - \mathcal{T}_{x_0}\bar{\lambda}\right\|_{\tau,c}^+ \overset{(3.3)}{\leq} C(c)\left\|g(\cdot, \lambda(\cdot)) - g(\cdot, \bar{\lambda}(\cdot))\right\|_{\tau,c}^+ \leq LC(c)\left\|\lambda - \bar{\lambda}\right\|_{\tau,c}^+;$$

hence $\mathcal{T}_{x_0}$ is a contraction on the Banach space $\mathcal{B}_{\tau,c}^+(\mathcal{X})$ by assumption (ii). Using the contraction principle (see e.g. LANG [14, p. 360, Lemma 1.1]), $\mathcal{T}_{x_0}$ has a unique fixed point $\lambda_* \in \mathcal{B}_{\tau,c}^+(\mathcal{X})$ with $(\tau, \lambda_*(\tau) - x_0) \in \mathcal{U}$. Moreover λ_* is a solution of (3.9) (for $\lambda = \lambda_*$) and thus also a solution of (3.5). Finally we get the estimate

$$\|\lambda_*\|_{\tau,c}^+ \overset{(3.3)}{\leq} K_1\,\|x_0\| + C(c)\,\|r_{\lambda_*}\|_{\tau,c}^+ \overset{(3.8)}{\leq} K_1\,\|x_0\| + C(c)\left[L\,\|\lambda_*\|_{\tau,c}^+ + \|g(\cdot, 0) + h\|_{\tau,c}^+\right],$$

which implies (3.7)

(b) One has to proceed analogously to step (a) and use Theorem 3.2(b) hereby. Consequently the proof is complete. □

4 Appendix: Parameter Integrals

Here we state two results about the differentiability of integrals, which depend on a parameter. So far they cannot be quoted from another reference concerning the calculus on measure chains.

Lemma 4.1 (parameter integrals)**:** *Assume $\tau, t \in \mathbb{T}^\kappa$, $\tau \preceq t$ and $r \in \mathcal{C}_{rd}(\mathbb{T}^\kappa, \mathcal{X})$. If $\Phi : \{(t,\tau) \in \mathbb{T} \times \mathbb{T}^\kappa : \tau \preceq t\} \to \mathcal{L}(\mathcal{X})$ is a continuous mapping, $\Phi(\cdot, t_0)$ possesses a continuous extension to a neighborhood of each right-dense $t_0 \in \mathbb{T}^\kappa$ and if $\Phi(\cdot, s)$ is rd-continuously differentiable such that the limit*

$$(\Delta_1 \Phi)(t_0, s) = \lim_{t \to t_0} \frac{\Phi(t,s) - \Phi(t_0,s)}{\mu(t,t_0)}$$

exists uniformly in s on compact subsets in each right-dense $t_0 \in \mathbb{T}^\kappa$. Then the function $F : \mathbb{T} \to \mathcal{X}$, $F(t) := \int_\tau^t \Phi(t,s) r(s)\, \Delta s$ is differentiable on $\mathbb{T}^\kappa$ with derivative

$$F^\Delta(t) = \Phi(\rho^+(t), t) r(t) + \int_\tau^t (\Delta_1 \Phi)(t,s) r(s)\, \Delta s.$$

Proof. The proof can be done similarly to Bohner & Peterson [4, Theorem 1.73]. □

Lemma 4.2 (improper parameter integrals)**:** *Assume that $\mathbb{T}$ is unbounded to the right, $\Phi : \mathbb{T} \times \mathbb{T} \to \mathcal{L}(\mathcal{X})$ is a continuous mapping, $r \in \mathcal{C}_{rd}(\mathbb{T}, \mathcal{X})$ and*

(i) for arbitrary $t \in \mathbb{T}$ it is $\int_t^\infty \Phi(t,s) r(s)\, \Delta s < \infty$,

(ii) $\Phi(\cdot, s)$ is rd-continuously differentiable, where the limit

$$(\Delta_1 \Phi)(t_0, s) = \lim_{t \to t_0} \frac{\Phi(t,s) - \Phi(t_0,s)}{\mu(t,t_0)}$$

exists uniformly in s on compact subsets of $\mathbb{T}$ in each right-dense $t_0 \in \mathbb{T}$,

(iii) there exists a locally bounded function $c : \mathbb{T} \to \mathbb{R}_0^+$ and a rd-continuous function $m : \mathbb{T} \to \mathbb{R}_0^+$ with $\int_\mathbb{T} m(s) \Delta s < \infty$ such that $\|(\Delta_1 \Phi)(t,s) r(s)\| \leq c(t) m(s)$ for $s, t \in \mathbb{T}$.

Then the mapping $F : \mathbb{T} \to \mathcal{X}$, $F(t) := \int_t^\infty \Phi(t,s) r(s)\, \Delta s$ is differentiable with derivative

$$F^\Delta(t) = -\Phi(\rho^+(t), t) r(t) + \int_t^\infty (\Delta_1 \Phi)(t,s) r(s)\, \Delta s.$$

Proof. The proof can be done using arguments from to the well-known case of the time scale $\mathbb{T} = \mathbb{R}$, which can be found in standard calculus textbooks on the Riemann integral. □

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