SLOW INTEGRAL MANIFOLDS FOR LAGRANGIAN FLUID DYNAMICS IN UNSTEADY GEOPHYSICAL FLOWS

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Abstract. The authors consider Lagrangian motion of fluid particles in unsteady gravity currents in geophysical flows. The vertical motion of fluid particles, especially the induced vertical mixing in these currents, is partially responsible for the ocean thermohaline circulation, and thus plays a role in the global climate dynamics.

First, a reduced dynamical system for slow variables is derived for a non-autonomous multiscale system. The reduced system, still non-autonomous, is the original system restricted to a center-like non-autonomous invariant manifold (so-called slow manifold) which holds slow motions of the system. An algorithm is also presented to obtain an approximation of the non-autonomous slow manifold. A novelty here is that the reduction principle applies to *non-autonomous* multiscale systems which satisfy conditions that are true only *locally in space* (as in many physical cases). This makes the reduction principle applicable to real physical systems.

Then, this invariant manifold reduction principle is applied to an approximate conceptual Lagrangian model of gravity currents and a reduced non-autonomous system for slow vertical motion is obtained. This reduced system may be useful as a conceptual tractable tool for understanding some features of vertical mixing in unsteady gravity currents.

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1. Introduction and Motivation

A gravity current (also called density current) is the flow of one fluid within another driven by the gravitational force acting on the density difference between the fluids. Gravity currents occur in a wide variety of circumstances.

Oceanic gravity currents are of particular importance because they are intimately related to the ocean's role in climate dynamics. The thermohaline circulation, also called meridional overturning circulation, in the ocean is strongly influenced by dense-water formation that takes place mainly in polar seas by cooling (e.g., Dickson et al. [10]; Borenäs and Lundberg [6]) and in marginal seas by evaporation (e.g., the Mediterranean Sea, Baringer and Price [1]). Such dense-water masses are released into the large-scale ocean circulation in the form of bottom gravity currents. It has been realized that the (vertical) mixing of gravity currents with the ambient fluid may be an important factor in the long-term behavior of large-scale ocean circulation, with a potential impact on climate [29].

Gravity currents are poorly represented in global ocean circulation simulations. To improve such representations, there has been recent research in numerical investigations of three-dimensional gravity currents themselves; see [24, 25] and references therein. Although numerical simulations of gravity currents provide valuable information, dynamical behavior of such currents still defies human comprehension [30]. The behavior of gravity currents in the ocean circulations (and thus in the global climate system) is complex and multifold. Understanding, as opposed to mere simulation, requires *conceptual models* [30] whose behavior can be grasped in its entirety, even if they are wrong or incomplete in some particulars. This would be true even if global ocean circulation simulations were perfect, and the necessity is even more pressing in the face of simulations which are imperfect. Conceptual models are needed for better understanding of basic mechanisms and paradigms for gravity currents dynamics. A goal of the present paper is to derive such a conceptual model for gravity currents.

The analysis of such physical problems leads to ordinary differential equations which explicitly depend on time. Frequently, this temporal dependence is quite arbitrary in the sense that it is not necessarily (quasi- or almost) periodic. In this paper, we would like to consider Lagrangian motion of fluid particles in unsteady gravity currents, as a slow-fast multiscale non-autonomous system, aiming at deriving a conceptual reduced model for vertical motion of fluid particles. A motivation for the occurrence of different time scales in such a model will be given in § 4.

To this end, we begin with an abstract singular perturbation approach and consider the following non-autonomous multiscale system

$$\left\{ \begin{array}{r} \dot{x} = f(t,x,y,\varepsilon) \\ \varepsilon\dot{y} = g(t,x,y,\varepsilon) \end{array} \right. , \tag{1.1}$$

where $0 < \varepsilon \leq \bar{\varepsilon}$ is a small parameter, the state variable $x \in \mathbb{R}^M$ is *slow*, and the state variable $y \in \mathbb{R}^N$ is *fast*. The vector field in x direction, f, is smaller, while vector field in y direction, $\frac{1}{\varepsilon}g$, is larger. For example, and as prime motivation for our work in gravity currents, the vertical velocity component is smaller than the horizontal component. Beyond that, such systems with multiple time scales appear canonically in many areas.

Both f and g depend explicitly on time t. Thus, we aim at deriving a reduced non-autonomous dynamical system for the slow variable x only, while y is being slaved by x. Namely, $y = s(t, x, \varepsilon)$ for some function s, which defines the so-called *slow manifold* $\mathcal{S}_\varepsilon$. In this paper, we rigorously establish the existence of the slow manifold under local assumptions which are easy to verify in real-world applications. In particular for the limit case $\varepsilon = 0$, we assume that the algebraic equation $g(t, x, y, 0) = 0$ can be solved only locally w.r.t. y.

Being aware of the vast literature dealing with invariant manifold theorems for singularly perturbed problems, our approach is basically driven by the desire for applicable results. Indeed, we found it problematic to quote one of the usual references (e.g., [12, 32]), since our setting is different from the typically studied classical set-up for various reasons:

First of all, as pointed out above, our assumptions on the right-hand side of (1.1) are only local in space. Such local slow manifold results can be deduced from global ones by an appropriate modification of f and g, where already the proof of a global result involves a cut-off technique. Instead of modifying each particular example in order to meet global assumptions of a general theory, we provide a more applicable tool in Theorem 2.2. Its proof is not marginal, since (1.1) needs to be modified only semi-locally, i.e., in a whole neighborhood of the so-called *reduced manifold* $\mathcal{S}_0$.

Secondly, (1.1) does not fit into the typically studied classical framework of autonomous dynamical systems from, e.g., [16, 17, 21, 23, 22, 32]. Indeed, many references on the existence of slow manifolds claim that our non-autonomous situation is only seemingly more general than the traditional autonomous setting, since one can consider t as a dependent variable and append the trivial equation $\dot{t} = 1$ to the first equation of (1.1). Such arguments, however, are only partially true: At first, while useful for proving the existence of geometrical properties, such a trick destroys the essential dynamical features of a system, since the resulting equation has no equilibria, only unbounded solutions and thus no compact invariant sets (e.g., all invariant sets are unbounded). Moreover, attaching t to the x-variable has the consequence that all local assumptions on x in the autonomous setting are assumed to hold for the pair (t, x) now, like, e.g., differentiability or boundedness. This makes the use of cut-off techniques questionable. As demonstrated in §2 we impose somehow minimal boundedness assumptions on the derivatives of (1.1) w.r.t. the t-variable. In addition, the "$\dot{t} = 1$"-approach does not apply, if one aims to obtain such results in a similar discrete situation of non-autonomous difference equations (cf. [28]).

We remark that results on singularly perturbed ordinary differential equations frequently impose global assumptions and deal with only Lipschitzian functions f, g in (1.1) (cf. [2, 3, 4, 19, 20, 31]. However, more smoothness, i.e. higher order differentiability, is the key ingredient to obtain approximations of the slow manifold $\mathcal{S}_\varepsilon$ in terms of Taylor series. Such approximations are indispensable for applications.

Finally, we point out recent general approaches to the analysis of multiscale systems on a bounded time interval by identifying Lagrangian coherent structures without using the particular form of equation (1.1) (cf. [13, 14, 15, 33]).

Summarizing these arguments, we think that our non-global approach to non-autonomous singularly perturbed ordinary differential equations is legitimate and necessary. Moreover, it equips us with a tool appropriate to study our model for Lagrangian motion in unsteady gravity currents.

This paper is organized as follows. In the next section, we derive a slow integral manifold reduction principle for non-autonomous systems under local assumptions

(Theorem 2.2). Then we present an approximation method for slow manifolds in §3 and illustrate it by two examples. The first one is a caricature for Lagrangian fluid motion, while the second one describes a non-autonomously perturbed van-der Pol oscillator. These tools allow an application to a reduced, simplified model for vertical motion of fluid particles in unsteady gravity currents in §4. The final summary and possible perspectives can be found in §5.

2. Slow integral manifolds

In this section, we first present a global and then a local integral manifold reduction principle.

2.1. **Global integral manifolds.** From a technical point of view it is advantageous to begin with an abstract integral manifold result for non-autonomous ordinary differential equations (ODEs) of the form

$$\begin{cases} \dot{x} = F(t,x,z,\varepsilon) \\ \varepsilon\dot{z} = A(t,x)z + G(t,x,z,\varepsilon) \end{cases} . \tag{2.1}$$

Although its assumptions (see below) are hardly ever met in real world applications, one is able to give quite transparent proofs for such systems. In addition, we will show in the next subsection, how the original ODE (1.1) can be modified to (2.1), using an appropriate cut-off technique.

We make the following *global assumptions*: Let $m \geq 2$ be an integer, suppose $\bar{\varepsilon}, b_0, C > 0$, $\bar{\delta} \in (0,1]$ are constants and define $B_{\bar{\delta}} := \mathbb{R} \times \mathbb{R}^M \times \{z \in \mathbb{R}^N : \|z\| < \bar{\delta}\}$.

(I) The continuous function $F : B_{\bar{\delta}} \times [0,\bar{\varepsilon}] \to \mathbb{R}^M$ possesses a continuous partial derivative $D^m_{(2,3,4)}F$ and the partial derivatives $D^n_{(2,3)}F, D_4F$ are globally bounded for $n = 0,\dots,m$.

(II) The continuous function $G : B_{\bar{\delta}} \times [0,\bar{\varepsilon}] \to \mathbb{R}^N$ possesses a continuous partial derivative $D^m_{(2,3,4)}G$, the partial derivatives $D^n_{(2,3)}G, D_4G$ are globally bounded for $n = 0,\dots,m$ and one has

$$\|G(t,x,z,\varepsilon)\| \leq C(\varepsilon + \|z\|^2), \tag{2.2}$$

$$\|D_2^n G(t,x,z,\varepsilon)\| \leq C(\varepsilon + \|z\|^2), \quad n = 1,2, \tag{2.3}$$

$$\|D_3 G(t,x,z,\varepsilon)\| \leq C(\varepsilon + \|z\|) \quad \text{for all } (t,x,z) \in B_{\bar{\delta}},\ \varepsilon \in [0,\bar{\varepsilon}]. \tag{2.4}$$

Note that (2.2) implies $G(t,x,0,0) \equiv 0$ on $\mathbb{R} \times \mathbb{R}^M$ and thus $z = 0$ is the algebraic solution of the degenerated fast equation in (2.1) when $\varepsilon = 0$.

(III) The C^1-function $A : \mathbb{R} \times \mathbb{R}^M \to \mathbb{R}^{N\times N}$ possesses globally bounded derivatives D^nA for $n = 0,1$, the partial derivatives D_2^nA exist, are continuous and globally bounded for $n = 2,\dots,m$, and all eigenvalues $\lambda(t,x) \in \mathbb{C}$ of $A(t,x)$ have real parts $\Re\lambda(t,x) \leq -\frac{b_0}{2}$ uniformly in $(t,x) \in \mathbb{R} \times \mathbb{R}^M$.

For $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}^M$, $\zeta \in \mathbb{R}^N$ and $\varepsilon > 0$, the solution of (2.1) satisfying the initial condition $x(\tau) = \xi$, $z(\tau) = \zeta$ will be denoted by $\tilde{\varphi}(t,\tau,\xi,\zeta,\varepsilon)$. Under the above assumptions it can be shown that for $\varepsilon > 0$ sufficiently small, (2.1) has a global smooth attractive integral manifold $\tilde{\mathcal{S}}_\varepsilon$ which is $O(\varepsilon)$-close to 0.

More general, for any nonempty subset $\mathcal{S} \subseteq \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N$ we define

$$\mathcal{S}(t) := \left\{(x,z) \in \mathbb{R}^M \times \mathbb{R}^N : (t,x,z) \in \mathcal{S}\right\}.$$

The precise result is stated as

Proposition 2.1 (Global integral manifold). *For every $\beta \in \left(0, \frac{b_0}{2}\right)$ there exist positive constants K, $\varepsilon_0 \leq \min\left\{\bar{\varepsilon}, \frac{\bar{\delta}}{2K}\right\}$, $\delta \leq \frac{\bar{\delta}}{2}$ and a C^1-function $\tilde{s} : \mathbb{R} \times \mathbb{R}^M \times [0, \varepsilon_0] \to \mathbb{R}^N$ such that the partial derivatives $D^n_{(2,3)}\tilde{s}$ exist and are continuous for $n = 0, \ldots, m$. Moreover, the following assertions hold for $\varepsilon \in (0, \varepsilon_0]$:*

(a) (Global invariance) *The graph*

$$\tilde{\mathcal{S}}_\varepsilon := \left\{(t, x, \tilde{s}(t, x, \varepsilon)) \in \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N : t \in \mathbb{R}, x \in \mathbb{R}^M\right\}$$

is an integral manifold *of* (2.1)*, i.e., we have $\tilde{\varphi}(t, \tau, \tilde{\mathcal{S}}_\varepsilon(\tau), \varepsilon) = \tilde{\mathcal{S}}_\varepsilon(t)$ for all $t, \tau \in \mathbb{R}$.*

(b) (Asymptotic phase) *For every $(\tau, \xi, \zeta) \in \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N$ with $\|\zeta\| \leq \delta$ there exists a unique point $(\xi_0, \zeta_0) \in \mathcal{S}_\varepsilon(\tau)$ and a constant $C = C(\tau, \xi, \zeta, \varepsilon) > 0$ such that*

$$\|\tilde{\varphi}(t, \tau, \xi, \zeta, \varepsilon) - \tilde{\varphi}(t, \tau, \xi_0, \zeta_0, \varepsilon)\| \leq Ce^{-\frac{\beta}{\varepsilon}(t-\tau)} \quad \text{for all } t \geq \tau,$$

where $C(\tau, \xi, \zeta, \varepsilon) > 0$ is bounded on bounded sets w.r.t. (ξ, ζ).

(c) (Closeness to 0) *One has*

$$\|\tilde{s}(t, x, \varepsilon)\| \leq K\varepsilon \quad \text{for all } t \in \mathbb{R},\ x \in \mathbb{R}^M$$

and in particular $\tilde{s}(t, x, 0) = 0$.

(d) (Maximality) *Every solution ϕ of* (2.1) *satisfying $\|\phi_2(t)\| \leq \delta$ for all $t \in \mathbb{R}$ lies in $\tilde{\mathcal{S}}_\varepsilon$, i.e., $\phi_2(t) = s(t, \phi_1(t), \varepsilon)$ for all $t \in \mathbb{R}$.*

(e) (Reduction) *The reduced system on the integral manifold $\tilde{\mathcal{S}}_\varepsilon$ has dimension N and given by*

$$\dot{x} = F(t, x, \tilde{s}(t, x, \varepsilon), \varepsilon).$$

Proof. An autonomous version of (2.1) is considered in [32, cf. equation $(S)'_\varepsilon$] and the explicit time-dependence in our set-up causes no essential additional difficulty. Therefore, the proof of Proposition 2.1(a), (c)–(e) basically follows the arguments given in reference [32, Theorem 2.1] and we restrict ourselves to the following comments:

- The boundedness of $D^n A$ for $n = 0, 1$ makes a "non-autonomous" (i.e. A depends explicitly on t here) version of [32, Lemma 2.3] work.
- In order to mimic the approach of [32] we have to extend the nonlinearities F, G to the whole set $\mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N$. We consequently introduce a C^∞-cut-off function $\rho : \mathbb{R} \to [0, 1]$; for later use we precisely define (see Figure 1)

$$\rho(a) := \begin{cases} 0 & , \quad a \leq 0 \\ \exp\left(1 - \frac{1}{a}\exp(a-1)\right) & , \quad 0 < a < 1 \\ 1 & , \quad 1 \leq a \end{cases} . \tag{2.5}$$

  To cut-off F, G in the z-variable, we define for a generic function $q(z, \cdot)$ depending on z with $\|z\| < \bar{\delta}$ and other arguments, the z-global extension

$$\overline{q}(z, \cdot) := \begin{cases} \rho\left(2 - \frac{2\|z\|}{\bar{\delta}}\right)q(z, \cdot) & , \|z\| < \bar{\delta} \\ 0 & , \|z\| > \bar{\delta} \end{cases}$$

  defined for all $z \in \mathbb{R}^N$ now. Then, $\overline{F} : \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N \times [0, \bar{\varepsilon}] \to \mathbb{R}^M$ satisfies assumption (I) globally. Furthermore, as easily seen, also (II) holds for the function $\overline{G} : \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N \times [0, \bar{\varepsilon}] \to \mathbb{R}^N$.

$\rho(a)$

1

0 1 a

FIGURE 1. C^∞ cut-off function $\rho : \mathbb{R} \to [0, 1]$

Hence, the globalized system

$$\begin{cases} \dot{x} = \overline{F}(t, x, z, \varepsilon) \\ \varepsilon \dot{z} = A(t, x)z + \overline{G}(t, x, z, \varepsilon) \end{cases} \tag{2.6}$$

is in the framework of [32, Theorem 2.1]. Keeping in mind that (2.6) and (2.1) coincide on the set $B_{\bar{\delta}/2}$, the integral manifold $\tilde{\mathcal{S}}_\varepsilon$ of (2.6) satisfies the above assertions (a), (c)–(e) also w.r.t. the system (2.1), since we can choose ε_0, δ so small that $K\varepsilon \leq \bar{\delta}/2$ and $\delta \leq \bar{\delta}/2$. Moreover, in a similar spirit the asymptotic phase property in Proposition 2.1(b) is a consequence of the stable foliation constructed in [32, Theorem 3.1]. □

2.2. **Local version.** Our approach in this section, to modify (1.1) outside a dynamically relevant region, is largely inspired by the autonomous situation from [22].

We make the following *local assumptions*: Let $m \geq 2$ be an integer and suppose $\bar{\varepsilon}, b_0 > 0$ are constants. In addition, let $\Omega_1 \subseteq \mathbb{R}^M$, $\Omega_2 \subseteq \mathbb{R}^N$ be bounded domains and assume Ω_1 has a C^m-boundary.

(i) The continuous function $f : \mathbb{R} \times \Omega_1 \times \Omega_2 \times [0, \bar{\varepsilon}] \to \mathbb{R}^M$ possesses continuous partial derivatives $D^n_{(2,3,4)} f$ for $n = 1, \dots, m$ and globally bounded partial derivatives $D^n_{(2,3)} f$ for $n = 0, \dots, m$ and $D_4 f$.

(ii) The continuous function $g : \mathbb{R} \times \Omega_1 \times \Omega_2 \times [0, \bar{\varepsilon}] \to \mathbb{R}^N$ possesses continuous partial derivatives $D^n_{(2,3,4)} g$ for $n = 1, \dots, m$ and globally bounded partial derivatives $D^n_{(2,3)} g$ for $n = 0, \dots, m$, $D_4 D^n_{(2,3)} g$ for $n = 0, 1, 2$ and $D^n_{(2,3)} g(\cdot, 0)$ for $n = 3, 4$.

(iii) There exists a C^1-function $s_0 : \mathbb{R} \times \Omega_1 \to \Omega_2$ possessing continuous and globally bounded partial derivatives $D^n_2 s_0$ for $n = 0, \dots, m+1$ and $D^n_2 D_1 s_0$ for $n = 0, \dots, m$, such that

$$g(t, x, s_0(t, x), 0) \equiv 0 \quad \text{on } \mathbb{R} \times \Omega_1. \tag{2.7}$$

The graph $\mathcal{S}_0 := \{(t, x, s_0(t, x)) \in \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N : t \in \mathbb{R}, x \in \Omega_1\}$ is called *reduced manifold* of (1.1). Suppose, a neighborhood of $\mathcal{S}_0$ w.r.t. $\mathbb{R}^N$ is contained in Ω_2, i.e. there exists $\bar{\delta} \in (0, 1]$ such that (cp. Figure 2)

$$s_0(t, x) + y \in \Omega_2 \quad \text{for all } t \in \mathbb{R}, x \in \Omega_1, y \in \mathbb{R}^N \text{ with } \|y\| < \bar{\delta}.$$

FIGURE 2. $\mathcal{S}_0$ is bounded away from $\partial\Omega_2$ uniformly in $t \in \mathbb{R}$.

(iv) The Jacobian $B : \mathbb{R} \times \Omega_1 \to \mathbb{R}^{N\times N}$,

$$B(t,x) := D_3 g(t,x,s_0(t,x),0)$$

is a C^1-function with globally bounded derivatives $D^n B$ for $n = 0,1$, the partial derivatives $D_2^n B$ exist, are continuous and globally bounded for $n = 2,\ldots,m$, and all eigenvalues $\lambda(t,x) \in \mathbb{C}$ of $B(t,x)$ have real parts $\Re\lambda(t,x) \leq -b_0$ uniformly in $(t,x) \in \mathbb{R} \times \Omega_1$.

Remark 2.1. In the setting of an autonomous ODE (1.1), the above smoothness and boundedness assumptions in (i)–(iv) can be simplified to: f is of class C^m, g is of class $C^{\max\{4,m\}}$ on $\bar{\Omega}_1 \times \bar{\Omega}_2 \times [0,\bar{\varepsilon}]$, and s_0, B are of class C^{m+1} on $\bar{\Omega}_1$.

In order to complete assumption (iv) we have to explicitly define an extension of $B(t,x)$ also for those (t,x) with $x \in \mathbb{R}^M \setminus \Omega_1$ and the property that the real parts of the eigenvalues are still bounded away from the imaginary axis. We therefore construct a set $\Omega_1^\theta \subset \Omega_1$ which is a little bit smaller than Ω_1, i.e. for $\theta > 0$ small enough we define (see Figure 3)

$$\Omega_1^\theta := \{x \in \Omega_1 \,:\, \mathrm{dist}(x,\partial\Omega_1) > \theta\}.$$

FIGURE 3. For $\theta > 0$ small enough the boundary of Ω_1^θ is C^m and $\Omega_1' \subset \Omega_1^\theta \subset \Omega_1$.

Due to the fact that for $\theta > 0$ small enough $\mathrm{dist}(\cdot,\partial\Omega_1) \in C_b^m(\Omega_1^\theta,\mathbb{R})$ and Ω_1 has a C^m-boundary, also the set Ω_1^θ has a C^m-boundary. Using the cut-off function (2.5) we now define the C^m-function

$$A(t,x) := \begin{cases} B(t,x) & ,x \in \Omega_1^\theta \\ \rho(\mathrm{dist}(x,\partial\Omega_1)/\theta)B(t,x) - b_0[1-\rho(\mathrm{dist}(x,\partial\Omega_1)/\theta)]I_N & ,x \in \Omega_1 \setminus \Omega_1^\theta \\ -b_0 I_N & ,x \in \mathbb{R}^N \setminus \Omega_1 \end{cases},$$

i.e. A equals B on the smaller set Ω_1^θ and equals $-b_0 I_N$ outside of Ω_1, whereas it is an interpolation between those matrices on the annulus $\Omega_1 \setminus \Omega_1^\theta$. Obviously, by construction the eigenvalue real parts of $A(t,x)$ are smaller than $-b_0$ for all $t \in \mathbb{R}$, if either $x \in \Omega_1^\theta$ or $x \in \mathbb{R}^N \setminus \Omega_1$. Note that for small $\theta > 0$ the function $A(t,x)$ is C^m as a function of θ and hence the spectrum changes continuously with θ. If e.g. $A(t,x)$ is periodic in t then the eigenvalue real parts of $A(t,x)$ are certainly smaller than $-b_0/2$ for $(t,x) \in \mathbb{R} \times \Omega_1$ if $\theta > 0$ is small enough. However, for $A(t,x)$ with general time dependence we have to uniformly bound the eigenvalue real parts for x on the annulus $\Omega_1 \setminus \Omega_1^\theta$ and therefore complete assumption (iv) by

(v) There is a $\theta > 0$ such that all eigenvalues of $A(t,x)$ have real parts smaller than $-b_0/2$ uniformly in $(t,x) \in \mathbb{R} \times \Omega_1 \setminus \Omega_1^\theta$.

Under the above assumptions on (1.1) one can derive the following localized version of Proposition 2.1. Thereto, with $\tau \in \mathbb{R}$, $\xi \in \Omega_1$, $\eta \in \Omega_2$ and $\varepsilon > 0$, $\varphi(t,\tau,\xi,\eta,\varepsilon)$ denotes the solution of (1.1) satisfying the initial condition $x(\tau) = \xi$, $y(\tau) = \eta$.

Theorem 2.2 (Local slow manifold)**.** *For every subdomain $\Omega_1' \subseteq \mathbb{R}^M$ with $\overline{\Omega_1'} \subset \Omega_1$ and for every $\beta \in \left(0, \frac{b_0}{2}\right)$ there are positive real constants K, $\varepsilon_0 \leq \min\left\{\bar{\varepsilon}, \frac{\bar{\delta}}{2K}\right\}$, $\delta \leq \frac{\bar{\delta}}{2}$ and a C^1-function $s : \mathbb{R} \times \Omega_1' \times [0, \varepsilon_0] \to \Omega_2$ such that the partial derivatives $D^n_{(2,3)} s$ exist and are continuous for $n = 0, \dots, m$. More specifically, the following assertions hold for $\varepsilon \in (0, \varepsilon_0]$:*

(a) (Local invariance) *The graph*
$$\mathcal{S}_\varepsilon := \left\{(t, x, s(t, x, \varepsilon)) \in \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N : t \in \mathbb{R}, x \in \Omega_1'\right\}$$
is a local integral manifold *of* (1.1)*, i.e., for $(\tau, \xi, \eta) \in \mathcal{S}_\varepsilon$ we have $\varphi(t, \tau, \xi, \eta, \varepsilon) \in \mathcal{S}_\varepsilon(t)$ for all $t, \tau \in \mathbb{R}$ satisfying $\varphi_1(t, \tau, \xi, \eta, \varepsilon) \in \Omega_1'$.*

(b) (Asymptotic phase) *For every $(\tau, \xi, \eta) \in \mathbb{R} \times \Omega_1' \times \Omega_2$ with $\|\eta - s_0(\tau, \xi)\| \leq \delta$ there exists a point $(\xi_0, \eta_0) \in \mathcal{S}_\varepsilon(\tau)$ and a constant $C = C(\tau, \xi, \eta, \varepsilon) > 0$ such that*
$$\|\varphi(t, \tau, \xi, \eta, \varepsilon) - \varphi(t, \tau, \xi_0, \eta_0, \varepsilon)\| \leq C e^{-\frac{\beta}{\varepsilon}(t-\tau)} \quad \textit{for all } t \geq \tau$$
satisfying $\varphi_1(s, \tau, \xi, \eta, \varepsilon), \varphi_1(s, \tau, \xi_0, \eta_0, \varepsilon) \in \Omega_1'$ for $\tau \leq s \leq t$.

(c) (Closeness to $\mathcal{S}_0$) *One has*
$$\|s(t, x, \varepsilon) - s_0(t, x)\| \leq K\varepsilon \quad \textit{for all } t \in \mathbb{R},\, x \in \Omega_1'$$
and in particular $s(t, x, 0) = s_0(t, x)$.

(d) (Maximality) *Every solution ϕ of* (1.1) *satisfying $\phi_1(t) \in \Omega_1'$ and*
$$\|\phi_2(t) - s_0(t, \phi_1(t))\| \leq \delta \quad \textit{for all } t \in \mathbb{R},$$
lies in $\mathcal{S}_\varepsilon$, i.e., $\phi_2(t) = s(t, \phi_1(t), \varepsilon)$ for all $t \in \mathbb{R}$.

(e) (Reduction) *The reduced system on the slow manifold $\mathcal{S}_\varepsilon$ has dimension N and is given by*
$$\dot{x} = f(t, x, s(t, x, \varepsilon), \varepsilon).$$

Proof. The proof is divided into two steps. In the first step we flatten the manifold $\mathcal{S}_0$ by introducing the new coordinate $z = y - s_0(t, x)$. In the second step we globalize the resulting equation by cutting off in x outside of $\Omega_1 \subset \mathbb{R}^M$ and apply Proposition 2.1. Above all, we can choose $\theta > 0$ sufficiently small that $\Omega_1' \subset \Omega_1^\theta$.

Step 1: We transform the second coordinate of the non-autonomous ODE (1.1) according to $z = y - s_0(t, x)$ and get

$$\left\{ \begin{aligned} \dot{x} &= \tilde{f}(t, x, z, \varepsilon) \\ \varepsilon \dot{z} &= \tilde{g}(t, x, z, \varepsilon) - \varepsilon D_1 s_0(t, x) - \varepsilon D_2 s_0(t, x) \tilde{f}(t, x, z, \varepsilon) \end{aligned} \right. \tag{2.8}$$

with
$$\tilde{f}(t, x, z, \varepsilon) := f(t, x, z + s_0(t, x), \varepsilon) \quad \text{and} \quad \tilde{g}(t, x, z, \varepsilon) := g(t, x, z + s_0(t, x), \varepsilon).$$

Note: These functions inherit their smoothness and boundedness properties from f, g and s_0 by virtue of the assumptions (i)–(iv). However, they are defined on a set of the form $\mathbb{R} \times \Omega_1 \times \{z \in \mathbb{R}^N : \|z\| < \bar{\delta}\}$ now. Moreover, $\tilde{g}(t, x, 0, 0) \equiv 0$ and the right hand side of the z-equation in (2.8) vanishes identically for $(z, \varepsilon) = (0, 0)$.

Step 2: We cut-off in $x \in \mathbb{R}^M$ by defining, for a generic function $q(x, \cdot)$ depending on $x \in \Omega_1$ and some other arguments, the notion of the "extended" function
$$\overline{q}(x, \cdot) := \begin{cases} q(x, \cdot) & , x \in \Omega_1^\theta \\ \rho(\operatorname{dist}(x, \partial\Omega_1)/\theta)\, q(x, \cdot) & , x \in \Omega_1 \setminus \Omega_1^\theta \\ 0 & , x \in \mathbb{R}^N \setminus \Omega_1 \end{cases} .$$

In particular, we get the extension $\overline{F} : B_{\bar{\delta}} \times [0, \bar{\varepsilon}] \to \mathbb{R}^M$ of $\tilde{f}$ given by

$$\overline{F}(t,x,z,\varepsilon) := \begin{cases} \tilde{f}(t,x,z,\varepsilon) & ,x \in \Omega_1^\theta \\ \rho(\operatorname{dist}(x, \partial\Omega_1)/\theta)\, \tilde{f}(t,x,z,\varepsilon) & ,x \in \Omega_1 \setminus \Omega_1^\theta \\ 0 & ,x \in \mathbb{R}^N \setminus \Omega_1 \end{cases} .$$

Next we rewrite the function $\tilde{g}(t,x,z,\varepsilon)$ as

$$\tilde{g}(t,x,z,\varepsilon) = B(t,x)z + G_1(t,x,z,\varepsilon)$$

with $B(t,x) = D_3\, g(t,x,s_0(t,x),0)$ as defined above. Recalling the definition of A from above, we have

$$A(t,x) = \overline{B}(t,x) - \begin{cases} 0 & ,x \in \Omega_1^\theta \\ b_0[1 - \rho(\operatorname{dist}(x,\partial\Omega_1)/\theta)]I_N & ,x \in \Omega_1 \setminus \Omega_1^\theta \\ b_0 I_N & ,x \in \mathbb{R}^N \setminus \Omega_1 \end{cases} .$$

By replacing the functions G_1 and s_0 with $\overline{G}_1$ and $\overline{s}_0$, resp., we arrive at the system

$$\text{(2.9)} \qquad \begin{cases} \dot{x} = \overline{F}(t,x,z,\varepsilon) \\ \varepsilon\dot{z} = A(t,x)z + \overline{G}_1(t,x,z,\varepsilon) - \varepsilon D_1 \overline{s}_0(t,x) - \varepsilon D_2 \overline{s}_0(t,x)\overline{F}(t,x,z,\varepsilon) \end{cases} .$$

Thus, the above modification provided a smooth global extension of the ODE (2.8) outside $\mathbb{R} \times \Omega_1^\theta \times \left\{ z \in \mathbb{R}^N : \|z\| \le \frac{\bar{\delta}}{2} \right\}$ to a set of the form $B_{\bar{\delta}/2}$. In particular, for the right hand side of (2.9) we can verify the following:

- $\overline{F} : B_{\bar{\delta}/2} \times [0,\bar{\varepsilon}] \to \mathbb{R}^M$ satisfies assumption (I) of Proposition 2.1. This can be derived from the global boundedness hypotheses on f and s_0 in (i) and (iii), resp.
- $A : \mathbb{R} \times \mathbb{R}^M \to \mathbb{R}^{N \times N}$ satisfies the assumption (III) of Proposition 2.1. This is a direct consequence of the conditions on $B(t,x)$ stated in (iv) and our cut-off procedure.
- Finally, the mapping $G : B_{\bar{\delta}/2} \times [0,\bar{\varepsilon}] \to \mathbb{R}^N$ defined by

  $$G(t,x,z,\varepsilon) := \overline{G}_1(t,x,z,\varepsilon) - \varepsilon D_1 \overline{s}_0(t,x) - \varepsilon D_2 \overline{s}_0(t,x)\overline{F}(t,x,z,\varepsilon)$$

  deserves a bit more care. From (i)–(iv) we obtain the boundedness assumptions in (II). In addition to that, we have to verify the estimates (2.2)–(2.4). Concerning (2.2), we observe $G(t,x,0,0) \equiv 0$ (cf. (2.7)). Then, without presenting the details here, boundedness of appropriate partial derivatives, estimates using the mean value theorem and the mean value inequality, guarantee that (2.2) holds true. In order to show inequality (2.3) one uses similar arguments and the fact that (2.7) implies the identity

  $$D_2 g(t,x,s_0(t,x),0) + D_3 g(t,x,s_0(t,x),0) D_2 s_0(t,x) \equiv 0 \quad \text{on } \mathbb{R} \times \Omega_1.$$

  This, together with (2.7), has to be used to deduce (2.3). Finally, (2.4) can be shown similarly.

Thus, (2.9) satisfies the assumptions of the global Proposition 2.1 with

$$F(t,x,z,\varepsilon) = \overline{F}(t,x,z,\varepsilon)$$

and, consequently, for sufficiently small $\varepsilon \in [0,\bar{\varepsilon}]$, say $\varepsilon \le \varepsilon_0$, the ODE (2.9) possesses an integral manifold $\tilde{\mathcal{S}}_\varepsilon$ given as graph of a smooth mapping $\tilde{s}(\cdot,\varepsilon)$ (see Proposition 2.1 for the precise properties). We then define $s(t,x,\varepsilon) := s_0(t,x) + \tilde{s}(t,x,\varepsilon)$ for $t \in \mathbb{R}$, $x \in \Omega_1'$ and $\varepsilon \in [0,\bar{\varepsilon}]$, which satisfies the assertions of Theorem 2.2.

$\square$

3. Approximations of Slow Manifolds

We now consider how to analytically approximate the slow manifold, i.e., approximate the function $s(t,x,\varepsilon)$. Throughout this Section, we assume that the hypotheses of Theorem 2.2 are satisfied and keep the subdomain $\Omega_1' \subset \mathbb{R}^M$ fixed. Then (1.1) possesses a slow manifold $\mathcal{S}_\varepsilon$ being graph of a smooth function $s : \mathbb{R} \times \Omega_1' \times [0,\varepsilon_0] \to \mathbb{R}^N$. From the local invariance of $\mathcal{S}_\varepsilon$ it is easy to see that s satisfies the following nonlinear first-order partial differential equation

$$\varepsilon D_1 s(t,x,\varepsilon) + \varepsilon D_2 s(t,x,\varepsilon) f(t,x,s(t,x,\varepsilon),\varepsilon) = g(t,x,s(t,x,\varepsilon),\varepsilon) \tag{3.1}$$

for all $t \in \mathbb{R}$, $x \in \Omega_1'$ and $\varepsilon \in (0,\varepsilon_0]$. We call (3.1) the *invariance equation*.

In order to apply the reduction principle contained in Theorem 2.2 to a specific problem, a central question is how to determine this function s. However, in general there is no hope for an exact calculation and one must rely on certain approximation techniques. Keeping in mind that $\varepsilon > 0$ is a small parameter, and that s is of class C^m also in ε, it is reasonable to expand s in powers of ε up to order m. This yields the ansatz

$$s(t,x,\varepsilon) = \sum_{n=0}^{m} s_n(t,x)\varepsilon^n + \hat{S}_m(t,x,\varepsilon), \tag{3.2}$$

where $\hat{S}_m(t,x,\varepsilon)$ is a remainder satisfying $\lim_{\varepsilon\searrow 0} \frac{\hat{S}_m(t,x,\varepsilon)}{\varepsilon^m} = 0$. Now we suppose for f the representation

$$f(t,x,s(t,x,\varepsilon),\varepsilon) = \sum_{n=0}^{m} F_n(t,x)\varepsilon^n + \hat{F}_m(t,x,\varepsilon),$$

with $F_n(t,x) := \left.\frac{d^n f(t,x,s(t,x,\varepsilon),\varepsilon)}{d\varepsilon^n}\right|_{\varepsilon=0}$. Taking into account $g(t,x,s_0(t,x),0) \equiv 0$ (cf. (2.7)), one has for g that

$$g(t,x,s(t,x,\varepsilon),\varepsilon) = B(t,x)\sum_{n=1}^{m} \varepsilon^n s_n(t,x) + \sum_{n=1}^{m} \varepsilon^n G_n(t,x) + \hat{G}_m(t,x,\varepsilon)$$

with certain functions G_n. Substituting this expressions into (3.1) and equating equal powers of ε, we obtain

$$D_1 s_{n-1}(t,x) + \sum_{i=0}^{n-1} D_2 s_i(t,x) F_{n-1-i}(t,x) = B(t,x)s_n(t,x) + G_n(t,x)$$

for $n = 1,...,m$. Since $B(t,x) \in \mathbb{R}^{N\times N}$ is invertible due to Hypothesis (iv) one has

$$s_n(t,x) = B(t,x)^{-1}\left[D_1 s_{n-1}(t,x) + \sum_{i=0}^{n-1} D_2 s_i(t,x)F_{n-1-i}(t,x) - G_n(t,x)\right] \tag{3.3}$$

and in particular $s_1(t,x) = B(t,x)^{-1}[D_1 s_0(t,x) + D_2 s_0(t,x)F_0(t,x) - G_1(t,x)]$. Hence, one can use (3.3) to determine s_n recursively for $n = 1,...,m$.

Let us look at a couple of examples.

Example 3.1. *Consider*

$$\begin{cases} \dot{x} = x + y^2 + \cos(t) \\ \varepsilon\dot{y} = -x^2 - y + \sin(t) \end{cases}.$$

for $t \in \mathbb{R}$, $(x, y) \in (0,1) \times (0,1)$. *Systems like this have been used as an approximate model for Lagrangian fluid motion in geophysical flows, e.g., large scale quasi-geostrophic flows* [26, 9, 11].

In this example, $M = N = 1$, $b_0 = \frac{1}{2}$, $\Omega_1 = \Omega_2 := (0,1)$, $f(t,x,y) = x + y^2 + \cos(t)$, $g(t,x,y) = -x^2 - y + \sin(t)$, *and*

$$\begin{aligned} s_0(t,x) &= -x^2 + \sin(t), \\ D_3 g(t,x,s_0(t,x)) &= -1 < -b_0 \quad \textit{for all } x \in \Omega_1. \end{aligned}$$

Therefore, all conclusions in Theorem 2.2 hold. In particular, there exists a local slow integral manifold

$$\mathcal{S}_\varepsilon = \{(t,x,s(t,x,\varepsilon)) : t \in \mathbb{R}, x \in \Omega_1'\},$$

where the smooth representation function $s(t,x,\varepsilon)$ *is close to* $s_0(t,x) = -x^2 + \sin(t)$ *in the sense that*

$$|s(t,x,\varepsilon) - [-x^2 + \sin(t)]| \le K\varepsilon \quad \textit{for all } t \in \mathbb{R},\ x \in \Omega_1'$$

with some constant $K > 0$.

Moreover, the reduced dynamics on this integral manifold is still a non-autonomous dynamical system, but with reduced dimension one:

$$\dot{x} = x + s(t,x,\varepsilon)^2 + \cos(t).$$

For s *we obtain the approximation*

$$\begin{aligned} s(t,x,\varepsilon) = &- x^2 + \sin(t) \\ &+ \varepsilon[-\cos(t) + 2x^2 + 2x^5 - 4x^3\sin(t) + 2x(\sin(t))^2 + 2x\cos(t)] \\ &+ \varepsilon^2[-4x^2 + 28x^3\sin(t) - 6x(\sin(t))^2 - 6x\cos(t) - \sin(t) - 22x^5 \\ &\quad - 8x\sin(t)\cos(t) + 24x^2\sin(t)\cos(t) + 8x^3\cos(t) + 2x\sin(t) \\ &\quad + 56x^6\sin(t) - 60x^4(\sin(t))^2 - 20x^4\cos(t) + 24x^2(\sin(t))^3 \\ &\quad - 4(\sin(t))^2\cos(t) - 2(\cos(t))^2 - 18x^8 - 2(\sin(t))^4] + O(\varepsilon^3). \end{aligned}$$

The following example of its own interest can be interpreted as a version of van-der-Pol's equation with large parameters and a non-autonomous forcing.

Example 3.2. *Choose* $\bar{\delta} \in [0,1)$ *and let* $\psi : \mathbb{R} \to \mathbb{R}$ *be a bounded* C^2*-function with bounded derivative* $\dot{\psi}$*; this will be our forcing function. Consider the planar ODE*

$$\begin{cases} \dot{x} = y \\ \varepsilon\dot{y} = (x^2 - 1)y + \varepsilon x + \psi(t) \end{cases},$$

where we choose $\Omega_1 = (\omega_-, \omega_+)$ *to be any nonempty interval with* $\bar{\Omega}_1 \subset (-1,1)$ *and* Ω_2 *be be an open bounded interval such that*

- $\frac{\psi(t)}{1-x^2} + d \in \Omega_2$ *for all* $t \in \mathbb{R}$, $x \in \Omega_1$ *and* $d \in [-\bar{\delta}, \bar{\delta}]$.

Adapting to the notation of Theorem 2.2 we have the functions

$$f(t,x,y,\varepsilon) = y, \qquad g(t,x,y,\varepsilon) = (x^2 - 1)y + \varepsilon x + \psi(t),$$

$$s_0(t,x) = \frac{\psi(t)}{1 - x^2}, \qquad B(t,x) = x^2 - 1$$

and choose a $b_0 > 0$ *such that* $B(t,x) < -b_0$ *for all* $t \in \mathbb{R}$, $x \in \Omega_1$. *Then the assumptions (i)–(iv) of Theorem 2.2 are satisfied. In our* 1*-dimensional setting, assumption (v) boils down to the obviously satisfied estimate*

$$\rho(\operatorname{dist}(x,\partial\Omega_1)/\theta)B(t,x) + b_0[1-\rho(\operatorname{dist}(x,\partial\Omega_1)/\theta)] \leq -\frac{b_0}{2}$$

for all $t \in \mathbb{R}$, $x \in \Omega_1 \setminus (\omega_- + \theta, \omega_+ - \theta)$ *and* $\theta \in \left(0, \frac{\omega_+ - \omega_-}{2}\right)$.

For any given subinterval Ω_1' *which lies compactly in* Ω_1, *Theorem 2.2 guarantees the existence of a local slow integral manifold*

$$\mathcal{S}_\varepsilon = \{(t,x,s(t,x,\varepsilon)) : t \in \mathbb{R}, x \in \Omega_1'\},$$

where the function $s(t,x,\varepsilon)$ *is close to* $s_0(t,x) = \frac{\psi(t)}{1-x^2}$ *in the sense that*

$$\left| s(t,x,\varepsilon) - \frac{\psi(t)}{1-x^2} \right| \leq K\varepsilon \quad \text{for all } t \in \mathbb{R},\ x \in \Omega_1'$$

with some constant $K > 0$. *Additionally, the reduced dynamics on this integral manifold is given by the scalar non-autonomous ODE*

$$\dot{x} = s(t,x,\varepsilon).$$

As suggested in §3 we approximate the right-hand side of this equation as follows:

$$s(t,x,\varepsilon) = \frac{\psi(t)}{1-x^2} - \varepsilon \frac{\dot{\psi}(t)(1-x^2)^2 + 2\psi(t)^2 x - (1-x^2)^3 x}{(1-x^2)^4} + O(\varepsilon^2).$$

4. Lagrangian dynamics of gravity currents

The three-dimensional quasi-geostrophic model is often used for theoretical, conceptual studies of oceanic flows. A three-dimensional model is derived here to model Lagrangian dynamics of fluid particles in gravity currents, based on the fact that the gravity currents is due to a shear instability, i.e., the Kelvin-Helmholtz instability. We then apply the above slow manifold reduction principle in §2 to this non-autonomous model for Lagrangian fluid motion in gravity currents, and derive a reduced model for vertical motion of fluid particles. This sets a stage for future investigation of some dynamical features of gravity currents.

As in Cushman-Roisin [8],Chapter 15, we model the dynamics of gravity currents by the stratified, three-dimensional baroclinic quasi-geostrophic equation in terms of the stream function $\psi(x,y,z,t)$:

$$q_t + J(\psi, q) = \nu \partial_{zz} \Delta\psi, \tag{4.1}$$

where q is the potential vorticity

$$q = \Delta\psi + \frac{f_0^2}{N^2}\psi_{zz} + \beta y.$$

Here x, y, z are Cartesian coordinates in zonal (east), meridional (north), vertical directions, respectively; $\Delta = \partial_{xx} + \partial_{yy}$ is the planar Laplace operator; $f_0 + \beta y$ (with f_0, β constants) is the Coriolis parameter. Moreover, $N > 0$ is the Brunt-Vaisala stratification frequency taking to be constant, $\nu > 0$ is viscosity; and $J(f,g) = f_x g_y - f_y g_x$ is the Jacobi (determinant) operator. Note that $\Delta\psi + \frac{f_0^2}{N^2}\psi_{zz}$

can be regarded as a modified three-dimensional Laplace operator where the coefficient in the vertical z direction is adjusted due to the density stratification, and the coefficients in x, y directions are constants due to the horizontal density homogeneity in the three-dimensional quasi-geostrophic flow model formulation. We consider this model in the cubic domain $D = (0, L) \times (0, L) \times (0, H) \subset \mathbb{R}^3$.

In this model, the three-dimensional fluid velocity field (u, v, w) is

$$\begin{aligned} u &= -\frac{\partial \psi}{\partial y}, \\ v &= +\frac{\partial \psi}{\partial x}, \\ w &= -\frac{f_0}{N^2}[\psi_{tz} + J(\psi, \psi_z)]. \end{aligned} \tag{4.2}$$

The pressure deviation from the mean pressure $\bar{p}$ and the density deviation from the basic density profile ("basic stratification") $\bar{\rho}(z)$ are also represented by the stream function. In terms of the basic density profile, the gravitational acceleration $g = 9.81\, ms^{-2}$, and the mean density ρ_0, the stratification frequency is $N^2 = -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz}$, which is positive in the usual case of static stability ($d\bar{\rho}/dz < 0$).

Now we present an approximate analytical non-autonomous model for Lagrangian fluid motion: $\dot{x} = u(x, y, z, t)$, $\dot{y} = v(x, y, z, t)$, $\dot{z} = w(x, y, z, t)$. Due to the shear instability, i.e., the Kelvin-Helmholtz instability of the gravity currents, we decompose their velocity field into two parts as in Cushman-Roisin [8]. The first part of the velocity field (u, v, w) is a horizontal shear flow

$$(\bar{u}(z), 0, 0), \tag{4.3}$$

with the corresponding stream function $\bar{\psi} = -\bar{u}(z)y$ and potential vorticity $\bar{q} = -\frac{f_0^2}{N^2}\bar{u}''(z)y + \beta y$. This is a solution of (4.1). For example in [24], $\bar{u}(z)$ is taken as a profile similar to $\sin(1 - 2z)\pi$.

The second part of the velocity field is represented by a perturbed stream function ψ'. Namely, $\psi = \bar{\psi} + \psi'$. Putting this into the equation (4.1) and linearize at the basic shear flow $\bar{\psi}$, we obtain

$$q'_t + J(\bar{\psi}, q') + J(\psi', \bar{q}) = \nu \partial_{zz} \Delta \psi',$$

where $q' = \Delta\psi' + \frac{f_0^2}{N^2}\psi'_{zz}$. This is further written as

$$q'_t + \bar{u}(z)q'_x + [\beta - \frac{f_0^2}{N^2}\bar{u}''(z)]\psi'_x = \nu \partial_{zz} \Delta \psi', \tag{4.4}$$

Since we are interested in vertical mixing, we seek solutions of the form $\psi' = \Re e^{i\,(nz - ct)} h(x, y)$ with wave speed c. Putting this into equation (4.4), we find that the horizontal profile $h(x, y)$ satisfies the elliptic partial differential equation

$$\nu n^2 \Delta h + [\beta - \frac{f_0^2}{N^2}\bar{u}''(z)]h_x = 0. \tag{4.5}$$

We assume that $\bar{u}''(z) = 0$, and h is solved with periodic boundary conditions in both x and y, with the period L. Thus we obtain the perturbed stream function $\psi' = \Re e^{i\,(nz-ct)} h(x, y) = h(x, y)\cos(nz - ct)$. The total stream function is then

$$\psi(x, y, z, t) = -\bar{u}(z)y + h(x, y)\cos(nz - ct).$$

Using (4.2), the corresponding velocity field is

$$\begin{aligned} u &= \bar{u}(z) - D_2 h(x,y)\ \cos(nz-ct), \\ v &= D_1 h(x,y)\ \cos(nz-ct), \\ w &= \frac{f_0}{N^2}[-c\,n\,h(x,y)\ \cos(nz-ct) + n\,\bar{u}(z) D_1 h(x,y) \sin(nz-ct) \\ &\quad + \bar{u}'(z) D_1 h(x,y) \cos(nz-ct)]. \end{aligned}$$

The equations of Lagrangian motion of fluid particles in this velocity field are

$$(4.6)\quad \begin{cases} \dot{z} = \frac{f_0}{N^2}[-c\,n\,h(x,y)\ \cos(nz-ct) + n\,\bar{u}(z) D_1 h(x,y) \sin(nz-ct) + \\ \qquad + \bar{u}'(z) D_1 h(x,y) \cos(nz-ct)]\ , \\ \dot{x} = \bar{u}(z) - D_2 h(x,y)\ \cos(nz-ct)\ , \\ \dot{y} = D_1 h(x,y)\ \cos(nz-ct) \end{cases}$$

We define a small parameter $0 < \varepsilon := \frac{f_0}{N^2} \ll 1$ for gravity currents. This parameter is small in the cases when the stratification is strong ($N \gg 1$) or rotation is weak ($f_0 \ll N$). (Due to the small spatial scales of gravity currents, the rotation, quantified by f_0 here, is less significant for their motion.) Now introduce the slow time variable $\tau = \varepsilon t$. Then $\frac{d}{d\tau} = \varepsilon^{-1}\frac{d}{dt}$. But in the following we still use t to denote this new time variable. The above Lagrangian model becomes the following form.

$$(4.7)\quad \begin{cases} \dot{z} = [-c\,n\,h(x,y)\ \cos(nz-ct) + n\,\bar{u}(z) D_1 h(x,y) \sin(nz-ct) + \\ \qquad + \bar{u}'(z) D_1 h(x,y) \cos(nz-ct)]\ , \\ \varepsilon\dot{x} = \bar{u}(z) - D_2 h(x,y)\ \cos(nz-ct)\ , \\ \varepsilon\dot{y} = D_1 h(x,y)\ \cos(nz-ct) \end{cases}$$

Note that, due to the convention used in geophysics, here x, y are the fast and z is the slow variable. This is different from the notation used in §2.

Now we apply the slow manifold reduction in §2 to this Lagrangian fluid model of stratified flows. The reduced manifold $x = s_0^x(t,z),\ y = s_0^y(t,z)$ satisfies

$$\begin{aligned} \bar{u}(z) - D_2 h(x,y)\ \cos(nz-ct) = 0, \\ D_1 h(x,y)\ \cos(nz-ct) = 0. \end{aligned}$$

Consider the matrix

$$B(t,z) = \begin{pmatrix} -D_{12}h & -D_{22}h \\ D_{11}h & D_{12}h \end{pmatrix} \cdot \cos(nz-ct)$$

evaluated at $x = s_0^x(t,z),\ y = s_0^y(t,z)$. If the eigenvalues $\lambda(t,z)$ satisfy $\Re\lambda(t,z) < -b_0$, for some positive constant b_0, uniformly in $(t,z) \in \mathbb{R} \times (0,H)$, then there exists a local slow integral manifold

$$\mathcal{S}_\varepsilon = \{(t, s^x(t,z,\varepsilon), s^y(t,z,\varepsilon), z) \mid t \in \mathbb{R}, z \in (0,H)\},$$

where the smooth representation functions $s^x(t,z,\varepsilon), s^y(t,z,\varepsilon)$ are close to $s_0^x(t,z), s_0^y(t,z)$ in the sense that

$$|s^x(t,z,\varepsilon) - s_0^x(t,z)| + |s^y(t,z,\varepsilon) - s_0^y(t,z)| \le K\varepsilon \quad \text{for all } t \in \mathbb{R},\ z \in \Omega_1' \subset (0,H)$$

and some constant $K > 0$.

Furthermore, the reduced dynamics is given by the non-autonomous dynamical system

$$\dot{z} \quad = [-c\,n\,h(s^x(t,z,\varepsilon), s^y(t,z,\varepsilon))\ \cos(nz-ct)$$

$$
\begin{aligned}
(4.8)\qquad &+n\,\bar{u}(z)D_1h(s^x(t,z,\varepsilon),s^y(t,z,\varepsilon))\sin(nz-ct)\\
&+\bar{u}'(z)D_1h(s^x(t,z,\varepsilon),s^y(t,z,\varepsilon))\cos(nz-ct)],
\end{aligned}
$$

where $s^x(t,z,\varepsilon), s^y(t,z,\varepsilon)$ are approximated as (see §3)

$$s^x(t,z,\varepsilon) = s_0^x(t,z) + \varepsilon\{\frac{\det(M_1)}{\det(D^2h)}\} + O(\varepsilon^2),$$

and

$$s^y(t,z,\varepsilon) = s_0^y(t,z) + \varepsilon\{\frac{\det(M_2)}{\det(D^2h)}\} + O(\varepsilon^2).$$

In the above formulas, the matrices

$$D^2h = \begin{pmatrix} D_{11}h & D_{12}h \\ D_{21}h & D_{22}h \end{pmatrix},$$

$$M_1 = \begin{pmatrix} F_1 & D_{12}h \\ F_2 & D_{22}h \end{pmatrix},$$

and

$$M_2 = \begin{pmatrix} D_{11}h & F_1 \\ D_{21}h & F_2 \end{pmatrix},$$

with

$$
\begin{aligned}
F_1 = \cos^{-1}(nz-ct)\ &\{D_1s_0^y + D_2s_0^y\ [-c\,n\,h(x,y)\ \cos(nz-ct)\\
&+n\,\bar{u}(z)D_1h(x,y)\sin(nz-ct) + \bar{u}'(z)D_1h(x,y)\cos(nz-ct)]\},
\end{aligned}
$$

and

$$
\begin{aligned}
F_2 = -\cos^{-1}(nz-ct)\ &\{D_1s_0^x + D_2s_0^x\ [-c\,n\,h(x,y)\ \cos(nz-ct)\\
&+n\,\bar{u}(z)D_1h(x,y)\sin(nz-ct) + \bar{u}'(z)D_1h(x,y)\cos(nz-ct)]\},
\end{aligned}
$$

are all evaluated at $x := s_0^x(t,z),\ y := s_0^y(t,z)$.

This scalar non-autonomous dynamical system (4.8) is easier to monitor than its two dimensional counterpart (4.7), and may be useful as a conceptual, tractable tool for understanding some features of vertical mixing in unsteady gravity currents. Note that the Coriolis parameter f_0 and stratification parameter N are scaled in the new slow time variable, which is $\frac{f_0}{N^2}t$, in this reduced model.

5. Conclusion and Perspectives

The aim of this paper is to derive a conceptual model for the three-dimensional Lagrangian motion of fluid particles in unsteady gravity currents in geophysical flows. A small parameter appears in the cases when the fluid stratification is strong or the Earth's rotation is less significant (at the spatial scales of gravity currents).

To tackle such a geophysical application, we have deduced a flexible slow manifold theorem (cf. Theorem 2.2) for nonautonomous ODEs. Its scope is not limited to the present paper and it should be useful for other slow-fast nonautonomous models in a variety of other fields, where usually only local (not global) assumptions (see §2) hold.

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