

TOPOLOGICAL DECOUPLING, LINEARIZATION AND PERTURBATION ON INHOMOGENEOUS TIME SCALES

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ABSTRACT. We derive a linearization theorem in the framework of dynamic equations on time scales. This extends a recent result from [Y. Xia, J. Cao & M. Han, A new analytical method for the linearization of dynamic equation on measure chains, *Journal of Differential Equations* 235 (2007), 527–543] in various directions: Firstly, in our setting the linear part needs not to be hyperbolic and due to the existence of a center manifold this leads to a generalized global Hartman-Grobman theorem for nonautonomous problems. Secondly, we investigate the behavior of the topological conjugacy under parameter variation.

These perturbation results are tailor-made for future applications in analytical discretization theory, i.e., to study the relationship between ODEs and numerical schemes applied to them.

1. INTRODUCTION, MOTIVATION AND TERMINOLOGY

Linearization of dynamical models given by differential or difference equations is a very successful and frequently used simplification concept in applied sciences, since linear equations are mathematically well-understood and problems can be approached on an analytical level. Indeed, the basic reason why such linear models actually yield to a realistic and successful description of real world nonlinear models, is the Hartman-Grobman theorem.

For that reason, the theorem of Hartman-Grobman, also known as linearization theorem, is one of the central results in the local theory of dynamical systems. Basically this result states that the behavior of a given dynamical system near a hyperbolic fixed point is qualitatively the same as the dynamics of its linearization close to the origin. Using a more technical terminology, this means that the nonlinear flow is *topologically conjugated* to the corresponding linear flow, i.e., both flows can be transformed into each other using a homeomorphism and the corresponding phase portraits are homeomorphic images of each other. Thus, when dealing with such fixed points the linearization of the system is sufficient to analyze its behavior. The classical Hartman-Grobman theorem dates back to [Gro59, Gro62, Ha60a] (ordinary differential equations) and [Ha60b] (maps, i.e., autonomous difference equations). Meanwhile it can be found in many textbooks on dynamical systems.

The central and generic assumption in the standard Hartman-Grobman setting is hyperbolicity of the linearization. If we have eigenvalues on the imaginary axis respectively on the unit circle, though, then the situation changes drastically due to the existence of a center manifold. This invariant manifold contains more complex dynamical objects, like periodic motions or homo-/heteroclinic orbits. Such an extended set-up is the playground for the generalized Hartman-Grobman theorem, which states that the dynamical behavior near a nonhyperbolic equilibrium is topologically conjugated to a saddle times the flow on the center manifold. This geometrically intuitive result is due to [Sos72] and [Pal73, Pal80]. The latter three references deal with ODEs, but a parallel treatment of

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the continuous and discrete case is contained in [KP90]. Finally, variants of the Hartman-Grobman theorem for different kinds of differential equations can be found in [LP99] (impulsive equations), [Far01] (retarded FDEs) and [Lu91] (scalar reaction diffusion equations).

The recent years saw an increasing interest in nonautonomous equations. They are well-motivated due to various reasons of mathematical nature (e.g., studies on the behavior near non-constant reference solutions, or a description of adaptive numerical schemes), as well as necessity from the applications (for instance, the desire to incorporate time-dependent parameters into models in order to obtain a more realistic description of phenomena under consideration). Accordingly, extensions of the (generalized) Hartman-Grobman theorem to nonautonomous equations go back to the thesis [Wan91]. Moreover, for nonautonomous ODEs of Carathéodory-type one can find them in [AW00] or [Sie99], whereas [AW06] is concerned with nonautonomous difference equations.

Another quite recent field of research is to investigate equations on *inhomogeneous time scales* (cf. [Hil90, BP01]). The corresponding calculus on measure chains or time scales (closed subsets \mathbb{T} of the reals) has two main motivations. On the one hand, it yields an elegant and accessible framework to describe discrete and continuous dynamics, i.e., differential ($\mathbb{T} = \mathbb{R}$) and difference equations ($\mathbb{T} = \mathbb{Z}$) in a unified manner. Moreover, it provides an extension of these two classical situations by allowing inhomogeneous time scales, where the time axis \mathbb{T} is different from the integers or the reals. For instance, in biological applications with hibernation effects it might be adequate to use a time scale consisting of the union of disjoint closed intervals. In discretization theory, the appropriate time scale is a grid of discrete points on the real axis. With these perspectives in vision, a generalized Hartman-Grobman theorem for dynamic equations on time scales already appeared in [Hil96]. Since the measure chain calculus has reached further maturity in the mean time, it is our intention to prepare an application and extend these results as follows:

- We present a transparent and geometric approach to global linearization and decoupling problems using invariant foliations – thus, our methods differ from [Hil96]. In particular, we harvest from technical preliminaries on pseudo-stable and -unstable foliations obtained in [Pöt06]. Additionally, the above references [Wan91, Sie99, AW00, AW06] and [Hil96] assume a decoupled linear part, while we, despite concepts like kinematical similarity, think it is more canonical and applicable to work with an exponentially trichotomic system.
- Not long ago, a Hartman-Grobman theorem on time scales appeared in the interesting paper [XCH07], with a proof based on admissibility properties for exponential dichotomies. We contribute to this nice result by addressing the nonhyperbolic situation, hence, allowing the presence of a center manifold and including a particular parameter dependence. Both improvements are not straight forward consequences of [XCH07].
- Finally, we prepare a theoretical perturbation framework for up-coming applications in analytical discretization theory, i.e., the question how topological conjugacy behaves under numerical discretization (see [Pöt07]). The time scales calculus is well-suited for such questions: Using [Pöt03] we obtained persistence and convergence results for invariant manifolds in [KP05]. The present paper plays a similar role as [Pöt03] did for invariant manifolds, but now to investigate the more complex problem of topological decoupling and linearization.

The mathematical part of this paper is divided into four sections. After introducing some notation, in Section 2 we lay down our essential set-up on semilinear dynamic equation with a Lipschitzian non-linearity and discuss our essential assumptions. The following Section 3 contains our construction of invariant foliations and fiber bundles, which generalize classical invariant manifolds; moreover, we deduce an asymptotic phase property. Section 4 suggests a nonautonomous notion for topological conjugacy and applies it in order to decouple dynamic equations. Our primary result, the generalized Hartman-Grobman theorem is featured in Section 5, which implies the main result of [XCH07] (cf. Corollary 5.6). Our corresponding perturbation results are supplemented as corollaries.

The Banach spaces \mathcal{X} of this paper are all real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$) and their norm is denoted by $\|\cdot\|$. $\mathcal{L}(\mathcal{X})$ is the Banach space of linear bounded endomorphisms, $I_{\mathcal{X}}$ the identity on \mathcal{X} , and $\mathcal{R}(T) := T\mathcal{X}$ the range of an operator $T \in \mathcal{L}(\mathcal{X})$.

If a mapping $f : \mathcal{Y} \rightarrow \mathcal{Z}$ between metric spaces \mathcal{Y} and \mathcal{Z} satisfies a Lipschitz condition, then its smallest Lipschitz constant is denoted by $\text{Lip } f$. When $f : \mathcal{Y} \times \mathcal{P} \rightarrow \mathcal{Z}$ additionally depends on a parameter from some set \mathcal{P} , we write

$$\text{Lip}_1 f := \sup_{p \in \mathcal{P}} \text{Lip } f(\cdot, p).$$

In case \mathcal{P} has a metric structure, we define $\text{Lip}_2 f$ accordingly, and proceed along these lines for mappings depending on more than two variables.

2. INHOMOGENEOUS TIME SCALES AND SEMILINEAR DYNAMIC EQUATIONS

Trying to keep this paper largely self-contained, we start the mathematical part by introducing some basic terminology from the calculus on measure chains (including time scales). For further details, see the pioneering paper [Hil90] or the monograph [BP01]). In all subsequent considerations we deal with a *measure chain* $(\mathbb{T}, \preceq, \mu)$, i.e. a conditionally complete totally ordered set (\mathbb{T}, \preceq) (see [Hil90, Axiom 2]) with growth calibration $\mu : \mathbb{T}^2 \rightarrow \mathbb{R}$ (see [Hil90, Axiom 3]). The most intuitive and relevant examples of measure chains are *time scales*, where \mathbb{T} is a canonically ordered closed subset of the real numbers and μ measures the oriented distance by $\mu(t, s) = t - s$. Furthermore, the function $\sigma : \mathbb{T} \rightarrow \mathbb{T}$, $\sigma(t) := \inf \{s \in \mathbb{T} : t \prec s\}$ defines the *forward jump operator* and $\mu^* : \mathbb{T} \rightarrow \mathbb{R}$, $\mu^*(t) := \mu(\sigma(t), t)$ the *graininess*. A \mathbb{T} -*interval* \mathbb{I} is a subset of \mathbb{T} with $\mathbb{I} \cup \mathbb{T} = \mathbb{I} \cap \mathbb{T}$ and for $\tau, T \in \mathbb{T}$ we write

$$[\tau, T]_{\mathbb{T}} := \{t \in \mathbb{T} : \tau \preceq t \preceq T\}, \quad \mathbb{T}_{\tau}^+ := \{s \in \mathbb{T} : \tau \preceq s\}, \quad \mathbb{T}_{\tau}^- := \{s \in \mathbb{T} : s \preceq \tau\}.$$

The classical time scales \mathbb{R}, \mathbb{Z} are *homogeneous* in the sense that their graininess is constant. We allow a much broader class of possibly inhomogeneous time scales. Indeed, since we deal with asymptotic behavior and stability questions, the following standing assumption is legitimate.

Hypothesis. $\mu(\mathbb{T}, \tau) \subseteq \mathbb{R}$, $\tau \in \mathbb{T}$, is unbounded above, and μ^* is bounded.

The set $\mathcal{C}_{rd}(\mathbb{T}, \mathcal{X})$ denotes the rd-continuous maps from \mathbb{T} to \mathcal{X} (cf. [Hil90, Section 4.1]). *Growth rates* are functions $a \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$ with $-1 < \inf_{t \in \mathbb{T}} \mu^*(t)a(t)$, $\sup_{t \in \mathbb{T}} \mu^*(t)a(t) < \infty$. Moreover, for $a, b \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$ we introduce the relations $[b - a] := \inf_{t \in \mathbb{T}} (b(t) - a(t))$,

$$a \triangleleft b \quad :\Leftrightarrow \quad 0 < [b - a], \quad a \trianglelefteq b \quad :\Leftrightarrow \quad 0 \leq [b - a]$$

and the set of *positively regressive functions*

$$\mathcal{C}_{rd}^+ \mathcal{R}(\mathbb{T}, \mathbb{R}) := \{a \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}) : a \text{ is a growth rate and } 1 + \mu^*(t)a(t) > 0 \text{ for } t \in \mathbb{T}\}.$$

This class is technically appropriate to describe exponential growth and for $a \in \mathcal{C}_{rd}^+ \mathcal{R}(\mathbb{T}, \mathbb{R})$ the *exponential function* on \mathbb{T} is denoted by $e_a(t, s) \in \mathbb{R}$, $s, t \in \mathbb{T}$ (cf. [Hil90, Theorem 7.3]).

To provide a flavor of these rather abstract notions, the following example might be helpful for readers unfamiliar with the realm of measure chains (or time scales).

Example 2.1. A variety of time scales is discussed in [BP01]. Of particular interest, though, are the time scales $\mathbb{T} = \mathbb{R}$ to describe ordinary differential equations, as well as discrete meshes

$$\mathbb{T} = \mathbb{D} := \left\{ t_k \in \mathbb{R} : \lim_{k \rightarrow \pm\infty} t_k = \pm\infty \text{ and } t_k < t_{k+1} \text{ for all } k \in \mathbb{Z} \right\}$$

to capture numerical schemes for temporal discretizations with varying step-sizes $t_{k+1} - t_k$ – or simply difference equations. On such time scales, the above objects are summarized in Figure 1.

\mathbb{T}	\mathbb{R}	\mathbb{D}
σ	$\sigma(t) = t$	$\sigma(t_k) = t_{k+1}$
μ^*	$\mu^*(t) \equiv 0$	$\mu^*(t_k) = t_{k+1} - t_k$
$\mathcal{C}_{rd}(\mathbb{T}, \mathcal{X})$	$\mathcal{C}(\mathbb{R}, \mathcal{X})$	$\{\phi : \mathbb{D} \rightarrow \mathcal{X}\}$
$e_a(t, \tau)$	$e_a(t, \tau) = \exp\left(\int_\tau^t a(s) ds\right)$	$e_a(t_k, t_n) = \prod_{l=n}^{k-1} [1 + (t_{l+1} - t_l)a(t_l)]$

FIGURE 1. Continuous and discrete time scales

Measure chain integrals of such mappings $\phi : \mathbb{T} \rightarrow \mathcal{X}$ are always understood in Lebesgue's sense and denoted by $\int_\tau^t \phi(s) \Delta s$ for $\tau, t \in \mathbb{T}$, provided they exist (cf. [Nei01]).

It is handy to introduce the so-called *quasiboundedness*, a convenient notion due to Bernd Aulbach describing exponential growth of functions. Thereto we keep $\tau \in \mathbb{T}$ fixed and choose growth rates $c, d \in \mathcal{C}_{rd}^+(\mathbb{R}, \mathbb{R})$. A function $\phi : \mathbb{T} \rightarrow \mathcal{X}$ is said to be *c^+ -quasibounded*, if

$$\|\phi\|_{\tau, c}^+ := \sup_{t \in \mathbb{T}^+} \|\phi(t)\| e_c(\tau, t) < \infty$$

and the set $\mathcal{X}_{\tau, c}^+ := \left\{ \phi \in \mathcal{C}_{rd}(\mathbb{T}^+, \mathcal{X}) : \|\phi\|_{\tau, c}^+ < \infty \right\}$ is a Banach space with norm $\|\cdot\|_{\tau, c}^+$ (this can be shown easily using [Hil90, Theorem 4.1(iii)]). Moreover, one has a continuous embedding $\mathcal{X}_{\tau, c}^+ \hookrightarrow \mathcal{X}_{\tau, d}^+$ for $c \leq d$ with

$$(2.1) \quad \|\phi\|_{\tau, d}^+ \leq \|\phi\|_{\tau, c}^+ \quad \text{for } \phi \in \mathcal{X}_{\tau, c}^+.$$

In a dual fashion, we define the Banach space of *c^- -quasibounded* functions, which is given by $\mathcal{X}_{\tau, c}^- := \left\{ \phi \in \mathcal{C}_{rd}(\mathbb{T}^-, \mathcal{X}) : \sup_{t \in \mathbb{T}^-} \|\phi(t)\| e_c(\tau, t) < \infty \right\}$ canonically equipped with the norm $\|\phi\|_{\tau, c}^- := \sup_{t \in \mathbb{T}^-} \|\phi(t)\| e_c(\tau, t)$. Here, one has $\mathcal{X}_{\tau, c}^- \hookrightarrow \mathcal{X}_{\tau, d}^-$ for $d \leq c$. Finally, a function ϕ is called *c^\pm -quasibounded*, if

$$\|\phi\|_{\tau, c}^\pm := \sup_{t \in \mathbb{T}} \|\phi(t)\| e_c(\tau, t) < \infty$$

and the set $\mathcal{X}_c^\pm := \left\{ \phi \in \mathcal{C}_{rd}(\mathbb{T}, \mathcal{X}) : \|\phi\|_{\tau, c}^\pm < \infty \right\}$ is a Banach space with norm $\|\cdot\|_{\tau, c}^\pm$.

Given $A \in \mathcal{C}_{rd}(\mathbb{T}, \mathcal{L}(\mathcal{X}))$, a *linear dynamic equation* is of the form

$$(2.2) \quad x^\Delta = A(t)x;$$

here the *transition operator* $\Phi_A(t, s) \in \mathcal{L}(\mathcal{X})$, $s \preceq t$, is the solution of the operator-valued initial value problem $X^\Delta = A(t)X$, $X(s) = I_{\mathcal{X}}$ in $\mathcal{L}(\mathcal{X})$.

Example 2.2. The \mathbb{T} -derivative $\phi^\Delta(t) \in \mathcal{X}$ of a function $\phi : \mathbb{T} \rightarrow \mathcal{X}$ reads as

$$\phi^\Delta(t) = \dot{\phi}(t) \quad \text{if } \mathbb{T} = \mathbb{R}, \quad \phi^\Delta(t_k) = \frac{\phi(t_{k+1}) - \phi(t_k)}{t_{k+1} - t_k} \quad \text{if } \mathbb{T} = \mathbb{D}.$$

However, note that the formula for $\phi^\Delta(t)$ is more involved than the above differential resp. difference quotient on more complicated time scales (think of, e.g., a Cantor set \mathbb{T}).

For a meaningful notion of pseudo-hyperbolicity in this framework, further notions are needed. A projection-valued mapping $P : \mathbb{T} \rightarrow \mathcal{L}(\mathcal{X})$ is an *invariant projector* of (2.2), if

$$(2.3) \quad P(t)\Phi_A(t, s) = \Phi_A(t, s)P(s) \quad \text{for } s, t \in \mathbb{T}, s \preceq t,$$

holds, and finally an invariant projector P is denoted as *regular*, if

$$I_{\mathcal{X}} + \mu^*(t)A(t)|_{\mathcal{R}(P(t))} : \mathcal{R}(P(t)) \rightarrow \mathcal{R}(P(\sigma(t))) \text{ is bijective for all } t \in \mathbb{T}.$$

This regularity condition enables us to deal with non-invertible dynamic equations. In fact, the restriction $\bar{\Phi}_A(t, s) := \Phi_A(t, s)|_{\mathcal{R}(P(s))} : \mathcal{R}(P(s)) \rightarrow \mathcal{R}(P(t))$, $s \preceq t$, is a well-defined isomorphism, and we write $\bar{\Phi}_A(s, t)$ for its inverse (cf. [Pöt02, p. 85, Lemma 2.1.8]). Having this at hand, for two invariant projectors P, Q we can define *Green's function* $G_Q^P : \mathbb{T} \times \mathbb{T} \rightarrow \mathcal{L}(\mathcal{X})$ of (2.2) by

$$(2.4) \quad G_Q^P(t, s) := \begin{cases} \bar{\Phi}_A(t, s)Q(s) & \text{for } s \preceq t \\ -\bar{\Phi}_A(t, s)P(s) & \text{for } t \prec s \end{cases}.$$

We close our discussion of invariant projectors by

Lemma 2.1. *A regular invariant projector P of (2.2) is rd-continuously differentiable with*

$$P^\Delta(t) = A(t)P(t) - P(\sigma(t))A(t) \quad \text{for } t \in \mathbb{T}.$$

Proof. See [Pöt02, p. 88, Satz 2.1.10]. \square

The results of this paper apply to a certain class of dynamic equations which are dominated by their linear parts. The advantage of dealing with such semilinear equations is that we obtain quantitative global results using transparent proofs. Local results, which hold under more realistic assumptions on the nonlinearities, can be deduced easily using standard cut-off techniques (cf., e.g., [Pöt06, Theorem 4.1]). Furthermore, for the mentioned applications in discretization theory it is crucial to deal with equations admitting a specific dependence on parameters $\theta \in \mathbb{F}$. As demonstrated in [KP05], θ serves as a homotopy parameter between a continuous flow and its discretization using a numerical scheme (e.g., an Euler or Runge-Kutta method). More precisely, we consider nonlinear perturbations of (2.2) given by

$$(2.5) \quad x^\Delta = A(t)x + H(t, x; \theta)$$

with the particular nonlinearity

$$H(t, x; \theta) := F_1(t, x) + \theta F_2(t, x)$$

and rd-continuous mappings $F_1, F_2 : \mathbb{T} \times \mathcal{X} \rightarrow \mathcal{X}$ (see [Hil90, Section 5.1]). Further assumptions on F_1, F_2 can be found below in Hypothesis 2.2. A *solution* of the nonlinear dynamic equation (2.5) is a function $\phi : \mathbb{I} \rightarrow \mathcal{X}$ satisfying the identity $\phi^\Delta(t) \equiv A(t)\phi(t) + F_1(t, \phi(t)) + \theta F_2(t, \phi(t))$ on a \mathbb{T} -interval \mathbb{I} . Provided it exists, φ denotes the *general solution* of (2.5), i.e., $\varphi(\cdot; \tau, \xi; \theta)$ solves (2.5) on \mathbb{T}_τ^+ and satisfies the initial condition $\varphi(\tau; \tau, \xi; \theta) = \xi$ for $\tau \in \mathbb{T}$, $\xi \in \mathcal{X}$.

In general, solutions of dynamic equations need not to exist or to be unique in backward time. However, in later results we have to enforce backward existence and uniqueness. Hence, we define the dynamic equation (2.5) to be *regressive* on a set $\Theta \subseteq \mathbb{F}$, if

$$(2.6) \quad T_{t, \theta} := I_{\mathcal{X}} + \mu^*(t) [A(t) + F_1(t, \cdot) + \theta F_2(t, \cdot)] : \mathcal{X} \rightarrow \mathcal{X} \quad \text{for } t \in \mathbb{T}, \theta \in \Theta$$

is a homeomorphism and $\tilde{T} : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$, $\tilde{T}(t, x; \theta) := T_{t, \theta}^{-1}(x)$ is rd-continuous.

Dynamic equations on measure chains (or time scales) are intrinsically nonautonomous. Accordingly, the following notions should provide some insight into their geometric behavior. We fix a certain parameter $\theta \in \Theta$. An arbitrary (nonempty) subset $S(\theta)$ of the *extended state space* $\mathbb{T} \times \mathcal{X}$ is called a *nonautonomous set with τ -fibers*

$$S(\theta)_\tau := \{x \in \mathcal{X} : (\tau, x) \in S(\theta)\} \quad \text{for } \tau \in \mathbb{T}.$$

We denote $S(\theta)$ as *forward invariant*, if for any pair $(\tau, \xi) \in S(\theta)$ one has the inclusion

$$\varphi(t; \tau, S(\theta)_\tau; \theta) \subseteq S(\theta)_t \quad \text{for } t \in \mathbb{T}_\tau^+$$

and $S(\theta)$ is called *invariant*, if equality holds. Presumed each fiber $S(\theta)_\tau$ is a submanifold of \mathcal{X} , we speak of a *fiber bundle*.

From now on we deal with nonlinearities satisfying global Lipschitz conditions with small constants. In this sense, the dynamic equations (2.5) are *semilinear* and we precisely assume

Hypothesis 2.2. Let $K_1, K_2 \geq 1$ be reals and $a, b \in \mathcal{C}_{rd}^+(\mathbb{T}, \mathbb{R})$ growth rates with $a \triangleleft b$.

- (i) Exponential dichotomy: There exists a regular invariant projector $P : \mathbb{T} \rightarrow \mathcal{L}(\mathcal{X})$ of (2.2) such that the estimates

$$(2.7) \quad \|\Phi_A(t, s)Q(s)\| \leq K_1 e_a(t, s), \quad \|\bar{\Phi}_A(s, t)P(t)\| \leq K_2 e_b(s, t) \quad \text{for } t \preceq s$$

are satisfied with the complementary projector $Q(t) := I_{\mathcal{X}} - P(t)$.

- (ii) Lipschitz perturbation: For $i = 1, 2$ the identities $F_i(t, 0) \equiv 0$ on \mathbb{T} hold and the mappings F_i satisfy the Lipschitz estimates

$$(2.8) \quad L_i := \sup_{t \in \mathbb{T}} \text{Lip } F_i(t, \cdot) < \infty.$$

Moreover, we set $K := 2(K_1 + K_2 + K_1 K_2 \max\{K_1, K_2\})$, require

$$(2.9) \quad L_1 < \frac{\lfloor b - a \rfloor}{2K},$$

choose a fixed $\delta \in \left(KL_1, \frac{\lfloor b - a \rfloor}{2}\right)$ and define

$$\Gamma := \{c \in \mathcal{C}_{rd}^+(\mathbb{T}, \mathbb{R}) : a + \delta \triangleleft c \triangleleft b - \delta\}, \quad \bar{\Gamma} := \{c \in \mathcal{C}_{rd}^+(\mathbb{T}, \mathbb{R}) : a + \delta \trianglelefteq c \trianglelefteq b - \delta\}.$$

Remark 2.1. (1) For the special case of ordinary differential equations ($\mathbb{T} = \mathbb{R}$) our Hypothesis 2.2(i) reduces to the generalized dichotomy notion introduced in [Mul84], allowing time-dependent decay rates (cf. Figure 1).

(2) In our considerations we sometimes have to restrict the parameter space Θ . As general convention, we define $\Theta \subseteq \mathbb{F}$ to be a compact neighborhood of $0 \in \mathbb{F}$ satisfying

$$\Theta \subseteq \{\theta \in \mathbb{F} : L_2 |\theta| \leq L_1\}.$$

- (3) The nonlinearity $H : \mathbb{T} \times \mathcal{X} \times \mathbb{F} \rightarrow \mathcal{X}$ satisfies the Lipschitz estimate

$$(2.10) \quad \text{Lip}_2 H(\cdot; \theta) \leq L_1 + |\theta| L_2 \leq 2L_1 \quad \text{for } \theta \in \Theta$$

and the existence of suitable values for δ yields from (2.9): Since we have $\delta < \frac{\lfloor b - a \rfloor}{2}$, there exist functions $c \in \Gamma$ and in addition $a + \delta, b - \delta$ are positively regressive. Furthermore, for later use we state the inequalities

$$(2.11) \quad L(\theta) := \frac{K_1 + K_2}{\delta} (L_1 + |\theta| L_2) \stackrel{(2.10)}{\leq} 2 \frac{K_1 + K_2}{\delta} L_1 \stackrel{(2.9)}{<} 1,$$

$$(2.12) \quad \ell(\theta) := \frac{K_1 K_2}{K_1 + K_2} \frac{L(\theta)}{1 - L(\theta)} < \min\left\{\frac{1}{K_1}, \frac{1}{K_2}\right\} < 1 \quad \text{for } \theta \in \Theta.$$

Note that the inequality (2.9) is stronger than the corresponding assumptions imposed in [Pöt07, Pöt06, KP05] to deduce variants of Theorem 3.5 (without the statement on the asymptotic phase) or Proposition 3.1. Nevertheless, we demand it for the sake of consistency.

(4) The general solution $\varphi : \{(t, \tau, \xi, \theta) \in \mathbb{T} \times \mathbb{T} \times \mathcal{X} \times \mathbb{F} : \tau \preceq t\} \rightarrow \mathcal{X}$ of the dynamic equation (2.5) exists uniquely and is continuous. If we additionally suppose that (2.5) is regressive on Θ , then even $\varphi : \mathbb{T} \times \mathbb{T} \times \mathcal{X} \times \mathbb{F} \rightarrow \mathcal{X}$ is well-defined and continuous (see [Pöt02, p. 38, Satz 1.2.17(a)]). In particular, referring to (2.10), a sufficient condition for regressivity is given by $\mu^*(t) [A(t) + 2L_1] < 1$ for all $t \in \mathbb{T}$.

As shown in our earlier papers [Pöt07, Pöt06, KP05], the above Hypothesis 2.2 is sufficient to derive a quite general version of the stable/unstable manifold theorem for dynamic equations of the form (2.5) – including attractivity properties of the invariant manifolds in terms of an asymptotic phase. In this paper our interest is focused on the geometrical behavior of (2.5) under variation of the parameter $\theta \in \Theta$. For our analysis concerning this matter, where we utilize the Lyapunov-Perron method, it is crucial to have an additional assumption controlling the exponential growth of solutions. At first glance it might seem purely technical and artificial, but can be justified in many applications, particularly if a certain “dissipativity” is present.

Hypothesis 2.3. *Assume there exist reals $C_i^+, C_i^-, \geq 0$ and functions $c_i^+, c_i^- \in \mathcal{C}_{rd}^+(\mathbb{T}, \mathbb{R})$ for $i \in \{1, 2\}$ such that for all $\tau \in \mathbb{T}$, $\xi \in \mathcal{X}$ and $\theta \in \Theta$ the following holds:*

(i) *The general solution of (2.5) satisfies*

$$(2.13) \quad \|\varphi(t; \tau, \xi; \theta)\| \leq C_1^+ e_{c_1^+}(t, \tau) \|\xi\|, \quad \text{Lip } \varphi(t; \tau, \xi; \cdot)|_{\Theta} \leq C_2^+ e_{c_2^+}(t, \tau) \|\xi\| \quad \text{for } t \in \mathbb{T}_\tau^+.$$

(ii) *The dynamic equation (2.5) is regressive on Θ and its general solution satisfies*

$$(2.14) \quad \|\varphi(t; \tau, \xi; \theta)\| \leq C_1^- e_{c_1^-}(t, \tau) \|\xi\|, \quad \text{Lip } \varphi(t; \tau, \xi; \cdot)|_{\Theta} \leq C_2^- e_{c_2^-}(t, \tau) \|\xi\| \quad \text{for } t \in \mathbb{T}_\tau^-.$$

The existence of classical Lipschitz estimates as given in Hypothesis 2.2(ii) is actually enough to deduce sufficient conditions for Hypothesis 2.3 to hold. Yet, the resulting growth rates c_i^+, c_i^- are often too pessimistic for applications.

3. INVARIANT FIBER BUNDLES AND FOLIATIONS

This section contains the key ingredients to understand the geometric behavior for semilinear dynamic equations of the form (2.5). We derive that there is a (pseudo-hyperbolic) saddle point structure around the trivial solution and obtain information on the corresponding pseudo-stable and -unstable sets (we call them invariant fiber bundles). They are given as Lipschitzian graphs which attract solutions in one time direction. This attraction will be given in terms of an asymptotic phase. All these properties, and their proofs, have already been prepared in [Pöt03] and [Pöt06].

Our earlier work [Pöt03] also contains precise knowledge on the behavior of the invariant fiber bundles under variation of the parameter θ in (2.5). Hence, the basic task of the section is to investigate how the asymptotic phase property is influenced, when θ is varied. Recapitulating [Pöt06], the asymptotic phase had been constructed using an invariant foliation of the extended state space. Thus, the main load will consist of obtaining corresponding perturbation results on this foliation.

Proposition 3.1 (invariant fibers). *Assume that Hypothesis 2.2 is fulfilled. Then for all $\tau \in \mathbb{T}$, $\xi \in \mathcal{X}$, $\theta \in \Theta$ the following holds:*

(a) *The pseudo-stable fiber through (τ, ξ) , given by*

$$S^+(\xi, \theta)_\tau := \{\zeta \in \mathcal{X} : \varphi(\cdot; \tau, \zeta; \theta) - \varphi(\cdot; \tau, \xi; \theta) \in \mathcal{X}_{\tau, c}^+ \text{ for all } c \in \Gamma\}$$

is forward invariant w.r.t. (2.5), i.e.,

$$(3.1) \quad \varphi(t; \tau, S^+(\xi, \theta)_\tau; \theta) \subseteq S^+(\varphi(t; \tau, \xi; \theta), \theta)_t \quad \text{for } t \in \mathbb{T}_\tau^+$$

and possesses the representation

$$(3.2) \quad S^+(\xi, \theta) = \{(\tau, \eta + s^+(\tau, \eta, \xi; \theta)) : \eta \in \mathcal{R}(Q(\tau))\}$$

as graph of a continuous mapping $s^+ : \mathbb{T} \times \mathcal{X} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ satisfying

$$s^+(\tau, \eta, \xi; \theta) = s^+(\tau, Q(\tau)\eta, \xi; \theta) \in \mathcal{R}(P(\tau)) \quad \text{for } \eta \in \mathcal{X}.$$

Furthermore, $s^+ : \mathbb{T} \times \mathcal{X} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ is linearly bounded

$$(3.3) \quad \|s^+(\tau, \eta, \xi; \theta)\| \leq \|P(\tau)\xi\| + \ell(\theta) \|\eta - \xi\| \quad \text{for } \eta \in \mathcal{X},$$

and satisfies $\text{Lip}_2 s^+(\cdot; \theta) \leq K_1 \ell(\theta)$.

- (b) For \mathbb{T} unbounded below and if (2.5) is regressive on Θ , then the pseudo-unstable fiber through (τ, ξ) , given by

$$R^-(\xi, \theta)_\tau := \{ \zeta \in \mathcal{X} : \varphi(\cdot; \tau, \zeta; \theta) - \varphi(\cdot; \tau, \xi; \theta) \in \mathcal{X}_{\tau, c}^- \text{ for all } c \in \Gamma \}$$

is invariant w.r.t. (2.5), i.e.,

$$(3.4) \quad \varphi(t; \tau, R^-(\xi, \theta)_\tau; \theta) = R^-(\varphi(t; \tau, \xi; \theta), \theta)_t \quad \text{for } t \in \mathbb{T}$$

and possesses the representation

$$R^-(\xi, \theta) = \{ (\tau, \eta + r^-(\tau, \eta, \xi; \theta)) : \eta \in \mathcal{R}(P(\tau)) \}$$

as graph of a continuous mapping $r^- : \mathbb{T} \times \mathcal{X} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ satisfying

$$r^+(\tau, \eta, \xi; \theta) = r^+(\tau, P(\tau)\eta, \xi; \theta) \in \mathcal{R}(Q(\tau)) \quad \text{for } \eta \in \mathcal{X}.$$

Furthermore, $r^- : \mathbb{T} \times \mathcal{X} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ is linearly bounded

$$(3.5) \quad \|r^-(\tau, \eta, \xi; \theta)\| \leq \|Q(\tau)\xi\| + \ell(\theta) \|\eta - \xi\| \quad \text{for } \eta \in \mathcal{X},$$

and satisfies $\text{Lip}_2 r^-(\cdot; \theta) \leq K_2 \ell(\theta)$.

- (c) For \mathbb{T} unbounded below and if (2.5) is regressive on Θ , then there exists a unique continuous mapping $\Pi : \mathbb{T} \times \mathcal{X} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ geometrically given by

$$(3.6) \quad S^+(x_1, \theta)_\tau \cap R^-(x_2, \theta)_\tau = \{ \Pi(\tau, x_1, x_2; \theta) \} \quad \text{for } \tau \in \mathbb{T}, x_1, x_2 \in \mathcal{X}, \theta \in \Theta.$$

Furthermore, Π is linearly bounded

$$(3.7) \quad \|\Pi(\tau, x_1, x_2; \theta)\| \leq \frac{(K_2 + \ell(\theta))(1 + 2\ell(\theta))}{1 - \ell(\theta)^2} \|x_1\| + \frac{(K_1 + \ell(\theta))(1 + 2\ell(\theta))}{1 - \ell(\theta)^2} \|x_2\|$$

for all $\tau \in \mathbb{T}$, $x_1, x_2 \in \mathcal{X}$ and $\theta \in \Theta$.

Remark 3.1. If the dynamic equation (2.5) is regressive on Θ , then the pseudo-stable fibers $S^+(\xi, \theta)$ are invariant w.r.t. (2.5), i.e., the inclusion (3.1) can be strengthened to

$$(3.8) \quad \varphi(t; \tau, S^+(\xi, \theta)_\tau; \theta) = S^+(\varphi(t; \tau, \xi; \theta), \theta)_t \quad \text{for } t \in \mathbb{T}.$$

Proof. The assertions (a) and (b) have already been shown in [Pöt06, Proposition 3.2]. Thus, it remains to establish (c). Thereto, let $\tau \in \mathbb{T}$, $x_1, x_2 \in \mathcal{X}$ and $\theta \in \Theta$. Having the inequality (2.12) at hand, we know from (a), or (b),

$$(3.9) \quad \text{Lip}_2 s^+(\cdot; \theta) \leq K_1 \ell(\theta) < 1, \quad \text{Lip}_2 r^-(\cdot; \theta) \leq K_2 \ell(\theta) < 1,$$

respectively. The intersection $S^+(x_1, \theta)_\tau \cap R^-(x_2, \theta)_\tau$ contains a point $y \in \mathcal{X}$, if and only if there exist $p \in \mathcal{R}(P(\tau))$, $q \in \mathcal{R}(Q(\tau))$ so that

$$y = q + s^+(\tau, q, x_1; \theta), \quad y = p + r^-(\tau, p, x_2; \theta),$$

which, in turn, is equivalent to the fact that y allows the representation $y = p + q$, where $p \in \mathcal{R}(P(\tau))$, $q \in \mathcal{R}(Q(\tau))$ solve the equations

$$(3.10) \quad p = s^+(\tau, q, x_1; \theta), \quad q = r^-(\tau, p, x_2; \theta).$$

Hence, we have to show that the equations (3.10) are uniquely solvable. Despite the fact that this is an easy consequence of the (uniform) contraction principle, we give the argument for later reference. Thanks to (a), (b) and (3.9), the two mappings

$$\begin{aligned} \Pi_\tau^- &: \mathcal{R}(P(\tau)) \times \mathcal{X} \times \mathcal{X} \times \Theta \rightarrow \mathcal{R}(P(\tau)), \\ &(p, x_1, x_2, \theta) \mapsto r^-(\tau, s^+(\tau, p, x_1; \theta), x_2; \theta), \\ \Pi_\tau^+ &: \mathcal{R}(Q(\tau)) \times \mathcal{X} \times \mathcal{X} \times \Theta \rightarrow \mathcal{R}(Q(\tau)), \end{aligned}$$

$$(q, x_1, x_2, \theta) \mapsto s^+(\tau, r^-(\tau, q, x_2; \theta), x_1; \theta)$$

are continuous and contractions in their first arguments (uniformly in the parameters τ, x_1, x_2 and θ). Consequently, the contraction mapping principle implies that there exist unique fixed point functions $p^* : \mathbb{T} \times \mathcal{X} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ and $q^* : \mathbb{T} \times \mathcal{X} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ for Π_τ^- and Π_τ^+ , respectively. Thus,

$$(3.11) \quad \Pi(\tau, x_1, x_2; \theta) := p^*(\tau, x_1, x_2; \theta) + q^*(\tau, x_1, x_2; \theta)$$

satisfies the geometric property (3.6). Given $\tau^0 \in \mathbb{T}$, $x_1^0, x_2^0 \in \mathcal{X}$, $\theta^0 \in \Theta$ arbitrarily, we have

$$(3.12) \quad \begin{aligned} & \|p^*(\tau, x_1, x_2; \theta) - p^*(\tau^0, x_1^0, x_2^0; \theta^0)\| \\ & \leq \frac{1}{1 - \text{Lip}_1 \Pi_\tau^+} \left\| \Pi_\tau^+(p^*(\tau^0, x_1^0, x_2^0; \theta^0), x_1, x_2; \theta) - \Pi_{\tau^0}^+(p^*(\tau^0, x_1^0, x_2^0; \theta^0), x_1^0, x_2^0; \theta^0) \right\|, \end{aligned}$$

$$(3.13) \quad \begin{aligned} & \|q^*(\tau, x_1, x_2; \theta) - q^*(\tau^0, x_1^0, x_2^0; \theta^0)\| \\ & \leq \frac{1}{1 - \text{Lip}_1 \Pi_\tau^-} \left\| \Pi_\tau^-(p^*(\tau^0, x_1^0, x_2^0; \theta^0), x_1, x_2; \theta) - \Pi_{\tau^0}^-(p^*(\tau^0, x_1^0, x_2^0; \theta^0), x_1^0, x_2^0; \theta^0) \right\|. \end{aligned}$$

and these inequalities immediately imply the continuity of Π inherited from the corresponding properties of s^+ and r^- . In order to prove that $\Pi(\tau, \cdot; \theta)$ is linearly bounded by (3.11), it suffices to show that $p^*(\tau, \cdot; \theta)$ and $q^*(\tau, \cdot; \theta)$ have this property. This can be seen as follows,

$$\begin{aligned} & \|p^*(\tau, x_1, x_2; \theta)\| \\ (3.10) \quad & \stackrel{=}{=} \|s^+(\tau, q^*(\tau, x_1, x_2; \theta), x_1; \theta)\| \\ (3.3) \quad & \leq \|P(\tau)x_1\| + \ell(\theta) \|q^*(\tau, x_1, x_2; \theta)\| + \ell(\theta) \|x_1\| \\ (3.5) \quad & \leq (K_2 + \ell(\theta)) \|x_1\| + \ell(\theta) \|Q(\tau)x_2\| + \ell(\theta)^2 \|x_2\| + \ell(\theta)^2 \|p^*(\tau, x_1, x_2; \theta)\|, \end{aligned}$$

hence,

$$\|p^*(\tau, x_1, x_2; \theta)\| \leq \frac{K_2 + \ell(\theta)}{1 - \ell(\theta)} \|x_1\| + \frac{\ell(\theta)(K_1 + \ell(\theta))}{1 - \ell(\theta)^2} \|x_2\|$$

and similarly

$$\|q^*(\tau, x_1, x_2; \theta)\| \leq \frac{K_1 + \ell(\theta)}{1 - \ell(\theta)} \|x_2\| + \frac{\ell(\theta)(K_2 + \ell(\theta))}{1 - \ell(\theta)^2} \|x_1\|,$$

which implies (3.7). We have established (c). \square

For later use, we accompany Proposition 3.1 by a corollary. It investigates the behavior of the functions s^+ , r^- and Π under perturbation, i.e., variation of the parameter θ . Having applications to discretization theory in mind, this can be considered as crucial result. It provides, in a certain sense, precise perturbation results for the invariant fibers of (2.5).

Corollary 3.2 (perturbed invariant fibers). *Let $\tau \in \mathbb{T}$. Assume that Hypotheses 2.2–2.3 are fulfilled. Then for all $\theta_0 \in \Theta$ the following holds:*

(a) *For any sets $B_1, B_2 \subseteq \mathcal{X}$ such that $Q(\tau)B_1$ and B_2 are bounded, one has*

$$(3.14) \quad \lim_{\theta \rightarrow \theta_0} s^+(\tau, \eta, \xi; \theta) = s^+(\tau, \eta, \xi; \theta_0) \quad \text{uniformly in } \eta \in B_1, \xi \in B_2,$$

and in case $a \triangleleft c_2^+ \triangleleft b - \delta$ there exist reals $\bar{C}_1^+, \bar{C}_2^+ \geq 0$ such that

$$(3.15) \quad \text{Lip } s^+(\tau, \eta, \xi, \cdot) \leq \bar{C}_1^+ \|\xi\| + \bar{C}_2^+ \|Q(\tau)\eta - \xi\| \quad \text{for } \tau \in \mathbb{T}, \xi, \eta \in \mathcal{X}.$$

(b) If \mathbb{T} is unbounded below and if Hypothesis 2.3(ii) holds, then for sets $B_1, B_2 \subseteq \mathcal{X}$ such that $P(\tau)B_1$ and B_2 are bounded, one has

$$(3.16) \quad \lim_{\theta \rightarrow \theta_0} r^-(\tau, \eta, \xi; \theta) = r^-(\tau, \eta, \xi; \theta_0) \quad \text{uniformly in } \eta \in B_1, \xi \in B_2,$$

and in case $a + \delta \triangleleft c_2^- \triangleleft b$ there exist reals $\bar{C}_1^-, \bar{C}_2^- \geq 0$ such that

$$(3.17) \quad \text{Lip } r^-(\tau, \eta, \xi, \cdot) \leq \bar{C}_1^- \|\xi\| + \bar{C}_2^- \|P(\tau)\eta - \xi\| \quad \text{for } \tau \in \mathbb{T}, \xi, \eta \in \mathcal{X}.$$

(c) If \mathbb{T} is unbounded below and if Hypothesis 2.3(ii) holds, then for bounded sets $B_1, B_2 \subseteq \mathcal{X}$, one has

$$(3.18) \quad \lim_{\theta \rightarrow \theta_0} \Pi(\tau, x_1, x_2; \theta) = \Pi(\tau, x_1, x_2; \theta_0) \quad \text{uniformly in } x_1 \in B_1, x_2 \in B_2,$$

and in case $a \triangleleft c_2^+ \triangleleft b - \delta$, $a + \delta \triangleleft c_2^- \triangleleft b$ there exist reals $\bar{C}_1, \bar{C}_2 \geq 0$ such that

$$(3.19) \quad \text{Lip } \Pi(\tau, x_1, x_2, \cdot) \leq \bar{C}_1 \|x_1\| + \bar{C}_2 \|x_2\| \quad \text{for } \tau \in \mathbb{T}, x_1, x_2 \in \mathcal{X}.$$

The proof of Corollary 3.2 is involved. Actually, before giving it, we have to recapitulate some basic concepts from our earlier paper [Pöt06] and have a closer look at them. Simply spoken, the basic tool for our analysis will be the equation of perturbed motion related to a solution of (2.5) starting in $(\tau, \xi) \in \mathbb{T} \times \mathcal{X}$. From a somehow more technical perspective, the general solution φ of (2.5) exists uniquely in forward time and consequently the mapping $G : \{(t, x, \tau, \xi, \theta) \in \mathbb{T} \times \mathcal{X} \times \mathbb{T} \times \mathbb{T} \times \mathbb{F} : \tau \in \mathbb{T}, t \in \mathbb{T}_\tau^+, x, \xi \in \mathcal{X}\} \rightarrow \mathcal{X}$,

$$G(t, x; \tau, \xi, \theta) := H(t, x + \varphi(t; \tau, \xi; \theta); \theta) - H(t, \varphi(t; \tau, \xi; \theta); \theta)$$

is well-defined under the above Hypothesis 2.2. Moreover, by Remark 2.1(4) the nonlinearity G is continuous in (τ, ξ, θ) , $G(t, 0; \tau, \xi, \theta) \equiv 0$ and $\text{Lip}_2 G(\cdot; \theta) \leq L_1 + |\theta| L_2$.

Lemma 3.3. Assume that Hypothesis 2.2 is fulfilled and choose $\tau \in \mathbb{T}$ fixed. Then for growth rates $c, d \in \mathcal{C}_{rd}^+ \mathcal{R}(\mathbb{T}, \mathbb{R})$, $a \triangleleft c \triangleleft b$, $c \triangleleft d$, the operator $\mathcal{S}_\tau^+ : \mathcal{X}_{\tau,c}^+ \times \mathcal{R}(Q(\tau)) \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}_{\tau,d}^+$,

$$(3.20) \quad \mathcal{S}_\tau^+(\psi; \eta, \xi, \theta) := \Phi_A(\cdot, \tau) [\eta - Q(\tau)\xi] s + \int_\tau^\infty G_Q^P(\cdot, \sigma(s)) G(s, \psi(s); \tau, \xi, \theta) \Delta s$$

is well-defined and has, for $\theta_0 \in \Theta$ and $c \in \bar{\Gamma}$, the following properties:

(a) $\mathcal{S}_\tau^+(\cdot; \eta, \xi, \theta_0) : \mathcal{X}_{\tau,c}^+ \rightarrow \mathcal{X}_{\tau,d}^+$ is a uniform contraction with Lipschitz constant

$$(3.21) \quad \text{Lip } \mathcal{S}_\tau^+(\cdot; \eta, \xi, \theta_0) \leq L(\theta_0) \quad \text{for } \eta \in \mathcal{R}(Q(\tau)), \xi \in \mathcal{X},$$

and, if additionally Hypothesis 2.3 holds, one furthermore obtains

(b) in case $c \triangleleft d \triangleleft b$ and for bounded sets $B_1 \subseteq \mathcal{X}_{\tau,c}^+$, $B_2 \subseteq \mathcal{X}$ one has

$$(3.22) \quad \lim_{\theta \rightarrow \theta_0} \|\mathcal{S}_\tau^+(\psi; \eta, \xi, \theta) - \mathcal{S}_\tau^+(\psi; \eta, \xi, \theta_0)\|_{\tau,d}^+ = 0$$

uniformly in $\psi \in B_1$, $\eta \in \mathcal{R}(Q(\tau))$ and $\xi \in B_2$,

(c) in case $a \triangleleft c_2^+ \triangleleft b$, $c_2^+ \triangleleft d$ one has

$$(3.23) \quad \text{Lip } \mathcal{S}_\tau^+(\psi; \eta, \xi, \cdot) \leq 4C_2^+ L_1 \lambda(c_2^+) \|\xi\| + 4L_2 \lambda(c) \|\psi\|_{\tau,c}^+$$

for all $\psi \in \mathcal{X}_{\tau,c}^+$, $\eta \in \mathcal{R}(Q(\tau))$, $\xi \in \mathcal{X}$, with $\lambda(c) := \frac{K_1}{[c-a]} + \frac{K_2}{[b-c]}$.

Proof. Referring to [Pöt06, Lemma 3.1(b)] it remains to deduce the assertions (b) and (c). Thereto, we keep $\tau \in \mathbb{T}$, $\xi \in \mathcal{X}$ arbitrary, but fixed. For notational reasons we write

$$\varphi(t) := \varphi(t; \tau, \xi; \theta), \quad \varphi_0(t) := \varphi(t; \tau, \xi; \theta_0) \quad \text{for } t \in \mathbb{T}_\tau^+.$$

(b) We begin with two preliminary inequalities and suppress the dependence of G on τ, ξ . A direct application of the triangle inequality to the definition of G gives us

$$(3.24) \quad \|G(t, x; \theta) - G(t, x; \theta_0)\| \stackrel{(2.10)}{\leq} 4L_1 \|x\| \quad \text{for } t \in \mathbb{T}_\tau^+, x \in \mathcal{X}, \theta \in \Theta,$$

and, on the other hand, one has

$$(3.25) \quad \begin{aligned} & \|G(t, x; \theta) - G(t, x; \theta_0)\| \\ & \leq \|H(t, x + \varphi(t); \theta) - H(t, x + \varphi_0(t); \theta)\| \\ & \quad + \|H(t, x + \varphi_0(t); \theta) - H(t, \varphi_0(t); \theta) + H(t, \varphi_0(t); \theta_0) - H(t, x + \varphi_0(t); \theta_0)\| \\ & \quad + \|H(t, \varphi_0(t); \theta) - H(t, \varphi(t); \theta_0)\| \\ & \leq 4L_1 \|\varphi(t) - \varphi_0(t)\| + L_2 \|x\| |\theta - \theta_0| \quad \text{for } t \in \mathbb{T}_\tau^+, x \in \mathcal{X}, \theta \in \Theta. \end{aligned}$$

Now let $c, d \in \mathcal{C}_{\tau d}^+ \mathcal{R}(\mathbb{T}, \mathbb{R})$, $c \triangleleft d$ and $\psi \in \mathcal{X}_{\tau, c}^+$. We define

$$\Delta(t) := G(t, \psi(t), \theta) - G(t, \psi(t), \theta_0) \quad \text{for } t \in \mathbb{T}_\tau^+$$

and obtain the inequalities

$$(3.26) \quad \|\Delta(t)\| \stackrel{(3.24)}{\leq} 4L_1 \|\psi(t)\| \leq 4L_1 e_c(t, \tau) \|\psi\|_{\tau, c}^+ \quad \text{for } t \in \mathbb{T}_\tau^+$$

and

$$(3.27) \quad \|\Delta(t)\| \stackrel{(3.25)}{\leq} 4L_1 \|\varphi(t) - \varphi_0(t)\| + L_2 e_c(t, \tau) \|\psi\|_{\tau, c}^+ |\theta - \theta_0| \quad \text{for } t \in \mathbb{T}_\tau^+.$$

In order to establish the limit relation (3.22) we have to estimate the difference

$$(3.28) \quad \|\mathcal{S}_\tau^+(\psi; \eta, \xi, \theta)(t) - \mathcal{S}_\tau^+(\psi; \eta, \xi, \theta_0)(t)\| e_d(\tau, t) \leq S_1(t) + S_2(t) \quad \text{for } t \in \mathbb{T}_\tau^+$$

with $\eta \in \mathcal{R}(Q(\tau))$ and (cf. (3.20))

$$\begin{aligned} S_1(t) &:= \int_\tau^t \|\Phi_A(t, \sigma(s))Q(\sigma(s))\Delta(s)\| \Delta s e_d(\tau, t), \\ S_2(t) &:= \int_t^\infty \|\bar{\Phi}_A(t, \sigma(s))P(\sigma(s))\Delta(s)\| \Delta s e_d(\tau, t) \quad \text{for } t \in \mathbb{T}_\tau^+. \end{aligned}$$

There to, let $\varepsilon > 0$. With (3.26) we immediately get from [Pöt02, p. 65, Lemma 1.3.29],

$$S_1(t) \leq \frac{4K_1 L_1}{[c - a]} \|\psi\|_{\tau, c}^+ e_{c \ominus d}(t, \tau), \quad S_2(t) \leq \frac{4K_2 L_1}{[b - c]} \|\psi\|_{\tau, c}^+ e_{c \ominus d}(t, \tau) \quad \text{for } t \in \mathbb{T}_\tau^+$$

using the estimates (2.7). Due to the limit relation $\lim_{t \rightarrow \infty} e_{c \ominus d}(t, \tau) = 0$ (see [Pöt02, p. 63, Lemma 1.3.26]) we can choose $T \in \mathbb{T}_\tau^+$ so large, and independent of $\eta \in \mathcal{R}(Q(\tau))$, $\xi \in \mathcal{X}$, that

$$\|\mathcal{S}_\tau^+(\psi; \eta, \xi, \theta)(t) - \mathcal{S}_\tau^+(\psi; \eta, \xi, \theta_0)(t)\| e_d(\tau, t) \leq \frac{\varepsilon}{2} \quad \text{for } t \in \mathbb{T}_T^+.$$

Thus, it remains to obtain an estimate for (3.28) on the compact \mathbb{T} -interval $[\tau, T]_{\mathbb{T}}$. There to, we apply (3.27) and obtain from (2.7) that

$$(3.29) \quad \begin{aligned} S_1(t) &\leq 4K_1 L_1 \int_\tau^t e_a(t, \sigma(s)) \|\varphi(s) - \varphi_0(s)\| \Delta s e_d(\tau, t) + \frac{4K_1 L_2}{[c - a]} \|\psi\|_{\tau, c}^+ e_{c \ominus d}(t, \tau) |\theta - \theta_0|, \\ (3.30) \quad S_2(t) &\leq 4K_2 L_1 \int_t^\infty e_b(t, \sigma(s)) \|\varphi(s) - \varphi_0(s)\| \Delta s e_d(\tau, t) + \frac{4K_2 L_2}{[b - c]} \|\psi\|_{\tau, c}^+ e_{c \ominus d}(t, \tau) |\theta - \theta_0| \end{aligned}$$

for all $t \in \mathbb{T}_\tau^+$. Concerning $S_1(t)$, this particularly implies

$$S_1(t) \leq 4K_1L_1 \int_\tau^T e_a(\tau, \sigma(s)) \|\varphi(s) - \varphi_0(s)\| \Delta s + \frac{4K_1L_2}{[c-a]} \|\psi\|_{\tau,c}^+ e_{c\ominus d}(t, \tau) |\theta - \theta_0|,$$

where we have used $e_{a\ominus d}(t, \tau) \leq 1$, and Hypothesis 2.13 leads to

$$S_1(t) \leq 4C_2^+K_1L_1 \int_\tau^T e_a(\tau, \sigma(s))e_{c_2^+}(s, \tau) \Delta s \|\xi\| |\theta - \theta_0| + \frac{4K_1L_2}{[c-a]} \|\psi\|_{\tau,c}^+ e_{c\ominus d}(t, \tau) |\theta - \theta_0|$$

for all $t \in [\tau, T]_{\mathbb{T}}$. Hence, for each bounded $B_1 \subseteq \mathcal{X}_{\tau,c}^+$ and $B_2 \subseteq \mathcal{X}$ we find a $\delta > 0$ such that

$$S_1(t) \leq \frac{\varepsilon}{4} \quad \text{for } t \in [\tau, t]_{\mathbb{T}}, \eta \in \mathcal{R}(Q(\tau)), \psi \in B_1, \xi \in B_2$$

and $\theta \in B_\delta(\theta_0)$. The corresponding estimate for $S_2(t)$ on $[\tau, T]_{\mathbb{T}}$ can be deduced using similar arguments and we arrive at

$$S_1(t) + S_2(t) \leq \frac{\varepsilon}{2} \quad \text{for } t \in [\tau, T]_{\mathbb{T}}$$

uniformly in $\eta \in \mathcal{R}(Q(\tau))$ and ξ, ψ from bounded sets. Consequently, we obtain

$$\|\mathcal{S}_\tau^+(\psi; \eta, \xi, \theta)(t) - \mathcal{S}_\tau^+(\psi; \eta, \xi, \theta_0)(t)\| e_d(\tau, t) \leq \frac{\varepsilon}{2} \quad \text{for } t \in \mathbb{T}_\tau^+$$

and taking the least upper bound over $t \in \mathbb{T}_\tau^+$ in this estimate yields assertion (b).

(c) We rely on the notation and the estimates inherited from (b). Indeed, if we substitute the right inequality in (2.13) into (3.29) and evaluate the integrals using [Pöt02, p. 65, Lemma 1.3.29], we get

$$S_1(t) \leq 4K_1 \left(\frac{C_2^+L_1}{[c_2^+ - a]} \|\xi\| + \frac{L_2}{[c-a]} \|\psi\|_{\tau,c}^+ \right) |\theta - \theta_0| \quad \text{for } t \in \mathbb{T}_\tau^+$$

and applying the same to (3.30) guarantees

$$S_2(t) \leq 4K_2 \left(\frac{C_2^+L_1}{[b - c_2^+]} \|\xi\| + \frac{L_2}{[b-c]} \|\psi\|_{\tau,c}^+ \right) |\theta - \theta_0| \quad \text{for } t \in \mathbb{T}_\tau^+.$$

Here, the assumptions $a \triangleleft c_2^+ \triangleleft b$, $c_2^+ \trianglelefteq d$ yield existence of the corresponding integrals. \square

With the preparations in Lemma 3.3 it is fairly standard to obtain results on the behavior of the fixed point of $\mathcal{S}_\tau^+(\cdot; \eta, \xi, \theta)$ under variation of the parameter $\theta \in \Theta$.

Lemma 3.4. *Assume that Hypothesis 2.2 is fulfilled and choose $\tau \in \mathbb{T}$ fixed. Then for $c \in \bar{\Gamma}$ the operator $\mathcal{S}_\tau^+(\cdot; \eta, \xi, \theta_0) : \mathcal{X}_{\tau,c}^+ \rightarrow \mathcal{X}_{\tau,c}^+$ from Lemma 3.3 possesses a unique fixed point $\psi_\tau^*(\eta, \xi, \theta_0) \in \mathcal{X}_{\tau,c}^+$ for all $\eta \in \mathcal{R}(Q(\tau))$, $\xi \in \mathcal{X}$, $\theta_0 \in \Theta$, which does not depend on the growth rate $c \in \bar{\Gamma}$, and has, for $\theta_0 \in \Theta$ and $c \in \Gamma$, the following properties:*

(a) *It satisfies the estimates*

$$(3.31) \quad \|\psi_\tau^*(\eta, \xi, \theta_0)\|_{\tau,c}^+ \leq \frac{K_1}{1 - L(\theta_0)} \|\eta - \xi\|,$$

$$(3.32) \quad \|P(\tau)\psi_\tau^*(\eta, \xi; \theta_0)(\tau)\| \leq \ell(\theta_0) \|\eta - \xi\|,$$

$$(3.33) \quad \text{Lip } P(\tau)\psi_\tau^*(\cdot, \xi; \theta_0)(\tau) \leq \ell(\theta_0) \quad \text{for } \eta \in \mathcal{R}(Q(\tau)), \xi \in \mathcal{X},$$

and the mapping $\psi_\tau^ : \mathcal{R}(Q(\tau)) \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}_{\tau,c}^+$ is continuous.*

If additionally Hypothesis 2.3 holds, one furthermore obtains

(b) *for every bounded $B_1 \subseteq \mathcal{R}(Q(\tau))$ and $B_2 \subseteq \mathcal{X}$ one has*

$$(3.34) \quad \lim_{\theta \rightarrow \theta_0} \|\psi_\tau^*(\eta, \xi; \theta) - \psi_\tau^*(\eta, \xi; \theta_0)\|_{\tau,c}^+ = 0 \quad \text{uniformly in } \eta \in B_1, \xi \in B_2,$$

(c) in case $a \triangleleft c_2^+ \triangleleft b$, $c_2^+ \trianglelefteq c$ one has

$$(3.35) \quad \text{Lip } \psi_\tau^*(\eta, \xi, \cdot) \leq \frac{4\delta}{\delta - 2(K_1 + K_2)L_1} \left(C_2^+ L_1 \lambda(c_2^+) \|\xi\| + \frac{\delta K_1 L_2 \lambda(c)}{\delta - 2(K_1 + K_2)L_1} \|\eta - \xi\| \right)$$

for all $\eta \in \mathcal{R}(Q(\tau))$, $\xi \in \mathcal{X}$.

Proof. Using our preparations in [Pöt06, Lemma 3.1(c), (d) and (3.9)] we only have to establish claims (b) and (c). Let $\tau \in \mathbb{T}$, $\xi \in \mathcal{X}$, $\eta \in \mathcal{R}(Q(\tau))$ and $c \in \Gamma$.

(b) Keep $\theta_0 \in \Theta$ fixed. Let $\bar{c} \in \Gamma$ with $\bar{c} \triangleleft c$ and, suppressing the dependence on η, ξ , we have the inclusion $\psi_\tau^*(\theta) \subseteq \mathcal{X}_{\tau, \bar{c}}^+ \subseteq \mathcal{X}_{\tau, c}^+$. Keeping in mind the fixed point relation $\psi_\tau^*(\theta) = \mathcal{S}_\tau^+(\psi_\tau^*(\theta); \theta)$ for all $\theta \in \Theta$, we obtain

$$\|\psi_\tau^*(\theta) - \psi_\tau^*(\theta_0)\|_{\tau, \bar{c}}^+ \stackrel{(3.21)}{\leq} L(\theta) \|\psi_\tau^*(\theta) - \psi_\tau^*(\theta_0)\|_{\tau, \bar{c}}^+ + \|\mathcal{S}_\tau^+(\psi_\tau^*(\theta_0); \theta) - \mathcal{S}_\tau^+(\psi_\tau^*(\theta_0); \theta_0)\|_{\tau, \bar{c}}^+$$

and consequently,

$$\|\psi_\tau^*(\theta) - \psi_\tau^*(\theta_0)\|_{\tau, \bar{c}}^+ \stackrel{(2.1)}{\leq} \frac{1}{1 - L(\theta)} \|\mathcal{S}_\tau^+(\psi_\tau^*(\theta_0); \theta) - \mathcal{S}_\tau^+(\psi_\tau^*(\theta_0); \theta_0)\|_{\tau, c}^+ \quad \text{for } \theta \in \Theta.$$

Now let $B_1 \subseteq \mathcal{R}(Q(\tau))$ and $B_2 \in \mathcal{X}$ be bounded. For $\eta \in B_1$, $\xi \in B_2$ we know from (3.31) that $\psi_\tau^*(\eta, \xi; \theta_0) \in \mathcal{X}_{\tau, c}^+$ is bounded. Consequently, passing over to the limit $\theta \rightarrow \theta_0$ yields the relation (3.34) by Lemma 3.3(b).

(c) For arbitrary $\theta, \theta_0 \in \Theta$ we suppose $a \triangleleft c_2^+ \triangleleft b$, $c_2^+ \trianglelefteq c$ and obtain just as in the above proof of (b) that

$$\begin{aligned} \|\psi_\tau^*(\theta) - \psi_\tau^*(\theta_0)\|_{\tau, c}^+ &\leq \frac{1}{1 - L(\theta)} \|\mathcal{S}_\tau^+(\psi_\tau^*(\theta_0); \theta) - \mathcal{S}_\tau^+(\psi_\tau^*(\theta_0); \theta_0)\|_{\tau, c}^+ \\ &\stackrel{(3.23)}{\leq} \frac{4}{1 - L(\theta)} \left(C_2^+ L_1 \lambda(c_2^+) \|\xi\| + L_2 \lambda(c) \|\psi_\tau^*(\theta_0)\|_{\tau, c}^+ \right) |\theta - \theta_0| \\ &\stackrel{(3.31)}{\leq} \frac{4}{1 - L(\theta)} \left(C_2^+ L_1 \lambda(c_2^+) \|\xi\| + \frac{K_1 L_2 \lambda(c)}{1 - L(\theta_0)} \|\eta - \xi\| \right) |\theta - \theta_0| \end{aligned}$$

for $\theta, \theta_0 \in \Theta$, which leads to assertion (c). \square

Proof of Corollary 3.2. Let $\tau \in \mathbb{T}$, $\xi, \eta \in \mathcal{X}$ and $\theta \in \Theta$ be given.

(a) Let $\psi_\tau^*(\eta, \xi; \theta)$ denote the unique fixed point of $\mathcal{S}_\tau^+(\cdot; \eta, \xi, \theta)$ from Lemma 3.3 and 3.4. From [Pöt06, (3.15)] we know that the function $s^+ : \mathbb{T} \times \mathcal{X} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ is given by

$$s^+(\tau, \xi, \eta; \theta) := P(\tau) [\xi + \psi_\tau^*(Q(\tau)\eta, \xi; \theta)(\tau)].$$

Then the claim is an easy consequence of Lemma 3.4(b) and (c), due to $\|P(\tau)\| \leq K_2$ for all $\tau \in \mathbb{T}$ and $\lambda(c) \leq \frac{K_1 + K_2}{\delta}$ for all $c \in \Gamma$.

(b) This can be shown analogously to (a).

(c) We adopt the notation from the proof of Proposition 3.1. Then the function Π is given by (3.11) and the components $p^*(\tau, x_1, x_2; \theta)$, $q^*(\tau, x_1, x_2; \theta)$ are fixed points of mappings Π_τ^- and Π_τ^+ , respectively. We note that Π_τ^- and Π_τ^+ are compositions of s^+ and r^- . Thus, if we apply assertions (a) and (b) to the estimates (3.12), (3.13), we obtain our claim. \square

Remark 3.2. As technical comment to the proof of Corollary 3.2 we like to point out that a direct estimate of $P(\tau)\psi_\tau^*(Q(\tau)\eta, \xi, \cdot)(\tau)$ (instead of using (3.34) or (3.35)) does not yield “better” estimates, i.e., estimates which are uniform in ξ and η .

We arrive at the main result of this section, which can be considered as a very general and quantitative version of the stable manifold theorem. Moreover, it provides, in a certain sense, precise perturbation results for the invariant fiber bundles of (2.5).

Theorem 3.5 (perturbed invariant fiber bundles). *Assume that Hypothesis 2.2 is fulfilled. Then for all $\theta \in \Theta$ the following statements are true:*

(a) *The pseudo-stable fiber bundle of (2.5), given by*

$$S(\theta) := \{(\tau, \xi) \in \mathbb{T} \times \mathcal{X} : \varphi(\cdot; \tau, \xi; \theta) \in \mathcal{X}_{\tau, c}^+ \text{ for all } c \in \Gamma\}$$

is a forward invariant fiber bundle of (2.5) possessing the representation

$$S(\theta) = \{(\tau, \xi + s(\tau, \xi; \theta)) \in \mathbb{T} \times \mathcal{X} : \tau \in \mathbb{T}, \xi \in \mathcal{R}(Q(\tau))\}$$

with a continuous mapping $s : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ satisfying

$$s(\tau, \xi; \theta) = s(\tau, Q(\tau)\xi; \theta) \in \mathcal{R}(P(\tau)) \quad \text{for } \tau \in \mathbb{T}, \xi \in \mathcal{X}.$$

Furthermore, for all $\tau \in \mathbb{T}, \xi \in \mathcal{X}$ it holds:

$$(a_1) \quad s(\tau, 0; \theta) \equiv 0,$$

(a₂) $s : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ satisfies the Lipschitz estimates

$$(3.36) \quad \text{Lip } s(\tau, \cdot; \theta) \leq \ell(\theta), \quad \text{Lip } s(\tau, \xi; \cdot) \leq \frac{\delta K_1 K_2 (K_1 + K_2) L_2}{[\delta - 2(K_1 + K_2) L_1]^2} \|\xi\|,$$

(a₃) for \mathbb{T} unbounded below and if the dynamic equation (2.5) is regressive on Θ , there exists a unique retraction $\pi^-(\tau, \cdot; \theta) : \mathcal{X} \rightarrow S(\theta)_\tau$ onto $S(\theta)_\tau$ with

$$\|\varphi(t; \tau, \xi; \theta) - \varphi(t; \tau, \pi^-(\tau, \xi; \theta); \theta)\| \leq \frac{K_2}{1 - L(\theta)} \frac{1 + (K_1 - 1)\ell(\theta)}{1 - \ell(\theta)} \|\xi\| e_c(t, \tau)$$

for all $t \in \mathbb{T}_\tau^-$. The map $\pi^- : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ is continuous, linearly bounded

$$(3.37) \quad \|\pi^-(\tau, \xi; \theta)\| \leq K_1 \frac{1 + \ell(\theta)}{1 - \ell(\theta)} \|\xi\|$$

and we denote $\pi^-(\cdot; \theta)$ as asymptotic (backward) phase of $S(\theta)$.

(b) *For \mathbb{T} unbounded below, the pseudo-unstable fiber bundle of (2.5), given by*

$$R(\theta) := \left\{ (\tau, \xi) \in \mathbb{T} \times \mathcal{X} : \begin{array}{l} \text{there exists a solution } \phi : \mathbb{T} \rightarrow \mathcal{X} \text{ of (2.5)} \\ \text{with } \phi(\tau) = \xi \text{ and } \phi \in \mathcal{X}_{\tau, c}^- \text{ for all } c \in \Gamma \end{array} \right\}$$

is an invariant fiber bundle of (2.5) possessing the representation

$$R(\theta) = \{(\tau, \eta + r(\tau, \eta; \theta)) \in \mathcal{X} : \tau \in \mathbb{T}, \eta \in \mathcal{R}(P(\tau))\}$$

with a continuous mapping $r : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ satisfying

$$(3.38) \quad r(\tau, \xi; \theta) = r(\tau, P(\tau)\xi; \theta) \in \mathcal{R}(Q(\tau)) \quad \text{for } \tau \in \mathbb{T}, \xi \in \mathcal{X}.$$

Furthermore, for all $\tau \in \mathbb{T}, \xi \in \mathcal{X}$ it holds:

$$(b_1) \quad r(\tau, 0; \theta) \equiv 0,$$

(b₂) $r : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ satisfies the Lipschitz estimates

$$(3.39) \quad \text{Lip } r(\tau, \cdot; \theta) \leq \ell(\theta), \quad \text{Lip } r(\tau, \xi; \cdot) \leq \frac{\delta K_1 K_2 (K_1 + K_2) L_2}{[\delta - 2(K_1 + K_2) L_1]^2} \|\xi\|,$$

(b₃) there exists a unique retraction $\pi^+(\tau, \cdot; \theta) : \mathcal{X} \rightarrow R(\theta)_\tau$ onto $R(\theta)_\tau$ with

$$\|\varphi(t; \tau, \xi; \theta) - \varphi(t; \tau, \pi^+(\tau, \xi; \theta); \theta)\| \leq \frac{K_1}{1 - L(\theta)} \frac{1 + (K_2 - 1)\ell(\theta)}{1 - \ell(\theta)} \|\xi\| e_c(t, \tau)$$

for all $t \in \mathbb{T}_\tau^+$. The map $\pi^+ : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ is continuous, linearly bounded

$$(3.40) \quad \|\pi^+(\tau, \xi; \theta)\| \leq K_2 \frac{1 + \ell(\theta)}{1 - \ell(\theta)} \|\xi\|$$

and we denote $\pi^+(\cdot; \theta)$ as asymptotic (forward) phase of $R(\theta)$.

(c) For \mathbb{T} unbounded below, one has $S(\theta) \cap R(\theta) = \mathbb{T} \times \{0\}$ and

$$S(\theta)_\tau \cap R^-(\xi, \theta)_\tau = \{\pi^-(\tau, \xi; \theta)\}, \quad R(\theta)_\tau \cap S^+(\xi, \theta)_\tau = \{\pi^+(\tau, \xi; \theta)\}$$

for all $\tau \in \mathbb{T}$, $\xi \in \mathcal{X}$.

Remark 3.3. For all $\theta \in \Theta$ the following holds:

- (1) If (2.5) is regressive on Θ , then $S(\theta)$ is an invariant fiber bundle.
- (2) From Proposition 3.1 and Theorem 3.5(c) we obviously have

$$\begin{aligned} S(\theta) &= S^+(0, \theta), & R(\theta) &= R^-(0, \theta), \\ \pi^+(\tau, \xi; \theta) &= \Pi(\tau, \xi, 0; \theta), & \pi^-(\tau, \xi; \theta) &= \Pi(\tau, 0, \xi; \theta); \end{aligned}$$

however, the right relations holds only if (2.5) is regressive.

(3) The pseudo-stable fibers $S^+(x_0, \theta)_\tau$ are the leaves of a (forward) invariant foliation over each fiber $R(\theta)_\tau$, i.e., for every $\tau \in \mathbb{T}$ we have

$$\mathcal{X} = \bigcup_{x_0 \in R(\theta)_\tau} S^+(x_0, \theta), \quad S^+(x_1, \theta) \cap S^+(x_2, \theta) = \emptyset \quad \text{for } x_1, x_2 \in R(\theta)_\tau, x_1 \neq x_2.$$

Similarly, the fibers $R^-(x_0, \theta)$ form a foliation over $S(\theta)_\tau$.

Proof. The properties of s and r have been shown in [Pöt03, Theorem 3.3], while the asymptotic (forward and backward) phases π^- and π^+ , resp., were constructed in [Pöt06, Theorem 3.3]. \square

It remains to obtain some additional information on the mappings π^- and π^+ .

Corollary 3.6 (perturbed asymptotic phase). *Let $\tau \in \mathbb{T}$. Assume that Hypotheses 2.2–2.3(i) are fulfilled and \mathbb{T} is unbounded below. Then for all $\theta_0 \in \Theta$ the following holds:*

(a) Under Hypothesis 2.3(ii), for every bounded $B \subseteq \mathcal{X}$ one has

$$(3.41) \quad \lim_{\theta \rightarrow \theta_0} \pi^-(\tau, \xi; \theta) = \pi^-(\tau, \xi; \theta_0) \quad \text{uniformly in } \xi \in B,$$

and in case $a \triangleleft c_2^+ \triangleleft b - \delta$ there exists a $C^- \geq 0$ such that

$$(3.42) \quad \text{Lip } \pi^-(\tau, \xi, \cdot) \leq C^-(L_1 + L_2) \|\xi\| \quad \text{for } \tau \in \mathbb{T}, \xi \in \mathcal{X}.$$

(b) For every bounded $B \subseteq \mathcal{X}$ one has

$$(3.43) \quad \lim_{\theta \rightarrow \theta_0} \pi^+(\tau, \xi; \theta) = \pi^+(\tau, \xi; \theta_0) \quad \text{uniformly in } \xi \in B,$$

and in case $a + \delta \triangleleft c_2^- \triangleleft b$ there exists a $C^+ \geq 0$ such that

$$(3.44) \quad \text{Lip } \pi^+(\tau, \xi, \cdot) \leq C^+(L_1 + L_2) \|\xi\| \quad \text{for } \tau \in \mathbb{T}, \xi \in \mathcal{X}.$$

Proof. Let $\tau \in \mathbb{T}$, $\xi \in \mathcal{X}$ and $\theta_0, \theta \in \Theta$ be given.

(a) Since $\pi^-(\tau, \xi; \theta) \in \mathcal{X}$ is the unique point in $S(\theta)_\tau \cap R^-(\xi, \theta)$ we can simply apply Corollary 3.2(c) with $\pi^-(\tau, \xi; \theta) := \Pi(\tau, \xi, 0; \theta)$. On the other hand, the relations (3.41) and (3.42) can be shown analogously to our proceeding in the following step (b).

(b) We do not need to assume that (2.5) is regressive on Θ . Hence, it is not legitimate to apply Corollary 3.2(c) with $x_2 = 0$. We only mimic the corresponding argument. First of all, from Theorem 3.5(c) we know that $\pi^+(\tau, \xi; \theta) \in \mathcal{X}$ is the unique element in $R(\theta)_\tau \cap S^+(\xi, \theta)$ for all $\theta \in \Theta$, i.e., we have the representation $\pi^+(\tau, \xi; \theta) = p(\tau, \xi; \theta) + q(\tau, \xi; \theta)$, where $p(\tau, \xi; \theta) \in \mathcal{R}(P(\tau))$ and $q(\tau, \xi; \theta) \in \mathcal{R}(Q(\tau))$ are the unique solutions of the equations

$$p = s^+(\tau, q, \xi; \theta), \quad q = r(\tau, p; \theta)$$

with the mapping $s^+ : \mathbb{T} \times \mathcal{X} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ from Proposition 3.1(a) defining the pseudo-stable fibers $S^+(\xi, \theta)$. Additionally, from [Pöt06, (3.23)] we get the estimates

$$(3.45) \quad \|p(\tau, \xi; \theta)\| \leq \frac{K_2}{1 - \ell(\theta)} \|\xi\|, \quad \|q(\tau, \xi; \theta)\| \leq \frac{K_2 \ell(\theta)}{1 - \ell(\theta)} \|\xi\|.$$

From now on we suppress the dependence on the fixed parameters $\tau \in \mathbb{T}$ and $\xi \in \mathcal{X}$. Thus, due to Proposition 3.1(a₁) we have

$$\begin{aligned} \|p(\theta) - p(\theta_0)\| &\leq \|s^+(q(\theta); \theta) - s^+(q(\theta_0); \theta)\| + \|s^+(q(\theta_0); \theta) - s^+(q(\theta_0); \theta_0)\| \\ &\leq K_1 \ell(\theta) \|q(\theta) - q(\theta_0)\| + \|s^+(q(\theta_0); \theta) - s^+(q(\theta_0); \theta_0)\| \end{aligned}$$

and the assertion (b₂) gives us

$$\begin{aligned} \|q(\theta) - q(\theta_0)\| &\leq \|r(p(\theta); \theta) - r(p(\theta_0); \theta)\| + \|r(p(\theta_0); \theta) - r(p(\theta_0); \theta_0)\| \\ &\stackrel{(3.39)}{\leq} \ell(\theta) \|p(\theta) - p(\theta_0)\| + \|r(p(\theta_0); \theta) - r(p(\theta_0); \theta_0)\|. \end{aligned}$$

Inserting these equations into each other, in combination with Theorem 3.5(b₂), leads to

$$(3.46) \quad \begin{aligned} \|p(\theta) - p(\theta_0)\| &\leq \frac{K_1 \ell(\theta)}{1 - K_1 \ell(\theta)^2} \|r(p(\theta_0); \theta) - r(p(\theta_0); \theta_0)\| \\ &\quad + \frac{1}{1 - K_1 \ell(\theta)^2} \|s^+(q(\theta_0); \theta) - s^+(q(\theta_0); \theta_0)\| \\ &\stackrel{(3.39)}{\leq} \frac{K_1 \ell(\theta)}{1 - K_1 \ell(\theta)^2} \frac{\delta K_1 K_2 (K_1 + K_2) L_2}{[\delta - 2(K_1 + K_2) L_1]^2} \|\xi\| |\theta - \theta_0| \\ &\quad + \frac{1}{1 - K_1 \ell(\theta)^2} \|s^+(q(\theta_0); \theta) - s^+(q(\theta_0); \theta_0)\| \quad \text{for } \theta \in \Theta \end{aligned}$$

and

$$(3.47) \quad \begin{aligned} \|q(\theta) - q(\theta_0)\| &\leq \frac{\ell(\theta)}{1 - K_1 \ell(\theta)^2} \|s^+(q(\theta_0); \theta) - s^+(q(\theta_0); \theta_0)\| \\ &\quad + \frac{1}{1 - K_1 \ell(\theta)^2} \|r(p(\theta_0); \theta) - r(p(\theta_0); \theta_0)\| \\ &\stackrel{(3.39)}{\leq} \frac{\ell(\theta)}{1 - K_1 \ell(\theta)^2} \|s^+(q(\theta_0); \theta) - s^+(q(\theta_0); \theta_0)\| \\ &\quad + \frac{1}{1 - K_1 \ell(\theta)^2} \frac{\delta K_1 K_2 (K_1 + K_2) L_2}{[\delta - 2(K_1 + K_2) L_1]^2} \|\xi\| |\theta - \theta_0| \quad \text{for } \theta \in \Theta. \end{aligned}$$

Thus, passing over to the limit $\theta \rightarrow \theta_0$ yields (3.43) by relation (3.14) from Corollary 3.2(a), where (3.45) guarantees that the convergence is uniform in $\xi \in B$ for bounded sets $B \subseteq \mathcal{X}$.

Moreover, having the Lipschitz estimate (3.15) from Corollary 3.2(a) available, also the assertion (3.44) immediately follows from the above inequalities (3.46) and (3.47), as well as the bound (3.45). Thus, Theorem 3.5 is established. \square

4. TOPOLOGICAL DECOUPLING

Throughout this section we suppose \mathbb{T} is unbounded above and below. Moreover, we assume the semilinear dynamic equation (2.5) is regressive on Θ . Without regressivity, the decoupling result of this section does not hold. Yet, one can transform nonregressive equations into a tridiagonal form – for difference equations this has been achieved in [AG94].

We begin with the notion of a transformation, suitable for parameter-dependent nonautonomous equations. For that purpose, we consider two dynamic equations

$$(4.1) \quad x^\Delta = A(t)x + H_1(t, x; \theta)$$

and

$$(4.2) \quad x^\Delta = A(t)x + H_2(t, x; \theta),$$

where the rd-continuous mappings $A : \mathbb{T} \rightarrow \mathcal{L}(\mathcal{X})$ and $H_1, H_2 : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ might ensure existence of unique forward solutions (this is guaranteed, for instance, by [Pöt02, p. 38, Satz 1.2.17(a)]). In addition, they are assumed to satisfy

$$H_1(t, 0; \theta) \equiv 0, \quad H_2(t, 0; \theta) \equiv 0 \quad \text{on } \mathbb{T} \times \Theta.$$

We denote the general solution of (4.1) by φ_1 and the general solution of (4.2) by φ_2 .

Definition 4.1. A continuous mapping $T : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ is said to be a *topological equivalence* between (4.1) and (4.2), if for every $\tau \in \mathbb{T}$, $\theta \in \Theta$ the mapping $T_{\tau, \theta} : \mathcal{X} \rightarrow \mathcal{X}$, $T_{\tau, \theta}(x) := T(\tau, x; \theta)$ is a homeomorphism, $\tilde{T} : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$, $\tilde{T}(\tau, x; \theta) := T_{\tau, \theta}^{-1}(x)$ is continuous, one has

$$(4.3) \quad \lim_{x \rightarrow 0} T(\tau, x; \theta) = \lim_{x \rightarrow 0} \tilde{T}(\tau, x; \theta) = 0 \quad \text{uniformly in } \tau \in \mathbb{T}, \theta \in \Theta$$

and the following properties hold:

- (i) For every solution ϕ_1 of (4.1) the function $\phi_2(t) := T(t, \phi_1(t); \theta)$ solves (4.2).
- (ii) For every solution ϕ_2 of (4.2) the function $\phi_1(t) := \tilde{T}(t, \phi_2(t); \theta)$ solves (4.1).

If such a mapping T exists, then (4.1) and (4.2) are called *topologically conjugated*.

Remark 4.1. Suppose the dynamic equations (4.1) and (4.2) are topologically conjugated. Then the trivial solution of (4.1) is stable (attractive, asymptotically stable, unstable), if and only if the trivial solution of (4.2) possesses the corresponding property.

Let us continue with a geometrical interpretation of the dynamics generated by the semilinear dynamic equation (2.5). From Theorem 3.5 we know that the pseudo-stable fiber bundle $S(\theta)$ possesses an asymptotic backward phase $\pi^-(\cdot; \theta)$ and, dually, the pseudo-unstable fiber bundle $R(\theta)$ admits an asymptotic forward phase $\pi^+(\cdot; \theta)$. These asymptotic phases assign to any given solution $\phi : \mathbb{T} \rightarrow \mathcal{X}$ of (2.5) two further solutions:

- A solution ϕ_- in $S(\theta)$, which is given by $\phi_-(t) := \varphi(t; \tau, \pi^-(\tau, \phi(\tau); \theta); \theta)$ and can be identified with its projection $\phi_-^-, \phi_-^-(t) := Q(t)\phi_-(t)$; it solves the dynamic equation

$$(4.4) \quad q^\Delta = A(t)q + Q(t)H(t, q + s(t, q; \theta); \theta).$$

- A solution ϕ_+ in $R(\theta)$ given by $\phi_+(t) := \varphi(t; \tau, \pi^+(\tau, \phi(\tau); \theta); \theta)$ and being identified with its projection $\phi_+^+, \phi_+^+(t) := P(t)\phi_+(t)$. It solves the dynamic equation

$$(4.5) \quad p^\Delta = A(t)p + P(t)H(t, p + r(t, p; \theta); \theta).$$

Thus, the assignment $\phi \mapsto (\phi_-^-, \phi_+^+)$ leads to a decoupling of (2.5) into components in the invariant fiber bundles given by the ranges of the invariant projections P and Q , respectively. The following proposition puts the above explanations into a more precise framework:

Proposition 4.1 (decoupling). *Assume that Hypothesis 2.2 is fulfilled. Then there exists a topological equivalence $T : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ between the dynamic equations (2.5) and*

$$(4.6) \quad x^\Delta = A(t)x + Q(t)H(t, Q(t)x + s(t, x; \theta); \theta) + P(t)H(t, P(t)x + r(t, x; \theta); \theta)$$

with the following properties:

(a) *One has the linear bounds*

$$(4.7) \quad \begin{aligned} \|T(\tau, \xi; \theta)\| &\leq (K_1^2 + K_2^2) \frac{1 + \ell(\theta)}{1 - \ell(\theta)} \|\xi\|, \\ \|\tilde{T}(\tau, \xi; \theta)\| &\leq [(K_1 + \ell(\theta))^2 + (K_2 + \ell(\theta))^2] \frac{1 + 2\ell(\theta)}{1 - \ell(\theta)^2} \|\xi\| \quad \text{for } \tau \in \mathbb{T}, \theta \in \Theta, \end{aligned}$$

(b) *the fiber bundles $S(\theta)$ and $R(\theta)$ of (2.5) are mapped to invariant fiber bundles*

$$S^0 := \{(t, x) \in \mathbb{T} \times \mathcal{X} : x \in \mathcal{R}(Q(t))\}, \quad R^0 := \{(t, x) \in \mathbb{T} \times \mathcal{X} : x \in \mathcal{R}(P(t))\}$$

of (4.6), respectively, i.e., for the corresponding fibers we have

$$T(\tau, S(\theta)_\tau; \theta) = S_\tau^0, \quad T(\tau, R(\theta)_\tau; \theta) = R_\tau^0 \quad \text{for } \tau \in \mathbb{T}, \theta \in \Theta.$$

Proof. Let $\tau \in \mathbb{T}$, $\xi \in \mathcal{X}$ and $\theta \in \Theta$. In our following notation we largely suppress the dependence on the parameter θ ; this issue is tackled in the subsequent Corollary 4.2. Using the asymptotic phases π^-, π^+ from Theorem 3.5 and the mapping Π from Proposition 3.1(c), we define mappings $T, \tilde{T} : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$,

$$\begin{aligned} T(\tau, \xi; \theta) &:= Q(\tau)\pi^-(\tau, \xi; \theta) + P(\tau)\pi^+(\tau, \xi; \theta), \\ \tilde{T}(\tau, \xi; \theta) &:= \Pi(\tau, P(\tau)\xi + r(\tau, \xi; \theta), Q(\tau)\xi + s(\tau, \xi; \theta); \theta) \end{aligned}$$

and use the notation introduced in Definition 4.1. From Lemma 2.1 and Theorem 3.1 we know that T is continuous, and thanks to Proposition 3.1(c) the same holds true for \tilde{T} .

In order to show that T is a topological equivalence between (2.5) and (4.6), we remark that $\pi^+(\tau, \xi) \in R(\theta)_\tau$ and $\pi^-(\tau, \xi) \in S(\theta)_\tau$ evidently imply

$$P(\tau)\pi^+(\tau, \xi) + r(\tau, \pi^+(\tau, \xi)) = \pi^+(\tau, \xi), \quad Q(\tau)\pi^-(\tau, \xi) + s(\tau, \pi^-(\tau, \xi)) = \pi^-(\tau, \xi),$$

respectively. This yields $\tilde{T}(\tau, T(\tau, \xi)) \equiv \Pi(\tau, \pi^+(\tau, \xi), \pi^-(\tau, \xi)) \equiv \xi$ on $\mathbb{T} \times \mathcal{X}$, since we also have $\pi^+(\tau, \xi) \in S^+(\xi, \theta)_\tau$ and $\pi^-(\tau, \xi) \in R^-(\xi, \theta)_\tau$; similarly one shows the identity $T(\tau, \tilde{T}(\tau, \xi)) \equiv \xi$ on $\mathbb{T} \times \mathcal{X}$, and the mappings $T(\tau, \cdot), \tilde{T}(\tau, \cdot)$ are inverse to each other.

Next we show the uniform limit relations (4.3), which immediately follow from (4.7). These relations, in turn, can be derived as follows. In case of T , it is an easy consequence of (3.37) and (3.40) from Theorem 3.5; concerning the inverse \tilde{T} , this follows from Proposition 3.1(c) and the Lipschitz estimates for $s(\tau, \cdot; \theta), r(\tau, \cdot; \theta)$ stated in Theorem 3.5.

Thus, it remains to establish the properties (i)–(ii) of Definition 4.1. Thereto, let the function $\phi : \mathbb{T} \rightarrow \mathcal{X}$ be a solution of (2.5) (say, for a fixed parameter $\theta \in \Theta$). We define $\phi^-(t) := \varphi(t; \tau, \pi^-(\tau, \phi(\tau)))$ and obtain

$$\phi^-(\tau) = \pi^-(\tau, \xi) \in S(\theta)_\tau \cap R^-(\phi(\tau), \theta)_\tau.$$

Due to the invariance of $R^-(\phi(\tau), \theta)$ (cf. (3.2)) and $S(\theta)$ (cf. Remark 3.3(1)) one has

$$\phi^-(t) \in S(\theta)_t \cap R^-(\phi(\tau), \theta)_t \quad \text{for } t \in \mathbb{T}$$

and by Theorem 3.5(a) and Proposition 3.1(b), respectively,

$$P(t)\phi^-(t) = s(t, \phi^-(t)), \quad Q(t)\phi^-(t) = Q(t)\pi^-(t, \phi(t)) \quad \text{for } t \in \mathbb{T}.$$

Hence, $Q(\cdot)\pi^-(\cdot, \phi(\cdot))$ is a solution of (4.4) and analogously $P(\cdot)\pi^+(\cdot, \phi(\cdot))$ solves the dynamic equation (4.5). We have established that $T(\cdot, \phi(\cdot))$ is a solution of (4.6) and, whence, (i) holds. Conversely, let $\tilde{\phi} : \mathbb{T} \rightarrow \mathcal{X}$ be a solution of (4.6). We define a solution of (2.5) by

$$\psi(t) := \varphi(t; \tau, \Pi(\tau, P(\tau)\tilde{\phi}(\tau) + r(\tau, \tilde{\phi}(\tau)), P(\tau)\tilde{\phi}(\tau) + r(\tau, \tilde{\phi}(\tau))))$$

and obtain from (3.8), (3.4) that

$$\phi(t) \in S^+(P(t)\tilde{\phi}(t) + r(t, \tilde{\phi}(t)); \theta)_t \cap R^-(Q(t)\tilde{\phi}(t) + s(t, \tilde{\phi}(t)); \theta)_t \quad \text{for } t \in \mathbb{T},$$

which implies $\psi(t) = \tilde{T}(t, \tilde{\phi}(t))$ and $\tilde{T}(\cdot, \tilde{\phi}(\cdot))$ is a solution of (2.5); we have shown (ii).

Referring to the estimates (4.7) we know that quasiboundedness of solutions for (2.5) (or (4.6)) is preserved under the mapping T (or \tilde{T} , resp.). Therefore, due to their dynamical characterization, the invariant fiber bundles $S(\theta)$ and $R(\theta)$ of (2.5) are bijectively mapped onto the respective invariant fiber bundles S_τ^0 and R_τ^0 of (4.6). This was claim (b) and we have shown Proposition 4.1. \square

Corollary 4.2. *Let $\tau \in \mathbb{T}$. Assume that Hypotheses 2.2–2.3 are fulfilled. Then for all $\theta_0 \in \Theta$ and every bounded $B \subseteq \mathcal{X}$ one has*

$$(4.8) \quad \lim_{\theta \rightarrow \theta_0} T(\tau, \xi; \theta) = T(\tau, \xi; \theta_0), \quad \lim_{\theta \rightarrow \theta_0} \tilde{T}(\tau, \xi; \theta) = \tilde{T}(\tau, \xi; \theta_0) \quad \text{uniformly in } \xi \in B,$$

and in case $a \triangleleft c_2^+ \triangleleft b - \delta$, $a + \delta \triangleleft c_2^- \triangleleft b$ there exists $C \geq 0$ such that

$$(4.9) \quad \text{Lip } T(\tau, \xi, \cdot) \leq C(L_1 + L_2) \|\xi\| \quad \text{for } \tau \in \mathbb{T}, \xi \in \mathcal{X}.$$

Proof. Due to the definition of T, \tilde{T} , the claims follows from Corollary 3.6 and 3.2(c). \square

5. TOPOLOGICAL LINEARIZATION

Now we are finally in the position to demonstrate how fruitful our preparations have been. In order to harvest a generalized topological linearization theorem, it remains to prove only one further result. Again, this section is based on the assumption that \mathbb{T} is unbounded in both direction. Moreover, we suppose the dynamic equation (2.5) is regressive on Θ . This is always fulfilled for ordinary differential equations (see Figure 1 and (2.6)). On discrete time scales, though, regressivity is an essential ingredient of linearization theory. E.g., for difference equations, a corresponding counterexample can be found in [Wan91, p. 140].

So far we dealt with dynamic equations, where we obtained single invariant splitting of their extended phase space into two invariant subsets – the pseudo-stable and the pseudo-unstable fiber bundle. This was guaranteed by (pseudo-) hyperbolicity of their linear part. In the hyperbolic case, this setting is sufficient to obtain a nonautonomous variant of the classical Hartman-Grobman theorem (cf. [XCH07]). We, nevertheless, are interested in the critical non-hyperbolic situation, where the linear part admits an exponential trichotomy.

We denote projectors $P_1, P_2, P_3 : \mathbb{T} \rightarrow \mathcal{L}(\mathcal{X})$ for (2.2) as *complementary*, if

$$(5.1) \quad P_1(t) + P_2(t) + P_3(t) = I_{\mathcal{X}}, \quad P_i(t)P_j(t) = 0 \quad \text{for } i \neq j, t \in \mathbb{T}.$$

Hypothesis 5.1. *Let $K_1^+, K_2^+, K_3^+, K_1^-, K_2^-, K_3^- \geq 1$ be reals and $a_1, a_2, b_1, b_2 \in \mathcal{C}_{rd}^+(\mathbb{T}, \mathbb{R})$ be growth rates with $a_1 \triangleleft b_1 \triangleleft a_2 \triangleleft b_2$.*

(i) *Exponential trichotomy: The linear part (2.2) is regressive and there are complementary projectors $P_1, P_2, P_3 : \mathbb{T} \rightarrow \mathcal{L}(\mathcal{X})$ such that P_2, P_3 are invariant with*

$$(5.2) \quad \|\Phi_A(t, s)P_1(s)\| \leq K_1^+ e_{a_1}(t, s), \quad \|\Phi_A(s, t)P_2(t)\| \leq K_1^- e_{b_1}(s, t) \quad \text{for } s \preceq t,$$

$$(5.3) \quad \|\Phi_A(t, s)P_2(s)\| \leq K_2^+ e_{a_2}(t, s), \quad \|\Phi_A(s, t)P_3(t)\| \leq K_2^- e_{b_2}(s, t) \quad \text{for } s \preceq t.$$

(ii) *Lipschitz perturbation: For $i \in \{1, 2\}$ the identities $F_i(t, 0) \equiv 0$ on $\mathbb{T} \times \Theta$ hold and the mappings F_i satisfy the Lipschitz estimates*

$$L_i := \sup_{t \in \mathbb{T}} \text{Lip } F_i(t, \cdot) < \infty.$$

Moreover, we set $K_1(j) := \sum_{k=1}^j K_k^+$, $K_2(j) := \sum_{k=j}^2 K_k^-$ for $j \in \{1, 2\}$, require

$$(5.4) \quad L_1 < \min_{i=1}^2 \frac{|b_i - a_i|}{4K_{\max}}, \quad K_{\max} := \max_{i=1}^2 (K_1(i) + K_2(i) + 3K_1(i)K_2(i)),$$

choose $\delta \in \left(2K_{\max}L_1, \min_{i=1}^2 \frac{|b_i - a_i|}{2}\right)$ and abbreviate $\Theta := \{\theta \in \mathbb{F} : L_2|\theta| \leq L_1\}$,

$$\Gamma_j := \{c \in \mathbb{C}_{r,d}^+ \mathcal{R}(\mathbb{T}, \mathbb{R}) : a_j + \delta \triangleleft c \triangleleft b_j - \delta\} \quad \text{for } j \in \{1, 2\}.$$

(ii) Bounded perturbation: *The mappings F_i satisfy*

$$(5.5) \quad M_{ij} := \sup_{(t,x) \in \mathbb{T} \times \mathcal{X}} \|P_j(t)F_i(t,x)\| < \infty \quad \text{for } i \in \{1, 2\}, j \in \{1, 3\}.$$

For $1 \leq i \leq j \leq 3$, we define the mappings $P_i^j : \mathbb{T} \rightarrow \mathcal{L}(\mathcal{X})$, $P_i^j(t) := P_i(t) + \dots + P_j(t)$ and using (5.1), also P_i^j is an invariant projector of (2.2); moreover, P_i^j is regular for $i \geq 2$. Defining the complementary subspaces

$$\mathcal{X}_i^j(\tau) := \mathcal{R}(P_i^j(\tau)) = \bigoplus_{k=i}^j \mathcal{R}(P_k(\tau)), \quad \bar{\mathcal{X}}_i^j(\tau) := \mathcal{N}(P_i^j(\tau)) = \bigcap_{k=i}^j \mathcal{N}(P_k(\tau)) \quad \text{for } \tau \in \mathbb{T},$$

we now formulate an important tool for our later linearization result, which is based on an admissibility property (cf. [Pöt02, p. 111, Satz 2.2.12]) of exponential dichotomies.

Proposition 5.2. *Let $M_1, M_2 \geq 0$ be reals. Assume both dynamic equations (4.1), (4.2) are regressive on Θ , that Hypothesis 5.1(i) is satisfied with*

$$a_1 \triangleleft 0 \triangleleft b_2$$

and that the functions H_i satisfy for all $t \in \mathbb{T}$, $x \in \bar{\mathcal{X}}_2^2(t)$, $\theta \in \Theta$ and $i \in \{1, 2\}$ that

$$(5.6) \quad \begin{aligned} \text{Lip}_2 H_1 &< \infty, & 2\nu(a_1, b_2) \text{Lip}_2 H_2 &< 1, \\ H_i(t, x; \theta) &\in \bar{\mathcal{X}}_2^2(t), & \|H_i(t, x; \theta)\| &\leq M_i. \end{aligned}$$

Then there exists a unique mapping $J : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ such that

$$\begin{aligned} J(\tau, \xi; \theta) &= J(\tau, [I_{\mathcal{X}} - P_2(\tau)]\xi; \theta) \in \bar{\mathcal{X}}_2^2(\tau) \quad \text{for } \tau \in \mathbb{T}, \xi \in \mathcal{X}, \theta \in \Theta, \\ \varphi_2(\cdot; \tau, J(\tau, \xi; \theta); \theta) - \varphi_1(\cdot; \tau, \xi) &\in \mathcal{X}_0^\pm \quad \text{for } \tau \in \mathbb{T}, \xi \in \bar{\mathcal{X}}_2^2(\tau), \theta \in \Theta. \end{aligned}$$

Moreover, the following holds for all $\theta \in \Theta$:

(a) $J : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ is continuous with

$$\lim_{\xi \rightarrow 0} J(\tau, \xi; \theta) = 0 \quad \text{uniformly in } \tau \in \mathbb{T}, \theta \in \Theta,$$

(b) J is “near identity” with

$$(5.7) \quad \|J(\tau, \xi; \theta) - \xi\| \leq \frac{2\nu(a_1, b_2)(M_1 + M_2)}{1 - 2\nu(a_1, b_2) \text{Lip}_2 H_2} \quad \text{for } \tau \in \mathbb{T}, \xi \in \bar{\mathcal{X}}_2^2(\tau),$$

(c) for every solution $\phi_1 : \mathbb{T} \rightarrow \mathcal{X}$ of (4.1) satisfying $\phi_1(\tau) \in \bar{\mathcal{X}}_2^2(\tau)$ for some $\tau \in \mathbb{T}$, the function $\phi_2(t) := J(t, \phi_1(t); \theta)$ solves (4.2),

where we have abbreviated $\nu(a, b) := \frac{K_1}{[-a]} + \frac{K_2}{[b]}$.

Proof. Referring to [Pöt02, p. 38, Satz 1.2.17] we know that the general solutions φ_1 and φ_2 of (4.1) and (4.2), resp., exist on $\mathbb{T}^2 \times \mathcal{X} \times \Theta$ as continuous functions. Now let $\tau \in \mathbb{T}$, $\xi \in \mathcal{X}$ and $\theta \in \Theta$. Central for our following considerations is the dynamic equation

$$(5.8) \quad x^\Delta = A(t)x + H_2(t, x + \varphi_1(t; \tau, \xi; \theta); \theta) - H_1(t, \varphi_1(t; \tau, \xi; \theta); \theta).$$

Its general solution φ_3 is continuous in $\tau \in \mathbb{T}$, $\xi \in \mathcal{X}$, $\theta \in \Theta$. Note that assumption (5.6) implies that the fiber bundle $\{(t, x) \in \mathbb{T} \times \mathcal{X} : x \in \bar{\mathcal{X}}_2^2(t)\}$ is invariant w.r.t. (4.1), (4.2) and (5.8).

Now we consider the operator $\mathcal{S}_\tau : \mathcal{X}_0^\pm \times \bar{\mathcal{X}}_2^2(\tau) \times \Theta \rightarrow \mathcal{X}_0^\pm$ given by

$$\mathcal{S}_\tau(\psi; \xi, \theta) := \int_{\mathbb{T}} G_{P_1}^{P_3}(\cdot, \sigma(s)) [H_2(s, \psi(s) + \varphi_1(s; \tau, \xi; \theta); \theta) - H_1(s, \varphi_1(s; \tau, \xi; \theta); \theta)] \Delta s;$$

quoting [Pöt02, p. 111, Satz 2.2.12] we know that \mathcal{S}_τ is well-defined and that $\mathcal{S}_\tau(\cdot; \xi, \theta)$ is a contraction uniformly in its parameters. Moreover, its unique fixed point is exactly the uniquely determined bounded solution $\phi^*(\tau, \xi; \theta) : \mathbb{T} \rightarrow \mathcal{X}$ of (5.8) with $\phi^*(\tau, \xi; \theta)(\tau) \in \bar{\mathcal{X}}_2^2(\tau)$, and additionally

$$(5.9) \quad \|\phi^*(\tau, \xi; \theta)(t)\| \leq \frac{2\nu(a_1, b_2)(M_1 + M_2)}{1 - 2\nu(a_1, b_2) \text{Lip}_2 H_2} \quad \text{for } \tau \in \mathbb{T}, \xi \in \bar{\mathcal{X}}_2^2(\tau), \theta \in \Theta.$$

Therefore, we define the mapping $J : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ by

$$J(\tau, \xi; \theta) := [P_1(\tau) + P_3(\tau)] \xi + \phi^*(\tau, [P_1(\tau) + P_3(\tau)] \xi; \theta)(\tau) \in \bar{\mathcal{X}}_2^2(\tau).$$

Evidently, $\varphi_2(\cdot; \tau, J(\tau, \xi; \theta); \theta)$ is a solution of (4.2), and due to construction the difference $\varphi_2(\cdot; \tau, J(\tau, \xi; \theta); \theta) - \varphi_1(\cdot; \tau, \xi; \theta)$ solves our initial equation (5.8). By uniqueness, this yields

$$\varphi_2(\cdot; \tau, J(\tau, \xi; \theta); \theta) - \varphi_1(\cdot; \tau, \xi; \theta) = \varphi_3(\cdot; \tau, \phi^*(\tau, \tau, \xi; \theta); \theta) \stackrel{(5.9)}{\in} \mathcal{X}_0^\pm \quad \text{for } \xi \in \bar{\mathcal{X}}_2^2(\tau).$$

(a) The continuity assertion on the function $\phi^* : \mathbb{T} \times \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ is not shown in [Pöt02, p. 111, Satz 2.2.12]. Nevertheless, it can be derived using very similar techniques as employed in Lemma 3.3 or 3.4 and we omit it here. In particular, the continuity of \mathcal{S}_τ can be shown as in [Pöt02, pp. 147–148, Lemma 3.2.10].

(b) Due to the definition of J , the claimed estimate in (b) immediately follows from (5.9).

(c) Let $\phi_1 : \mathbb{T} \rightarrow \mathcal{X}$ be a solution of the dynamic equation (4.1) with $\phi_1(\tau_0) \in \bar{\mathcal{X}}_2^2(\tau_0)$ for $\tau_0 \in \mathbb{T}$. Then $\phi_2 := \varphi_2(\cdot; \tau_0, J(\tau_0, \phi_1(\tau_0); \theta); \theta)$ solves the dynamic equation (4.2) and, by construction, the difference $\phi_2 - \phi_1$ is bounded. On the other hand, for an arbitrary $\tau \in \mathbb{T}$, $J(\tau, \phi_1(\tau); \theta)$ is the unique element of $\bar{\mathcal{X}}_2^2(\tau)$ such that

$$\varphi_2(\cdot; \tau, J(\tau, \phi_1(\tau); \theta); \theta) - \varphi_1(\cdot; \tau, \phi_1(\tau); \theta) \in \mathcal{X}_0^\pm.$$

Therefore, the identity $\varphi_1(\cdot; \tau, \phi_1(\tau); \theta) = \phi_1$ implies $\varphi_2(\cdot; \tau, J(\tau, \phi_1(\tau); \theta); \theta) = \phi_2$ and $\phi_2 = J(\cdot, \phi_1(\cdot); \theta)$, which in turn yields that $J(\cdot, \phi_1(\cdot); \theta)$ solves the dynamic equation (4.2). \square

Corollary 5.3. *Let $\tau \in \mathbb{T}$. Suppose the assumptions of Proposition 5.2 hold and that the general solutions φ_1, φ_2 of (4.1), (4.2), resp., satisfy Hypothesis 2.3. Then for all $\theta_0 \in \Theta$ and every $B \subseteq \mathcal{X}$ such that $[P_1(\tau) + P_3(\tau)] B$ is bounded one has*

$$(5.10) \quad \lim_{\theta \rightarrow \theta_0} J(\tau, \xi; \theta) = J(\tau, \xi; \theta_0) \quad \text{uniformly in } \xi \in B,$$

and in case $a_1 + \delta \triangleleft c_2^-, c_2^+ \triangleleft b_2 - \delta$ and $\text{Lip}_3 H_i < \infty$ for $i = 1, 2$ there exists a $C \geq 0$ such that

$$(5.11) \quad \text{Lip } J(\tau, \xi, \cdot) \leq C(L_1 + L_2) \|\xi\| \quad \text{for } \tau \in \mathbb{T}, \xi \in \mathcal{X}.$$

Proof. We use the notation from the proof of Proposition 5.2. The mapping J is constructed via the fixed point $\phi^* : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}_0^\pm$ of the operator $\mathcal{S}_\tau : \mathcal{X}_0^\pm \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}_0^\pm$. This operator has a very similar structure as the operator \mathcal{S}_τ^+ introduced in Lemma 3.3. Thus, ϕ^* can be handled analogously to the fixed point mapping of \mathcal{S}_τ^+ in Lemma 3.4. To avoid redundancy, we omit the details and leave them to the interested reader. \square

We now head for the main result in this paper, the generalized Hartman-Grobman theorem:

Theorem 5.4 (Palmer-Šošitašvili). *Assume Hypothesis 5.1 holds with*

$$b_1 \leq 0 \leq a_2$$

and that (2.5) is regressive on Θ . Then for all $\theta \in \Theta$ the following statements are true:

(a) *The center fiber bundle of (2.5), given by*

$$C(\theta) := \left\{ (\tau, x_0) \in \mathbb{T} \times \mathcal{X} : \begin{array}{l} \varphi(\cdot; \tau, x_0; \theta) \in \mathcal{X}_{\tau, c_2}^+ \text{ for all } c_2 \in \Gamma_2 \text{ and} \\ \text{there exists a solution } \phi : \mathbb{T} \rightarrow \mathcal{X} \text{ of (2.5)} \\ \text{with } \phi(\tau) = x_0 \text{ and } \phi \in \mathcal{X}_{\tau, c_1}^- \text{ for all } c_1 \in \Gamma_1 \end{array} \right\}$$

is a forward invariant fiber bundle of (2.5) possessing the representation

$$(5.12) \quad C(\theta) = \{(\tau, \eta + c(\tau, \eta; \theta)) \in \mathbb{T} \times \mathcal{X} : \tau \in \mathbb{T}, \eta \in \mathcal{X}_2^2(\tau)\}$$

with a uniquely determined continuous mapping $c : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ satisfying

$$(5.13) \quad c(\tau, x_0; \theta) = c(\tau, P_2^2(\tau)x_0; \theta) \in \tilde{\mathcal{X}}_2^2(\tau) \quad \text{for } \tau \in \mathbb{T}, x_0 \in \mathcal{X}$$

and $c(\tau, 0; \theta) \equiv 0$ on $\mathbb{T} \times \Theta$.

(b) *Under the additional assumptions*

$$(5.14) \quad 8(K_2^+ + K_2^-)(1 + K_2^+ + K_2^-)L_1 < \frac{\lfloor b_2 - a_1 \rfloor}{K_2^+ + K_2^- + K_2^+ K_2^- \max\{K_2^+, K_2^-\}},$$

$$(5.15) \quad 4\nu(a_1, b_2) [K_1^+(1 + K_1^+) + K_2^-(1 + K_2^-)]L_1 < 1,$$

there exists a topological equivalence $T : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ between (2.5) and the reduced dynamic equation

$$(5.16) \quad x^\Delta = A(t)x + P_2(t)H(t, P_2(t)x + c(t, x; \theta); \theta),$$

where $\nu(a, b)$ is defined in Proposition 5.2.

Remark 5.1. (1) As demonstrated in [Pöt03, Theorem 4.3], the regressivity of the dynamic equations (2.2) and (2.5) is not needed in the proof of Theorem 5.4(a). Moreover, this reference also discussed the dependence of the center fiber bundle $C(\theta)$ under variation of θ .

(2) An interesting special case is the situation where the measure chain (\mathbb{T}, \leq, μ) and the dynamic equation (2.5) are T -periodic in time (cf. [Pöt02, p. 3]). Then this T -periodicity in the first component carries over to the mapping c defining the center fiber bundle $C(\theta)$, as well as to the topological equivalence T . This can be seen as in the special cases of differential equations (see [Wan91, Sie99, AW00]) or difference equations (see [Wan91, AW06]). Nonetheless, for periodic dynamic equations on time scales we refer to [Pöt03] and [XCH07].

Proof. The proof brings most of our previous results together. We begin with some preparations. Thereto, let $1 \leq j \leq i \leq 3$, $(j, i) \neq (1, 3)$ be integers and choose $\theta \in \Theta$. From [Pöt03, Theorem 4.3], which is shown via a successive application of Theorem 3.5, we obtain that the sets

$$C_{i,j}(\theta) := \begin{cases} \left\{ (\tau, x_0) \in \mathbb{T} \times \mathcal{X} : \varphi(\cdot; \tau, x_0; \theta) \in \mathcal{X}_{\tau, c}^+ \text{ for all } c \in \Gamma_i \right\} & \text{for } j = 1 \\ \left\{ (\tau, x_0) \in \mathbb{T} \times \mathcal{X} : \begin{array}{l} \text{there exists a solution } \phi : \mathbb{T} \rightarrow \mathcal{X} \text{ of (2.5)} \\ \text{with } \phi(\tau) = x_0 \text{ and } \phi \in \mathcal{X}_{\tau, c}^- \text{ for all } c \in \Gamma_{j-1} \end{array} \right\} & \text{for } i = N \\ C_{i,1}(\theta) \cap C_{N,j}(\theta) & \text{else} \end{cases}$$

are forward invariant fiber bundles of (2.5) admitting the extended hierarchy

$$(5.17) \quad \begin{array}{ccccc} C_{1,1}(\theta) & \subset & C_{2,1}(\theta) & \subset & \mathbb{T} \times \Theta \\ & & \cup & & \cup \\ & & C_{2,2}(\theta) & \subset & C_{3,2}(\theta) \\ & & & & \cup \\ & & & & C_{3,1}(\theta). \end{array}$$

Each $C_{i,j}(\theta) \subseteq \mathbb{T} \times \mathcal{X}$ possesses the representation

$$(5.18) \quad C_{i,j}(\theta) = \{(\tau, \eta + c_{i,j}(\tau, \eta; \theta)) \in \mathbb{T} \times \mathcal{X} : \tau \in \mathbb{T}, \eta \in \mathcal{X}_j^i(\tau)\}$$

with a uniquely determined continuous mapping $c_{i,j} : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ satisfying

$$(5.19) \quad c_{i,j}(\tau, x_0; \theta) = c_{i,j}(\tau, P_j^i(\tau)x_0; \theta) \in \bar{\mathcal{X}}_j^i(\tau) \quad \text{for } \tau \in \mathbb{T}, x_0 \in \mathcal{X}.$$

Furthermore, $c_{i,j}(\tau, 0; \theta) \equiv 0$ on $\mathbb{T} \times \Theta$ and $c_{i,j} : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ satisfies the estimate

$$(5.20) \quad \text{Lip } c_{i,j}(\tau, \cdot; \theta) \leq \begin{cases} \frac{K_1(i)K_2(i)(L_1+|\theta|L_2)}{\delta-(K_1(i)+K_2(i))(L_1+|\theta|L_2)} & \text{for } j = 1 \\ \frac{K_1(j-1)K_2(j-1)(L_1+|\theta|L_2)}{\delta-(K_1(j-1)+K_2(j-1))(L_1+|\theta|L_2)} & \text{for } i = 3 \\ \max_{k \in \{i,j-1\}} \frac{2K_1(k)K_2(k)(L_1+|\theta|L_2)}{\delta-(K_1(k)+K_2(k)+K_1(k)K_2(k))(L_1+|\theta|L_2)} & \text{else} \end{cases}$$

for $\tau \in \mathbb{T}, \theta \in \Theta$. Our assumptions (5.4) yield $\text{Lip } c_{ij}(\tau, \cdot; \theta) \leq 1$. More detailed, $C_{1,1}(\theta)$ is the pseudo-stable, and $C_{3,2}(\theta)$ the pseudo-unstable fiber bundle of (2.5) obtained from Theorem 3.5 by choosing the invariant projector $Q := P_1$. Similarly, we get $C_{2,1}(\theta)$ as pseudo-stable and $C_{3,1}(\theta)$ as pseudo-unstable fiber bundle of (2.5) using Theorem 3.5 with $Q := P_1^2$.

For notational convenience, we abbreviate $C_i(\theta) := C_{i,i}(\theta)$ and $c_i := c_{i,i}$ for $1 \leq i \leq 3$.

(a) From the above we obtain the assertion by defining $C(\theta) := C_{2,2}(\theta)$ and $c := c_{2,2}$.

(b) The proof of part (b) is subdivided into two steps. We suppress the dependence on θ .

(I) Claim: *There exists a topological equivalence $U : \mathbb{T} \times \mathcal{X} \rightarrow \mathcal{X}$ between (2.5) and*

$$(5.21) \quad x^\Delta = A(t)x + \sum_{i \in \{1,2,3\}} P_i(t)H(t, P_i(t)x + c_i(t, x)).$$

We can apply Proposition 4.1 with the invariant projector $Q = P_1$ to (2.5) and obtain topological equivalence to the decoupled dynamic equation

$$x^\Delta = A(t)x + P_1(t)H(t, P_1(t)x + c_1(t, x)) + P_2^3(t)H(t, P_2^3(t)x + c_{2,3}(t, x))$$

by virtue of a mapping $U_1 : \mathbb{T} \times \mathcal{X} \rightarrow \mathcal{X}$. Next, thanks to (5.14), we are able to apply Proposition 4.1 with $Q = P_2$ to the reduced dynamic equation

$$x^\Delta = A(t)P_2^3(t)x + P_2^3(t)H(t, P_2^3(t)x + c_{2,3}(t, x)),$$

which lives in $\{(t, x) \in \mathbb{T} \times \mathcal{X} : x \in \mathcal{X}_2^3(t)\}$, and obtain a topological equivalence U_2 to

$$x^\Delta = A(t)P_2^3(t)x + \sum_{i \in \{2,3\}} P_i(t)H(t, P_i(t)x + c_i(t, x)).$$

Therefore, the composition $U(t, x) := P_1(t)U_1(t, x) + U_2(t, P_2^3(t)U_1(t, x))$ provides a topological equivalence between the initial equation (2.5) and (5.21). The inverse of U is given by $\tilde{U}(t, x) := \tilde{U}_1(t, P_1(t)x + \tilde{U}_2(t, P_2^3(t)x))$, where the tilde indicates inverse mappings.

(II) Claim: *There exists a topological equivalence $V : \mathbb{T} \times \mathcal{X} \rightarrow \mathcal{X}$ between (5.21) and the reduced dynamic equation (5.16).*

The basic tool in this step is Proposition 5.2, which will be successively applied to the following dynamic equations

$$(5.22) \quad x^\Delta = A(t) \sum_{i \in \{1,3\}} P_i(t)x + \sum_{i \in \{1,3\}} P_i(t)H(t, P_i(t)x + c_i(t, x))$$

and its linearization

$$(5.23) \quad x^\Delta = A(t) \sum_{i \in \{1,3\}} P_i(t)x,$$

both living in $\{(t, x) \in \mathbb{T} \times \mathcal{X} : x \in \bar{\mathcal{X}}_2^2(t)\}$, with miscellaneous nonlinearities:

- For $H_1(t, x) := \sum_{i \in \{1,3\}} P_i(t)H(t, P_i(t)x + c_i(t, x))$ and $H_2(t, x) := 0$ we obtain a unique continuous mapping $W : \mathbb{T} \times \mathcal{X} \rightarrow \mathcal{X}$ satisfying

$$\Phi_A(\cdot, \tau)W(\tau, \xi) - \varphi(\cdot; \tau, \xi) \in \mathcal{X}_0^\pm \quad \text{for } \tau \in \mathbb{T}, \xi \in \bar{\mathcal{X}}_2^2(\tau).$$

- For $H_1(t, x) := 0$ and $H_2(t, x) := \sum_{i \in \{1,3\}} P_i(t)H(t, P_i(t)x + c_i(t, x))$ we obtain a unique continuous $\tilde{W} : \mathbb{T} \times \mathcal{X} \rightarrow \mathcal{X}$ satisfying

$$\varphi(\cdot; \tau, \tilde{W}(\tau, W(\tau, \xi))) - \Phi_A(\cdot, \tau)W(\tau, \xi) \in \mathcal{X}_0^\pm \quad \text{for } \tau \in \mathbb{T}, \xi \in \bar{\mathcal{X}}_2^2(\tau).$$

- Finally, $H_1(t, x) := H_2(t, x) := \sum_{i \in \{1,3\}} P_i(t)H(t, P_i(t)x + c_i(t, x))$ leads to a unique continuous mapping $J : \mathbb{T} \times \mathcal{X} \rightarrow \mathcal{X}$ satisfying

$$\varphi(\cdot; \tau, J(\tau, \xi)) - \varphi(\cdot; \tau, \xi) \in \mathcal{X}_0^\pm \quad \text{for } \tau \in \mathbb{T}, \xi \in \bar{\mathcal{X}}_2^2(\tau),$$

where obviously $J(\tau, \xi) = \xi$.

Note that Proposition 5.2 is applicable due to (5.15). Since the bounded functions \mathcal{X}_0^\pm form a linear space, this implies the inclusion

$$\varphi(\cdot; \tau, \tilde{W}(\tau, W(\tau, \xi))) - \varphi(\cdot; \tau, \xi) \in \mathcal{X}_0^\pm \quad \text{for } \tau \in \mathbb{T}, \xi \in \bar{\mathcal{X}}_2^2(\tau)$$

and consequently, due to the uniqueness assertion in Proposition 5.2,

$$\tilde{W}(\tau, W(\tau, \xi)) = J(\tau, \xi) = \xi \quad \text{for } \tau \in \mathbb{T}, \xi \in \bar{\mathcal{X}}_2^2(\tau).$$

Analogously, one shows the identity $W(\tau, \tilde{W}(\tau, \xi)) = \xi$ and thus the mappings W, \tilde{W} are inverse to each other. The remaining properties to show that W is a topological equivalence between (5.22) and (5.23) directly follow from Proposition 5.2. Hence, the desired topological equivalence between (5.21) and (5.16) is given by $V(t, x) = P_2(t)x + W(t, [P_1(t) + P_3(t)]x)$.

Summarizing step (I) and (II), the composition $T(t, x) = V(t, U(t, x))$ is the claimed topological equivalence between (2.5) and (5.21). \square

Corollary 5.5. *Let $\tau \in \mathbb{T}$. Assume that Hypothesis 5.1 and 2.3 are fulfilled. Then for all $\theta_0 \in \Theta$ and every bounded $B \subseteq \mathcal{X}$ one has*

$$(5.24) \quad \lim_{\theta \rightarrow \theta_0} T(\tau, \xi; \theta) = T(\tau, \xi; \theta_0), \quad \lim_{\theta \rightarrow \theta_0} \tilde{T}(\tau, \xi; \theta) = \tilde{T}(\tau, \xi; \theta_0) \quad \text{uniformly in } \xi \in B.$$

Proof. We borrow notation from the proof of Theorem 5.4(b). First of all, the mappings U, \tilde{U} from step (I) satisfy the uniform limit relations (4.8) from Corollary 4.2. Additionally, we show that also the mappings W, \tilde{W} of step (II) satisfy the limit relation (5.10) from the first part of Corollary 5.3. Since this property is preserved under composition, we have established Corollary 5.5. \square

Corollary 5.6 (Hartman-Grobman). *Assume that Hypothesis 5.1 holds with*

$$P_2(t) \equiv 0 \quad \text{on } \mathbb{T}, \quad a_2 = b_1 = 0$$

and that (2.5) is regressive on Θ . Then there exists a topological equivalence $T : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$ between (2.5) and the linear dynamic equation (2.2), which is “near identity” in the following sense:

$$(5.25) \quad \|T(\tau, \xi; \theta) - \xi\| \leq 2\nu(a_1, b_2)M, \quad \left\| \tilde{T}(\tau, \xi; \theta) - \xi \right\| \leq \frac{2\nu(a_1, b_2)M}{1 - 2\nu(a_1, b_2)L}$$

for all $\tau \in \mathbb{T}, \xi \in \mathcal{X}$ and $\theta \in \Theta$, with $M := M_{11} + M_{13} + |\theta|(M_{21} + M_{23})$,

$$L := K_1^+ \left(K_1^+ + \frac{K_1(1)K_2(1)(L_1 + |\theta|L_2)}{\delta - (K_1(1) + K_2(1))(L_1 + |\theta|L_2)} \right) (L_1 + |\theta|L_2) \\ + K_2^- \left(K_2^- + \frac{K_1(2)K_2(2)(L_1 + |\theta|L_2)}{\delta - (K_1(2) + K_2(2))(L_1 + |\theta|L_2)} \right) (L_1 + |\theta|L_2),$$

where $\nu(a, b)$ is defined in Proposition 5.2.

Proof. We apply Theorem 5.4 with $P_2 = 0$ and rely on the notation of its proof. Our present assumptions guarantee that U_1 decouples (2.5) into the dynamic equation (5.21), which degenerates into (5.22). Accordingly, the topological equivalence between (2.2) and (2.5) is given by the mapping $T : \mathbb{T} \times \mathcal{X} \times \Theta \rightarrow \mathcal{X}$, $T(t, x; \theta) := W(t, U_1(t, x; \theta); \theta)$. Thus, it remains to establish (5.25).

Referring to Proposition 5.2 we know, by construction, that W satisfies the estimate (5.7). Then the particular choice of the functions H_1 and H_2 in step (II) together with (5.20) implies the left inequality in (5.25). In addition, we know from step (II) that the inverse of W is given by \tilde{W} , and using the same arguments, the right estimate of (5.25) follows from Proposition 5.2 and (5.20). \square

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