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A LIMIT SET TRICHOTOMY FOR ABSTRACT 2-PARAMETER SEMIFLOWS ON TIME SCALES

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Abstract. Under certain contractivity conditions, we study the asymptotic behavior of abstract 2-parameter semiflows on normal cones in Banach spaces, and show that there are only three possible scenarios for their limit behavior.

Dedicated to Prof. István Győri on the occasion of this 60th birthday.

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1. Introduction. In certain relevant situations, e.g. in biological applications from population dynamics, it frequently happens that a dynamical system preserves a (partial) order relation on its state space. Such systems are called *order-preserving* or *monotone* and Krasnosel'skii laid the basics for their qualitative theory in [Kra64, Kra68]. Meanwhile many others made important contributions for different types of monotone (semi-)dynamical systems and in this small note we simply refer to [PS04, Chu02] for further references.

The essential property of order-preserving dynamical systems is that they possess a surprisingly simple asymptotic behavior. In fact Krause et al. [KN93, KR92] proved a so-called *limit set trichotomy* (cf. also [Nes99] for nonautonomous difference equations or [Chu02] for random dynamical systems), describing the only three possible asymptotic scenarios of difference equations under a certain kind of concavity.

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In an earlier paper (cf. [PS04]), Siegmund and the author proved such a limit set trichotomy for a general model of nonexpansive dynamical processes, namely 2-*parameter semiflows* in normal cones on time scales. They include, for example, solution operators of dynamic equations on time scales (cf. [BP01]) and in particular of nonautonomous difference and differential equations. Here we impose different contractivity conditions on the 2-parameter semiflows and obtain a stronger limit set trichotomy in this situation, leading to the asymptotic equivalence of all bounded solutions. Despite the more general setting, our arguments follow closely those of [Nes99].

2. Semiflows, Cones, and the Part Metric. Let $\mathbb{T}$ be an arbitrary *time scale*, i.e., a canonically ordered closed subset of the real axis $\mathbb{R}$. Since we are interested in the asymptotic behavior of evolutionary processes on such sets $\mathbb{T}$, it is reasonable to assume that $\mathbb{T}$ is unbounded above in the whole paper. (X, d) stands for a metric space from now on.

We begin with a very elementary result.

LEMMA 1. *Let $x_0 \in X$, $T > 0$ and $f : \mathbb{T} \to X$. Then* $\lim_{t\to\infty} f(t) = x_0$ *holds, if and only if one has* $\lim_{n\to\infty} f(t_n) = x_0$ *for every sequence* $(t_n)_{n\in\mathbb{N}}$ *in $\mathbb{T}$ satisfying $T \leq t_{n+1} - t_n$ for all $n \in \mathbb{N}$.*

Proof. We leave the easy proof to the reader. □

Now we are in the position to define an abstract concept to describe nonautonomous evolutionary processes.

DEFINITION 1. *A mapping $\varphi : \{(t, \tau) \in \mathbb{T}^2 : \tau \leq t\} \times X \to X$ is denoted as a* 2-parameter semiflow *on X, if the mappings $\varphi(t, \tau, \cdot) = \varphi(t, \tau) : X \to X$, $\tau \leq t$, satisfy the following properties:*

(i) $\varphi(\tau, \tau)x = x$ for all $\tau \in \mathbb{T}$, $x \in X$,

(ii) $\varphi(t, s)\varphi(s, \tau) = \varphi(t, \tau)$ for all $\tau, s, t \in \mathbb{T}$, $\tau \leq s \leq t$,

(iii) $\varphi(\cdot, \cdot)x : \{(t, \tau) \in \mathbb{T}^2 : \tau \leq t\} \to X$ is continuous for all $x \in X$.

For explicit examples of 2-parameter semiflows we only mention strongly-continuous 1-parameter semiflows, as well as solution operators of nonautonomous difference equations ($\mathbb{T} = \mathbb{Z}$) or ordinary and functional differential equations ($\mathbb{T} = \mathbb{R}$) under certain canonical assumptions on their right-hand side (cf. [PS04, Example 2.3]).

To provide some concepts from the classical theory of (autonomous) dynamical systems, we denote a point $x_0 \in X$ as an *equilibrium* of φ, if $\varphi(t, \tau)x_0 = x_0$ holds for all $\tau \leq t$. Moreover, for $\tau \in \mathbb{T}$ and $x \in X$, the *orbit emanating from* (τ, x) is $\gamma_\tau^+(x) := \{\varphi(t, \tau)x \in X : \tau \leq t\}$ and the *ω-limit set of* (τ, x) is given by $\omega_\tau^+(x) := \bigcap_{\tau\leq t} \operatorname{cl}\{\varphi(s, \tau)x \in X : t \leq s\}$. Equivalently, $\omega_\tau^+(x)$ consists of all points $x^* \in X$ such that there exists a sequence $t_n \to \infty$ in $\mathbb{T}$ with $x^* = \lim_{n\to\infty} \varphi(t_n, \tau)x$.

We say a self-mapping $\Phi : X \to X$ is *nonexpansive* (on (X,d)), if $d(\Phi x, \Phi\bar{x}) \leq d(x,\bar{x})$ for all $x, \bar{x} \in X$. The set of nonexpansive self-mappings is closed under composition. If $P \neq \emptyset$ is a set, then a family of parameter-dependent self-mappings $\Phi(p) : X \to X$, $p \in P$, is called *uniformly contractive*, if there exists a continuous function $c : X \times X \to [0,\infty)$, such that the following two conditions are fulfilled (cf. [Nes99]):

(i) $c(x,\bar{x}) < d(x,\bar{x})$ for all $x, \bar{x} \in X$, $x \neq \bar{x}$,

(ii) $d(\Phi(p)x, \Phi(p)\bar{x}) \leq c(x,\bar{x})$ for all $p \in P$, $x, \bar{x} \in X$.

In particular, each $\Phi(p)$ is nonexpansive. Moreover, in case, the mappings $\Phi_1(p), \Phi_2(p) : X \to X$, $p \in P$, are uniformly contractive (with contractivity function c) and $\Psi : X \to X$ is nonexpansive, then the compositions $\Phi_1(p) \circ \Phi_2(p)$ and $\Psi \circ \Phi_1(p)$ are uniformly contractive (with the same contractivity function c).

Assume from now on that the metric space X is a cone V_+ in a real Banach space $(V, \|\cdot\|)$. Recall that a *cone* is a nonempty closed convex set $V_+ \subset V$ such that $\alpha V_+ \subset V_+$ for $\alpha \geq 0$ and $V_+ \cap (-V_+) = \{0\}$. Moreover, define $V_+^* := V_+ \setminus \{0\}$. Any cone induces a partial order relation on V via $u \leq v$, if $v - u \in V_+$, which is preserved under addition and scalar multiplication with nonnegative reals. A cone V_+ is called *normal*, if there exists an equivalent norm $\|\cdot\|'$ on V such that $\|u\|' \leq \|v\|'$, if $u \leq v$.

Although forthcoming results on the boundedness of orbits are stated in the norm topology on V_+, our contractivity condition for 2-parameter semiflows will be formulated in a different metric topology:

Definition 2. *If $\lambda(u,v) := \sup\{\alpha \in [0,\infty) : \alpha u \leq v\}$ for $u, v \in V_+$, then the mapping $p : V_+^* \times V_+^* \to [0,\infty)$, $p(u,v) := -\log\min\{\lambda(u,v), \lambda(v,u)\}$ for $u, v \in V_+^*$ defines a quasi-metric on V_+^*, called the* part metric.

Remark 1. *(1) One easily sees $p(u,v) = \inf\{\log\alpha : \alpha^{-1}u \leq v \leq \alpha u\}$ for all $u, v \in V_+^*$ and, therefore, the part metric defined in [PS04, Definition 2.4(ii)] coincides with the one from Definition 2.*

(2) If the cone V_+ is normal, then $\operatorname{int} V_+$ *is a complete metric space w.r.t. the part metric p (cf. [Tho63]).*

Lemma 2. *If $V_+ \subset V$ is a normal cone with monotone norm, then*

$$\|u - v\| \leq \left(2e^{p(u,v)} - e^{-p(u,v)} - 1\right)\min\{\|u\|, \|v\|\} \quad \text{for all } u, v \in V_+^*.$$

Proof. See [KN93, Lemma 2.3]. □

The subsequent result is an adaption from [Nes99, Lemma 4]. Thereto, let $P \neq \emptyset$ be a set again, and we denote $\Phi(p) : V_+ \to V_+$, $p \in P$, as *uniformly ascending* on $A \subset V_+$, if there exists a continuous mapping $\phi : [0,1] \to [0,1]$

with $\alpha < \phi(\alpha)$ for all $\alpha \in (0, 1)$ such that

$$\alpha v \le u \quad \Rightarrow \quad \phi(\alpha)\Phi(p)v \le \Phi(p)u \quad \text{for all } \alpha \in [0, 1]\,,\; p \in P,\; u, v \in A.$$

Evidently, each such operator $\Phi(p)$ is order-preserving and subhomogeneous on A; latter means that $\alpha\Phi(p)v \le \Phi(p)\alpha v$ holds for $\alpha \in (0, 1)$, $v \in A$ and $p \in P$. Moreover, if $\Psi : V_+ \to V_+$ is a mapping satisfying $\Psi(A) \subset A$ and

$$\alpha v \le u \quad \Rightarrow \quad \alpha\Psi v \le \Psi u \quad \text{for all } \alpha \in [0, 1]\,,\; u, v \in A,$$

then also the composition $\Phi(p) \circ \Psi : V_+ \to V_+$, $p \in P$, is uniformly ascending with ϕ. In particular, the composition $\Phi_1(p) \circ \Phi_2(p)$ of two uniformly ascending mappings $\Phi_1(p), \Phi_2(p) : V_+ \to V_+$, $p \in P$, is uniformly ascending on A, if $\Phi_2(p)A \subset A$.

LEMMA 3. *Let $V_+ \subset V$ be a normal cone with $\operatorname{int} V_+ \neq \emptyset$ and assume that the mapping $\Phi(p) : \operatorname{int} V_+ \to \operatorname{int} V_+$, $p \in P$, is uniformly ascending w.r.t. ϕ. Then $\Phi(p)$ is uniformly contractive on $\operatorname{int} V_+$ for the part metric, where the contractivity function c is given by*

$$c(u, v) := -\log \phi(\min\{\lambda(u, v), \lambda(v, u)\}) \quad \textit{for all } u, v \in \operatorname{int} V_+. \tag{1}$$

Proof. Let $u, v \in \operatorname{int} V_+$ be given arbitrarily. Since p is a metric on $\operatorname{int} V_+$, one has $p(u, v) \ge 0$ and the definition of p yields $\min\{\lambda(u, v), \lambda(v, u)\} \le 1$, where $\lambda(u, v)$ is given in Definition 2. Therefore, w.l.o.g. we can assume $\lambda(u, v) = \min\{\lambda(u, v), \lambda(v, u)\} \le 1$. Since $\lambda(u, v)u \le v$ and $\Phi(p)$ is uniformly ascending, it follows that $\phi(\lambda(u, v))\Phi(p)u \le \Phi(p)v$, and consequently $\lambda(\Phi(p)u, \Phi(p)v) \ge \phi(\lambda(u, v))$ for $p \in P$. One gets $\phi\left(\min\{\lambda(u, v), \lambda(v, u)\}\right) \le \lambda(\Phi(p)u, \Phi(p)v)$ and exchanging u and v in the proof of the above estimate, yields that $\Phi(p)$ satisfies property (ii) of a uniformly contractive mapping. On the other hand, due to the metric properties of p, one has $0 < \min\{\lambda(u, v), \lambda(v, u)\} < 1$ for all $u, v \in V_+^*$, $u \neq v$, and thus we obtain the inequality $\phi(\min\{\lambda(u, v), \lambda(v, u)\}) > \min\{\lambda(u, v), \lambda(v, u)\}$ for $u, v \in \operatorname{int} V_+$, $u \neq v$. Hence, c satisfies both conditions in the definition of uniform contractivity w.r.t. the part metric. As in [Nes99, Proof of Lemma 4], one sees that c is continuous under the part metric, and this implies the assertion. $\square$

3. Limit Set Trichotomies. The following theorem is a clear manifestation of the intuition that contractivity drastically simplifies the possible long-term behavior of a dynamical system — in fact, only three asymptotic scenarios are possible. In the autonomous discrete time case, these limit set trichotomy was discovered (and so named) by Krause and Ranft [KR92] and generalized in [KN93] to infinite-dimensional autonomous difference equations; in addition, [Nes99] considers such nonautonomous systems, while [PS04] prove a limit set trichotomy for general nonexpansive 2-parameter semiflows. Now the nonexpansiveness of $\varphi(t,\tau)$ is strengthened to the ssumption that $\varphi(t,\tau)$ is uniformly ascending.

THEOREM 1 (LIMIT SET TRICHOTOMY). *Let $V_+ \subset V$ be a normal cone, $\operatorname{int} V_+ \neq \emptyset$ and assume that φ is a 2-parameter semiflow on V_+ with the following properties:*

(i) There exists a real $T > 0$ such that for all $t, \tau \in \mathbb{T}$ with $T \leq t - \tau$, one has $\varphi(t,\tau)V_+^* \subset \operatorname{int} V_+$ and that $\varphi(t,\tau)$ is uniformly ascending on $\operatorname{int} V_+$,

(ii) for all $(\tau, v) \in \mathbb{T} \times V_+$ every bounded orbit $\gamma_\tau^+(v)$ is relatively compact in the norm topology.

Then for every $\tau \in \mathbb{T}$ the following trichotomy holds, i.e., precisely one of the following three cases applies:

(a) For all $v \in V_+^*$ the orbits $\gamma_\tau^+(v)$ are unbounded in norm,

(b) for all $v \in V_+$ the orbits $\gamma_\tau^+(v)$ are bounded in norm and for all $v \in V_+^*$ we have $\lim_{t\to\infty} \|\varphi(t,\tau)v\| = 0$,

(c) for all $v \in V_+$ the orbits $\gamma_\tau^+(v)$ are bounded in norm, for $v \in V_+^*$ they have a nontrivial accumulation point, and

$$\lim_{t\to\infty} \|\varphi(t,\tau)u - \varphi(t,\tau)v\| = 0 \quad \textit{for all } u, v \in V_+^*. \tag{2}$$

REMARK 2. *(1) The above limit relation (2) implies that all ω-limit sets $\omega_\tau^+(v)$, $v \in V_+^*$, are identical, and it excludes the existence of two different equilibria of φ in V_+^*. In fact, if φ possesses an equilibrium $v_0 \in V_+^*$, then (2) guarantees $\omega_\tau^+(v) = \{v_0\}$ for all $v \in V_+^*$.*

(2) Let $T_{\max} \geq T$ and suppose $\mathbb{T}$ is a time scale such that for all $t, \tau \in \mathbb{T}$, $T \leq t-\tau$, there exist finitely many points $t_0 := \tau < t_1 < \ldots < t_{N-1} < t_N := t$ in $\mathbb{T}$ satisfying $T \leq t_{n+1} - t_n \leq T_{\max}$ for all $n \in \{0, \ldots, N-1\}$. Then it is sufficient in hypothesis (i) to assume that $\varphi(t,\tau)$ is uniformly ascending on $\operatorname{int} V_+$ for all $t, \tau \in \mathbb{T}$ with $T \leq t - \tau \leq T_{\max}$. This can be seen as follows: For arbitrary $t, \tau \in \mathbb{T}$, $T \leq t-\tau$, choose $t_0, \ldots, t_N$ as above. Then, due to the 2-parameter semiflow property, one has that $\varphi(t,\tau) = \varphi(t_N, t_{N-1}) \ldots \varphi(t_1, t_0)$ is a composition of uniformly ascending operators $\varphi(t_n, t_{n-1})$, $n = 1, \ldots, N$, with $\varphi(t_n, t_{n-1}) \operatorname{int} V_+ \subset \operatorname{int} V_+$ and functions ϕ not depending on n. Hence, $\varphi(t,\tau)$, $T \leq t - \tau$, itself is uniformly ascending on $\operatorname{int} V_+$.

(3) A remark similar to (2) holds for the nonexpansiveness and uniform contractivity assumptions from [PS04, Theorem 3.1].

Proof. Let $\tau \in \mathbb{T}$ be fixed. Because of Lemma 3, we know that the mapping $\varphi(t,\tau)$, $T \leq t-\tau$, is nonexpansive, and all assumptions of [PS04, Theorem 3.1] are satisfied. To obtain (2), we show that in case (c) the relation

$$\lim_{t\to\infty} p(\varphi(t,\tau)u, \varphi(t,\tau)v) = 0 \quad \text{for all } u, v \in V_+^* \tag{3}$$

holds. By Lemma 2 this implies (2), since all orbits are bounded in norm.

To verify (3), let $u, v \in V_+^*$ and $(t_n)_{n\in\mathbb{N}_0}$ be a sequence in $\mathbb{T}$ with $t_0 = \tau$ and $\lim_{n\to\infty} t_n = \infty$, where w.l.o.g. we may assume $T \leq t_{n+1} - t_n$ for all $n \in \mathbb{N}_0$ (cf. Lemma 1). Now let neither (a) nor (b) hold. Then the orbits $\gamma_\tau^+(u), \gamma_\tau^+(v)$ are norm-bounded (cf. [PS04, Theorem 3.1]), and furthermore, by (i), one has $\varphi(t_n,\tau)u, \varphi(t_n,\tau)v \in \operatorname{int} V_+$ for $n \in \mathbb{N}$. With a view to assumption (ii), this implies that the set $F := \{(\varphi(t_n,\tau)u, \varphi(t_n,\tau)v) : n \in \mathbb{N}\} \subset (\operatorname{int} V_+)^2$ is relatively compact.

Due to Lemma 3, we know that $\varphi(t,\tau)$, $T \leq t-\tau$, is uniformly contractive on $\operatorname{int} V_+$ and it follows from the 2-parameter semiflow property that there exists a constant $\gamma \geq 0$ with

$$\begin{aligned} p\left(\varphi(t_n,\tau)u, \varphi(t_n,\tau)v\right) &\geq c(\varphi(t_n,\tau)u, \varphi(t_n,\tau)v) \\ &\geq p(\varphi(t_{n+1},\tau)u, \varphi(t_{n+1},\tau)v) \geq \gamma \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{4}$$

From now on, we assume that (3) does not hold, which yields $0 < \gamma \leq p(\xi_1,\xi_2) \leq \Gamma$ for $(\xi_1,\xi_2) \in F$, with some real $\Gamma > 0$; note here that (4) implies $\Gamma < \infty$. Setting $\psi(\xi_1,\xi_2) := e^{-p(\xi_1,\xi_2)}$, we therefore obtain $0 < e^{-\Gamma} \leq \psi(\xi_1,\xi_2) \leq e^{-\gamma} < 1$ for $(\xi_1,\xi_2) \in F$, and hence the continuity of ϕ and $\alpha < \phi(\alpha)$ for $\alpha \in (0,1)$ implies the existence of a $C > 0$ with $\frac{\phi(\psi(\xi_1,\xi_2))}{\psi(\xi_1,\xi_2)} \geq C > 1$ for $(\xi_1,\xi_2) \in F$. Thus, using $\psi(\xi_1,\xi_2) = \min\{\lambda(\xi_1,\xi_2), \lambda(\xi_2,\xi_1)\}$ we arrive at the estimate

$$\begin{aligned} \gamma &\leq p(\varphi(t_{n+1},\tau)u, \varphi(t_{n+1},\tau)v) \overset{(4)}{\leq} p(\varphi(t_n,\tau)u, \varphi(t_n,\tau)v) \\ &\overset{(1)}{=} -\log \phi(\psi(\varphi(t_n,\tau)u, \varphi(t_n,\tau)v)) \\ &\leq -\log\left(C\psi(\varphi(t_n,\tau)u, \varphi(t_n,\tau)v)\right) \\ &= p(\varphi(t_n,\tau)u, \varphi(t_n,\tau)v) - \log C \\ &\cdots \\ &\leq p(\varphi(t_1,\tau)u, \varphi(t_1,\tau)v) - n\log C \quad \text{for all } n \in \mathbb{N}, \end{aligned}$$

yielding a contradiction for $n \to \infty$, since $C > 1$. So we must have $\gamma = 0$ and in the light of Lemma 1, the limit relation (3) holds true. □

Now we switch to a finite-dimensional situation.

Theorem 2 (Limit Set Trichotomy). *Let $V_+ \subset [0,\infty)^d$ be a normal cone, $\operatorname{int} V_+ \neq \emptyset$ and assume that φ is a 2-parameter semiflow on V_+ with the following properties:*

(i) There exists a real $T > 0$ such that one has $\varphi(t,\tau)V_+^* \subset \operatorname{int} V_+$ for all $t,\tau \in \mathbb{T}$ with $T \leq t-\tau$,

(ii) $\varphi(t,\tau)|_{\operatorname{int} V_+}$, $T \leq t - \tau$, is continuously differentiable, and

$$\sum_{k=1}^{d} v_k \left| \frac{\partial \varphi_j(t,\tau,v)}{\partial v_k} \right| \leq a(v)\varphi_j(t,\tau,v) \quad \textit{for all } T \leq t-\tau,\ v \in \operatorname{int} V_+$$

and $j = 1,\dots,d$, where $a : V_+ \to [0,1)$ is a continuous mapping.

Then for every $\tau \in \mathbb{T}$ the following trichotomy holds, i.e., precisely one of the following three cases applies:

(a) For all $v \in V_+^*$ the orbits $\gamma_\tau^+(v)$ are unbounded in norm,

(b) for all $v \in V_+$ the orbits $\gamma_\tau^+(v)$ are bounded in norm and for all $v \in V_+^*$ we have $\lim_{t\to\infty} \|\varphi(t,\tau)v\| = 0$,

(c) for all $v \in V_+$ the orbits $\gamma_\tau^+(v)$ are bounded in norm, for $v \in V_+^*$ they have a nontrivial accumulation point, and

$$\lim_{t\to\infty} \|\varphi(t,\tau)u - \varphi(t,\tau)v\| = 0 \quad \textit{for all } u,v \in V_+^*. \tag{5}$$

Proof. Let $\tau \in \mathbb{T}$ be fixed. Now we define the mapping

$$c(u,v) := \sup_{\theta\in[0,1]} a(u^\theta v^{1-\theta})p(u,v) \tag{6}$$

for all $u,v \in V_+^*$ with $p(u,v) < \infty$, where $u^\theta v^{1-\theta} \in [0,\infty)^d$ abbreviates the vector with components $u_i^\theta v_i^{1-\theta} \in [0,\infty)$, $i = 1,\dots,d$. By assumption we have $c(u,v) < p(u,v)$ for $u,v \in V_+^*$, $u \neq v$, with $p(u,v) < \infty$, and [Nes99, Lemma 6] applied to $\varphi(t,\tau)$, $T \leq t-\tau$, gives us $p(\varphi(t,\tau)u, \varphi(t,\tau)v) \leq c(u,v)$ for all $t,\tau \in \mathbb{T}$, $T \leq t-\tau$, and $u,v \in \operatorname{int} V_+$. The definition of c readily implies its continuity w.r.t. the part metric and, therefore, $\varphi(t,\tau)$, $T \leq t-\tau$, is a uniform contraction on $\operatorname{int} V_+$. Since we are in a finite-dimensional setting, each bounded orbit of φ is relatively compact and the limit set trichotomy from [PS04, Theorem 3.1] applies. It remains to strengthen the assertion in case (c) of this trichotomy, by showing the limit relation (5).

Thereto, let $u,v \in V_+^*$ and assume that neither (a) nor (b) of the limit set trichotomy in [PS04, Theorem 3.1] holds. Then the orbits $\gamma_\tau^+(u), \gamma_\tau^+(v)$ are bounded in norm and one has $\varphi(t,\tau)u, \varphi(t,\tau)v \in \operatorname{int} V_+$ for $T \leq t-\tau$. Now choose an arbitrary sequence $(t_n)_{n\in\mathbb{N}_0}$ in $\mathbb{T}$ with $t_0 := \tau$, $\lim_{n\to\infty} t_n = \infty$, and w.l.o.g. we suppose $T \leq t_{n+1} - t_n$ for $n \in \mathbb{N}_0$ (cf. Lemma 1). In case, the sequence $(\varphi(t_n,\tau)u)_{n\in\mathbb{N}}$ has a trivial accumulation point, then there exists an infinite set $N \subset \mathbb{N}$ with $\lim_{n\to\infty, n\in N}\|\varphi(t_n,\tau)u\| = 0$; otherwise we set $N := \emptyset$. Using mathematical induction, one obtains from (4) the inequality $p(\varphi(t_n,\tau)u, \varphi(t_n,\tau)v) \le p(\varphi(t_1,\tau)u, \varphi(t_1,\tau)v)$ for all $n \in \mathbb{N}$. Hence we can find some real $\lambda > 0$ with $0 \le \varphi(t_n,\tau)v \le \lambda\varphi(t_n,\tau)u$ for all $n \in \mathbb{N}$ and consequently one has $\lim_{n\to\infty,n\in N}\|\varphi(t_n,\tau)v\| = 0$, which immediately implies $\lim_{n\to\infty,n\in N}\|\varphi(t_n,\tau)u - \varphi(t_n,\tau)v\| = 0$. It remains to prove

$$\lim_{n\to\infty, n\notin N}\|\varphi(t_n,\tau)u - \varphi(t_n,\tau)v\| = 0. \tag{7}$$

Due to the construction of N, the set $F := \{(\varphi(t_n,\tau)u, \varphi(t_n,\tau)v) : n \in N\} \subset (\operatorname{int} V_+)^2$ has compact closure in $\left(V_+^*\right)^2$. Consequently, there exists an $\alpha < 1$ with $\sup_{\theta\in[0,1]} a(u^\theta v^{1-\theta}) \le \alpha$ for $(u,v) \in F$, we obtain from the definition of c and the 2-parameter semiflow property

$$p(\varphi(t_{n+1},\tau)u, \varphi(t_{n+1},\tau)v) \le c(\varphi(t_n,\tau)u, \varphi(t_n,\tau)v) \le \alpha p(\varphi(t_n,\tau)u, \varphi(t_n,\tau)v)$$

and inductively

$$0 \le p(\varphi(t_n,\tau)u, \varphi(t_n,\tau)v) \le \alpha^{n-1} p(\varphi(t_1,\tau)u, \varphi(t_1,\tau)v) \xrightarrow[n\to\infty, n\notin N]{} 0.$$

Finally, Lemma 2 and the norm-compactness of $\operatorname{cl} F$ implies the limit relation (7), which concludes our present proof. $\square$

REFERENCES

[BP01] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales — An Introduction with Applications*, Birkhäuser, Boston, 2001.

[Chu02] I. Chueshov, *Monotone Random Systems. Theory and Applications*, Lecture Notes in Mathematics 1779, Springer-Verlag, Berlin, 2002.

[Kra64] M.A. Krasnosel'skii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[Kra68] ———, *The Operator of Translation Along Trajectories of Differential Equations*, Translations of Mathematical Monographs, Vol. 19, American Mathematical Society, Providence, Rhode Island, 1968.

[KN93] U. Krause and R.D. Nussbaum, *A limit set trichotomy for self-mappings of normal cones in Banach spaces*, Nonlinear Analysis (TMA) **20(7)** (1993), 855–870.

[KR92] U. Krause and P. Ranft, *A limit set trichotomy for monotone nonlinear dynamical systems*, Nonlinear Analysis (TMA) **16(4)** (1992), 375–392.

[Nes99] T. Nesemann, *A limit set trichotomy for positive nonautonomous discrete dynamical systems*, J. Math. Anal. Appl. **237** (1999), 55–73.

[PS04] C. Pötzsche and S. Siegmund, *A limit set trichotomy for order-preserving systems on time scales*, Electronic J. Diff. Equations **64** (2004) 1–18.

[Tho63] A.C. Thompson, *On certain contraction mappings on a partially ordered vector space*, Proc. of the Amer. Math. Soc. **14** (1963), 438–443.