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# Order-preserving nonautonomous discrete dynamics

## Attractors and entire solutions

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**Abstract** The concept of pullback convergence turned out to be a central idea to describe the long-term behavior of nonautonomous dynamical systems. This paper provides a general framework for the existence and structure of pullback attractors capturing the asymptotics of nonautonomous and order-preserving difference equations in Banach spaces. Furthermore we obtain criteria for the convergence to bounded entire solutions and additionally discuss various applications.

**Keywords** Nonautonomous dynamical system · Order-preserving dynamical system · Pullback convergence · Difference equation

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## 1 Introduction and basics

The feature that a dynamical system preserves an order relation (induced e.g. by a cone) on its state space has far-reaching consequences concerning its long-term behavior. To name several examples, under natural assumptions (for instance, relatively compact forward orbits), such problems possess the property of *generic quasi-convergence*, i.e. the fact that a typical orbit converges to an equilibrium or more general a periodic orbit (cf. [20, pp. 8ff] or [9] for references). The *order interval trichotomy* (see [9, Thm. 5.1]) guarantees the existence of fixed-points and heteroclinic connections, or finally limit sets (and attractors) are contained between extremal equilibria (cf. [9, Thm. 5.7]). These properties particularly hold for discrete-time dynamical systems and we refer to [8,9] for an extensive survey.

Given these fundamental results, the recent years brought an increasing interest in dynamical systems under random or aperiodic deterministic driving (cf. [3,18,11,2]) inducing *nonautonomous dynamical systems*. In such a setting, the evolutionary equations of interest have explicitly time-dependent right-hand sides. For this reason the simplifying effect of order-preservation on the asymptotics turns out to be drastically lessened. Actually it

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is already unrealistic to expect the existence of equilibria or periodic solutions. So first of all an ambient counterpart to fixed points is required, which ideally reflects the particular time-dependence. For instance, when describing nonautonomous dynamics via skew-product flows, [19] show that the generic convergence property fails in most almost periodic systems even in the class of almost automorphic motions. On the other hand, for random dynamical systems a generic convergence result can be found in [1].

This paper deals with discrete nonautonomous dynamical systems, i.e. difference equations (maps), whose time-dependent right-hand side is order-preserving in the state space. Although historically one of the prime reasons to study problems in discrete time was their importance as Poincaré maps of periodic differential equations, there are definitely various further applications of their own right. As typical examples, monotone maps do occur naturally in population dynamics by means of, e.g. the Leslie-Gower model [4] or integro-difference equations [12, 7], when also spatial effects matter. Precisely for these applications it is well-motivated to allow time-varying parameters in order to incorporate a realistic description of external influences. The required theoretical basics for a corresponding mathematical analysis of nonautonomous dynamical systems can be found in for example [11, 18] or [3] concerning the random situation. In this endeavor two aspects should be taken into account: Firstly, rather than forward behavior as classically investigated in e.g. [14, 16], it turned out that *pullback convergence* is a more natural concept since it yields invariant limit sets. Secondly, due to the absence of equilibria, bounded entire solutions often appear to be an appropriate substitute.

Our presentation is subdivided into four further sections. Above all, we introduce the required basics on nonautonomous difference equations in Banach spaces and their attractors. The subsequent concept of sub- and super-solutions turns out to be crucial to identify invariant sets and bounded entire solutions of order-preserving problems, while particularly Thm. 3 can be seen as a nonautonomous counterpart to the above mentioned results on generic quasi-convergence. The following Sect. 4 shows that the global attractor of an order-preserving dynamical system can be bounded between two extremal entire solutions, which in turn attract upper and lower solutions. To illuminate these results, a 2-species model from population dynamics, a delay-difference equation and a class of integro-difference equations is considered. We finally conclude our analysis in Sect. 6 indicating possible extensions to general nonautonomous dynamical systems and in particular underline obvious parallels to the theory of random dynamical systems (cf. [3]).

We begin with some standard terminology. Let $X$ be a real normed space with norm $\|\cdot\|$ in which $B_r(x)$ denotes the open $r$-ball with radius $r > 0$ centered around $x \in X$. A closed subset $X_+ \subseteq X$ is called a *cone*, if it fulfills $X_+ \cap (-X_+) = \{0\}$ and is compatible with the vector space structure on $X$ in the sense of

$$x, y \in X_+ \quad \Rightarrow \quad tx + sy \in X_+ \quad \text{for all } t, s \geq 0.$$

A *solid cone* $X_+$ moreover has nonempty interior, i.e. $X_+^\circ \neq \emptyset$. One says that $X$ is an *ordered space*, if there exists a cone $X_+ \subseteq X$ and a *strongly ordered space*, provided $X_+$ is solid. Every cone $X_+$ induces an order on $X$ via

$$x \preceq y \quad :\Leftrightarrow \quad y - x \in X_+ \quad \text{for all } x, y \in X$$

and on a strongly ordered space one can additionally define

$$x \ll y \quad :\Leftrightarrow \quad y - x \in X_+^\circ \quad \text{for all } x, y \in X.$$

| $X_+$ | solid | normal | regular | minihedral | strongly minihedral |
|---|---|---|---|---|---|
| $\mathbb{R}^d_+$ | + | + | + | + | + |
| $C_+(\Omega)$ | + | + | - | + | - |
| $C^1_+(\Omega)$ | + | - | - | - | - |
| $L^p_+(\Omega)$ | - | + | + | + | + |

**Table 1** The cones from Ex. 1 and their properties

An *order interval* is a set of the form $[x^-, x^+] := \{x \in X : x^- \preceq x \preceq x^+\}$ with $x^-, x^+ \in X$. A set $\Omega \subseteq X$ is called *order-bounded above* (resp. *below*), if there exists some $x^+ \in X$ (or $x^- \in X$) so that $x \preceq x^+$ (resp. $x^- \preceq x$) holds for every $x \in \Omega$. The smallest such $x^+$ (or the largest $x^-$) is denoted the *supremum* $\sup \Omega$ (resp. the *infimum* $\inf \Omega$) of $\Omega$ — if they exist.

Finally, one refers to a cone $X_+$ as

- *normal*, if the norm on $X$ is semi-monotone, i.e. that there exists a real $C \geq 0$ such that

$$0 \preceq x \preceq y \quad \Rightarrow \quad \|x\| \leq C\|y\| \quad \text{for all } x, y \in X \tag{1}$$

- *regular*, if a sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ converges (in norm), provided it is increasing (i.e. $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$) and order-bounded above
- *minihedral*, if every finite order-bounded subset of $X$ has a supremum, and *strongly minihedral*, provided all order-bounded subsets have a supremum

In [6, p. 220, Prop. 19.2] one finds a proof that every regular cone in a Banach space is normal. Note that cones in finite-dimensional spaces are always regular and concerning concrete examples in infinite-dimensional spaces, we refer to [6, p. 217ff]. Yet, for the reader's convenience frequently used cones are summarized in

*Example 1* Given $p \in [1, \infty)$ and a subset $\Omega \subseteq \mathbb{R}^d$ let us define the cones

$$\begin{aligned}
\mathbb{R}^d_+ &:= \{x \in \mathbb{R}^d : x_j \geq 0 \text{ for } 1 \leq j \leq d\}, \\
C_+(\Omega) &:= \{u \in C(\Omega) : u(x) \geq 0 \text{ for all } x \in \Omega\}, \\
C^1_+(\Omega) &:= \left\{u \in C^1(\Omega) : u(x) \geq 0 \text{ for all } x \in \Omega\right\}, \\
L^p_+(\Omega) &:= \{u \in L^p(\Omega) : u(x) \geq 0 \text{ a.e. in } \Omega\}.
\end{aligned}$$

It is understood that we use the natural norm on the spaces of nonnegative functions; the domain $\Omega$ is assumed to be compact in case of the $C$-spaces, resp. of finite measure in case of the $L^p$-spaces. Properties of these cones are listed in Tab. 1 and give rise to Banach lattices on the Euklidean space $\mathbb{R}^d$, resp. on $C(\Omega)$ or $L^p(\Omega)$.

For later use we quote

**Lemma 1** *([20, p. 3, Lemma 1.2]) Every increasing (or decreasing) sequence contained in a compact subset of an ordered space converges.*

**Lemma 2** *([3, p. 87, Thm. 3.1.2]) Every compact subset of a strongly ordered Banach space with a normal minihedral cone has supremum and infimum.*

## 2 Nonautonomous dynamics

Let $\mathbb{I}$ denote a discrete interval, i.e. the intersection of a real interval with the integers $\mathbb{Z}$, $\mathbb{I}' := \{k \in \mathbb{I} : k+1 \in \mathbb{I}\}$ and $[a,b]_{\mathbb{Z}} := [a,b] \cap \mathbb{Z}$ for reals $a \leq b$. A subset $\mathscr{A} \subseteq \mathbb{I} \times X$ is called *nonautonomous set* with the *k-fibers*

$$\mathscr{A}(k) := \{x \in X : (k,x) \in \mathscr{A}\} \quad \text{for all } k \in \mathbb{I}$$

and it is convenient to abbreviate $\mathscr{A}'(k) := \mathscr{A}(k+1)$ for all $k \in \mathbb{I}'$.

One speaks of a *bounded* nonautonomous set $\mathscr{A}$, when every fiber $\mathscr{A}(k)$ is bounded and of a *uniformly bounded* set, if there exists a $R > 0$ such that $\mathscr{A}(k) \subseteq B_R(0)$ holds for all $k \in \mathbb{I}$. One says $\mathscr{A}$ possesses some topological property (open, connected, etc.), provided every fiber has this particular property.

We consistently use greek letters to denote sequences $\xi = (\xi_k)_{k\in\mathbb{I}}$ (in e.g. $X$), and sometimes identify $\xi$ with the nonautonomous set $\{(k,\xi_k) : k \in \mathbb{I}\} \subseteq \mathbb{I} \times X$. It turns out useful to have *nonautonomous balls*

$$\mathscr{B}_\rho(\xi) := \{(k,x) \in \mathbb{I} \times X : \|x - \xi_k\| < \rho_k\}$$

at hand, where $\rho = (\rho_k)_{k\in\mathbb{I}}$ is a given sequence of positive reals. Furthermore, a *(nonautonomous) order interval* stands for a nonautonomous set of the form

$$\left[\xi^-,\xi^+\right] := \left\{(k,x) \in \mathbb{I} \times X : \xi_k^- \preceq x \preceq \xi_k^+\right\}$$

with sequences $\xi^- = (\xi_k^-)_{k\in\mathbb{I}}$, $\xi^+ = (\xi_k^+)_{k\in\mathbb{I}}$ in $X$, while half-sided order intervals are defined as

$$\left[\xi^-,\infty\right) := \left\{(k,x) \in \mathbb{I} \times X : \xi_k^- \preceq x\right\},$$
$$\left(\infty,\xi^+\right] := \left\{(k,x) \in \mathbb{I} \times X : x \preceq \xi_k^+\right\}.$$

At first we note that boundedness and order-boundedness of nonautonomous sets coincides in strongly ordered spaces:

**Lemma 3** *Let $X$ be strongly ordered.*

*(a) If a nonautonomous set $\mathscr{A}$ is bounded, then there exists a sequence $\xi$ in $X$ satisfying $0 \ll \xi_k$ for all $k \in \mathbb{I}$ such that*

$$\mathscr{A} \subseteq [-\xi,\xi]. \tag{2}$$

*For a uniformly bounded set $\mathscr{A}$ the sequence $\xi$ can be chosen as constant.*

*(b) If $X_+$ is normal and (2) holds with $0 \preceq \xi_k$ for all $k \in \mathbb{I}$, then $\mathscr{A}$ is bounded. In case of a bounded sequence $\xi$, the set $\mathscr{A}$ is even uniformly bounded.*

*Proof* (a) Suppose that $\mathscr{A}$ is bounded. Then there exists a sequence $(\rho_k)_{k\in\mathbb{I}}$ of positive real numbers such that $\mathscr{A}(k) \subseteq B_{\rho_k}(0)$ holds for all $k \in \mathbb{I}$. Since the cone $X_+$ is solid one finds some $y \in X_+^\circ$ with $B_1(0) \subseteq [-y,y] \subset X$ and (2) is satisfied with the sequence $\xi_k := \rho_k y$. On a uniformly bounded set $\mathscr{A}$ one chooses $r := \sup_{k\in\mathbb{I}} \rho_k$ and defines $\xi_k :\equiv ry$ for all $k \in \mathbb{I}$.

(b) Conversely, under the inclusion (2) one has $0 \preceq x + \xi_k \preceq 2\xi_k$ for arbitrary pairs $(k,x) \in \mathscr{A}$. Because $X_+$ is normal, it follows

$$\|x\| \leq \|x + \xi_k\| + \|\xi_k\| \stackrel{(1)}{\leq} 2C\|\xi_k\| + \|\xi_k\| = (1+2C)\|\xi_k\| \quad \text{for all } (k,x) \in \mathscr{A}$$

and so $x \in B_{(1+2C)\|\xi_k\|}(0)$. Thus, $\mathscr{A}$ is bounded and one obtains that a bounded sequence $\xi$ yields a uniformly bounded nonautonomous set $\mathscr{A}$. $\square$

## 2.1 Nonautonomous difference equations and attractors

From now on we assume that $\mathscr{D} \subseteq \mathbb{I} \times X$ is a nonautonomous set, serving as extended state space of a nonautonomous difference equation

$$\boxed{u' = f_k(u)} \tag{$\Delta_f$}$$

with right-hand side $f_k : \mathscr{D}(k) \to \mathscr{D}'(k)$, $k \in \mathbb{I}'$. An *entire solution* is a sequence $\phi \subseteq \mathscr{D}$ satisfying the solution identity $\phi_{k+1} \equiv f_k(\phi_k)$ on the discrete interval $\mathbb{I}'$. The forward solution to $(\Delta_f)$ fulfilling the initial condition $u_{k_0} = u$ with arbitrary pairs $(k_0, u) \in \mathscr{D}$ is called *general solution* $\varphi(\cdot; k_0, u) : \mathscr{D}(k_0) \to \mathscr{D}(k)$ and reads as

$$\varphi(k; k_0, \cdot) := \begin{cases} I_{\mathscr{D}(k_0)}, & k = k_0, \\ f_{k-1} \circ \ldots \circ f_{k_0}, & k_0 < k \end{cases} \quad \text{for all } k_0 \leq k; \tag{3}$$

here, $I_{\mathscr{D}(k_0)}$ is the identity mapping in $\mathscr{D}(k_0) \subseteq X$.

In order to emphasize the dependence on the right-hand sides $f_k$, we sometimes write $\varphi_f$ instead of $\varphi$. The relation (3) immediately implies the *cocycle property*

$$\varphi(k; k_0, u) = \varphi(k; l, \varphi(l; k_0, u)) \quad \text{for all } k_0 \leq l \leq k,\, u \in \mathscr{D}(k_0). \tag{4}$$

After this basic terminology, we arrive at a central concept (cf. [2, 11, 18]):

**Theorem 1 (pullback limit)** *Let $\mathbb{I}$ be unbounded below and suppose the right-hand sides $f_k$ are continuous. If $\xi$ denotes a sequence in $\mathscr{D}$ such that the so-called* pullback limits

$$\phi_k^* := \lim_{n \to -\infty} \varphi(k; n, \xi_n)$$

*exist in $\mathscr{D}(k)$ for all $k \in \mathbb{I}$, then $\phi^*$ is an entire solution to $(\Delta_f)$.*

*Proof* Due to the continuity of every right-hand side $f_k$ one has

$$\begin{aligned} \phi_{k+1}^* &= \lim_{n \to -\infty} \varphi(k+1; n, \xi_n) \overset{(\Delta_f)}{=} \lim_{n \to -\infty} f_k(\varphi(k; n, \xi_n)) \\ &= f_k\left(\lim_{n \to -\infty} \varphi(k; n, \xi_n)\right) = f_k(\phi_k^*) \quad \text{for all } k \in \mathbb{I}' \end{aligned}$$

and this completes the proof. □

Pullback convergence plays a crucial role throughout our following analysis, in particular when asking for an appropriate attractor concept. For this, an *(attraction) universe* $\hat{\mathscr{B}}$ is a family of nonautonomous sets, like e.g. the family of bounded or uniformly bounded such sets. One says a nonautonomous set $\mathscr{A}$ is

- *$\hat{\mathscr{B}}$-absorbing*, if for all $k \in \mathbb{I}$ and $\mathscr{B} \in \hat{\mathscr{B}}$ there is an $N = N(k, \mathscr{B}) \in \mathbb{N}_0$ with

$$\varphi(k; k-n, \mathscr{B}(k-n)) \subseteq \mathscr{A}(k) \quad \text{for all } n \geq N$$

- *$\hat{\mathscr{B}}$-attracting*, if for all $k \in \mathbb{I}$ one has the limit relation

$$\lim_{n \to \infty} h(\varphi(k; k-n, \mathscr{B}(k-n)), \mathscr{A}(k)) = 0 \quad \text{for all } \mathscr{B} \in \hat{\mathscr{B}}. \tag{5}$$

Here, we have used the *Hausdorff semidistance*

$$h(A,B) := \sup_{a\in A}\operatorname{dist}(a,B), \qquad \operatorname{dist}(x,B) := \inf_{b\in B}\|x-b\| \quad \text{for all } A,B\subseteq X.$$

It is clear that every $\hat{\mathscr{B}}$-absorbing set is $\hat{\mathscr{B}}$-attracting.

Beyond that, $(\Delta_f)$ is called *$\hat{\mathscr{B}}$-asymptotically compact*, if for all $k\in\mathbb{I}$ and $\mathscr{B}\in\hat{\mathscr{B}}$ and all strictly increasing sequences $(n_l)_{l\in\mathbb{N}}$ in $\mathbb{N}_0$, $x_l\in\mathscr{B}(k-n_l)$ the sequence $(\varphi(k;k-n_l,x_l))_{l\in\mathbb{N}_0}$ in $\mathscr{D}(k)$ has a convergent subsequence.

For the reader's convenience we next formulate a standard result:

**Lemma 4** *([2, p. 25, Lemma 2.3]) If $C\subseteq X$ is compact and a sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ satisfies the limit relation $\lim_{n\to\infty}\operatorname{dist}(x_n,C)=0$, then there exists a convergent subsequence $(x_{n_l})_{l\in\mathbb{N}}$ with limit in $C$.*

An immediate consequence is the next criterion for asymptotic compactness:

**Proposition 1** *Let $\mathbb{I}$ be unbounded below. If $(\Delta_f)$ has a compact $\hat{\mathscr{B}}$-attracting set, then it is $\hat{\mathscr{B}}$-asymptotically compact.*

*Proof* Let $\mathscr{B}\in\hat{\mathscr{B}}$ be given. For a compact $\hat{\mathscr{B}}$-attracting set $\mathscr{A}$ and sequences $k_n\to\infty$, $x_n\in\mathscr{B}(k-k_n)$ we obtain for every $k\in\mathbb{I}$ that

$$\operatorname{dist}(\varphi(k;k-k_n,x_n),\mathscr{A}(k)) \leq h(\varphi(k;k-k_n,\mathscr{B}(k-k_n)),\mathscr{A}(k)) \xrightarrow[n\to\infty]{(5)} 0.$$

Since every fiber $\mathscr{A}(k)$ is compact, Lemma 4 guarantees that there exists a subsequence $(\varphi(k;k-k_{n_l},x_{n_l}))_{l\in\mathbb{N}}$ with limit in $\mathscr{A}(k)$. □

## 2.2 Linear difference equations

Given a sequence of bounded linear operators $A_k\in L(X)$, $k\in\mathbb{I}'$, a *homogeneous linear difference equation* is of the form

$$\boxed{u' = A_k u.} \tag{$L_0$}$$

Its general solution can be written as $\varphi(k;k_0,u)=\Phi(k,k_0)u$ for all $k_0\leq k$, $u\in X$ with the *evolution operator*

$$\Phi(k,k_0) := \begin{cases} I_X, & k=k_0,\\ A_{k-1}\cdots A_{k_0}, & k_0<k.\end{cases}$$

Furthermore, an *inhomogeneous linear difference equation* reads as

$$\boxed{u' = A_k u + b_k} \tag{$L$}$$

with a sequence $b_k\in X$, $k\in\mathbb{I}'$. Due to the variation of constants formula (cf. [18, p. 100, Thm. 3.1.16(a)]) the general solution to $(L)$ becomes

$$\varphi(k;k_0,u) = \Phi(k,k_0)u + \sum_{j=k_0}^{k-1}\Phi(k,j+1)b_j \quad \text{for all } k_0\leq k,\ u\in X. \tag{6}$$

After these basics, we subsequently address preparations on the dynamics of nonautonomous linear equations: A universe $\hat{\mathscr{B}}$ is called *scaling invariant*, if for every $\mathscr{B} \in \hat{\mathscr{B}}$ and $t > 0$ also the nonautonomous set $\mathscr{B}_t$ fiber-wise given by

$$\mathscr{B}_t(k) := \left\{x \in X : \tfrac{1}{t}x \in \mathscr{B}(k)\right\} \quad \text{for all } k \in \mathbb{I}$$

belongs to $\hat{\mathscr{B}}$. For instance, the universe $\{\mathscr{B}_r(0) : r > 0\}$ of nonautonomous $r$-balls is scaling invariant; so is the family of (uniformly) bounded nonautonomous sets.

**Proposition 2** *Let $\hat{\mathscr{B}}$ be a scaling invariant universe and $\mathbb{I}$ be unbounded below. A homogeneous linear equation $(L_0)$ possesses a $\hat{\mathscr{B}}$-absorbing set $\mathscr{A} \subseteq \mathscr{B}_\rho(0)$ for some sequence $\rho = (\rho_k)_{k\in\mathbb{I}}$ of positive reals, if and only if*

$$\lim_{n\to\infty} \sup_{u\in\mathscr{B}(k-n)} \|\Phi(k,k-n)u\| = 0 \quad \text{for all } k \in \mathbb{I}, \mathscr{B} \in \hat{\mathscr{B}}. \tag{7}$$

*Proof* Let $k \in \mathbb{I}$ be fixed.

($\Rightarrow$) Suppose $(L_0)$ has a $\hat{\mathscr{B}}$-absorbing set $\mathscr{A} \subseteq \mathscr{B}_\rho(0)$ and choose a $\mathscr{B} \in \hat{\mathscr{B}}$. By the scaling invariance for every $t > 0$ there is a $N_t = N_t(k,\mathscr{B}) \in \mathbb{N}_0$ with

$$\Phi(k,k-n)\mathscr{B}_t(k-n) \subseteq \mathscr{A}(k) \subseteq B_{\rho_k}(0) \quad \text{for all } n \geq N_t.$$

This yields $\|\Phi(k,k-n)u\| \leq \rho_k$ for all $u \in \mathscr{B}_t(k-n)$, consequently

$$\sup_{u\in\mathscr{B}(k-n)} \|\Phi(k,k-n)u\| \leq \frac{\rho_k}{t} \quad \text{for all } n \geq N_t$$

and hence $\limsup_{n\to\infty} \sup_{u\in\mathscr{B}(k-n)} \|\Phi(k,k-n)u\| \leq \frac{\rho_k}{t}$. Since this estimate holds for arbitrary $t > 0$, we arrive at (7).

($\Leftarrow$) Conversely, from the limit relation (7) we know that for every $k \in \mathbb{I}$ and $\mathscr{B} \in \hat{\mathscr{B}}$ there is a $N = N(k,\mathscr{B}) \in \mathbb{N}_0$ such that $\sup_{u\in\mathscr{B}(k-n)} \|\Phi(k,k-n)u\| < 1$ holds for all $n \geq N$. This implies the inclusion $\Phi(k,k-n)\mathscr{B}(k-n) \subseteq B_1(0)$ and thus the nonautonomous ball $\mathscr{B}_1(0)$ is a $\hat{\mathscr{B}}$-absorbing set. □

In [18, p. 106, Prop. 3.1.26] we have characterized the asymptotic behavior of inhomogeneous equations $(L)$ given that $(L_0)$ is uniformly asymptotically stable. The following result replaces this assumption by merely attractivity:

**Proposition 3 (asymptotics of linear equations)** *Let $\hat{\mathscr{B}}$ be a scaling invariant universe with $\mathbb{I} \times \{0\} \in \hat{\mathscr{B}}$ and $\mathbb{I}$ be unbounded below. If $(L)$ possesses a compact $\hat{\mathscr{B}}$-attracting set $\mathscr{A} \in \hat{\mathscr{B}}$, then the pullback limit*

$$\phi_k^* := \sum_{j=-\infty}^{k-1} \Phi(k,j+1)b_j \quad \text{for all } k \in \mathbb{I} \tag{8}$$

*is an entire solution to $(L)$ in $\hat{\mathscr{B}}$ with the following properties:*

*(a) $\phi^*$ is globally pullback attracting in $\hat{\mathscr{B}}$, i.e.*

$$\lim_{n\to\infty} \sup_{u\in\mathscr{B}(k-n)} \|\varphi(k;k-n,u) - \phi_k^*\| = 0 \quad \text{for all } \mathscr{B} \in \hat{\mathscr{B}}. \tag{9}$$

*(b) $\phi^*$ is the unique entire solution of $(L)$ in $\hat{\mathscr{B}}$.*

*Proof* By assumption the zero sequence is contained in $\hat{\mathscr{B}}$ and therefore

$$\operatorname{dist}\left(\sum_{j=k-n}^{m-1}\Phi(m,j+1)b_j,\mathscr{A}(m)\right)\stackrel{(6)}{=}\operatorname{dist}\left(\varphi(m;m-n,0),\mathscr{A}(m)\right)\xrightarrow[n\to\infty]{}0 \tag{10}$$

holds for all $m\in\mathbb{I}$. Thus, due to Lemma 4 we can extract a strictly increasing subsequence $(n_l)_{l\in\mathbb{N}}$ in $\mathbb{N}$ and obtain an $a_m\in\mathscr{A}(m)$ with

$$a_m:=\lim_{l\to\infty}\sum_{j=k-n_l}^{m-1}\Phi(m,j+1)b_j\in\mathscr{A}(m). \tag{11}$$

Moreover, we similarly have the limit relation

$$\operatorname{dist}\left(\sum_{j=k-n_l}^{k-1}\Phi(k,j+1)b_j,\mathscr{A}(k)\right)\stackrel{(6)}{=}\operatorname{dist}(\varphi(k;k-n_l,0),\mathscr{A}(k))\xrightarrow[l\to\infty]{}0\quad\text{for all } k\in\mathbb{I}$$

and as above there exists a further subsequence, w.l.o.g. also denoted by $(n_l)_{l\in\mathbb{N}}$, such that

$$a_k^*:=\lim_{l\to\infty}\sum_{j=k-n_l}^{k-1}\Phi(k,j+1)b_j\in\mathscr{A}(k)\quad\text{for all } k\in\mathbb{I}.$$

With arbitrary integers $k_0\leq k\leq m$ one observes

$$\sum_{j=k_0}^{m-1}\Phi(m,j+1)b_j=\Phi(m,k)\sum_{j=k_0}^{k-1}\Phi(k,j+1)b_j+\sum_{j=k}^{m-1}\Phi(m,j+1)b_j$$

and setting $k_0=k-n_l$ in this relation yields in the limit $l\to\infty$ that

$$a_m\stackrel{(11)}{=}\Phi(m,k)a_k^*+\sum_{j=k}^{m-1}\Phi(m,j+1)b_j. \tag{12}$$

Since $(L)$ has a compact $\hat{\mathscr{B}}$-attracting set, we conclude from (10) that also $(L_0)$ possesses a $\hat{\mathscr{B}}$-attracting set $\mathscr{A}_0\in\hat{\mathscr{B}}$. Due to (5) the 1-neighborhood

$$\mathscr{A}_1:=\{(k,x)\in\mathscr{D}:\operatorname{dist}(x,\mathscr{A}_0(k))<1\}$$

of $\mathscr{A}_0$ is a $\hat{\mathscr{B}}$-absorbing set for the homogeneous equation $(L_0)$ and thus Prop. 2 yields the limit relation $\lim_{k\to-\infty}\Phi(m,k)a_k=0$. Passing over to $k\to-\infty$ in (12) implies that the limit defining $\phi_m^*$ exists for all $m\in\mathbb{I}$. Thanks to Thm. 1 it defines an entire solution to $(L)$ in $\hat{\mathscr{B}}$.

(a) Since the difference of two solutions for $(L)$ always solves $(L_0)$, we deduce

$$\varphi(k;k-n,u)-\phi_k^*=\Phi(k,k-n)(u-\phi_{k-n}^*)\quad\text{for all } k\in\mathbb{I}, n\in\mathbb{N}_0$$

and consequently assertion (a) follows from (7).

(b) If $\psi^*$ is a further entire solution to $(L)$ fulfilling (7), then we have

$$\begin{aligned}\|\psi_k^*-\phi_k^*\|&=\left\|\varphi(k;k-n,\psi_{k-n}^*)-\phi_k^*\right\|\\&=\sup_{u\in\{\psi_{k-n}^*\}}\|\varphi(k;k-n,u)-\phi_k^*\|\xrightarrow[n\to\infty]{(9)}0\quad\text{for all } k\in\mathbb{I}\end{aligned}$$

and therefore $\phi^*=\psi^*$. $\square$

An almost trivial, but still illustrative and explicit application of Prop. 3 is

*Example 2* Let us suppose that $\alpha \in (-1,1)$ and $(b_k)_{k\in\mathbb{I}'}$ denotes a bounded real sequence with $\beta := \sup_{k\in\mathbb{I}'}|b_k|$. Using the variation of constants (6) the scalar difference equation

$$u' = \alpha u + b_k \tag{13}$$

has the general solution

$$\varphi(k;k_0,u_0) = \alpha^{k-k_0}u_0 + \sum_{j=k_0}^{k-1}\alpha^{k-j-1}b_j \quad \text{for all } k_0 \le k,\, u_0 \in \mathbb{R}.$$

Given the scaling invariant universe $\hat{\mathscr{B}} := \{\mathbb{I}\times[-r,r] : r>0\}$ we choose the uniformly bounded nonautonomous set $\mathscr{B} := \mathbb{I}\times[-\rho,\rho] \in \hat{\mathscr{B}}$, some $u_0 \in \mathscr{B}(k-n) = [-\rho,\rho]$ and $N = N(\rho) \in \mathbb{N}$ so large that $|\alpha|^n\rho \le \beta$ for all $n \ge N$. This readily implies

$$|\varphi(k;k-n,u_0)| \le |\alpha|^n|u_0| + \sum_{j=k-n}^{k-1}|\alpha|^{k-j-1}\left|b_j\right| \le |\alpha|^n\rho + \frac{\beta}{1-|\alpha|} \le \frac{2-|\alpha|}{1-|\alpha|}\beta$$

for all $n \ge N$ and we arrive at the inclusion

$$\varphi(k;k-n,\mathscr{B}(k-n)) \subseteq \left[\tfrac{|\alpha|-2}{1-|\alpha|}\beta, \tfrac{2-|\alpha|}{1-|\alpha|}\beta\right] =: \mathscr{A}(k) \quad \text{for all } k\in\mathbb{I},\, n\ge N.$$

This defines a compact nonautonomous set $\mathscr{A} \in \hat{\mathscr{B}}$, which is even $\hat{\mathscr{B}}$-absorbing. Consequently, $\mathscr{A}$ is $\hat{\mathscr{B}}$-attracting and Prop. 3 applies: It yields a globally pullback attracting and unique entire solution $\phi^*$ in $\hat{\mathscr{B}}$ (it is bounded) given by (8). For a constant inhomogeneity $(b_k)_{k\in\mathbb{I}'}$ the equation (13) becomes autonomous and $\phi^*$ is simply the globally attracting equilibrium $\phi^* = \frac{b}{1-\alpha}$. For a periodic sequence $(b_k)_{k\in\mathbb{I}'}$, also the pullback limit $\phi^*$ is periodic with the same period, while the remaining solutions are unbounded on $\mathbb{I}$. Thus, the solution $\phi^*$ reflects e.g. periodicity properties of the inhomogeneity $(b_k)_{k\in\mathbb{I}'}$.

## 3 Order-preserving difference equations

Assume that $X$ is an ordered normed space with cone $X_+$. One denotes a linear bounded operator $T \in L(X)$ as *positive* (on $X_+$), provided $TX_+ \subseteq X_+$, i.e. it leaves the cone $X_+$ invariant.

**Definition 1** An equation $(\Delta_f)$ is said to be *order-preserving*, if one has

$$u \preceq v \quad\Rightarrow\quad f_k(u) \preceq f_k(v) \quad \text{for all } k\in\mathbb{I}',\, u,v\in\mathscr{D}(k). \tag{14}$$

*Remark 1* (1) It immediately follows by mathematical induction that a difference equation $(\Delta_f)$ is order-preserving, if and only if

$$u \preceq v \quad\Rightarrow\quad \varphi(k;k_0,u) \preceq \varphi(k;k_0,v) \quad \text{for all } k_0\le k,\, u,v\in\mathscr{D}(k_0). \tag{15}$$

(2) A linear difference equation $(L)$ is order-preserving, if and only if all coefficient operators $A_k \in L(X)$, $k\in\mathbb{I}'$, are positive.

We continue by quoting a sufficient criterion for difference equations $(\Delta_f)$ with continuously Fréchet-differentiable right-hand sides to be order-preserving. Thereto, a subset $\Omega \subseteq X$ is called $X_+$-*convex*, if the implication

$$x \preceq y \quad \Rightarrow \quad \{x + t(y-x) \in X : t \in [0,1]\} \subseteq \Omega \quad \text{for all } x, y \in \Omega$$

holds, i.e. $\Omega$ always contains the line joining any elements $x, y \in \Omega$ with $x \preceq y$.

**Lemma 5** *([9, Lemma 2.2]) Let $\mathscr{D}$ be an open $X_+$-convex nonautonomous set and suppose the right-hand sides $f_k$ are of class $C^1$. If the Fréchet derivative $Df_k(x)$ is positive for all pairs $(k,x) \in \mathscr{D}$, $k \in \mathbb{I}'$, then $(\Delta_f)$ is order-preserving.*

Given right-hand sides $g_k : \mathscr{D}(k) \to \mathscr{D}'(k)$, $k \in \mathbb{I}'$, we say a further difference equation $(\Delta_g)$ *dominates* $(\Delta_f)$, if one has

$$f_k(x) \preceq g_k(x) \quad \text{for all } (k,x) \in \mathscr{D}.$$

This is equivalent to the fact that the corresponding general solutions fulfill

$$\varphi_f(k;k_0,u) \preceq \varphi_g(k;k_0,u) \quad \text{for all } (k_0,u) \in \mathscr{D},\ k_0 \leq k.$$

The following concepts turn out to be essential to enclose attractors of nonautonomous difference equations between their entire solutions:

**Definition 2** A sequence $\xi = (\xi_k)_{k\in\mathbb{I}}$ in $\mathscr{D}$ is said to be a

(i) *sub-solution* of $(\Delta_f)$, if

$$\xi_{k+1} \preceq f_k(\xi_k) \quad \text{for all } k \in \mathbb{I}' \tag{16}$$

(ii) *super-solution* of $(\Delta_f)$, if

$$f_k(\xi_k) \preceq \xi_{k+1} \quad \text{for all } k \in \mathbb{I}'.$$

Every entire solution to $(\Delta_f)$ is both a sub- and a super-solution. Conversely, a sequence being both a sub- and a super-solution is an entire solution. Nevertheless, for practical purposes sub- or super-solutions are easier to construct then entire (bounded) solutions — particularly in our nonautonomous set-up.

*Remark 2* For a minihedral cone $X_+$ the following holds:

- With sub-solutions $\xi, \bar{\xi}$ to $(\Delta_f)$ also $(\sup\{\xi_k, \bar{\xi}_k\})_{k\in\mathbb{I}}$ is a sub-solution
- With super-solutions $\xi, \bar{\xi}$ to $(\Delta_f)$ also $(\inf\{\xi_k, \bar{\xi}_k\})_{k\in\mathbb{I}}$ is a super-solution

**Corollary 1** *Suppose $(\Delta_f)$ is order-preserving and let $\xi^-, \xi^+$ be sequences in $\mathscr{D}$:*

*(a) $\xi^-$ is a sub-solution of $(\Delta_f)$, if and only if*

$$\xi_k^- \preceq \varphi(k;k_0,\xi_{k_0}^-) \quad \textit{for all } k_0 \leq k. \tag{17}$$

*(b) $\xi^+$ is a super-solution of $(\Delta_f)$, if and only if*

$$\varphi(k;k_0,\xi_{k_0}^+) \preceq \xi_k^+ \quad \textit{for all } k_0 \leq k.$$

*Proof* (a) Obviously, (17) is true for $k = k_0$. To proceed inductively, suppose (17) holds for some fixed $k \geq k_0$. Then the induction step $k \to k+1$ becomes

$$\xi_{k+1}^- \overset{(16)}{\preceq} f_k(\xi_k^-) \overset{(17)}{\preceq} f_k(\varphi(k;k_0,\xi_{k_0}^-)) = \varphi(k+1;k_0,\xi_{k_0}^-)$$

yielding the claim. The converse immediately follows from (17) for $k = k_0 + 1$.

(b) can be shown analogously. □

One calls a nonautonomous set $\mathscr{A}$ *forward invariant* (w.r.t. $(\Delta_f)$), if $f_k(\mathscr{A}(k)) \subseteq \mathscr{A}'(k)$ for all $k \in \mathbb{I}'$ and in case of equality one speaks of an *invariant* set.

**Corollary 2** *Suppose $(\Delta_f)$ is order-preserving and let $\xi^-, \xi^+$ denote sequences in $\mathscr{D}$ with $\xi_k^- \preceq \xi_k^+$ for all $k \in \mathbb{I}$:*

*(a) $[\xi^-,\infty) \cap \mathscr{D}$ is forward invariant, if and only if $\xi^-$ is a sub-solution.*
*(b) $(-\infty,\xi^+] \cap \mathscr{D}$ is forward invariant, if and only if $\xi^+$ is a super-solution.*
*(c) $[\xi^-,\xi^+] \cap \mathscr{D}$ is forward invariant, if and only if $\xi^-$ is a sub- and $\xi^+$ is a super-solution.*

*Proof* (a) The forward invariance of $[\xi^-,\infty) \cap \mathscr{D}$ means that the fibers fulfill

$$f_k([\xi_k^-,\infty) \cap \mathscr{D}(k)) \subseteq \left[\xi_{k+1}^-,\infty\right) \cap \mathscr{D}'(k)$$

and thus $\xi_{k+1}^- \preceq f_k(\xi_k^-)$ for all $k \in \mathbb{I}'$. Hence, $\xi^-$ is a sub-solution of $(\Delta_f)$. Conversely, for every $x \in \mathscr{D}(k)$ satisfying $\xi_k^- \preceq x$ one has the inequality $f_k(\xi_k^-) \preceq f_k(x)$. Due to (14) and the fact that $\xi^-$ is a sub-solution, one obtains $\xi_{k+1}^- \preceq f_k(x)$. We therefore have the inclusion $f_k(\left[\xi_k^-,\infty\right) \cap \mathscr{D}(k)) \subseteq \left[\xi_{k+1}^-,\infty\right) \cap \mathscr{D}'(k)$ and so $[\xi^-,\infty) \cap \mathscr{D}$ is forward invariant.

(b) and (c) can be shown analogously. □

An inhomogeneous linear difference equation $(L)$ is said to be *positive*, if $A_k \in L(X)$ is positive and $0 \preceq b_k$ holds for all $k \in \mathbb{I}'$. This yields

**Corollary 3** *The set $\mathbb{I} \times X_+$ is forward invariant w.r.t. $(L)$, if and only if $(L)$ is positive.*

*Proof* $(\Rightarrow)$ Let $\mathbb{I} \times X_+$ be forward invariant and $x \in X_+$. Then the relation

$$0 \preceq \varphi(k+1;k,0) = A_k 0 + b_k \quad \text{for all } k \in \mathbb{I}'$$

on the one hand yields $0 \preceq b_k$, and passing over to the limit $t \to \infty$ in

$$0 \preceq \tfrac{1}{t}\varphi(k+1;k,tx) = \tfrac{1}{t}\left(A_k(tx) + b_k\right) = A_k x + \tfrac{1}{t} b_k \quad \text{for all } t > 0,\, k \in \mathbb{I}'$$

on the other hand implies $0 \preceq A_k x$, hence the claim.

$(\Leftarrow)$ Since the positive linear operators are closed under composition, $\Phi(k,l) \in L(X)$, $l \leq k$, is positive and then the proof follows from the variation of constants formula (6). □

Let us go on with a sufficient condition for order-intervals to be both forward invariant and absorbing:

**Proposition 4** *Suppose $(\Delta_f)$ is order-preserving. If $\xi^-$ is a sub-solution and $\xi^+$ is a super-solution of $(\Delta_f)$ in $\hat{\mathscr{B}}$ such that both order intervals $(-\infty,\xi^+]$ and $[\xi^-,\infty)$ are $\hat{\mathscr{B}}$-absorbing, then $\xi_k^- \preceq \xi_k^+$ for all $k \in \mathbb{I}$ and $[\xi^-,\xi^+]$ is a forward invariant, $\hat{\mathscr{B}}$-absorbing set w.r.t. $(\Delta_f)$.*

*Proof* On the one hand, because the order interval $(-\infty,\xi^+]$ is $\hat{\mathscr{B}}$-absorbing and $\xi^- \in \hat{\mathscr{B}}$ holds, for every $k \in \mathbb{I}$ there exists a $N_k(\xi^-) \in \mathbb{N}$ such that

$$\varphi(k;k-n,\xi^-_{k-n}) \in \left(-\infty,\xi^+_k\right] \quad \text{for all } n \geq N_k(\xi^-). \tag{18}$$

On the other hand, thanks to Cor. 1(a) it is $\xi^-_k \preceq \varphi(k;k-n,\xi^-_{k-n})$ for all $n \in \mathbb{N}_0$ and in combination we arrive at $\xi^-_k \preceq \xi^+_k$ for all $k \in \mathbb{I}$. The forward invariance of $[\xi^-,\xi^+]$ follows from Cor. 2(c). In order to show that $[\xi^-,\xi^+]$ is $\hat{\mathscr{B}}$-absorbing, we choose a nonautonomous set $\mathscr{B} \in \hat{\mathscr{B}}$ and deduce from our assumptions that there exist $N^-_k, N^+_k \in \mathbb{N}$ with

$$\begin{aligned}\varphi(k;k-n,\mathscr{B}(k-n)) &\subseteq \left[\xi^-_k,\infty\right) \quad \text{for all } n \geq N^-_k,\\ \varphi(k;k-n,\mathscr{B}(k-n)) &\subseteq \left(-\infty,\xi^+_k\right] \quad \text{for all } n \geq N^+_k.\end{aligned}$$

Hence, for every sequence $\xi$ in $\mathscr{B}$ one gets $\xi^-_k \preceq \varphi(k;k-n,\xi_{k-n}) \preceq \xi^+_k$, and this in turn means $\varphi(k;k-n,\mathscr{B}(k-n)) \subseteq [\xi^-_k,\xi^+_k]$ for all $n \geq \max\left\{N^+_k,N^-_k\right\}$. □

In order to prepare our first main result, let us formulate

**Lemma 6** *Suppose* $(\Delta_f)$ *is order-preserving.*

*(a) For every sub-solution* $\xi^-$ *and every fixed* $n \in \mathbb{N}_0$ *also the sequence*

$$\phi^-_{k,n} := \varphi(k;k-n,\xi^-_{k-n}) \quad \text{for all } k \in \mathbb{I}$$

*is a sub-solution of* $(\Delta_f)$ *satisfying*

$$\xi^-_k \preceq \phi^-_{k,m} \preceq \phi^-_{k,n} \quad \text{for all } k \in \mathbb{I},\ 0 \leq m \leq n. \tag{19}$$

*(b) For every super-solution* $\xi^+$ *and every fixed* $n \in \mathbb{N}_0$ *also the sequence*

$$\phi^+_{k,n} := \varphi(k;k-n,\xi^+_{k-n}) \quad \text{for all } k \in \mathbb{I}$$

*is a super-solution of* $(\Delta_f)$ *satisfying*

$$\phi^+_{k,n} \preceq \phi^+_{k,m} \preceq \xi^+_k \quad \text{for all } k \in \mathbb{I},\ 0 \leq m \leq n. \tag{20}$$

*Proof* Let $n \in \mathbb{N}_0$ be fixed.

(a) Since $\xi^-$ is a sub-solution to $(\Delta_f)$, we deduce

$$\begin{aligned}\phi^-_{k+1,n} &= \varphi(k+1;k+1-n,\xi^-_{k+1-n}) \overset{(15)}{\preceq} \varphi(k+1;k+1-n,f_{k-n}(\xi^-_{k-n}))\\ &\overset{(3)}{=} f_k(\varphi(k;k-n,\xi^-_{k-n})) \overset{(4)}{=} f_k(\phi^-_{k,n}) \quad \text{for all } k \in \mathbb{I}'\end{aligned}$$

and thus also $\phi^-_{\cdot,n}$ is a sub-solution. One furthermore obtains

$$\begin{aligned}\phi^-_{k,m} &= \varphi(k;k-m,\xi^-_{k-m}) \overset{(17)}{\preceq} \varphi(k;k-m,\varphi(k-m;k-n,\xi^-_{k-n}))\\ &\overset{(4)}{=} \varphi(k;k-n,\xi^-_{k-n}) = \phi^-_{k,n} \quad \text{for all } 0 \leq m \leq n,\ k \in \mathbb{I}.\end{aligned}$$

In particular, setting $m = 0$ yields $\xi^-_k = \varphi(k;k,\xi^-_k) = \phi^-_{k,0} \preceq \phi^-_{k,n}$ and the assertion.

(b) can be proved analogously. □

This leads us to the useful result that between a sub- and a super-solution of an order-preserving equation, and under appropriate compactness assumptions, there always exists at least one entire solution for $(\Delta_f)$. By Cor. 2(c) this means that forward invariant order intervals contain entire solutions:

**Theorem 2 (pullback solutions)** *Let $\mathbb{I}$ be unbounded below and suppose $(\Delta_f)$ is order-preserving with continuous right-hand sides $f_k$ and closed $\mathscr{D}$. If there exists a sub-solution $\xi^-$ and a super-solution $\xi^+$ fulfilling $[\xi^-,\xi^+]\subseteq\mathscr{D}$, and one of the assumptions*

*(i) for all $k\in\mathbb{I}$ there is an $N=N_k\in\mathbb{N}$ so that the images $\varphi(k;k-N,[\xi^-_{k-N},\xi^+_{k-N}])\subseteq\mathscr{D}(k)$ are relatively compact,*
*(ii) the cone $X_+$ is regular*

*holds, then the monotone limits*

$$\begin{aligned}\phi_k^- &:= \lim_{n\to\infty}\varphi(k;k-n,\xi^-_{k-n}) = \sup_{n\in\mathbb{N}_0}\varphi(k;k-n,\xi^-_{k-n}),\\ \phi_k^+ &:= \lim_{n\to\infty}\varphi(k;k-n,\xi^+_{k-n}) = \inf_{n\in\mathbb{N}_0}\varphi(k;k-n,\xi^+_{k-n})\end{aligned}\tag{21}$$

*exist for all $k\in\mathbb{I}$ and define entire solutions $\phi^-$, $\phi^+$ of $(\Delta_f)$ satisfying*

$$\xi_k^-\preceq\phi_k^-\preceq\phi_k^+\preceq\xi_k^+\quad\textit{for all } k\in\mathbb{I}.\tag{22}$$

In a way, Thm. 2 resembles classical fixed-point results of e.g. Schauder-type (see, for instance, [6, p. 60, Thm. 8.8]) or for monotone mappings like [6, p. 224, Thm. 19.1], where our nonautonomous set-up requires fixed-points to be replaced by the entire solutions $\phi^-,\phi^+$.

*Proof* Let $k\in\mathbb{I}$ be fixed. Mimicking the notation of Lemma 6 we define

$$\phi^\pm_{k,n} := \varphi(k;k-n,\xi^\pm_{k-n})\quad\text{for all } n\in\mathbb{N}_0,$$

where $(\phi^-_{k,n})_{n\in\mathbb{N}_0}$ is increasing (see (19)), while $(\phi^+_{k,n})_{n\in\mathbb{N}_0}$ decreases (cf. (20)).

(I) Under assumption (i) we define $\hat\phi^\pm_{k,n} := \varphi(k;k-nN,\xi^\pm_{k-nN})$ and get

$$\begin{aligned}\hat\phi^\pm_{k,n+1} &= \varphi(k;k-(n+1)N,\xi^\pm_{k-(n+1)N})\\ &\overset{(4)}{=} \varphi(k;k-N,\varphi(k-N;k-(n+1)N,\xi^\pm_{k-(n+1)N}))\\ &\in \varphi\left(k;k-N,[\xi^-_{k-N},\xi^+_{k-N}]\right)\quad\text{for all } n\in\mathbb{N}_0\end{aligned}$$

due to Cor. 2(c). Hence, the subsets $\{\hat\phi^\pm_{k,n}: n\in\mathbb{N}_0\}\subseteq\mathscr{D}(k)$ are relatively compact. Thanks to Lemma 1 we get convergence of the sequences $(\hat\phi^\pm_{k,n})_{n\in\mathbb{N}_0}$. Because every $n\in\mathbb{N}_0$ is contained in the discrete interval $\left[N\left[\frac{n}{N}\right],N\left[\frac{n}{N}\right]+N\right]_{\mathbb{Z}}$, we use Lemma 6(a) and (19) to obtain

$$\hat\phi^-_{k,[\frac{n}{N}]}\preceq\phi^-_{k,n}\preceq\hat\phi^-_{k,[\frac{n}{N}]+1}\quad\text{for all } n\in\mathbb{N};$$

this guarantees that also $(\phi^-_{k,n})_{n\in\mathbb{N}_0}$ converges as $n\to\infty$. Analogously, one makes use of Lemma 6(b) in order to establish the convergence of $(\phi^+_{k,n})_{n\in\mathbb{N}_0}$.

(II) Given assumption (ii) one proceeds as follows: By Cor. 1(b) it is

$$\phi^-_{k,n} = \varphi(k;k-n,\xi^-_{k-n}) \overset{(15)}{\preceq} \varphi(k;k-n,\xi^+_{k-n})\preceq\xi^+_k\quad\text{for all } n\in\mathbb{N}_0$$

and thus the increasing sequence $(\phi^-_{k,n})_{n\in\mathbb{N}_0}$ is order-bounded above. Since $X_+$ is regular, we obtain its convergence. Dually, with Cor. 1(a) and (17) one obtains that the decreasing sequence $(\phi^+_{k,n})_{n\in\mathbb{N}_0}$ is convergent.

(III) By Thm. 1 the pullback limits (21) yield entire solutions to $(\Delta_f)$. The inequalities (22) follow by passing to the limit $n\to\infty$ in (19), (20) and

$$\varphi(k;k-n,\xi^-_{k-n}) \overset{(15)}{\preceq} \varphi(k;k-n,\xi^+_{k-n}) \quad \text{for all } k\in\mathbb{I},\, n\in\mathbb{N}_0,$$

which concludes the proof. □

For $m\in\mathbb{N}_0$ the *truncated orbit* of a nonautonomous set $\mathscr{A}$ is fiber-wise given as

$$\gamma^m_{\mathscr{A}}(k) := \bigcup_{n\geq m} \varphi(k;k-n,\mathscr{A}(k-n)) \subseteq \mathscr{D}(k) \quad \text{for all } k\in\mathbb{I}$$

and we define the $\omega$*-limit set* $\omega_{\mathscr{A}} := \bigcap_{m\geq 0}\overline{\gamma^m_{\mathscr{B}}}$ of $\mathscr{A}$.

**Proposition 5** *Let $\mathbb{I}$ be unbounded below and suppose $\xi^*$ is a sub- or super-solution of an order-preserving equation $(\Delta_f)$ with continuous right-hand sides $f_k$ and closed $\mathscr{D}$. If for every $k\in\mathbb{I}$ one of the assumptions*

*(i) for some $N_k\in\mathbb{N}$ the truncated orbit $\gamma^{N_k}_{\xi^*}(k)\subseteq\mathscr{D}(k)$ is relatively compact,*

*(ii) $X_+$ is regular and $\left\{\varphi(k;k-n,\xi^*_{k-n})\right\}_{n\in\mathbb{N}_0}$ is order-bounded above (or below), if $\xi^*$ is a sub-solution (resp. a super-solution)*

*holds, then $\phi^*_k := \lim_{n\to\infty}\varphi(k;k-n,\xi^*_{k-n})$ (monotonically) is an entire solution of $(\Delta_f)$ satisfying $\omega_{\xi^*}=\phi^*$.*

*Proof* Using Lemma 6 we see that $\phi^*_{k,n} := \varphi(k;k-n,\xi^*_{k-n})$ is monotone in $n\in\mathbb{N}_0$ and one can repeat the argument from the above proof of Thm. 2. □

**Corollary 4 (linear positive equations)** *Suppose $(L)$ is positive and define*

$$\phi^*_{k,n} := \sum_{j=k-n}^{k-1}\Phi(k,j+1)b_j \quad \textit{for all } k\in\mathbb{I},\, n\in\mathbb{N}.$$

*If for all $k\in\mathbb{I}$ one of the assumptions*

*(i) there exists an $N_k\in\mathbb{N}$ so that $\left\{\phi^*_{k,n} : N_k\leq n\right\}\subseteq X$ is relatively compact,*

*(ii) $X_+$ is regular and $\left\{\phi^*_{k,n} : n\in\mathbb{N}_0\right\}\subseteq X$ is order-bounded above*

*holds, then the pullback limit* (8) *is an entire solution of* $(L)$ *in* $\mathbb{I}\times X_+$.

*Proof* Due to $0\preceq b_k = A_k 0 + b_k$ for all $k\in\mathbb{I}'$ the zero sequence is a sub-solution of $(L)$. Moreover, since $A_k\in L(X)$ is positive, $(L)$ becomes order-preserving and since we have

$$\varphi(k;k-n,0) \overset{(6)}{=} \sum_{j=k-n}^{k-1}\Phi(k,j+1)b_j \quad \text{for all } k\in\mathbb{I},\, n\in\mathbb{N}_0$$

the claim follows from Prop. 5 with the sub-solution $\xi^*=0$.

Let $\hat{\mathscr{B}}$ again denote a universe. A difference equation $(\Delta_f)$ is called *$\hat{\mathscr{B}}$-eventually compact*, if for all $k \in \mathbb{I}$ and $\mathscr{B} \in \hat{\mathscr{B}}$ there exists an $N = N_k(\mathscr{B}) \in \mathbb{N}_0$ such that the truncated orbit $\gamma_{\mathscr{B}}^N$ is relatively compact. It has been shown in [18, p. 13, Cor. 1.2.22] that a $\hat{\mathscr{B}}$-eventually compact equation $(\Delta_f)$ is $\hat{\mathscr{B}}$-asymptotically compact.

The next result is a nonautonomous counterpart to [1, Thm. 3.2]. It guarantees that solutions to initial values from a given universe $\hat{\mathscr{B}}$ always pullback converge to entire solutions. Precisely, one arrives at

**Theorem 3 (convergence)** *Let $\mathbb{I}$ be unbounded below, $X$ be a strongly ordered Banach space with a normal minihedral cone and suppose $(\Delta_f)$ is order-preserving with continuous right-hand sides $f_k$ and closed $\mathscr{D}$. If $(\Delta_f)$ is $\hat{\mathscr{B}}$-eventually compact with a universe $\hat{\mathscr{B}}$ containing singleton nonautonomous sets, then for every sequence $\xi \in \hat{\mathscr{B}}$ there exists an entire solution $\phi^* \in \hat{\mathscr{B}}$ to $(\Delta_f)$ such that*

$$\lim_{n\to\infty} \|\varphi(k;k-n,\xi_{k-n}) - \phi_k^*\| = 0 \quad \textit{for all } k \in \mathbb{I}. \tag{23}$$

*Proof* Let $k \in \mathbb{I}$ be fixed. For every $\xi \in \hat{\mathscr{B}}$ we obtain from [18, p. 14, Thm. 1.2.25] that the limit set $\omega_\xi \subseteq \mathscr{D}$ is compact and invariant. We define the sequences

$$\xi_k^- := \inf \omega_\xi(k), \qquad \xi_k^+ := \sup \omega_\xi(k),$$

whose existence is implied by Lemma 2, and our assumption yields $\xi^+, \xi^- \in \hat{\mathscr{B}}$. Since $(\Delta_f)$ is order-preserving, the inequality $\xi_k^- \preceq x \preceq \xi_k^+$ implies

$$\varphi(k;k_0,\xi_{k_0}^-) \overset{(15)}{\preceq} \varphi(k;k_0,x) \overset{(15)}{\preceq} \varphi(k;k_0,\xi_{k_0}^+) \quad \text{for all } (k_0,x) \in \omega_\xi,\, k_0 \leq k.$$

Because $\omega_\xi$ is invariant, we arrive at the estimate

$$\varphi(k;k-n,\xi_{k-n}^-) \preceq x \preceq \varphi(k;k-n,\xi_{k-n}^+) \quad \text{for all } (k,x) \in \omega_\xi,\, n \in \mathbb{N}_0 \tag{24}$$

and by definition of $\xi_k^-$ as infimum resp. $\xi_k^+$ as supremum, it is

$$\varphi(k;k-n,\xi_{k-n}^-) \preceq \xi_k^-, \qquad \xi_k^+ \preceq \varphi(k;k-n,\xi_{k-n}^+) \quad \text{for all } n \in \mathbb{N}_0.$$

Thus, Cor. 1 shows that $\xi^-$ is a super- and $\xi^+$ is a sub-solution. The $\hat{\mathscr{B}}$-eventual compactness of $(\Delta_f)$ and $\xi^-, \xi^+ \in \hat{\mathscr{B}}$ imply that assumption (i) of Prop. 5 is satisfied, which yields that the pullback limits $\phi_k^\pm := \lim_{n\to\infty} \varphi(k;k-n,\xi_{k-n}^\pm)$ are well-defined and entire solutions to $(\Delta_f)$ with $\omega_{\xi^\pm} = \phi^\pm \in \hat{\mathscr{B}}$. The claim follows by choosing $\phi^*$ as one of the entire solutions $\phi^-, \phi^+$. The fact that $\omega_\xi$ is $\xi$-attracting (cf. [18, p. 14, Thm. 1.2.25(c)]) gives (23). □

**Corollary 5** *If $(\Delta_f)$ has a unique entire solution $\phi^* \in \hat{\mathscr{B}}$, then $\phi^*$ is globally pullback-attractive, i.e. for all sequences $\xi \in \hat{\mathscr{B}}$ one has the limit relation* (23).

*Proof* With the notation from the proof of Thm. 3, the uniqueness assumption on the entire solutions to $(\Delta_f)$ in $\hat{\mathscr{B}}$ implies $\phi^- = \phi^+ =: \phi_\xi^*$. It remains to establish that $\phi_\xi^*$ is independent of $\xi \in \hat{\mathscr{B}}$. Thereto, repeating the proof of Thm. 3 with another $\eta \in \hat{\mathscr{B}}$ yields an entire solution $\phi_\eta^* \in \hat{\mathscr{B}}$ and the uniqueness assumption on such solutions guarantees $\phi_\xi^* = \phi_\eta^*$. □

We conclude this section by presenting a method how sub- and super-solutions of difference equations can be constructed. Thereto, we restrict to extended state spaces $\mathscr{D} = \mathbb{I} \times X_+$ and say that $(\Delta_f)$ is *sub-homogeneous*, if

$$t f_k(x) \preceq f_k(tx) \quad \text{for all } k \in \mathbb{I}', x \in X_+, t \geq 1,$$

while $(\Delta_f)$ is called *super-homogeneous*, provided

$$f_k(tx) \preceq t f_k(x) \quad \text{for all } k \in \mathbb{I}', x \in X_+, t \geq 1.$$

Now, if a difference equation $(\Delta_f)$ is dominated by a

- sub-homogeneous equation $(\Delta_g)$, then given an entire solution $\phi$ to $(\Delta_f)$, every scaled sequence $t\phi$, $t \geq 1$, is a sub-solution of $(\Delta_g)$, since

$$t\phi_{k+1} = t f_k(\phi_k) \preceq t g_k(\phi_k) \preceq g_k(t\phi_k) \quad \text{for all } k \in \mathbb{I}', t \geq 1$$

- super-homogenous equation $(\Delta_g)$, then given an entire solution $\psi$ of $(\Delta_g)$, every scaled sequence $t\psi$, $t \geq 1$, is a super-solution of $(\Delta_f)$, because

$$f_k(t\psi_k) \preceq g_k(t\psi_k) \preceq t g_k(\psi_k) = t\psi_{k+1} \quad \text{for all } k \in \mathbb{I}', t \geq 1$$

Positive linear equation $(L)$ are always super-homogenous, because with reals $t \geq 1$ it is

$$A_k(tx) + b_k = t(A_k x + b_k) + (1-t)b_k \preceq t(A_k x + b_k) \quad \text{for all } k \in \mathbb{I}', x \in X_+.$$

## 4 Attractors

Throughout this section, we suppose the discrete interval $\mathbb{I}$ is unbounded below.

A nonautonomous set $\mathscr{A}$ is said to *attract* a sequence $\xi$ in $\mathscr{D}$, if

$$\lim_{n\to\infty} \operatorname{dist}(\varphi(k; k-n, \xi_{k-n}), \mathscr{A}(k)) = 0 \quad \text{for all } k \in \mathbb{I}. \tag{25}$$

The following observation can be seen as a nonautonomous version of [9, Prop. 3.2]; this result for autonomous strongly order-preserving equations ensures that supremum and infimum of invariant sets attracting upper and lower bounds are always equilibria. Note that we merely assume order-preservation here, but need compactness of the invariant set:

**Proposition 6** *Suppose $(\Delta_f)$ is order-preserving with a compact invariant nonautonomous set $\mathscr{A}$.*

*(a) If $\mathscr{A}$ attracts a sequence $\xi^+$ in $\mathscr{D}$ fulfilling $\alpha_k \preceq \xi_k^+$ for all sequences $\alpha \subseteq \mathscr{A}$, then $\phi_k^+ := \sup \mathscr{A}(k)$ defines an entire solution of $(\Delta_f)$ satisfying*

$$\phi_k^+ = \lim_{n\to\infty} \varphi(k; k-n, \xi_{k-n}^+) \quad \textit{for all } k \in \mathbb{I}. \tag{26}$$

*(b) If $\mathscr{A}$ attracts a sequence $\xi^-$ in $\mathscr{D}$ fulfilling $\xi_k^- \preceq \alpha_k$ for all sequences $\alpha \subseteq \mathscr{A}$, then $\phi_k^- := \inf \mathscr{A}(k)$ defines an entire solution of $(\Delta_f)$ satisfying*

$$\phi_k^- = \lim_{n\to\infty} \varphi(k; k-n, \xi_{k-n}^-) \quad \textit{for all } k \in \mathbb{I}. \tag{27}$$

*Proof* Let $k \in \mathbb{I}$ be fixed.

(a) Since $\mathscr{A}$ is invariant, our assumption on the sequence $\alpha$ implies

$$\alpha_k \preceq \varphi(k;k-n,\xi^+_{k-n}) \quad \text{for all } n \in \mathbb{N}_0,\ \alpha_k \in \mathscr{A}(k). \tag{28}$$

Due to the compactness of each fiber $\mathscr{A}(k)$ and the fact that $\xi^+$ fulfills (25), we find a strictly increasing sequence $(n_l)_{l\in\mathbb{N}}$ in $\mathbb{N}_0$ such that

$$\psi^+_k := \lim_{l\to\infty} \varphi(k;k-n_l,\xi^+_{k-n_l}) \in \mathscr{A}(k)$$

(see Lemma 4). Since this limit fulfills (28) one has $\psi^+_k = \sup \mathscr{A}(k)$. To show that $\psi^+_k$ is the pullback limit of $\xi^+$, i.e. $\psi^+ = \phi^+$, we proceed indirectly and assume (26) does not hold. This means there is a $j \in \mathbb{I}$, a $r > 0$ and a strictly increasing sequence $(m_l)_{l\in\mathbb{N}}$ in $\mathbb{N}_0$ with

$$\left\| \varphi(j;j-m_l,\xi^+_{j-m_l}) - \psi^+_j \right\| \geq r \quad \text{for all } l \in \mathbb{N}. \tag{29}$$

Again, the compactness of $\mathscr{A}(j)$ and (25) guarantee that the limit

$$\psi^*_j := \lim_{n\to\infty} \varphi(j;j-m_{l_n},\xi^+_{j-m_{l_n}}) \in \mathscr{A}(j)$$

exists for a further subsequence $(m_{l_n})_{n\in\mathbb{N}}$ (see again Lemma 4). Thanks to (28) one has the inequality $\alpha_j \preceq \psi^*_j$ for all $\alpha_j \in \mathscr{A}(j)$ and consequently $\psi^*_j = \sup \mathscr{A}(j) = \phi^+_j$. This, however, contradicts (29) and thus (26) is established. It remains to show that $\psi^+$ is an entire solution to $(\Delta_f)$. Since $\psi^+_k = \sup \mathscr{A}(k)$, the invariance of $\mathscr{A}$ yields

$$\alpha_k \preceq \varphi(k;k_0,\psi^+_{k_0}) \quad \text{for all } k_0 \leq k,\ \alpha_k \in \mathscr{A}(k)$$

and therefore

$$\psi^+_k \preceq \varphi(k;k_0,\psi^+_{k_0}).$$

Hence, since $\psi^+_k \in \mathscr{A}(k)$ for $k \in \mathbb{I}$ one finally arrives at $\varphi(k;k_0,\psi^+_{k_0}) = \psi^+_k$ for all $k_0 \leq k$ and so $\psi^+$ solves $(\Delta_f)$, i.e. coincides with $\phi^+$.

(b) can be shown dually. □

In the following, we assume that $\hat{\mathscr{B}}$ is a universe. A compact nonautonomous set $\mathscr{A}^*$ is called a *$\hat{\mathscr{B}}$-attractor* of $(\Delta_f)$, if it is invariant and $\hat{\mathscr{B}}$-attracting. Provided a $\hat{\mathscr{B}}$-attractor is included in an order-interval, we next show that $\mathscr{A}^*$ contains two extremal entire solutions being attracting from above resp. below. For autonomous equations, [5, Thm. 1] proves a related result that limit sets (of points) can always be bracketed between fixed points.

**Theorem 4** *Suppose $(\Delta_f)$ is order-preserving with continuous right-hand sides $f_k$ and a closed $\mathscr{D}$. If a $\hat{\mathscr{B}}$-attractor $\mathscr{A}^*$ satisfies $\mathscr{A}^* \subseteq [\xi^-,\xi^+]$ with sequences $\xi^-,\xi^+ \in \hat{\mathscr{B}}$, then there exist entire solutions $\phi^-,\phi^+$ of $(\Delta_f)$ in $\mathscr{A}^*$ with*

$$\phi^-_k \preceq x \preceq \phi^+_k \quad \textit{for all } (k,x) \in \mathscr{A}^* \tag{30}$$

*and the following properties:*

*(a) $\phi^-$ is* globally pullback attracting from below, *i.e. sequences $\xi \in \hat{\mathscr{B}}$ with $\xi_k \preceq \phi^-_k$ fulfill*

$$\lim_{n\to\infty} \varphi(k;k-n,\xi_{k-n}) = \phi^-_k \quad \textit{for all } k \in \mathbb{I}. \tag{31}$$

*(b)* $\phi^+$ *is* globally pullback attracting from above*, i.e. sequences* $\xi \in \hat{\mathscr{B}}$ *with* $\phi_k^+ \preceq \xi_k$ *fulfill*

$$\lim_{n\to\infty} \varphi(k; k-n, \xi_{k-n}) = \phi_k^+ \quad \text{for all } k \in \mathbb{I}. \tag{32}$$

*Remark 3 (global attractor)* A $\hat{\mathscr{B}}$-attractor is called *global attractor* of $(\Delta_f)$, if the universe $\hat{\mathscr{B}}$ contains all uniformly bounded sets. Due to (30) a global attractor $\mathscr{A}^*$ satisfies

$$\mathscr{A}^* \subseteq [\phi^-, \phi^+].$$

*Proof* Let $k \in \mathbb{I}$. Thanks to Prop. 6 applied to the compact invariant set $\mathscr{A}^*$, we obtain the entire solutions $\phi^\pm$.

(a) If $\xi \in \hat{\mathscr{B}}$ is a sequence with $\xi_k \preceq \phi_k^-$, then the monotone convergence in (27) yields

$$\varphi(k; k-n, \xi_{k-n}) \preceq \phi_k^- \quad \text{for all } n \in \mathbb{N}_0$$

and in addition $\xi$ satisfies (25). The compactness of $\mathscr{A}^*$ and an argument as in the previous proof of Prop. 6 now implies (31).

(b) Similarly, one establishes the assertion (b). □

**Corollary 6** *Suppose* $(\Delta_f)$ *is* $\hat{\mathscr{B}}$*-asymptotically compact and that* $\hat{\mathscr{B}}$ *contains a* $\hat{\mathscr{B}}$*-absorbing order interval. If* $(\Delta_f)$ *has a unique entire solution* $\phi^* \in \hat{\mathscr{B}}$*, then* $\phi^*$ *is a* $\hat{\mathscr{B}}$*-attractor.*

*Proof* Due to [18, p. 19, Thm. 1.3.9] the difference equation $(\Delta_f)$ has a $\hat{\mathscr{B}}$-attractor $\mathscr{A}^*$ being contained in the $\hat{\mathscr{B}}$-absorbing order interval. Thus, Thm. 4 applies and yields the two entire solutions $\phi^-, \phi^+ \in \hat{\mathscr{B}}$. Since we assumed a unique entire solution to $(\Delta_f)$ in $\hat{\mathscr{B}}$, it is $\phi^- = \phi^+$ and (30) implies the claim. □

## 5 Applications

We abbreviate $\mathbb{R}_+ := [0, \infty)$ and suppose that $\mathbb{I}$ is unbounded below.

### 5.1 Leslie-Gower equation

In the Banach space $X = \mathbb{R}^2$ define the closed extended state space $\mathscr{D} = \mathbb{I} \times \mathbb{R}_+^2$. Let us consider $(\Delta_f)$ with $C^1$-right-hand sides $f_k : \mathbb{R}_+^2 \to \mathbb{R}_+^2$, $k \in \mathbb{I}'$,

$$f_k(u) := \begin{pmatrix} \frac{\alpha_k u_1}{1+u_1+\beta_k u_2} \\ \frac{\gamma_k u_2}{1+\delta_k u_1+u_2} \end{pmatrix} \tag{33}$$

and real parameter sequences $\alpha_k, \beta_k, \gamma_k, \delta_k > 0$, $k \in \mathbb{I}'$; one speaks of the *Leslie-Gower equation*. This is a nonautonomous generalization of a planar competition model from [4].

For every $u$ from the convex set $\mathbb{R}_+^2$ the derivative

$$Df_k(u) := \begin{pmatrix} \frac{\alpha_k}{1+u_1+\beta_k u_2}\left(1 - \frac{u_1}{1+u_1+\beta_k u_2}\right) & -\frac{\beta_k}{(1+u_1+\beta_k u_2)^2} \\ -\frac{\delta_k}{(1+\delta_k u_1+u_2)^2} & \frac{\gamma_k}{1+\delta_k u_1+u_2}\left(1 - \frac{u_2}{1+\delta_k u_1+u_2}\right) \end{pmatrix}$$

leaves the south east quadrant invariant and Lemma 5 ensures that $(\Delta_f)$ is order-preserving w.r.t. the cone $X_+ := \mathbb{R}_+ \times (-\mathbb{R}_+)$. Explicitly, the order-relation is

$$u \preceq \bar{u} \quad \Leftrightarrow \quad u_1 \le \bar{u}_1 \text{ and } \bar{u}_2 \le u_2.$$

**Lemma 7** *The nonautonomous Leslie-Gower equation* $(\Delta_f)$ *with right-hand side* (33) *possesses a sub-solution* $\xi^-$ *and a super-solution* $\xi^+$ *satisfying* $\xi_k^- \preceq \xi_k^+$, $k \in \mathbb{I}$, *given by*

$$\xi_k^- := \begin{pmatrix} 0 \\ \gamma_{k-1} \end{pmatrix}, \qquad \xi_k^+ := \begin{pmatrix} \alpha_{k-1} \\ 0 \end{pmatrix} \quad \textit{for all } k \in \mathbb{I}.$$

*Proof* For every $u \in \mathbb{R}_+^2$ and $k \in \mathbb{I}'$ one has $\xi_k^- \preceq \xi_k^+$,

$$0 \le \frac{\alpha_k u_1}{1+u_1+\beta_k u_2} \le \frac{\alpha_k u_1}{1+u_1} \le \alpha_k, \qquad 0 \le \frac{\gamma_k u_2}{1+\delta_k u_1+u_2} \le \frac{\gamma_k u_2}{1+u_2} \le \gamma_k.$$

Consequently, it is

$$\xi_{k+1}^- = \begin{pmatrix} 0 \\ \gamma_k \end{pmatrix} \preceq \begin{pmatrix} 0 \\ \frac{\gamma_k \gamma_{k-1}}{1+\gamma_{k-1}} \end{pmatrix} = f_k(\xi_k^-), \qquad f_k(\xi_k^+) = \begin{pmatrix} \frac{\alpha_k \alpha_{k-1}}{1+\alpha_{k-1}} \\ 0 \end{pmatrix} \preceq \begin{pmatrix} \alpha_k \\ 0 \end{pmatrix} = \xi_{k+1}^+$$

for all $k \in \mathbb{I}'$ and therefore the claim holds. □

This simple observation discloses several consequences: Firstly, from Cor. 2(c) we at once deduce the forward invariance of the order interval

$$[\xi^-, \xi^+] := \left\{(k,u) \in \mathbb{I} \times \mathbb{R}_+^2 : u \in [0, \alpha_{k-1}] \times [0, \gamma_{k-1}]\right\}$$

and since the cone $X_+$ is regular, Thm. 2 yields the two extremal pullback solutions

$$\phi_k^- := \lim_{n\to\infty} \varphi(k; k-n, (0, \gamma_{k-n-1})) = \sup_{n\in\mathbb{N}_0} \varphi(k; k-n, (0, \gamma_{k-n-1})),$$
$$\phi_k^+ := \lim_{n\to\infty} \varphi(k; k-n, (\alpha_{k-n-1}, 0)) = \inf_{n\in\mathbb{N}_0} \varphi(k; k-n, (\alpha_{k-n-1}, 0))$$

for all $k \in \mathbb{I}$. Secondly, since the mapping $f_k$ is globally bounded, the order interval $[\xi^-, \xi^+]$ also pullback attracts all bounded nonautonomous subsets of $\mathscr{D}$ and contains every bounded entire solutions to $(\Delta_f)$. In particular, $(\Delta_f)$ has a global attractor $\mathscr{A}^*$ being embraced by the order interval $[\phi^-, \phi^+]$, where $\phi^-$ is globally pullback attracting from below, while $\phi^+$ turns out to be globally pullback attracting from above due to Thm. 4. Ultimately, because the assumptions of Thm. 3 are fulfilled, one even obtains that every bounded sequence $(\xi_k)_{k\in\mathbb{I}}$ is pullback attracted to an entire solution $\phi^* \in \mathscr{A}^*$ in the sense of (23).

### 5.2 Mackey-Glass equations

Let $d \in \mathbb{N}$ be given. In the Banach space $X = \mathbb{R}^{d+1}$ we use the *discrete exponential ordering* (cf. [13,15]) induced by the solid regular cone

$$X_\mu^+ := \left\{(x_0, \dots, x_d) \in \mathbb{R}^{d+1} : 0 \le x_d,\, \mu x_{j+1} \le x_j \quad \text{for all } 0 \le j < d\right\}$$

with some real parameter $\mu \ge 0$. It is clear that $X_\mu^+ \subseteq \mathbb{R}_+^{d+1}$ is regular and thus normal. If we denote the $X_\mu^+$-induced order on $\mathbb{R}^{d+1}$ by $\preceq_\mu$, then one easily shows

$$x \preceq_\mu y \quad \Rightarrow \quad x_0 \le y_0 \quad \text{for all } x, y \in \mathbb{R}^{d+1}, \tag{34}$$

where we enumerate the components of $x$ by $x = (x_0, x_1, \dots, x_d)$ and similarly for $y$.

Our interest focusses on the so-called *Mackey-Glass equation*, i.e. the nonautonomous delay difference equation

$$x_{k+1} = ax_k + \frac{\beta_k}{1+x_{k-d}^p}. \tag{35}$$

The growth factor $a > 0$ is a real number, $p \in \mathbb{N}$ and on the positive real sequence $(\beta_k)_{k\in\mathbb{I}'}$ we impose the boundedness assumption

$$b := \sup_{k\in\mathbb{I}'} \beta_k < \infty.$$

As a higher-order difference equation, (35) does not immediately fit into the setting of our paper. However, its dynamical behavior is fully characterized by an equivalent first order equation $(\Delta_f)$ with the right-hand side $f_k : \mathbb{R}_+^{d+1} \to \mathbb{R}_+^{d+1}$,

$$f_k(u_0,\dots,u_k) := \begin{pmatrix} au_0 + \frac{\beta_k}{1+u_d^p} \\ u_0 \\ \vdots \\ u_{d-1} \end{pmatrix} \quad \text{for all } k \in \mathbb{I}', \tag{36}$$

whose extended state space $\mathscr{D} = \mathbb{I} \times \mathbb{R}_+^{d+1}$ is closed. Indeed, given an initial time $k_0 \in \mathbb{I}$ and an initial value $u_{k_0} = (\xi_0, \xi_{-1}, \dots, \xi_{-d})$, then the solution $(x_k)_{k\geq k_0-d}$ to (35) satisfying

$$x_{k_0} = \xi_0, \qquad x_{k_0-1} = \xi_{-1}, \dots \qquad x_{k_0-d} = \xi_{-d}$$

is determined by the 0th component of the general solution to $(\Delta_f)$, namely

$$x_k = \varphi_0(k; k_0, (\xi_0, \xi_{-1}, \dots, \xi_{-d})) \quad \text{for all } k \geq k_0. \tag{37}$$

**Lemma 8** *If there exists a $\mu \geq 0$ such that*

$$\mu^{d+1} + pb \leq a\mu^d, \tag{38}$$

*then the nonautonomous Mackey-Glass equation $(\Delta_f)$ with right-hand side* (36) *is order-preserving and super-homogeneous.*

Fig. 1 illustrates that the assumption (38) can be fulfilled, but large delays $d$ or powers $p$ require small values of the bound $b$.

*Proof* For the right-hand side of (35) we abbreviate $\phi(u_0, u_d) := au_0 + \frac{\beta_k}{1+u_d^p}$.

(I) Let $x, y \in \mathbb{R}_+^{d+1}$ fulfill $x_j \leq y_j$ for all $0 \leq j \leq d$ and we obtain

$$\begin{aligned} \phi(y_0,y_d) - \phi(x_0,x_d) &\geq \inf_{(\xi,\eta)\in\mathbb{R}_+^2} D_1\phi(\xi,\eta)(y_0-x_0) + \inf_{(\xi,\eta)\in\mathbb{R}_+^2} D_2\phi(\xi,\eta)(y_d-x_d) \\ &\geq a(y_0-x_0) - p\beta_k \sup_{\eta\geq 0} \frac{\eta^{p-1}}{(1+\eta^p)^2}(y_d-x_d). \end{aligned}$$

Due to the elementary estimates $\frac{\eta^{p-1}}{(1+\eta^p)^2} = \frac{\eta^{-1}}{\eta^{-p}+2+\eta^p} \leq \frac{1}{3}$ for all $\eta \geq 1$ and $\frac{\eta^{p-1}}{(1+\eta^p)^2} \leq 1$ for every $\eta \leq 1$, one consequently has

$$\phi(y_0,y_d) - \phi(x_0,x_d) \geq a(y_0-x_0) - pb(y_d-x_d).$$

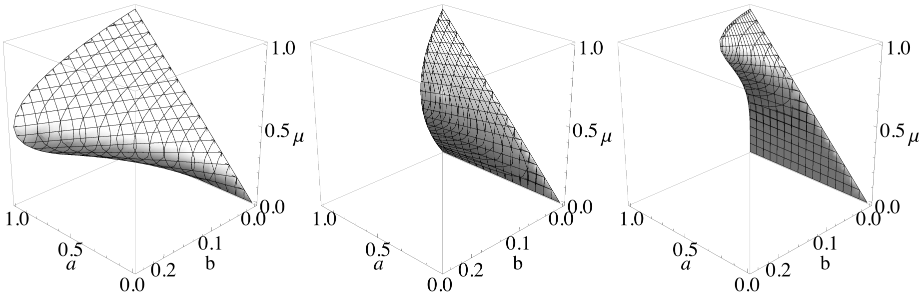


**Fig. 1** Triples $(a,b,\mu)$ for which the assumed inequality (38) of Lemma 8 is satisfied with $d=p=1$ (left), $d=1,p=5$ (center) and $d=5,p=1$ (right)

Now it follows from [13, Prop. 2] that $(\Delta_f)$ is order-preserving w.r.t. $X_\mu^+$.

(II) Suppose $k\in\mathbb{I}'$, $t\geq 1$ and $x\in\mathbb{R}_+^{d+1}$. The only non-zero component of $tf_k(x)-f_k(tx)$ is the zeroth, namely $\frac{\beta_k t}{1+x_d^p}-\frac{\beta_k}{1+(tx_d)^p}$ and thus the inclusion $tf_k(x)-f_k(tx)\in X_\mu^+$ holds if and only if $\frac{\beta_k}{1+(tx_d)^p}\leq\frac{\beta_k t}{1+x_d^p}$, which is true. □

From now on we choose a fixed $\mu\geq 0$ satisfying the assumption (38) from Lemma 8 and a growth rate $a\in(0,1)$. It is clear that the nonautonomous delay equation (35) gets dominated by its autonomous counterpart

$$x_{k+1}=ax_k+\frac{b}{1+x_{k-d}^p}. \tag{39}$$

Now it is not difficult to see that (39) has a unique equilibrium $x^*>0$ which, in turn, yields a family $tx^*$, $t\geq 1$, of constant super-solutions to (35). Moreover, the zero sequence is a sub-solution to (35), thus

$$\xi_k^-:\equiv\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix},\qquad\qquad \xi_k^+:\equiv tx^*e\quad\text{for all } k\in\mathbb{I}$$

are sub- resp. super-solutions to $(\Delta_f)$, where $e:=(1,\dots,1)\in\mathbb{R}^{d+1}$. By Cor. 2(c)

$$[0,\xi^+]=\left\{(k,x)\in\mathbb{I}\times\mathbb{R}_+^{d+1}: x_d\in[0,tx^*], x_j-\mu x_{j+1}\in[0,(1-\mu)tx^*]\text{ for all } 0\leq j<d\right\}$$

are forward invariant order intervals w.r.t. $(\Delta_f)$ for every $t\geq 1$ and $\mu\in[0,1)$.

We next investigate the (pullback-) asymptotics to $(\Delta_f)$ and (35). For every entire solution $(\phi_k)_{k\in\mathbb{I}}$ to $(\Delta_f)$ its 0th component $((\phi_k)_0)_{k\in\mathbb{I}}$ yields an entire solution to the nonautonomous Mackey-Glass equation (35). Due to Thm. 2 there exist two extremal pullback solutions

$$\begin{aligned}\phi_k^-&:=\lim_{n\to\infty}\varphi(k;k-n,0)=\sup_{n\in\mathbb{N}_0}\varphi(k;k-n,0),\\ \phi_k^+&:=\lim_{n\to\infty}\varphi(k;k-n,tx^*e)=\inf_{n\in\mathbb{N}_0}\varphi(k;k-n,tx^*e)\quad\text{for all } k\in\mathbb{I}\end{aligned}$$

to $(\Delta_f)$, whose 0th components $x_k^\pm := (\phi_k^\pm)_0$ define entire solutions to (35). Due to the implication (34) the estimate (22) yields

$$0 \le x_k^- \le x_k^+ \le tx^* \quad \text{for all } k \in \mathbb{I}.$$

Let $\hat{\mathscr{B}}$ be the universe of all uniformly bounded nonautonomous subsets of $\mathscr{D}$. Our next result guarantees that (35) (i.e. $(\Delta_f)$) has a global attractor, which contains a bounded entire solution attracting all sequences above the absorbing set w.r.t. $\preceq_\mu$.

**Proposition 7** *If $a \in (0,1)$ and $\mu > 0$ fulfills the assumption of Lemma 8, then the nonautonomous Mackay-Glass equation* (35) *possesses a global attractor $\mathscr{A}^*$ contained in the nonautonomous set $\mathbb{I} \times \left[0, \frac{b}{1-a}\right]^{d+1}$. The sequence $x_k^* := (\sup \mathscr{A}^*(k))_0$ defines a bounded entire solution to* (35) *satisfying*

$$x_k^* = \lim_{n\to\infty} \varphi_0(k; k-n, \xi_{k-n}) \quad \textit{for all } k \in \mathbb{I}$$

*and every sequence $(\xi_k)_{k\in\mathbb{I}}$ with $\frac{b}{1-a} \preceq_\mu \xi_k$.*

*Proof* First, we deduce from [17, Ex. 3.4] that $(\Delta_f)$ has $\hat{\mathscr{B}}$-absorbing set $\mathbb{I} \times \left[0, \frac{b}{1-a}\right]^{d+1}$. Due to [18, p. 19, Thm. 1.3.9] there exists a global attractor $\mathscr{A}^*$ for $(\Delta_f)$, which is contained in the above absorbing set. By means of Lemma 8 we can apply Prop. 6(a) to this compact invariant set $\mathscr{A}^*$, which particularly attracts all bounded sequences $\xi \subseteq \mathscr{D}$. It guarantees that $\phi_k^* := \sup \mathscr{A}^*(k)$ defines an entire bounded solution to $(\Delta_f)$ fulfilling (26), notably for sequences $(\xi_k)_{k\in\mathbb{I}}$ in $\mathbb{R}_+^{d+1}$ with $\frac{b}{1-a} e \preceq_\mu \xi_k$ for all $k \in \mathbb{I}$. The limit relation (26) implies convergence in the first component and we obtain the assertion. □

### 5.3 Integro-difference equations

In this closing example, we assume that $\Omega \subseteq \mathbb{R}^d$ is a compact set with nonempty interior. On the space $X = C(\Omega)$ of continuous real-valued functions equipped with the sup-norm one considers the solid, normal and minihedral cone (cf. Tab. 1)

$$C_+(\Omega) := \{u : \Omega \to \mathbb{R} \,|\, 0 \le u(x) \text{ for all } x \in \Omega\}$$

of nonnegative functions.

This subsection investigates an integro-difference equations $(\Delta_f)$ with the right-hand side $f_k : C_+(\Omega) \to C_+(\Omega)$, $k \in \mathbb{I}'$, given by

$$f_k(u) := \int_\Omega K_k(\cdot, y) F_k(y, u(y))\, dy : \Omega \to \mathbb{R}. \tag{40}$$

For the sake of well-defined nonlinear Fredholm operators $f_k$ being ambient for our analysis, let us impose the following standing assumptions:

- Both the densities $K_k : \Omega \times \Omega \to \mathbb{R}_+$, as well as the nonlinearities $F_k : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous with nonnegative values. This shows that $f_k : C_+(\Omega) \to C_+(\Omega)$, $k \in \mathbb{I}'$, are well-defined and completely continuous.
- $F_k(y, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing for all $k \in \mathbb{I}'$ and $y \in \Omega$. The monotonicity of the integral implies that $(\Delta_f)$ is an order-preserving equation w.r.t. the cone $C_+(\Omega)$. Moreover,

$$F_k(y, 0) = 0 \quad \text{for all } k \in \mathbb{I}', y \in \Omega \tag{41}$$

guarantees that $(\Delta_f)$ has the trivial solution.

– There exist continuous functions $\gamma_k : \Omega \to \mathbb{R}$ such that

$$K_k(x,y)F_k(y,u) \le \gamma_k(y) \quad \text{for all } y \in \Omega,\, u \ge 0$$

and $\sup_{k\in\mathbb{I}'} \int_\Omega \gamma_k(y)\,dy < \infty$.

Boundedness assumptions on the integrand in (40) give rise to an absorbing set in the universe $\hat{\mathscr{B}}$ of all uniformly bounded nonautonomous subsets of $\mathbb{I} \times C_+(\Omega)$:

**Lemma 9** *The nonautonomous order interval* $[0,\rho] \subseteq \mathbb{I} \times C_+(\Omega)$ *with constant functions*

$$\rho_k : \Omega \to \mathbb{R}^+, \qquad\qquad \rho_k(x) :\equiv \int_\Omega \gamma_{k-1}(y)\,dy \tag{42}$$

*is uniformly bounded, forward invariant and* $\hat{\mathscr{B}}$*-absorbing w.r.t.* $(\Delta_f)$.

*Proof* Let $k \in \mathbb{I}'$ be fixed. For every continuous function $u : \Omega \to \mathbb{R}_+$ we have

$$0 \le \int_\Omega K_k(x,y)F_k(y,u(y))\,dy \le \int_\Omega \gamma_k(y)\,dy \equiv \rho_{k+1} \quad \text{for all } x \in \Omega$$

and this implies $f_k(C_+(\Omega)) \subseteq [0,\rho_{k+1}]$. Hence, for every bounded set $B \subset C_+(\Omega)$ one obtains $f_k(B) \subseteq [0,\rho_{k+1}]$ and in particular $f_k([0,\rho_k]) \subseteq [0,\rho_{k+1}]$, i.e. $[0,\rho]$ is forward invariant and $\hat{\mathscr{B}}$-absorbing. By assumption, the constant function $\sup_{k\in\mathbb{I}'}\rho_k$ is an upper bound for every order interval $[0,\rho_k] \subseteq C_+(\Omega)$ and since the cone $C_+(\Omega)$ is solid and normal, Lemma 3(b) implies that the nonautonomous set $[0,\rho]$ is uniformly bounded. □

**Proposition 8** *Given the sequence* $\rho_k : \Omega \to \mathbb{R}_+$, $k \in \mathbb{I}'$, *of constant functions from* (42)*, the integro-difference equation* $(\Delta_f)$ *with right-hand side* (40) *possesses a* $\hat{\mathscr{B}}$*-attractor* $\mathscr{A}^* \in \hat{\mathscr{B}}$ *with the following properties:*

*(a)* $\phi^+ = \omega_\rho \subseteq \mathscr{A}^* \subseteq [0,\phi^+] \subseteq [0,\rho]$, *where*

$$\phi_k^+ := \lim_{n\to-\infty} \varphi\left(k; k-n, \int_\Omega \gamma_{k-n-1}(y)\,dy\right) \quad \textit{for all } k \in \mathbb{I}', \tag{43}$$

*(b)* $\phi^+$ *is globally pullback-attracting from above, i.e. every sequence* $(\xi_k)_{k\in\mathbb{I}}$ *in* $\hat{\mathscr{B}}$ *satisfying* $\phi_k^+(x) \le \xi_k(x)$ *on* $x \in \Omega$ *fulfills*

$$\lim_{n\to\infty} \varphi(k; k-n, \xi_{k-n}) = \phi_k^+ \quad \textit{for all } k \in \mathbb{I}'.$$

*Proof* Let $k \in \mathbb{I}'$ be fixed. Thanks to Lemma 9 the equation $(\Delta_f)$ has the uniformly bounded $\hat{\mathscr{B}}$-absorbing set $\mathscr{A} := [0,\rho]$ and thus $\mathscr{A} \in \hat{\mathscr{B}}$. On the other and, the right-hand sides $f_k$ are completely continuous and consequently $(\Delta_f)$ is $\hat{\mathscr{B}}$-asymptotically compact by [18, p. 13, Cor. 1.2.22]. Hence, [18, p. 19, Cor. 1.3.9] guarantees that there exists a uniformly bounded $\hat{\mathscr{B}}$-attractor $\mathscr{A}^*$. Obviously, the trivial solution $\xi_k^- := 0$ is a sub-solution, while Cor. 2(b) shows that $\xi_k^+ := \rho_k$ is a super-solution to $(\Delta_f)$. The limit (43) exists by Thm. 2.

(a) The relation $\phi^+ = \omega_\rho$ follows from Prop. 5 and since $\phi^+$ is a bounded entire solution, [18, p. 17, Cor. 1.3.4] implies $\phi^+ \subseteq \mathscr{A}^*$. Because $\xi^+$ is a super-solution, Thm. 2 additionally implies $\phi_k^+ \preceq \rho_k$ and $\mathscr{A}^* \subseteq [0,\rho^+]$ yields by Thm. 4.

(b) is an immediate consequence of Thm. 4(b). □

*Example 3 (Beverton-Holt nonlinearity)* We choose the domain $\Omega = \left[-\frac{L}{2}, \frac{L}{2}\right]$ for some length $L > 0$ and nonlinearities in (40) of Beverton-Holt form

$$F_k(y,u) := \frac{a_k u}{1+u}$$

with a positive real bounded sequence $(a_k)_{k\in\mathbb{I}'}$. Concerning the kernel in (40), the following choices were suggested by [12]:

- $K^1(x,y) := \frac{\alpha}{2}e^{-\alpha|x-y|}$ (*Laplace kernel*, $\alpha > 0$)
- $K^2(x,y) := \frac{\pi}{4R}\cos\left(\frac{\pi}{2R}(x-y)\right)$ (*cosine kernel*, $R \geq L$)

It is clear that the assumptions of Prop. 8 are satisfied with

$$\rho_k(y) :\equiv LC_i a_k, \qquad C_i := \begin{cases} \frac{\alpha}{2}, & i=1, \\ \frac{\pi}{4L}, & i=2, \end{cases}$$

where the index $i \in \{1,2\}$ refers to the respective kernel $K^i$. In Fig. 2 (Laplace kernel) and Fig. 3 (cosine kernel) we depicted numerical approximations of the sets

$$A_k := \left\{(x,y) \in \mathbb{R}^2 : |x| \leq \tfrac{L}{2}, 0 \leq y \leq \phi_k^+(x)\right\} \quad \text{for all } k \in \mathbb{I},$$

where $\phi_k^+ : \Omega \to \mathbb{R}_+$, $k \in \mathbb{I}$, denotes the entire solution to $(\Delta_f)$ from (8). Due to Prop. 8, the functions contained in the attractor fibers $\mathscr{A}^*(k)$ have graphs within $A_k \subseteq \mathbb{R}^2$ for all $k \in \mathbb{I}$. Note that the particular sequence $a_k := 2 - \tanh k$ is asymptotically constant, i.e. one has the limit relations $\lim_{k\to-\infty} a_k = 3$ and $\lim_{k\to\infty} a_k = 1$.

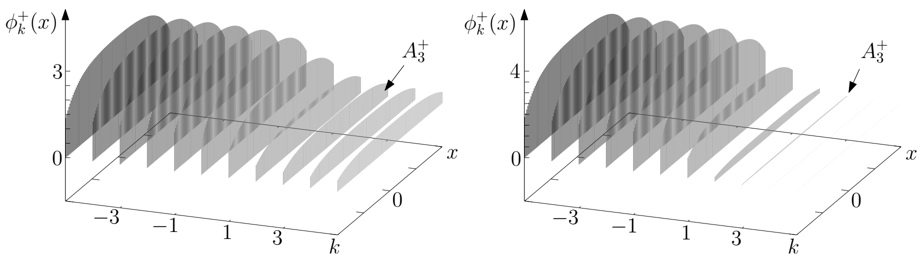


**Fig. 2** Numerically computed sets $A_k$ for $k \in \{-5,\dots,5\}$ containing the graphs of the functions in the attractor $\mathscr{A}^*$ of $(\Delta_f)$ with right-hand side (40) for the Laplace kernel with $\alpha = 4$, $L = 2$ given the asymptotically constant sequences $a_k := 2 - \tanh k$ (left) and $a_k := 2 - 2\tanh k$ (right)

## 6 Conclusion and context

We obtained several results on the long-term behavior of order-preserving nonautonomous difference equations $(\Delta_f)$ in Banach spaces. Our overall approach was based on the fact that their general solutions are continuous mappings $\varphi(k;k_0,\cdot)$ fulfilling the cocycle property (4). In this case, one also speaks of a *process* — a notion representing one possibility to describe nonautonomous dynamics. Here it is conceptionally irrelevant, if $k_0 \leq k$ are integers ($\mathbb{T} = \mathbb{Z}$) or reals ($\mathbb{T} = \mathbb{R}$) (see [11, p. 24, Def. 2.1]).

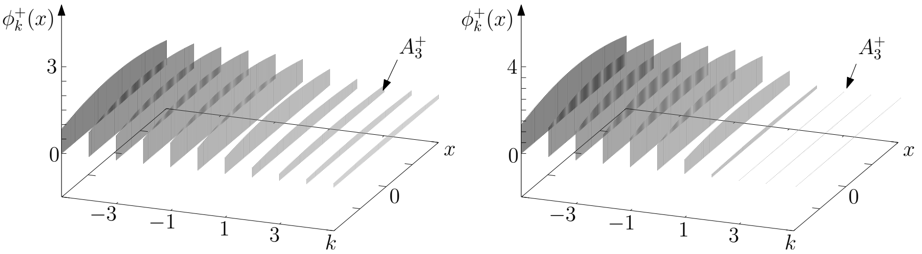


**Fig. 3** Numerically computed sets $A_k$ for $k \in \{-5,\dots,5\}$ containing the graphs of the functions in the attractor $\mathscr{A}^*$ of $(\Delta_f)$ with right-hand side (40) for the cosine kernel with $R = 2$, $L = 2$ given the asymptotically constant sequences $a_k := 2 - \tanh k$ (left) and $a_k := 2 - 2\tanh k$ (right)

Alternatively, a generalization and unification can be given by means of the following notion having its origin both in skew-product flows (see [19, 11]), as well as metric and random dynamical systems (RDSs for short, cf. [3]). For the sake of a precise definition, let $P \neq \emptyset$ be a set. A *nonautonomous dynamical system* on a metric space $X$ with *base space* $P$ is a pair of mappings $\theta : \mathbb{T} \times P \to P$, $\Phi : \mathbb{T} \times P \times X \to X$ satisfying:

(i) The *base flow* $\theta$ fulfills the flow properties

$$\theta(0,p) = 0, \qquad \theta(k+l,p) = \theta(k,\theta(l,p)) \quad \text{for all } p \in P, k,l \in \mathbb{T},$$

(ii) $\Phi$ is a *cocycle* over $\theta$, i.e. for all $k,l \in \mathbb{T}$ and $p \in P$, $x \in X$ we have

$$\Phi(0,p,x) = x, \qquad \Phi(k+l,p,x) = \Phi(k,\theta(l,p),\Phi(l,p,x)),$$

(iii) $\Phi$ is continuous in the third argument.

We point out two important special cases:

- A continuous process $\varphi$ gives rise to a nonautonomous dynamical system on the (topological) base space $P = \mathbb{T} \in \{\mathbb{R},\mathbb{Z}\}$ via the base flow $\theta(k,l) := k + l$ and the cocycle $\Phi(k,l,x) := \varphi(k+l;l,x)$ for all $x \in X$. For $\mathbb{T} = \mathbb{Z}$ this particularly captures nonautonomous difference equations as considered throughout this paper.
- In case of RDSs the base flow is a metric dynamical system on a probability space $(P,\mathscr{F},\mathbb{P})$ (cf. [3, p. 10, Def. 1.1.1]). A corresponding theory for order-preserving RDSs was carefully developed in [3].

Given this observation, on the one hand various of our results clearly allow an extension to nonautonomous dynamical systems as above in discrete and continuous time. Nonetheless, for the sake of a non-technical presentation we restricted to the difference equations situation. In addition, we prepare applications given in [10]. On the other hand, we deduced *deterministic nonautonomous* counterparts to related results obtained in [3] for RDSs.

## References


1. F. Cao and J. Jiang: On the global attractivity of monotone random dynamical systems, Proc. Am. Math. Soc. 138(3), 891–898 (2009)
2. A. Carvalho, J. Langa and J. Robinson: Attractors for infinite-dimensional non-autonomous dynamical systems, Applied Mathematical Sciences 182, Springer, Berlin etc., 2012
3. I. Chueshov: Monotone Random Systems. Theory and Applications, Lect. Notes Math. 1779, Springer, Berlin etc., 2002

4. J. Cushing, S. LeVarge, N. Chitnis and S. Henson: Some discrete competition models and the competitive exclusion principle, J. Difference Equ. Appl. 10(13–15), 1139–1151 (2004)
5. E.N. Dancer: Some remarks on a boundedness assumption for monotone dynamical systems, Proc. Am. Math. Soc. 126(3), 801–807 (1998)
6. K. Deimling: Nonlinear Functional Analysis, Springer, Berlin etc., 1985
7. D.P. Hardin, P. Takáč and G.F. Webb: A comparison of dispersal strategies for survival of spatially heterogeneous populations, SIAM J. Appl. Math. 48(6), 1396–1423 (1988)
8. M. Hirsch and H. Smith: Monotone dynamical systems, in A. Cañada (editor), *Ordinary Differential Equations II*, pages 239–357. Elsevier, Amsterdam, 2005
9. M. Hirsch and H. Smith: Monotone maps: A review, J. Difference Equ. Appl. 11(4–5), 379–398 (2005)
10. T. Hüls and C. Pötzsche: Qualitative analysis of a nonautonomous Beverton-Holt Ricker model, SIAM J. on Applied Dynamical Systems, to appear (2014)
11. P. Kloeden and M. Rasmussen: Nonautonomous Dynamical Systems, Mathematical Surveys and Monographs 176, AMS, Providence RI, 2011
12. M. Kot and W. Schaefer: Discrete-time growth-dispersal models, Math. Biosci. 80, 109–136 (1986)
13. U. Krause and M. Pituk: Boundedness and stability for higher-order difference equations. J. Difference Equ. Appl. 10(4), 343–356 (2004)
14. U. Krause: Stability of non-autonomous population models with bounded and periodic enforcement, J. Difference Equ. Appl. 15(7), 649–658 (2009)
15. E. Liz and M. Pituk: Asymptotic estimates and exponential stability for higher order monotone difference equations, Adv. Difference Equ. 2005(1), 41–55 (2005)
16. T. Nesemann: A nonlinear extension of the Coale-Lopez theorem, Positivity 3, 135–148 (1999)
17. C. Pötzsche: Dissipative delay endomorphisms and asymptotic equivalence, Advanced Studies in Pure Mathematics 53, 249–271 (2009)
18. C. Pötzsche: Geometric theory of discrete nonautonomous dynamical systems, Lect. Notes Math. 2002, Springer, Berlin etc., 2010
19. W. Shen and Y. Yi: Almost automorphic and almost periodic dynamics in skew-product semiflows, Memoirs of the AMS 647, AMS, Providence RI, 1998
20. H. Smith: Monotone Dynamical Systems. An Introduction to the Theory of Competitive and Cooperative Systems, AMS, Providence RI, 1996