Slow and Fast Variables in Nonautonomous Difference Equations*

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Abstract

In this paper we present an existence and smoothness result for center-like invariant manifolds of nonautonomous difference equations with slow and fast state-space variables. This result can be seen as a first step to obtain Fenichel's geometric theory for difference equations. Hereby, our basic tool is an abstract integral manifold theorem.
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1 Introduction and Preliminaries

Many problems from the applied sciences lead to dynamical systems, where the state space variables have certain components which vary rapidly, and others which vary relatively slowly in time. Usually one approaches such problems within the framework of singular perturbations and integral manifolds. In fact, the investigation of singularly perturbed ordinary differential equations (ODEs) using the method of integral manifolds is a classical matter, since its origins reach back to Baris [3] or even Zadiraka [22]. Meanwhile their approach has been developed further in various directions, and we only mention the contributions of Henry [8], Sacker & Sell [17], Sakamoto [18], Rybakowski [15], as well as the very applicable considerations in Nipp [11, 12] here. Moreover, Fenichel [7] provided a largely complete geometric theory of singular perturbations in continuous time over 20 years ago.

In the case of difference equations the situation is significantly different, because the existing literature on singular perturbations is comparatively small. Using asymptotic methods, Comstock & Hsiao [4], Kelley [9], Suzuki [21], or in a simple situation Agarwal [2, pp. 92–95], obtain results concerning individual solutions instead of a whole set of solutions. Other approaches based on invariant manifolds have their origins in discretization theory of ODEs, namely in the

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occurrence of singularly perturbed Poincaré-mappings (cf. STIEFENHOFER [20]) or in numerical approximation of singularly perturbed differential equations (cf. e.g. NIPP & STOFFER [13]).

The aim of this paper is to show the existence and smoothness of a center-like invariant manifold for certain types of difference equations, which can be interpreted as discrete counterparts of singularly perturbed ODEs with fast time. Such a result in the continuous autonomous situation and the corresponding geometric ideas are already contained in [7], but with a proof heavily depending on a "differential-geometric" approach developed in [6]. However, our technical machinery is purely analytical and basically inspired by [8] and [18, 19].

The present paper is organized in a straight-forward manner: Section 2 contains the two basic technical tools, namely a perturbation result for exponential dichotomies (Theorem 2.1) and the quite general Theorem 2.2 about invariant fiber bundles — the discrete pendant of integral manifolds. Both are new for difference equations and of independent interest, but the proofs are lengthly and technical, and therefore omitted here. The main result (Theorem 3.6) is deduced in Section 3 after some preparations. As a general philosophy, and although this article belongs to the qualitative theory of (nonautonomous) dynamical systems, we have tried to state our results as quantitatively as possible.

Now we introduce our basic terminology. $\mathbb{Z}$ denotes the integers and $\mathbb{N}$ the set of positive integers. The Banach spaces $\mathcal{X}$, $\mathcal{Y}$ are all real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$) throughout this paper and their norm is denoted by $\|\cdot\|_{\mathcal{X}}$, $\|\cdot\|_{\mathcal{Y}}$ or simply by $\|\cdot\|$. On the cartesian product $\mathcal{X} \times \mathcal{Y}$ we always use the norm $\|(x,y)\|_{\mathcal{X}\times\mathcal{Y}} := \max\left\{\|x\|_{\mathcal{X}}, \|y\|_{\mathcal{Y}}\right\}$ and $B_\rho(x)$ is the ball in a normed space with center x and radius $\rho > 0$. $\mathcal{L}_n(\mathcal{X};\mathcal{Y})$ is the Banach space of n-linear continuous operators from $\mathcal{X}^n$ to $\mathcal{Y}$ for $n \in \mathbb{N}$, $\mathcal{L}_0(\mathcal{X};\mathcal{Y}) := \mathcal{Y}$, $\mathcal{L}(\mathcal{X}) := \mathcal{L}_1(\mathcal{X};\mathcal{X})$; $I_{\mathcal{X}}$ is the identity map on $\mathcal{X}$ and $\mathcal{GL}(\mathcal{X})$ the multiplicative group of bijective mappings in $\mathcal{L}(\mathcal{X})$. We write $\mathcal{N}(T) := T^{-1}(\{0\})$ for the kernel and $\mathcal{R}(T) := T(\mathcal{X})$ for the range of an operator $T \in \mathcal{L}(\mathcal{X})$.

DF stands for the Frechét-derivative of a mapping F, and if $F : (x,y) \mapsto F(x,y)$ depends differentiable on more than one variable, then its partial derivatives are denoted by D_1F and D_2F, respectively, provided they exist. Higher order derivatives, like e.g. D_1^nF, $n \in \mathbb{N}_0$, are defined inductively.

Finally, we use the notation

$$\boxed{\Delta x = f(k,x)} \tag{1.1}$$

to denote the difference equation $\Delta x(k) = f(k, x(k))$, with the *forward difference operator* $(\Delta x)(k) := x(k+1) - x(k)$, $k \in \mathbb{Z}$, and the right-hand side $f : \mathbb{Z} \times \mathcal{X} \to \mathcal{X}$. Let $\lambda(\cdot;\kappa,\xi)$ be the *general solution* of equation (1.1), i.e., it solves (1.1) and satisfies the initial condition $\lambda(\kappa;\kappa,\xi) = \xi$ for $\kappa \in \mathbb{Z}$, $\xi \in \mathcal{X}$. In forward time ($k \geq \kappa$), $\lambda(\cdot;\kappa,\xi)$ can be defined recursively as

$$\lambda(k;\kappa,\xi) := \begin{cases} \xi & \text{for } k = \kappa \\ \lambda(k-1;\kappa,\xi) + f(k-1, \lambda(k-1;\kappa,\xi)) & \text{for } k > \kappa \end{cases},$$

and if the inverse mapping $[I_{\mathcal{X}} + f(k-1,\cdot)]^{-1} : \mathcal{X} \to \mathcal{X}$ exists for $k \leq \kappa$, then $\lambda(\cdot;\kappa,\xi)$ can be extended on $\mathbb{Z}$ by defining recursively

$$\lambda(k-1;\kappa,\xi) := [I_{\mathcal{X}} + f(k-1,\cdot)]^{-1}\,\lambda(k;\kappa,\xi) \quad \text{for } k \leq \kappa.$$

A subset $\mathcal{W}$ of the *extended state space* $\mathbb{Z} \times \mathcal{X}$ is called an *invariant fiber bundle* of (1.1), if for any pair $(\kappa,\xi) \in \mathcal{W}$, the solution $\lambda(\cdot;\kappa,\xi)$ exists on $\mathbb{Z}$, and if it follows that $(k, \lambda(k;\kappa,\xi)) \in \mathcal{W}$

for all $k \in \mathbb{Z}$. Given an operator sequence $A : \mathbb{Z} \to \mathcal{L}(\mathcal{X})$, we define the *evolution operator* $\Phi(k, \kappa) \in \mathcal{L}(\mathcal{X})$ of the linear difference equation $\Delta x = A(k)x$ as the mapping given by

$$\Phi(k,\kappa) := \begin{cases} I_{\mathcal{X}} & \text{for } k = \kappa \\ [I_{\mathcal{X}} + A(k-1)] \cdots [I_{\mathcal{X}} + A(\kappa)] & \text{for } k > \kappa \end{cases},$$

and if $I_{\mathcal{X}} + A(k)$ is invertible (in $\mathcal{L}(\mathcal{X})$) for $k < \kappa$, then

$$\Phi(k,\kappa) := [I_{\mathcal{X}} + A(k)]^{-1} \cdots [I_{\mathcal{X}} + A(\kappa-1)]^{-1} \quad \text{for } k < \kappa.$$

2 Exponential Dichotomies and Invariant Fiber Bundles

We need some preparations to establish the announced results in Section 3, namely a nonautonomous concept of hyperbolicity, its robustness under perturbations, and a generalized center-manifold theorem. Thereto consider an operator-valued sequence $A : \mathbb{Z} \to \mathcal{L}(\mathcal{X})$. Then a nonautonomous linear difference equation

$$\boxed{\Delta x = A(k)x} \tag{2.1}$$

is said to have

(i) γ^+-*bounded growth* (with constant C), if there exist real numbers $\gamma > 0$, $C \geq 1$ such that $\|\Phi(k,l)\| \leq C\gamma^{k-l}$ for $k \geq l$,

(ii) (γ, δ)-*bounded growth* (with constant C), if $I_{\mathcal{X}} + A(k) \in \mathcal{GL}(\mathcal{X})$, $k \in \mathbb{Z}$, and if there exist reals $\gamma, \delta > 0$, $C \geq 1$ such that $\|\Phi(k,l)\| \leq C\gamma^{k-l}$ for $k \geq l$ and $\|\Phi(k,l)\| \leq C\delta^{k-l}$ for $l \geq k$.

Differing from the situation of ODEs (cf. Coppel [5, pp. 8–9]), bounded growth of (2.1) can be characterized in terms of boundedness of the coefficient operator A. Precisely, the equation (2.1) has (γ, δ)-bounded growth with constant 1, if and only if

$$\|I_{\mathcal{X}} + A(k)\| \leq \gamma, \qquad \left\| [I_{\mathcal{X}} + A(k)]^{-1} \right\| \leq \delta \quad \text{for } k \in \mathbb{Z}$$

holds. Accordingly, the left inequality is necessary and sufficient for γ^+-bounded growth of (2.1) with constant 1. Furthermore, in the finite dimensional setting $\mathcal{X} = \mathbb{F}^N$, the boundedness of $I_{\mathcal{X}} + A : \mathbb{Z} \to \mathcal{GL}(\mathbb{F}^N)$ implies that $[I_{\mathcal{X}} + A(\cdot)]^{-1} : \mathbb{Z} \to \mathcal{GL}(\mathbb{F}^N)$ is bounded (cf. [5, p. 47, Lemma 1]). Autonomous linear difference equations evidently have bounded growth.

A sequence of projections $P : \mathbb{Z} \to \mathcal{L}(\mathcal{X})$ is called a *regular invariant projector* of (2.1), if

$$P(k+1)A(k) = A(k)P(k), \qquad \mathcal{N}(P(k+1)) \subseteq \mathcal{R}(I_{\mathcal{X}} + A(k)) \quad \text{for } k \in \mathbb{Z}$$

holds. Then it is not difficult to show that the restriction $\bar{\Phi}(k,l) := \Phi(k,l)|_{\mathcal{N}(P(l))} : \mathcal{N}(P(l)) \to \mathcal{N}(P(k))$, $k \geq l$, is a well-defined isomorphism, and we denote its inverse by $\bar{\Phi}(l,k)$. The system (2.1) is said to possess an *exponential dichotomy* with α, β, K_1, K_2 on $\mathbb{Z}$, if there exists a regular invariant projector $P : \mathbb{Z} \to \mathcal{L}(\mathcal{X})$ of (2.1) satisfying

$$\|\Phi(k,l)P(l)\| \leq K_1\alpha^{k-l} \quad \text{for } k \geq l, \qquad \left\|\bar{\Phi}(k,l)\,[I_{\mathcal{X}} - P(l)]\right\| \leq K_2\beta^{k-l} \quad \text{for } l \geq k,$$

where $0 < \alpha < \beta$, $K_1, K_2 \geq 1$ are real constants. In the autonomous situation $A(k) \equiv A$, the equation (2.1) has an exponential dichotomy with α, β, if the spectrum of $I_{\mathcal{X}} + A \in \mathcal{L}(\mathcal{X})$ can be separated into spectral sets disjoint from the annulus $\{\lambda \in \mathbb{C} : \alpha \leq |\lambda| \leq \beta\}$ in the complex plane $\mathbb{C}$. A comprehensive introduction to exponential dichotomies for difference equations can be found in, for example, Henry [8, pp. 229–237].

The next result is crucial and concerns the robustness of exponential dichotomies under slowly varying parameters. Related results for ODEs are provided by e.g., [5, p. 50, Proposition 1], Sacker & Sell [16, Theorem 6], [8, pp. 240–241, Theorem 7.6.12] or Sakamoto [19, Theorem 1]. However, it is new for difference equations to our knowledge.

Theorem 2.1 (slowly varying coefficients)**:** *Let $(\mathcal{Q}, d)$ be a metric space, $A : \mathbb{Z} \times \mathcal{Q} \to \mathcal{L}(\mathcal{X})$ be a mapping, and let $C_1, C_2, L \geq 0$, $K_1, K_2 \geq 1$, $\gamma_1, \gamma_2, \delta_2 > 0$, $0 < \alpha < \beta$ be reals such that for any $q \in \mathcal{Q}$ the following holds:*

(i) A satisfies the Lipschitz condition

$$\|A(k, q) - A(k, \bar{q})\| \leq L d(q, \bar{q}) \quad \textit{for } k \in \mathbb{Z},\ \bar{q} \in \mathcal{Q}, \tag{2.2}$$

(ii) the linear difference equation

$$\boxed{\Delta x = A(k, q)x} \tag{2.3}$$

has γ_1^+-bounded growth with constant C_1,

(iii) (2.3) *possesses an exponential dichotomy with α, β, K_1, K_1 and the regular invariant projector $P_q : \mathbb{Z} \to \mathcal{L}(\mathcal{X})$ on $\mathbb{Z}$.*

Now, if we fix reals γ, δ with $\alpha < \gamma < \delta < \beta$ and choose $h \in \mathbb{N}$ so large that

$$\max\left\{\log_{\frac{\beta}{\alpha}}(K_1 K_2), \log_{\frac{\gamma}{\alpha}}(K_1), \log_{\frac{\beta}{\delta}}(K_2)\right\} < h,$$

then for arbitrary fixed real numbers $\theta_1 < 1 < \theta_2$ with $\frac{\theta_2}{\theta_1} = \frac{1}{2}\left[1 + \left(\frac{\beta}{\alpha}\right)^h\right]$ and any sequence $q^* : \mathbb{Z} \to \mathcal{Q}$ satisfying

(iv) the following Lipschitz condition is fulfilled

$$d(q^*(k), q^*(l)) \leq \vartheta\, |k - l| \quad \textit{for } k, l \in \mathbb{Z}, \tag{2.4}$$

where the Lipschitz constant $\vartheta \geq 0$ is so small that

$$L h \vartheta \leq \varepsilon_0, \qquad L h \max\{K_1, K_2\}\, C(\gamma, \delta) \vartheta \leq \varepsilon_1$$

and $\varepsilon_0, \varepsilon_1 > 0$ suffice the estimates

$$2K_1 \varepsilon_1 \leq \min\{1 - \theta_1, \theta_2 - 1\}, \qquad \bar{C}(\gamma, \delta)\varepsilon < 1,$$

whereby $C(\gamma, \delta) := \frac{K_1}{\delta - \alpha} + \frac{K_2}{\beta - \gamma} + \max\left\{\frac{K_1}{\gamma - \alpha}, \frac{K_2}{\beta - \delta}\right\}$,

$$\begin{aligned} \bar{C}(\gamma, \delta) &:= \frac{h\theta_1 K_1}{\theta_1 \delta^h - K_1 \alpha^h} + \frac{h\theta_2 K_2^2}{\beta^h - \theta_2 K_2 \gamma^h} + \max\left\{\frac{h\theta_1 K_1}{\theta_1 \gamma^h - K_1 \alpha^h}, \frac{h\theta_2 K_2^2}{\beta^h - \theta_2 K_2 \delta^h}\right\}, \\ \varepsilon &:= C_1\left[\varepsilon_0 C_1 (\gamma_1 + C_1 \varepsilon_0)^{h-1} + \frac{1}{h}\frac{2\varepsilon_1 K_1 \gamma_1^h}{1 - 2\varepsilon_1 K_1}\right], \end{aligned}$$

(v) *the linear difference equation*

$$\boxed{\Delta x = A(k, q^*(k))x} \tag{2.5}$$

has (γ_2, δ_2)*-bounded growth with constant* C_2*,*

also the system (2.5) *possesses an exponential dichotomy with* γ, δ*,*

$$\begin{aligned} L_1(\gamma,\delta) &:= \max\left\{1, C_2\left(\frac{\gamma_2}{\gamma}\right)^h\left(\frac{\bar{C}(\gamma,\delta)\delta}{1-\varepsilon\bar{C}(\gamma,\delta)}\right)^2\right\}, \\ L_2(\gamma,\delta) &:= \max\left\{1, C_2\left(\frac{\delta}{\delta_2}\right)^h\left(1+\frac{\bar{C}(\gamma,\delta)\delta}{1-\varepsilon\bar{C}(\gamma,\delta)}\right)\frac{\bar{C}(\gamma,\delta)\delta}{1-\varepsilon\bar{C}(\gamma,\delta)}\right\} \end{aligned} \tag{2.6}$$

and a regular invariant projector $Q : \mathbb{Z} \to \mathcal{L}(\mathcal{X})$ *satisfying*

$$\left\|Q(k) - P_{q^*(k)}(k)\right\| \le \varepsilon_1 + \varepsilon\left(1+\frac{\bar{C}(\gamma,\delta)\delta}{1-\varepsilon\bar{C}(\gamma,\delta)}\right)\frac{\bar{C}(\gamma,\delta)^2\delta}{1-\varepsilon\bar{C}(\gamma,\delta)} \quad \text{for } k \in \mathbb{Z}.$$

Proof. The detailed proof is technically involved and can be found in Pötzsche [14, pp. 121–128, Satz 2.3.6, Korollar 2.3.10 and Korollar 2.3.11]. Nonetheless it is roughly modeled after [8, pp. 240–241, Theorem 7.6.12] or [19, Theorem 1]. □

Now we turn our attention to invariant fiber bundles of nonlinear systems. The related theory on integral manifolds of ODEs is wide-spread, and for the Lipschitz-smooth case we only refer to [19, Theorem 6], whereas [8, pp. 275–277, Theorem 9.1.1] also treats C^1-smoothness. Thereto let $\mathcal{X}, \mathcal{Y}$ be Banach spaces over $\mathbb{F}$.

Theorem 2.2 (invariant fiber bundles)**:** *Let* $m \in \mathbb{N}$*,* $\rho_0 > 0$*,* $K_1, K_2 \ge 1$ *and* $0 < \alpha < 1 < \beta$*,* $\gamma_1 = \gamma_2, \delta_2 > 0$*,* $C_1 = C_2 \ge 1$ *be reals. Consider a nonautonomous difference equation*

$$\boxed{\begin{cases} \Delta x = A(k,y)x + f(k,x,y) \\ \Delta y = \qquad\qquad\quad g(k,x,y) \end{cases}} \tag{2.7}$$

under the following assumptions:

(i) $A : \mathbb{Z} \times \mathcal{Y} \to \mathcal{L}(\mathcal{X})$ *is* m*-times continuously differentiable in* $y \in \mathcal{Y}$ *with globally bounded partial derivatives:*

$$|A|_n := \sup_{(k,y)\in\mathbb{Z}\times\mathcal{Y}} \|D_2^n A(k,y)\|_{\mathcal{L}_n(\mathcal{Y};\mathcal{L}(\mathcal{X}))} < \infty \quad \text{for } n \in \{0,\dots,m\},$$

(ii) $f : \mathbb{Z} \times \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ *is* m*-times continuously differentiable in* $(x,y) \in \mathcal{X} \times \mathcal{Y}$ *with globally bounded partial derivatives:*

$$|f|_n := \sup_{(k,x,y)\in\mathbb{Z}\times\mathcal{X}\times\mathcal{Y}} \left\|D_{(2,3)}^n f(k,x,y)\right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y};\mathcal{X})} < \infty \quad \text{for } n \in \{0,\dots,m\},$$

(iii) $g : \mathbb{Z} \times \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ *is* m*-times continuously differentiable in* $(x,y) \in \mathcal{X} \times \mathcal{Y}$ *with globally bounded partial derivatives:*

$$|g|_n := \sup_{(k,x,y)\in\mathbb{Z}\times\mathcal{X}\times\mathcal{Y}} \left\|D_{(2,3)}^n g(k,x,y)\right\|_{\mathcal{L}_n(\mathcal{X}\times\mathcal{Y};\mathcal{Y})} < \infty \quad \text{for } n \in \{0,\dots,m\},$$

(iv) *if* $\psi : \mathbb{Z} \to \mathcal{X}$ *is an arbitrary sequence with* $\sup_{k\in\mathbb{Z}} \|\psi(k)\| \leq \rho_0$*, and if* ν_ψ *denotes the general solution of the difference equation* $\Delta y = g(k, \psi(k), y)$*, then the linear difference equation*

$$\boxed{\Delta x = A(k, \nu_\psi(k; \kappa, \eta))x}$$

has (γ_2, δ_2)*-bounded growth with constant* C_2 *(uniformly in* $\kappa \in \mathbb{Z}$*,* $\eta \in \mathcal{Y}$*,* ψ*), and the linear difference equation*

$$\boxed{\Delta x = A(k, \nu_0(k; \kappa, \eta))x}$$

possesses an exponential dichotomy with α, β*,* K_1, K_2 *on* $\mathbb{Z}$ *(uniformly in* $\kappa \in \mathbb{Z}$*,* $\eta \in \mathcal{Y}$*),*

(v) *the mappings* A*,* f, g *satisfy the following estimates:*

$$\begin{aligned} |g|_1 &< \min\left\{\frac{1}{2}, 1 - \frac{\alpha}{\beta}, \frac{\beta}{\alpha} - 1\right\}, \\ |g|_1 &< \frac{1}{2}\min\left\{\frac{\beta-1}{4}, \frac{1-\alpha}{4}, \sqrt[m]{\frac{1+\beta}{2}} - 1, 1 - \sqrt[m]{\frac{1+\alpha}{2}}\right\} \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} |A|_1 |g|_1 (1 + |g|_1)^h h\rho_0 &\leq \varepsilon_0, \\ |A|_1 |g|_1 \left(\frac{K_1K_2(1+|g|_1)^h}{\beta - \alpha - \beta|g|_1} + \frac{K_1K_2(1+|g|_1)^h}{\beta - \alpha - \alpha|g|_1}\right) h\rho_0 &\leq \varepsilon_1, \\ 2C^1_{K_1,K_2}(\alpha,\beta)|A|_1|f|_0 + 4C^2_{K_1,K_2}(\alpha,\beta)|f|_1 &< 1, \\ 2C^1_{K_1,K_2}(\alpha,\beta)|f|_0 &\leq \rho_0, \\ 4C^2_{K_1,K_2}(\alpha,\beta)\left(2C^1_{K_1,K_2}(\alpha,\beta)|A|_1|f|_0 + |f|_1\right) &< 1, \end{aligned}$$

with an integer $h > \max\left\{\log_{\frac{\beta}{\alpha}}(K_1K_2), \log_{\frac{1+\alpha}{2\alpha}}(K_1), \log_{\frac{2\beta}{1+\beta}}(K_2)\right\}$*,*

$$\begin{aligned} C^1_{K_1,K_2}(\alpha,\beta) &:= \frac{C_2\left(\frac{2\gamma_2}{1+\alpha}\right)^h L_1\left(\frac{1+\alpha}{2}, \frac{1+\beta}{2}\right)}{1-\alpha} + \frac{C_2\left(\frac{1+\beta}{2\delta_2}\right)^h L_2\left(\frac{1+\alpha}{2}, \frac{1+\beta}{2}\right)}{\beta - 1} + \\ &\quad + \max\left\{\frac{C_2\left(\frac{2\gamma_2}{1+\alpha}\right)^h L_1\left(\frac{1+\alpha}{2}, \frac{1+\beta}{2}\right)}{1-\alpha}, \frac{C_2\left(\frac{1+\beta}{2\delta_2}\right)^h L_2\left(\frac{1+\alpha}{2}, \frac{1+\beta}{2}\right)}{\beta-1}\right\}, \\ C^2_{K_1,K_2}(\alpha,\beta) &:= C^1_{K_1,K_2}(\alpha,\beta) + \frac{C_2\left(\frac{2\gamma_2}{1+\alpha}\right)^h L_1\left(\frac{1+\alpha}{2}, \frac{1+\beta}{2}\right)}{1-\alpha} + \frac{C_2\left(\frac{1+\beta}{2\delta_2}\right)^h L_2\left(\frac{1+\alpha}{2}, \frac{1+\beta}{2}\right)}{\beta-1} \end{aligned}$$

and the reals $\varepsilon_0, \varepsilon_1 > 0$*,* $L_1\left(\frac{1+\alpha}{2}, \frac{1+\beta}{2}\right), L_2\left(\frac{1+\alpha}{2}, \frac{1+\beta}{2}\right) \geq 1$ *from Theorem 2.1.*

Then there exists a uniquely determined mapping $w : \mathbb{Z} \times \mathcal{Y} \to \mathcal{X}$*, whose graph*

$$\mathcal{W} := \{(\kappa, w(\kappa,\eta), \eta) \in \mathbb{Z} \times \mathcal{X} \times \mathcal{Y} : \kappa \in \mathbb{Z}, \eta \in \mathcal{Y}\}$$

can be characterized dynamically as

$$\mathcal{W} = \{(\kappa, \xi, \eta) \in \mathbb{Z} \times \mathcal{X} \times \mathcal{Y} : \|\lambda_1(k; \kappa, \xi, \eta)\| \leq \rho_0 \text{ for } k \in \mathbb{Z}\},$$

where $\lambda = (\lambda_1, \lambda_2)$ *is the general solution of* (2.7)*. Moreover, we obtain:*

(a) $w : \mathbb{Z} \times \mathcal{Y} \to \mathcal{X}$ *is globally bounded:* $\|w(\kappa, \eta)\| \leq 2C^1_{K_1,K_2}(\alpha, \beta)\, |f|_0$ *for* $\kappa \in \mathbb{Z}$*,* $\eta \in \mathcal{Y}$*,*

(b) $w : \mathbb{Z} \times \mathcal{Y} \to \mathcal{X}$ *is* m*-times continuously differentiable in* $\eta \in \mathcal{Y}$ *with globally bounded partial derivatives,*

(c) the graph $\mathcal{W}$ *is an invariant fiber bundle of the difference equation* (2.7).

Remark 2.3: (1) Under the condition (2.8) it is an easy application of the contraction mapping principle (cf. LANG [10, p. 360, Lemma 1.1]) to show that the solutions $\nu_\psi(\cdot; \kappa, \eta)$ exists on $\mathbb{Z}$, $\kappa \in \mathbb{Z}$, $\eta \in \mathcal{Y}$.

(2) The uniformity of the exponential dichotomy in (iv) means that the dichotomy constants α, β, K_1, K_2 are independent of $\kappa \in \mathbb{Z}$, $\eta \in \mathcal{Y}$, whilst the corresponding invariant projector is allowed to depend on these parameters.

(3) If one replaces (i), (ii), (iii) by global Lipschitz conditions for the mappings A, f, g, which are uniform in $k \in \mathbb{Z}$, then the existence of Lipschitz-smooth invariant fiber bundles yields analogously (cf. [14, p. 154, Satz 3.2.15]).

Proof. The above result is a special case of [14, pp. 210–211, Satz 3.3.15]. □

3 Slow Invariant Fiber Bundles

For this main section and for the rest of the paper let $\mathcal{Y}$ be an arbitrary Banach space, and let $\mathcal{X}$ be a C^m-Banach space, $m \in \mathbb{N}_0$, i.e., a complete normed space, where $\|\cdot\|_{\mathcal{X}} \setminus \{0\} \to \mathbb{R}$ is m-times continuously Frechét-differentiable. The setting within we work is a system of nonautonomous difference equations in $\mathcal{X} \times \mathcal{Y}$ under the subsequent standing

Hypothesis 3.1: Let $m \in \mathbb{N} \setminus \{1\}$, $K_1, K_2 \geq 1$, $0 < \alpha < 1 < \beta$ and $\varepsilon \in \mathbb{R}$ be reals. Consider a nonautonomous difference equation

$$\boxed{\begin{cases} \Delta x = F(k, x, y) \\ \Delta y = \varepsilon G(k, x, y) \end{cases}} \tag{3.1}$$

under the following assumptions:

(i) $F : \mathbb{Z} \times \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ is $(m+1)$-times continuously differentiable in $(x, y) \in \mathcal{X} \times \mathcal{Y}$ with globally bounded partial derivatives:

$$|F|_n := \sup_{(k,x,y) \in \mathbb{Z} \times \mathcal{X} \times \mathcal{Y}} \left\| D^n_{(2,3)} F(k, x, y) \right\|_{\mathcal{L}_n(\mathcal{X} \times \mathcal{Y}; \mathcal{X})} < \infty, \tag{3.2}$$

$$|F|^*_n := \sup_{(k,x,y) \in \mathbb{Z} \times \mathcal{X} \times \mathcal{Y}} \left\| D^n_2 F(k, x, y) \right\|_{\mathcal{L}_n(\mathcal{X}; \mathcal{X})} \leq |F|_n$$

for $n \in \{0, \ldots, m+1\}$,

(ii) $G : \mathbb{Z} \times \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ is m-times continuously differentiable in $(x, y) \in \mathcal{X} \times \mathcal{Y}$ with globally bounded partial derivatives:

$$|G|_n := \sup_{(k,x,y) \in \mathbb{Z} \times \mathcal{X} \times \mathcal{Y}} \left\| D^n_{(2,3)} G(k, x, y) \right\|_{\mathcal{L}_n(\mathcal{X} \times \mathcal{Y}; \mathcal{Y})} < \infty \tag{3.3}$$

for $n \in \{0, \ldots, m\}$,

(iii) there exists a mapping $w_0 : \mathbb{Z} \times \mathcal{Y} \to \mathcal{X}$ such that

$$w_0(k, y) + F(k, w_0(k, y), y) \equiv w_0(k + 1, y) \quad \text{on } \mathbb{Z} \times \mathcal{Y}, \tag{3.4}$$

and w_0 is m-times continuously differentiable in $y \in \mathcal{Y}$ with globally bounded partial derivatives

$$|w_0|_n := \sup_{(k,y)\in\mathbb{Z}\times\mathcal{Y}} \|D_2^n w_0(k, y)\|_{\mathcal{L}_n(\mathcal{Y};\mathcal{X})} < \infty \tag{3.5}$$

for $n \in \{1, \dots, m\}$,

(iv) the linear difference equation

$$\boxed{\Delta x = D_2F(k, w_0(k, y), y)x} \tag{3.6}$$

possesses an exponential dichotomy with $\alpha, \beta,\ K_1, K_2$ on $\mathbb{Z}$ (uniformly in $y \in \mathcal{Y}$) and $I_{\mathcal{X}} + D_2F(k, w_0(k, y), y) \in \mathcal{GL}(\mathcal{X})$, $k \in \mathbb{Z}$, $y \in \mathcal{Y}$, such that

$$\delta_2 := \sup_{(k,y)\in\mathbb{Z}\times\mathcal{Y}} \left\| [I_{\mathcal{X}} + D_2F(k, w_0(k, y), y)]^{-1} \right\| < \infty.$$

Remark 3.2: (1) Finite dimensional Banach spaces $\mathcal{X}$ are C^∞-Banach spaces, like Hilbert spaces as well. Nevertheless, in general infinite dimensional Banach spaces $\mathcal{X}$, norms are only C^m-mappings on $\mathcal{X} \setminus \{0\}$ for some finite $m \in \mathbb{N}_0$. Explicit examples and further results can be found in Abraham, Marsden & Ratiu [1, pp. 387–388].

(2) In case of an autonomous difference equation (3.1) every mapping $w_0 : \mathcal{Y} \to \mathcal{X}$ parameterizing the zeros of $F(\cdot, \eta)$, $\eta \in \mathcal{Y}$, satisfies the identity (3.4). Hence, in such a setting, the graph of w_0 is a manifold of equilibria for (3.1) in the limit case $\varepsilon = 0$.

(3) For simplicity reasons only, we have omitted the dependence of the mappings F, G on the parameter $\varepsilon \in \mathbb{R}$. If F and G depend on ε and are of class C^{m+1} and C^m, respectively, in $(x, y, \varepsilon) \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}$ with globally bounded derivatives, then the following results yield similarly.

(4) Using Theorem 2.1 it is possible to derive sufficient conditions for an exponential dichotomy of (3.6) with α, β by making assumptions on the spectrum of $I_{\mathcal{X}} + D_2F(k, w_0(k, y), y) \in \mathcal{L}(\mathcal{X})$ for fixed $k \in \mathbb{Z}$, $y \in \mathcal{Y}$. Actually, one considers the parameter space $\mathcal{Q} = \mathbb{Z} \times \mathcal{Y}$, the sequence $q^*(k) := (k, y)$, $y \in \mathcal{Y}$, and has to require:

(iv)$_1$ There exists a neighborhood $N \subseteq \mathbb{C}$ of the annulus $\{\lambda \in \mathbb{C} : \alpha \le |\lambda| \le \beta\}$ such that for any $k \in \mathbb{Z}$, $y \in \mathcal{Y}$ the spectrum of $I_{\mathcal{X}} + D_2F(k, w_0(k, y), y)$ is disjoint from N.

(iv)$_2$ The coefficient operator $D_2F(k, w_0(k, y), y)$ of (3.6) satisfies a Lipschitz condition in $(k, y) \in \mathbb{Z} \times \mathcal{Y}$ with a sufficiently small Lipschitz constant $L \ge 0$ (cf. (2.2)).

Difference equations of the form (3.1) are discrete counterparts of singularly perturbed ODEs with fast time (see e.g. [7]). Here ε is real parameter with small absolute value, and for obvious reasons, the x variable is called the *fast variable*, while the y variable is called *slow*. The system (3.1) in the limit case $\varepsilon = 0$, namely

$$\boxed{\left\{ \begin{array}{l} \Delta x = F(k, x, y) \\ \Delta y = \qquad\quad 0 \end{array} \right. ,} \tag{3.7}$$

will be denoted as the *singular system.* Unlike the situation of singularly perturbed ODEs, we are confronted with a nonautonomous equation here. By Hypothesis 3.1(iii) each sequence $\mu_\eta : \mathbb{Z} \to \mathcal{X} \times \mathcal{Y}$, $\mu_\eta(k) := (w_0(k,\eta),\eta)$, $\eta \in \mathcal{Y}$, is a solution of (3.7), and hence the set

$$\mathcal{W}_0 := \{(\kappa, w_0(\kappa,\eta),\eta) \in \mathbb{Z} \times \mathcal{X} \times \mathcal{Y} : \, \kappa \in \mathbb{Z}, \, \eta \in \mathcal{Y}\}$$

is an invariant fiber bundle of the singular system (3.7). Because of the particular y-equation in (3.7) the extended state space $\mathbb{Z} \times \mathcal{X} \times \mathcal{Y}$ near the invariant fiber bundle $\mathcal{W}_0$ is foliated by "horizontal fiber bundles" which are parameterized by $y \in \mathcal{Y}$. On each of those fibers bundles, the behavior of the solutions of system (3.7) is described by the parameter depending x-equation

$$\boxed{\Delta x = F(k,x,y).} \tag{3.8}$$

The qualitative behavior of the solutions of this system (3.8) near the solution $w_0(\cdot,\eta) : \mathbb{Z} \to \mathcal{X}$, $\eta \in \mathcal{Y}$, is determined its linearization, i.e., the dichotomy properties of equation (3.6) in Hypothesis 3.1(iv). Now from a perturbation point of view, it is reasonable to conjecture the existence of an invariant fiber bundle $\mathcal{W}_\varepsilon \subseteq \mathbb{Z} \times \mathcal{X} \times \mathcal{Y}$ for (3.1) near $\mathcal{W}_0$, assumed that $|\varepsilon|$ is small. More precise, $\mathcal{W}_\varepsilon$ allows the representation

$$\mathcal{W}_\varepsilon = \{(\kappa, w_0(\kappa,\eta) + w_\varepsilon(\kappa,\eta),\eta) \in \mathbb{Z} \times \mathcal{X} \times \mathcal{Y} : \, \kappa \in \mathbb{Z}, \, \eta \in \mathcal{Y}\},$$

where the mapping $w_\varepsilon : \mathbb{Z} \times \mathcal{Y} \to \mathcal{X}$ satisfies $\lim_{\varepsilon \to 0} w_\varepsilon(\kappa,\eta) = 0$ for any $\kappa \in \mathbb{Z}$, $\eta \in \mathcal{Y}$. The relevance of this invariant fiber bundle $\mathcal{W}_\varepsilon$ is due to the fact that any solution of (3.1) which remains sufficiently close to the manifold $\mathcal{W}_0$ for all $k \in \mathbb{Z}$ lies entirely on $\mathcal{W}_\varepsilon$.

To verify these statements using our Theorem 2.2 we have to transform the difference equation (3.1) into a certain normal form:

Lemma 3.3 (normal form)**:** *Assume that the Hypothesis 3.1 holds. Then the change of variables* $x \mapsto x - w_0(k,y)$ *transforms* (3.1) *into the nonautonomous difference equation*

$$\begin{cases} \Delta x = A(k,y)x + F_1(k,x,y;\varepsilon) \\ \Delta y = \qquad\qquad\quad \varepsilon G_1(k,x,y) \end{cases}, \tag{3.9}$$

where the mappings $A : \mathbb{Z} \times \mathcal{Y} \to \mathcal{L}(\mathcal{X})$, $F_1 : \mathbb{Z} \times \mathcal{X} \times \mathcal{Y} \times \mathbb{R} \to \mathcal{X}$, $G_1 : \mathbb{Z} \times \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ *are given by*

$$\begin{aligned} A(k,y) &:= D_2F(k,w_0(k,y),y), && (3.10)\\ F_1(k,x,y;\varepsilon) &:= F(k,x+w_0(k,y),y) - D_2F(k,w_0(k,y),y)x - w_0(k+1,y) + w_0(k,y) - \\ &\quad -\varepsilon \int_0^1 D_2w_0(k+1,y+h\varepsilon G(k,x+w_0(k,y),y))\,dh\,G(k,x+w_0(k,y),y), \\ G_1(k,x,y) &:= G(k,x+w_0(k,y),y) \end{aligned}$$

and possess the following properties:

$$\begin{aligned} \|D_2A(k,y)\|_{\mathcal{L}_1(\mathcal{Y};\mathcal{L}(\mathcal{X}))} &\le \max\{1, |w_0|_1\} |F|_2\,, && (3.11)\\ \|F_1(k,x,y;\varepsilon)\| &\le \tfrac{1}{2}|F|_2^* \|x\|^2 + |\varepsilon|\,|w_0|_1\,|G|_0\,, && (3.12)\\ \|D_2F_1(k,x,y;\varepsilon)\|_{\mathcal{L}_1(\mathcal{X})} &\le |F|_2\|x\| + \tfrac{1}{2}|\varepsilon|^2 |w_0|_2\,|G|_1\,|G|_0 + |\varepsilon|\,|w_0|_1\,|G|_1 && (3.13)\\ \|D_3F_1(k,x,y;\varepsilon)\|_{\mathcal{L}_1(\mathcal{Y};\mathcal{X})} &\le 2(1+|w_0|_1)\,|F|_2\|x\| + \tfrac{1}{2}|\varepsilon|^2 |w_0|_2\,|G|_0\,|G|_1(1+|w_0|_1) + \\ &\quad + |\varepsilon|\left[|w_0|_2\,|G|_0 + |w_0|_1\,|G|_1(1+|w_0|_1)\right], && (3.14)\\ \left\|D_{(2,3)}G_1(k,x,y)\right\|_{\mathcal{L}_1(\mathcal{X}\times\mathcal{Y};\mathcal{Y})} &\le (1+|w_0|_1)\,|G|_1 && (3.15) \end{aligned}$$

for all $k \in \mathbb{Z}$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$ *and* $\varepsilon \in \mathbb{R}$.

Proof. Apparently the mapping $(x,y) \mapsto (x - w_0(k,y), y)$ is a C^m-diffeomorphism on $\mathcal{X} \times \mathcal{Y}$ for each fixed $k \in \mathbb{Z}$, by Hypothesis 3.1(iii). Furthermore, it transforms (3.1) into the difference equation (3.9). Now Hypothesis 3.1(i), [10, p. 342, Corollary 4.3]), and (3.5) yields the Lipschitz estimate

$$\begin{aligned} \|A(k,y) - A(k,\bar{y})\| &\overset{(3.10)}{=} \|D_2F(k, w_0(k,y), y) - D_2F(k, w_0(k,\bar{y}), \bar{y})\| \leq \\ &\overset{(3.2)}{\leq} |F|_2 \left\| \begin{pmatrix} w_0(k,y) - w_0(k,\bar{y}) \\ y - \bar{y} \end{pmatrix} \right\| \leq \max\{1, |w_0|_1\} |F|_2 \|y - \bar{y}\| \end{aligned}$$

for $k \in \mathbb{Z}$, $y, \bar{y} \in \mathcal{Y}$, which in turn implies (3.11). With a quite similar argument one can show (3.15). The estimate (3.12) is a consequence of Taylor's formula (cf. [10, p. 349]) applied to $F_1(k,x,y;\varepsilon)$ using (3.4), (3.2), (3.3) and (3.5). We can derive (3.13) by differentiation and a direct estimate using [10, p. 342, Corollary 4.3] and (3.2), (3.3), (3.5). Finally, the remaining inequality (3.14) yields by analogous arguments. □

Lemma 3.4: *Assume that the Hypothesis 3.1 holds. Now if ν_0 denotes the general solution of the difference equation $\Delta y = \varepsilon G(k, w_0(k,y), y)$, and if $\varepsilon \in \mathbb{R}$ satisfies the estimates*

$$\begin{aligned} \max\{1, |w_0|_1\} |F|_2 |G|_0 \, h \, |\varepsilon| &\leq \varepsilon_0, \\ \max\{K_1, K_2\} C\left(\frac{1+\alpha}{2}, \frac{1+\beta}{2}\right) \max\{1, |w_0|_1\} |F|_2 |G|_0 \, h \, |\varepsilon| &\leq \varepsilon_1, \end{aligned} \tag{3.16}$$

with the real constants $h \in \mathbb{N}$ from Theorem 2.2, and $\varepsilon_0, \varepsilon_1 > 0$, $C\left(\frac{1+\alpha}{2}, \frac{1+\beta}{2}\right) > 0$ from Theorem 2.1, then for any $\kappa \in \mathbb{Z}$, $\eta \in \mathcal{Y}$ the linear difference equation

$$\Delta x = A(k, \nu_0(k;\kappa,\eta))x \tag{3.17}$$

possesses an exponential dichotomy on $\mathbb{Z}$ with $\frac{1+\alpha}{2}, \frac{1+\beta}{2}$, $L_1\left(\frac{1+\alpha}{2}, \frac{1+\beta}{2}\right), L_2\left(\frac{1+\alpha}{2}, \frac{1+\beta}{2}\right) \geq 1$ (uniformly in $\kappa \in \mathbb{Z}$, $\eta \in \mathcal{Y}$).

Remark 3.5: Due to Hypotheses 3.1(i) and (iv), equation (3.17) has (γ_2, δ_2)-bounded growth with constant $C_2 = 1$, where $\gamma_2 = 1 + |F|_1^*$. To determine the size of $\varepsilon_0, \varepsilon_1 > 0$ in Theorem 2.1, one in addition has to set $\gamma_1 = 1 + |F|_1^*$, $C_1 = 1$. These values of $\gamma_1, \gamma_2, \delta_2 > 0$ and $C_1, C_2 \geq 1$ should be used subsequently to calculate the constants $C^1_{K_1,K_2}(\alpha,\beta), C^1_{K_1,K_2}(\alpha,\beta) > 0$ as well.

Proof. We apply Theorem 2.1 with $\mathcal{Q} = \mathcal{Y}$, $\gamma = \frac{1+\alpha}{2}$, $\delta = \frac{1+\beta}{2}$ to the linear difference equation

$$\Delta x = A(k,y)x, \tag{3.18}$$

depending on the parameter $y \in \mathcal{Y}$. Then due to (3.11) the mapping A satisfies (2.2) with the Lipschitz constant $L = \max\{1, |w_0|_1\} |F|_2$ and (3.18) has γ_1^+-bounded growth with $\gamma_1 = 1 + |F|_1^*$ and constant 1. Hypothesis 3.1(iv) guarantees that (3.18) possesses an exponential dichotomy with α, β, K_1, K_2 uniformly in $y \in \mathcal{Y}$. For arbitrary $\kappa \in \mathbb{Z}$, $\eta \in \mathcal{Y}$ we define the mapping $q_* : \mathbb{Z} \to \mathcal{Y}$, $q_*(k) := \nu_0(k;\kappa,\eta)$ and using the solution property of ν_0 we readily obtain by "telescope summation"

$$\|q^*(k) - q^*(l)\| = \left\| \sum_{n=l}^{k-1} \Delta q^*(n) \right\| \leq \sum_{n=l}^{k-1} \|\varepsilon G(n, w_0(n, q^*(n)), q^*(n))\| \overset{(3.3)}{\leq} |\varepsilon| \, |G|_0 \, (k-l)$$

for $k \geq l$; an analogous estimate for $l \geq k$ implies that the Lipschitz condition (2.4) holds with Lipschitz constant $\vartheta = |\varepsilon| \, |G|_0$. Under the assumptions (3.16) together with Remark 3.5 we ultimately obtain the assertion of Lemma 3.4. □

Theorem 3.6 (slow fiber bundles)**:** *Assume that the Hypothesis 3.1 and*

$$4C^1_{L_1,L_2}\left(\tfrac{1+\alpha}{2},\tfrac{1+\beta}{2}\right)|F|^*_2 < 1 \tag{3.19}$$

holds, and set

$$\rho_0(\varepsilon) := \frac{2C^1_{L_1,L_2}\left(\tfrac{1+\alpha}{2},\tfrac{1+\beta}{2}\right)|w_0|_1\,|G|_0}{1-4C^1_{L_1,L_2}\left(\tfrac{1+\alpha}{2},\tfrac{1+\beta}{2}\right)|F|^*_2}\,|\varepsilon|\,,$$

whereby $C^1_{K_1,K_2}(\alpha,\beta) > 0$ *is defined in Theorem 2.2 and the real numbers* $L_1 = L_1\left(\tfrac{1+\alpha}{2},\tfrac{1+\beta}{2}\right)$, $L_2 = L_2\left(\tfrac{1+\alpha}{2},\tfrac{1+\beta}{2}\right)$ *from Theorem 2.1. Then there is a real constant* $\varepsilon_{\max} > 0$*, depending on* $\alpha,\beta,\ \delta_2,\ K_1,K_2,\ |F|_1,|F|_2,\ |G|_0,|G|_1,\ |w_0|_1,|w_0|_2$*, as well as on* $m \geq 2$ *and on the space* $\mathcal{X}$*, such that for any* $\varepsilon \in [-\varepsilon_{\max},\varepsilon_{\max}]$ *there exists a mapping* $w_\varepsilon : \mathbb{Z}\times\mathcal{Y}\to\mathcal{X}$*, such that the graph*

$$\mathcal{W}_\varepsilon := \{(\kappa, w_0(\kappa,\eta)+w_\varepsilon(\kappa,\eta),\eta)\in\mathbb{Z}\times\mathcal{X}\times\mathcal{Y} :\ \kappa\in\mathbb{Z},\ \eta\in\mathcal{Y}\}$$

is denoted as slow fiber bundle *of* (3.1) *and can be characterized dynamically as*

$$\mathcal{W}_\varepsilon = \{(\kappa,\xi,\eta)\in\mathbb{Z}\times\mathcal{X}\times\mathcal{Y} :\ \|\lambda_1(k;\kappa,\xi,\eta)-w_0(k,\lambda_2(k;\kappa,\xi,\eta))\| \leq \rho_0(\varepsilon) \quad \text{for } k\in\mathbb{Z}\}\,,$$

where $\lambda = (\lambda_1,\lambda_2)$ *denotes the general solution of* (3.1)*. Moreover, we obtain:*

- *(a)* $w_\varepsilon : \mathbb{Z}\times\mathcal{Y}\to\mathcal{X}$ *is globally bounded:* $\|w_\varepsilon(\kappa,\eta)\| \leq \rho_0(\varepsilon)$ *for* $\kappa\in\mathbb{Z}$, $\eta\in\mathcal{Y}$,
- *(b)* $w_\varepsilon : \mathbb{Z}\times\mathcal{Y}\to\mathcal{X}$ *is* m*-times continuously differentiable in* $\eta\in\mathcal{Y}$ *with globally bounded partial derivatives,*
- *(c) the graph* $\mathcal{W}_\varepsilon$ *is an invariant fiber bundle of the difference equation* (3.1)*.*

Remark 3.7: (1) Due to the transformation in Lemma 3.3 we loose one order in the smoothness of $\mathcal{W}_\varepsilon$ compared to the mapping $F(k,\cdot) : \mathcal{X}\times\mathcal{Y}\to\mathcal{X}$, $k\in\mathbb{Z}$. In case of ODEs, [11] or [18, 15] are confronted with the same deficit, while the method provided in [12] eludes this problem.

(2) During the proof of Theorem 3.6 we will see that the constant $\varepsilon_{\max} > 0$ in particular depends on the desired order of smoothness m for $\mathcal{W}_\varepsilon$. The larger $m \in \mathbb{N}\setminus\{1\}$ is, the smaller one has to choose $\varepsilon_{\max}$. Hence, if $F(k,\cdot)$ and $G(k,\cdot)$ are e.g. C^∞-functions, then w_ε is the smoother the smaller $\varepsilon_{\max}$ is taken. A precise estimate illuminating this fact is given in (3.23) below.

(3) The global Hypotheses 3.1 are hardly ever met in applications. Nevertheless, in case of autonomous ODEs, e.g. [11, 12] shows the existence of invariant manifolds in singular perturbation problems under much more reasonable assumptions, like e.g., a bounded domain of w_0. Here one modifies the right hand side of (3.1) outside a neighborhood of $\mathcal{W}_0$ such that certain global hypotheses are fulfilled, and applies a result like Theorem 3.6 afterwards.

Proof. We subdivide the proof in two steps:

(I) First of all we introduce the abbreviation $\omega := (1+|w_0|_1)\,|G|_1$ and choose $\varepsilon_{\max} > 0$ so small that $\varepsilon\in[-\varepsilon_{\max},\varepsilon_{\max}]$ satisfies the estimates (3.16). Consequently, we can use Lemma 3.4 to verify that the linear system (3.17) possesses an exponential dichotomy with $\frac{1+\alpha}{2}$, $\frac{1+\beta}{2}$, $L_1 = L_1\left(\tfrac{1+\alpha}{2},\tfrac{1+\beta}{2}\right)$, $L_2 = L_2\left(\tfrac{1+\alpha}{2},\tfrac{1+\beta}{2}\right)$ on $\mathbb{Z}$. Before we can apply Theorem 2.2 to the transformed difference equation (3.9), one has to modify the mapping $F_1 : \mathbb{Z}\times\mathcal{X}\times\mathcal{Y}\times\mathbb{R}\to\mathcal{X}$ in an appropriate way. Thereto let $\rho > 0$ be given. Since $\mathcal{X}$ is assumed to be a C^m-Banach space, because of [1, p. 473, Lemma 4.2.13] there exists a C^m-cut-off-function $\Theta : \mathcal{X} \to [0,1]$ with $\Theta(x) \equiv 1$ on $B_1(0)$ and $\Theta(x) \equiv 0$ on $\mathcal{X} \setminus B_2(0)$. Hence, due to Hypothesis 3.1 and Lemma 3.3 also $f_\rho : \mathbb{Z} \times \mathcal{X} \times \mathcal{Y} \times \mathbb{R} \to \mathcal{X}$, $f_\rho(k,x,y;\varepsilon) := \Theta\left(\frac{x}{\rho}\right) F_1(k,x,y;\varepsilon)$ defines a mapping such that $f_\rho(k,\cdot;\varepsilon)$, $k \in \mathbb{Z}$, $|\varepsilon| \le \varepsilon_{\max}$, is of class C^m with globally bounded derivatives. Furthermore, Lemma 3.3 yields

$$\|f_\rho(k,x,y;\varepsilon)\| = \Theta\left(\frac{x}{\rho}\right) \|F_1(k,x,y;\varepsilon)\| \stackrel{(3.12)}{\le} 2\,|F|_2^*\,\rho^2 + |\varepsilon|\,|w_0|_1\,|G|_0 \tag{3.20}$$

for $k \in \mathbb{Z}$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $\varepsilon \in \mathbb{R}$. Additionally, by reproducing the explicit construction of $\Theta : \mathcal{X} \to [0,1]_\mathbb{R}$ from [1, p. 473, Lemma 4.2.13], one can see that $|\Theta|_1 := \sup_{x\in\mathcal{X}} \|D\Theta(x)\|_{\mathcal{L}_1(\mathcal{X};\mathbb{R})}$ exists and depends on the smooth norm $\|\cdot\|_\mathcal{X}$, therefore on the normed space $\mathcal{X}$. Using the product rule (cf. [10, p. 336]) and Lemma 3.3 we see that

$$\begin{aligned}
&\|D_2 f_\rho(k,x,y;\varepsilon)\| \le \frac{\left\|D\Theta\left(\frac{x}{\rho}\right)\right\|}{\rho} \|F_1(k,x,y;\varepsilon)\| + \Theta\left(\frac{x}{\rho}\right) \|D_2 F_1(k,x,y;\varepsilon)\| \le \\
\stackrel{(3.12)}{\le}\;& |\Theta|_1 \left(2\,|F|_2\,\rho + |w_0|_1\,|G|_0\,\frac{|\varepsilon|}{\rho}\right) + \Theta\left(\frac{x}{\rho}\right) \|D_2 F_1(k,x,y;\varepsilon)\| \le \\
\stackrel{(3.13)}{\le}\;& |\Theta|_1 \left(2\,|F|_2\,\rho + |w_0|_1\,|G|_0\,\frac{|\varepsilon|}{\rho}\right) + 2\,|F|_2\,\rho + \tfrac{1}{2}\,|\varepsilon|^2\,|w_0|_2\,|G|_1\,|G|_0 + |\varepsilon|\,|w_0|_1\,|G|_1\,, \\
&\|D_3 f_\rho(k,x,y;\varepsilon)\| \le \Theta\left(\frac{x}{\rho}\right) \|D_3 F_1(k,x,y;\varepsilon)\| \le \\
\stackrel{(3.14)}{\le}\;& 4(1+|w_0|_1)\,|F|_2\,\rho + \tfrac{1}{2}\,|\varepsilon|^2\,|w_0|_2\,|G|_0\,\omega + |\varepsilon|\,(|w_0|_2\,|G|_0 + |w_0|_1\,\omega)
\end{aligned}$$

for $k \in \mathbb{Z}$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $\varepsilon \in \mathbb{R}$, which in turn implies the existence of the least upper bounds $\left|f_{\sqrt{\rho_0(\varepsilon)}}\right|_n := \sup_{(k,x,y)\in\mathbb{Z}\times\mathcal{X}\times\mathcal{Y}} \left\|D^n_{(2,3)} f_{\sqrt{\rho_0(\varepsilon)}}(k,x,y;\varepsilon)\right\|$ for $n \in \{0,1\}$, as well as the limit relations

$$\lim_{\varepsilon\to 0} \left|f_{\sqrt{\rho_0(\varepsilon)}}\right|_n = 0 \quad \text{for } n \in \{0,1\}\,. \tag{3.21}$$

Additionally, the modified difference equation

$$\left\{\begin{array}{ll} \Delta x = A(k,y)x + f_{\sqrt{\rho_0(\varepsilon)}}(k,x,y;\varepsilon) \\ \Delta y = \qquad\qquad \varepsilon G_1(k,x,y) \end{array}\right. \tag{3.22}$$

coincides with (3.9) on the set $\mathbb{Z} \times B_{\sqrt{\rho_0(\varepsilon)}}(0) \times \mathcal{Y} \subseteq \mathbb{Z} \times \mathcal{X} \times \mathcal{Y}$.

(II) With a view to the above step (I), the difference equation (3.22) satisfies all the hypotheses of Theorem 2.2. Beyond step (I), we choose $\varepsilon_{\max} > 0$ so small that

$$\begin{aligned}
\varepsilon_{\max}\omega &< \min\left\{\frac{1}{2}, \frac{\beta-\alpha}{1+\beta}\right\}, \\
\varepsilon_{\max}\omega &< \frac{1}{2}\min\left\{\frac{\beta-1}{8}, \frac{1-\alpha}{8}, \sqrt[m]{\frac{3+\beta}{4}} - 1, 1 - \sqrt[m]{\frac{3+\alpha}{4}}\right\}
\end{aligned} \tag{3.23}$$

and $\rho_0(\varepsilon_{\max}) < 1$,

$$\varepsilon_{\max}\max\{1,|w_0|_1\}\,|F|_2\,\omega\,(1+\varepsilon_{\max}\omega)^h\,h\rho_0(\varepsilon_{\max}) \;\le\; \varepsilon_0,$$

$$2\varepsilon_{\max}\max\{1,|w_0|_1\}|F|_2\,\omega\left(\frac{L_1L_2(1+\varepsilon_{\max}\omega)^h}{\beta-\alpha-(1+\beta)\varepsilon_{\max}\omega}+\frac{L_1L_2(1+\varepsilon_{\max}\omega)^h}{\beta-\alpha-(1+\alpha)\varepsilon_{\max}\omega}\right)h\rho_0(\varepsilon_{\max})\ \le\ \varepsilon_1,$$

$$2C^1_{L_1,L_2}\left(\tfrac{1+\alpha}{2},\tfrac{1+\beta}{2}\right)\max\{1,|w_0|_1\}|F|_2\left|f_{\sqrt{\rho_0(\varepsilon)}}\right|_0+4C^2_{L_1,L_2}\left(\tfrac{1+\alpha}{2},\tfrac{1+\beta}{2}\right)\left|f_{\sqrt{\rho_0(\varepsilon)}}\right|_1\ <\ 1,$$

$$4C^2_{L_1,L_2}\left(\tfrac{1+\alpha}{2},\tfrac{1+\beta}{2}\right)\left(2C^1_{L_1,L_2}\left(\tfrac{1+\alpha}{2},\tfrac{1+\beta}{2}\right)\max\{1,|w_0|_1\}|F|_2\left|f_{\sqrt{\rho_0(\varepsilon)}}\right|_0+\left|f_{\sqrt{\rho_0(\varepsilon)}}\right|_1\right)\ <\ 1,$$

which is possible due to $\lim_{s\to 0}\rho_0(s)=0$ and the limit relations (3.21); here we have applied the abbreviations $h\in\mathbb{N}$ such that

$$h>\max\left\{\log_{\frac{1+\beta}{1+\alpha}}(L_1L_2),\log_{\frac{3+\alpha}{2+2\alpha}}(L_1),\log_{\frac{2+2\beta}{3+\beta}}(L_2)\right\}$$

and $C^1_{K_1,K_2}(\alpha,\beta),C^2_{K_1,K_2}(\alpha,\beta)>0$ from Theorem 2.2. Using assumption (3.19) and the estimate (3.20) we additionally obtain $2C^1_{L_1,L_2}\left(\tfrac{1+\alpha}{2},\tfrac{1+\beta}{2}\right)\left|f_{\sqrt{\rho_0(\varepsilon)}}\right|_0\le\rho_0(\varepsilon)$. Consequently, we have completely verified the assumptions of Theorem 2.2. Hence, for each fixed $\varepsilon\in[-\varepsilon_{\max},\varepsilon_{\max}]$, there exists an invariant fiber bundle $\mathcal{W}_\varepsilon\subseteq\mathbb{Z}\times\mathcal{X}\times\mathcal{Y}$ of the modified difference equation (3.22), which can be characterized dynamically as

$$\mathcal{W}_\varepsilon=\left\{(\kappa,\xi,\eta)\in\mathbb{Z}\times\mathcal{X}\times\mathcal{Y}:\ \left\|\bar\lambda_1(k;\kappa,\xi,\eta)\right\|\le\rho_0(\varepsilon)\quad\text{for }k\in\mathbb{Z}\right\},$$

where $\bar\lambda=(\bar\lambda_1,\bar\lambda_2)$ is the general solution of (3.22). Moreover, $\mathcal{W}_\varepsilon$ is the graph of a mapping $w_\varepsilon:\mathbb{Z}\times\mathcal{Y}\to\mathcal{X}$ with the properties

- $\|w_\varepsilon(k,y)\|\le\rho_0(\varepsilon)$ for $k\in\mathbb{Z}$, $y\in\mathcal{Y}$,
- $w_\varepsilon:\mathbb{Z}\times\mathcal{Y}\to\mathcal{X}$ is m-times continuously differentiable in $\eta\in\mathcal{Y}$ with globally bounded partial derivatives.

Since the modified difference equation (3.22) and the system (3.9) in normal form coincide on the set $\mathbb{Z}\times B_{\rho_0(\varepsilon)}(0)\times\mathcal{Y}\subseteq\mathbb{Z}\times\mathcal{X}\times\mathcal{Y}$, also (3.9) possesses the invariant fiber bundle $\mathcal{W}_\varepsilon$ with the above properties, as well as its dynamical characterization. In regard to the transformation from Lemma 3.3 we obtain that the graph of the mapping $w_\varepsilon+w_0:\mathbb{Z}\times\mathcal{Y}\to\mathcal{X}$ is an invariant fiber bundle of the initial difference equation (3.1) and the proof of Theorem 3.6 is complete. □

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