Nonautonomous bifurcation scenarios in SIR models

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Abstract

The standard obstacles in developing a bifurcation theory for nonautonomous differential equations are the lack of steady state equilibria and the insignificance of eigenvalues in stability investigations. For this reason, various different techniques have been proposed to specify changes in the qualitative behavior of time-dependent dynamical systems. In this paper, we investigate and compare several approaches to nonautonomous bifurcations using SIR-like models from epidemiology as a paradigm. These models are sufficiently simple to allow explicit solutions to a large extent and consequently enable a detailed discussion of the different results.

Keywords: Nonautonomous dynamical system, nonautonomous bifurcation, SIR-like model, Bohl exponents, dichotomy spectrum, nonautonomous center manifold, pullback attractor.
2010 MSC: 37C60; 34C23; 37G35; 78A70

1. Introduction

Many phenomena in the life sciences crucially depend on time-varying factors. They might be intrinsic caused by a temporally fluctuating environment, or extrinsic due to control or regulation strategies. Provided these phenomena allow a description in form of evolutionary differential equations, the mathematical models consequently have to be nonautonomous. Hence, a thorough analysis of such problems requires a generalization of the classical dynamical systems theory due to several of its limitations. Among them are the facts that eigenvalues do not yield stability information anymore or that steady state equilibria

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[1]Partially supported the DFG grant KL 1203/7-1, the Ministerio de Ciencia e Innovación (Spain) grant MTM2008-00088 and the Junta de Andalucía grant P07-FQM-02468

might not exist. In particular, the concept of a bifurcation in nonautonomous systems is far from being well-understood and various different approaches appear in the literature. They range from stability transitions [26] over changes in the structure of minimal sets [19] to essentially analytical approaches [22].

To illustrate this we consider a classical and well-known model from mathematical epidemiology in form of the SIR equations (cf., e.g., [18, 16, 20, 14]), in which the coefficients are allowed to be time-dependent. There is a strong biological motivation to include time-dependent coefficients into epidemiological models. Certainly the probability to contract a disease is hardly constant over time and it depends rather on the season (indicating periodic coefficients in $(*)$ below). Moreover, in childhood diseases, for instance, the school calendar must also be taken into account, thus making it necessary to consider more general time dependencies (see [30, 8, 29]).

The goal of the present paper is two-fold. Firstly, the SIR equations are sufficiently simple and therefore an ideal vehicle to exemplify and illustrate some fundamental principles from the recently developed theory of (deterministic) nonautonomous dynamical systems (see, for instance, [13]). Secondly, and more specifically, they allow different corresponding concepts of a bifurcation for nonautonomous ordinary differential equations (ODEs) to be presented clearly without technical distractions.

An immediate issue is the lack of steady state solutions, which requires the set of possible candidates for bifurcating objects to be enlarged. Three approaches have been proposed in the literature: [17] and [24] investigate entire bounded solutions, while [26] is based of changes in attraction rates, and finally [19] looks at changes in dynamically minimal sets reflecting the characteristics of the time-forcing.

Our aim is to apply these approaches as well as geometrical tools such as nonautonomous center manifold reductions to appropriate SIR models. Four appendices provide background material on basic concepts from the field of nonautonomous dynamics, including Bohl exponents as a substitute for eigenvalues, the process and skew-product formulations of nonautonomous systems and their attractors, as well as a transcritical bifurcation result due to [26] and center integral manifolds. We consider both the intuitive process (two-parameter semi-group) formulation of nonautonomous dynamical systems, since it is the most natural generalization of the (semi-)groups known from the autonomous theory, and the alternative formulation of skew-product flows (cf. [28]). Here the references [9, 10, 22] deal with bifurcation problems in nonautonomous systems; the "small-coefficient assumption" in [11, 7] does not, however, fit into our setting, while the results in [17, 26, 19, 24] are tailor-made for it. A nonautonomous center manifold theory [3, 21, 25] becomes important since [17, 26, 19] are limited to scalar ODEs. Finally, our approach is based on preparations from [14] providing some required stability conditions, which were generalized in [2] to equations under the effect of diffusion.

The SIR equations are a classical model describing the spread of an epidemic in a population, which is partitioned into S (susceptibles), I (infected) and R (recovered) individuals. Such models will be investigated here under time-variable parameters or deterministic forcing in the model, so that the total population $N(t) = S(t) + I(t) + R(t)$ need not be constant. In a temporally fluctuating environment it turns out to be crucial, which temporal horizon is of importance, i.e., on which time interval J the dynamical behavior is of interest. Biologically, J might be finite, but mathematics often requires J to be unbounded in at least one time direction, i.e. of the form

$$\mathbb{R}_+ := [0, \infty), \qquad\qquad \mathbb{R}_- := (-\infty, 0].$$

Fluctuations will be achieved first through a temporal forcing term given by a function $q : J \to \mathbb{R}$ taking nonnegative bounded values, i.e., $q(t) \in [q^-, q^+]$ for all $t \in J$ for some $0 \le q^- \le q^+$, and later with a time varying interaction coefficient γ between the S and I variables.

The general SIR model investigated here is a 3-dimensional system of ODEs

$$\begin{cases} \dot{S} = aq(t) - aS + bI - \gamma(t)\dfrac{SI}{N(t)}, \\ \dot{I} = -(a+b+c)I + \gamma(t)\dfrac{SI}{N(t)}, \\ \dot{R} = cI - aR, \end{cases} \tag{$*$}$$

where the parameters a, b and c are positive constants. Later they will also be allowed to vary in time within suitable nonnegative bounds. This means that solutions with nonnegative initial values remain nonnegative. Moreover, from a technical perspective, the functions q, $\gamma : J \to \mathbb{R}$ are typically assumed to be continuous or essentially bounded.

Adding both sides of the system $(*)$ gives the scalar nonautonomous ODE

$$\dot{N} = a(q(t) - N) \tag{1.1}$$

for the total population N, which has the general solution

$$N(t) = N_0 e^{-a(t-t_0)} + ae^{-at} \int_{t_0}^{t} q(s)e^{as}\, ds \tag{1.2}$$

satisfying $N(t_0) = N_0$. Since $ae^{-at} \int_{t_0}^{t} e^{as}\, ds = 1 - e^{-a(t-t_0)}$, by the positivity bounds on $q(t)$ the integral in (1.2) takes values between

$$q^- \left(1 - e^{-a(t-t_0)}\right) \le ae^{-at} \int_{t_0}^{t} q(s)e^{as}\, ds \le q^+ \left(1 - e^{-a(t-t_0)}\right) \quad \text{for all } t \ge t_0.$$

This means the total population is bounded from above and below, specifically

$$q^- + \left(N_0 - q^-\right) e^{-a(t-t_0)} \le N(t) \le q^+ + \left(N_0 - q^+\right) e^{-a(t-t_0)}.$$

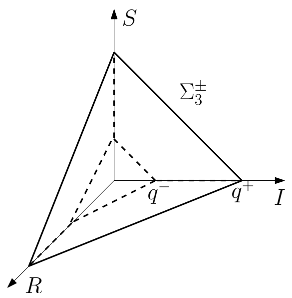

Figure 1: Simplex slap $\Sigma_3^{\pm}$

Hence the simplex slab (see Fig. 1)

$$\Sigma_3^{\pm} = \left\{(S, I, R) \in \mathbb{R}^3 : S, I, R \geq 0,\, N = S + I + R \in [q^-, q^+]\right\}$$

attracts all populations starting outside it and populations originating within it remain there. When the forcing term $q(t)$ is not identically equal to a constant, the simplex slab $\Sigma_3^{\pm}$ will, in fact, absorb outside populations in a finite time and will be positively invariant (rather than strictly invariant), so attention can be restricted to the dynamics in $\Sigma_3^{\pm}$. Moreover, $\Sigma_3^{\pm}$ then contains an attractor, which, in the present time-varying framework, is a *pullback attractor* (cf. [13, pp. 37ff, Chapt. 3]).

Due to its nonautonomous character the ODE (1.1) has no steady state solutions, but has what Chueshov [5] called a *nonautonomous equilibrium solution* in a random dynamical systems set-up. It is found by taking the *pullback limit* in (1.2) (i.e., as the initial $t_0 \to -\infty$ with the current time t held fixed, see [13]), namely

$$N_a^\star(t) = ae^{-at} \int_{-\infty}^{t} q(s)e^{as}\, ds. \tag{1.3}$$

This concept requires the interval J to be unbounded from below. The nonautonomous equilibrium $N_a^\star$ also forward attracts all other solution of the linear ODE (1.1), i.e.,

$$|N(t) - N_a^\star(t)| = e^{-a(t-t_0)}\, |N(t_0) - N_a^\star(t_0)| \to 0 \quad \text{as} \quad t \to \infty,$$

if J is also unbounded from above, so $J = \mathbb{R}$. In particular, (1.3) reduces to the nontrivial equilibrium $N_a^\star(t) \equiv q$ on J for constant functions q, which clearly reflects the leitmotiv that the steady state equilibria of autonomous equations persist as bounded entire solutions for time-varying parameters (cf. [23]).

2. The SI equations with variable population

Suppose that the interaction coefficient γ in $(*)$ is constant and, for simplicity, assume that the R component is not present in $(*)$ and $c = 0$, which means solutions can then be determined explicitly. The equations (∗) reduce to

$$\begin{cases} \dot{S} = aq(t) - aS + bI - \gamma \dfrac{SI}{N(t)}, \\ \dot{I} = -(a+b)I + \gamma \dfrac{SI}{N(t)}. \end{cases} \tag{$*_2$}$$

Without an infected population $I_0 = 0$, then $I(t) \equiv 0$. Hence the S face of

$$\Sigma_2^{\pm} = \left\{ (S, I) \in \mathbb{R}^2 : \, S, I \geq 0, \, S + I \in [q^-, q^+] \right\}$$

(see Fig. 2) is invariant and the equations reduce to the scalar linear problem

$$\dot{S} = aq(t) - aS, \tag{2.1}$$

which has the general solution

$$S(t) = e^{-a(t-t_0)} S_0 + ae^{-at} \int_{t_0}^{t} q(s) e^{as} \, ds$$

with $S(t_0) = S_0$. As with (1.1) above, (2.1) has no steady state solution (unless $q(t)$ is constant), but it possesses a nonautonomous equilibrium $S_a^\star$ that is again found by taking the pullback limit (i.e., as $t_0 \to -\infty$ with t held fixed), namely

$$S_a^\star(t) = ae^{-at} \int_{-\infty}^{t} q(s) e^{as} \, ds,$$

provided J is unbounded below. When $J = \mathbb{R}$, this also forward attracts all other solution of the S equation in the $I = 0$ face exponentially, i.e.,

$$|S(t) - S_a^\star(t)| \to 0 \quad \text{as} \quad t \to \infty.$$

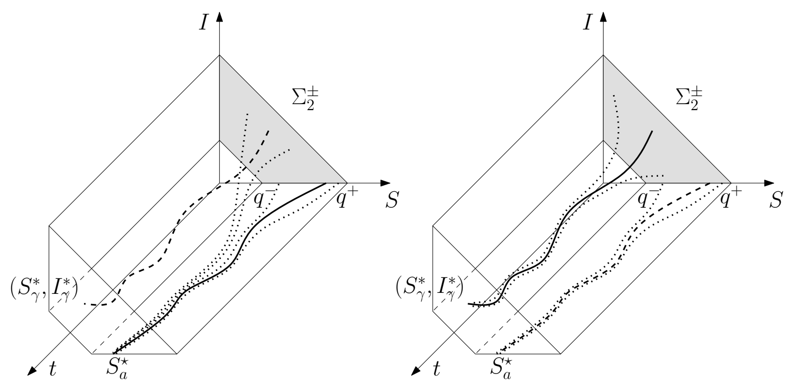

Figure 2: The simplex slap $\Sigma_2^{\pm}$ (shaded in grey) and nonautonomous equilibria to ($*_2$):
Left: $(S_a^\star, 0)$ is globally asymptotically stable for $\gamma \leq a + b$
Right: $(S_\gamma^*(t), I_\gamma^*(t))$ is globally asymptotically stable for $\gamma > a + b$

In particular, the full SI dynamics in ($*_2$) has no steady state solution, but a nonautonomous equilibrium solution $(S_a^\star(t), 0)$. We say such a solution is *(globally asymptotically) stable w.r.t. a set* $A \subseteq \mathbb{R}^2$, if it fulfills the usual corresponding stability definition with initial values taken from A rather than the full state space. The following results are taken from [14], where merely stability rather than bifurcation issues were tackled.

Proposition 2.1. *[14, Lemma 1] The nonautonomous equilibrium* $(S_a^\star(t), 0)$ *of* $(*_2)$ *is globally asymptotically stable w.r.t.* $\Sigma_2^\pm$ *when* $\gamma \leq a + b$*. It is unstable w.r.t.* $\Sigma_2^\pm$*, when* $\gamma > a + b$*.*

Proposition 2.2. *[14, Lemma 2] The nonautonomous equilibrium* $(S_\gamma^*(t), I_\gamma^*(t))$ *of* $(*_2)$*, where*

$$S_\gamma^*(t) = N_a^\star(t) - I_\gamma^*(t), \qquad I_\gamma^*(t) = \frac{e^{(\gamma-a-b)t}}{\gamma \int_{-\infty}^t \frac{e^{(\gamma-a-b)s}}{N_a^\star(s)}\, ds}$$

and $N_a^\star(t)$ *is from* (1.2)*, is globally asymptotically stable w.r.t. the interior of* $\Sigma_2^\pm$ *for* $\gamma > a + b$*.*

If the contact rate γ increases through the critical value $a + b$, there is thus a change of stability from the infection-free solution $(S_a^\star(t), 0)$ to the nontrivial solution $(S_\gamma^*(t), I_\gamma^*(t))$ (cf. Fig. 2). It is a folklore result from the autonomous theory that this goes hand in hand with a bifurcation occurring at $\gamma = a+b$. Indeed, for the autonomous special case with constant q the asymptotically stable steady state equilibrium $(S_a^\star, 0) = (q, 0)$ undergoes a transcritical bifurcation into the steady state equilibrium

$$(S_\gamma^*, I_\gamma^*) = \frac{q}{\gamma}(a + b, \gamma - a - b).$$

For time-varying q, a similar classification is possible. First, this requires a nonautonomous center manifold reduction described in Appendix D (and thus with J unbounded below) together with a suitable bifurcation result allowing time-dependent parameters. We illustrate this using the approach from [26], where a bifurcation means a change in the attraction and repulsion radii. An application of the terminology from Appendix B.1 to the problem $(*_2)$ yields:

Theorem 2.3 (transcritical bifurcation in $(*_2)$)**.** *If* J *is unbounded below, then there exists a neighborhood* $\Gamma \subseteq \mathbb{R}$ *of the critical parameter* $\gamma^* = a + b$ *such that for all* $\gamma \in \Gamma$ *the disease-free nonautonomous equilibrium* $(S_a^\star(t), 0)$ *to* $(*_2)$

(a) is J*-attractive for* $\gamma < a+b$ *and* J*-repulsive for* $\gamma > a+b$*. In case* $\gamma = a+b$ *and* $J = \mathbb{R}$*, it is unstable.*

(b) undergoes a bifurcation in the sense that its corresponding radii of J*-attraction and -repulsion satisfy*

$$\lim_{\gamma \nearrow a+b} \rho_\gamma^+(S_a^\star(t), 0) = 0 = \lim_{\gamma \searrow a+b} \rho_\gamma^-(S_a^\star(t), 0).$$

The fact that the disease-free solution is unstable for $\gamma = a + b$ does not contradict Proposition 2.1, since Theorem 2.3 takes a whole $\mathbb{R}^2$-neighborhood of $(S_a^\star(t), 0)$ into account, rather than only initial values in the simplex slab $\Sigma_2^\pm$.

Proof. After some preliminaries, the proof consists of three steps. By assumption, the nonautonomous equilibrium $S_a^\star(t)$ of (2.1) satisfies $q^- \le S_a^\star(t) \le q^+$ for all $t \in J$. Denote the right-hand side of $(*_2)$ by

$$F(t,S,I) := \begin{pmatrix} aq(t) - aS + bI - \gamma \dfrac{SI}{S+I} \\ -(a+b)I + \gamma \dfrac{SI}{S+I} \end{pmatrix}.$$

Then the equation of perturbed motion for $(*_2)$ corresponding to the entire bounded solution $(S_a^\star(t), 0)$ is given by the planar nonautonomous system

$$\begin{pmatrix} \dot S \\ \dot I \end{pmatrix} = f(t,S,I), \quad f(t,S,I) := \begin{pmatrix} -aS + bI - (a+b+\lambda)\dfrac{(S+S_a^*(t))I}{S+S_a^*(t)+I} \\ -(a+b)I + (a+b+\lambda)\dfrac{(S+S_a^*(t))I}{S+S_a^*(t)+I} \end{pmatrix},$$

where we substituted $\gamma = a+b+\lambda$. It has the trivial solution for all $\lambda \in \mathbb{R}$.

(I) To perform a nonautonomous center manifold reduction, as in Appendix D we augment the above ODE with $\dot\lambda = 0$ and apply the transformation

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S \\ I \\ \lambda \end{pmatrix}$$

which gives the 3-dimensional system

$$\dot y = Ay + F(t,y) \tag{2.2}$$

with $A := \operatorname{diag}(-a,0,0)$ and the nonlinearity

$$F(t,y) := \begin{pmatrix} 0 \\ y_2 \dfrac{-(a+b)y_2 + y_3(y_1 - y_2 + S_a^\star(t))}{y_1 + S_a^\star(t)} \\ 0 \end{pmatrix}.$$

By Appendix D the nonautonomous system (2.2) in $\mathbb{R}^3$ has a 2-dimensional center integral manifold $\mathcal{W} \subseteq J \times \mathbb{R}^3$ given as the graph $y_1 = w(t,y_2,y_3)$ of a smooth mapping w and the ODE (2.2) reduced to $\mathcal{W}$ therefore becomes

$$\dot y_2 = y_2 \frac{y_3(w(t,y_2,y_3) - y_2 + S_a^\star(t)) - (a+b)y_2}{w(t,y_2,y_3) + S_a^\star(t)} =: g(t,y_2,y_3).$$

Taking into account $y_3 = \lambda$ (constant) and the identity $w(t,0,\lambda) \equiv 0$ as well as $D_{(2,3)}w(t,0,0) \equiv 0$ on J (cf. (d_1)), the reduced equation is

$$\dot y_2 = \lambda y_2 - \frac{a+b+\lambda}{S_a^\star(t)} y_2^2 + r(t,y_2,\lambda) \tag{2.3}$$

with

$$r(t,y_2,\lambda) := \frac{\lambda(w(t,y_2,\lambda) - y_2 + S_a^\star(t)) - (a+b)y_2}{w(t,y_2,\lambda) + S_a^\star(t)} y_2 - \lambda y_2 + \frac{a+b+\lambda}{S_a^\star(t)} y_2^2.$$

(II) To verify a transcritical bifurcation in (2.3) we make use of Theorem C.1, whose notation we mimic in the following. First, $r(t,0,\lambda) \equiv 0$. We now successively verify the assumptions of Theorem C.1.

(i) The linearized equation $\dot{y}_2 = \lambda y_2$ has the transition mapping $\Phi_\lambda(t,s) = e^{\lambda(t-s)}$, we choose $K = 1$ and the functions $\gamma_+(\lambda) = \gamma_-(\lambda) = \lambda$ are monotone increasing.

(ii) With (D.7) we compute the second order term $D_2^2 g(t,0,\lambda) = -2\,(a+b+\lambda)/S_a^\star(t)$, which is strictly negative for λ near 0. This gives the estimates

$$-\frac{a+b+\lambda}{q^-} \leq \inf_{t\in J}\left(-\frac{a+b+\lambda}{S_a^\star(t)}\right) \leq \sup_{t\in J}\frac{a+b+\lambda}{S_a^\star(t)} \leq -\frac{a+b+\lambda}{q^+}$$

and therefore

$$-\infty < \lim_{\lambda\to 0}\inf_{t\in J}\left(-\frac{a+b+\lambda}{S_a^\star(t)}\right) \leq \lim_{\lambda\to 0}\sup_{t\in J}\left(-\frac{a+b+\lambda}{S_a^\star(t)}\right) \leq -\frac{a+b}{q^+} < 0.$$

(iii) It remains to check the conditions on the remainder in the third order Taylor approximation of g near $x = 0$. For this purpose, we deduce

$$\begin{aligned} D_2^3 g(t,x,\lambda) = \frac{a+b+\lambda}{(w_0+S_a^\star(t))^4}\Big(&-6x^2 S_a^\star(t) w_2 w_1 + x^2 S_a^\star(t)^2 w_3 \\ &+ 2x^2 S_a^\star(t) w_0 w_3 - 12x S_a^\star(t) w_1^2 + 6 S_a^\star(t)^2 w_1 + 12 S_a^\star(t) w_0 w_1 \\ &+ 6x S_a^\star(t)^2 w_2 + 12x S_a^\star(t) w_0 w_2 + 6x^2 w_1^3 - 6x^2 w_0 w_2 w_1 \\ &+ x^2 w_0^2 w_3 - 12x w_0 w_1^2 + 6 w_0^2 w_1 + 6x w_0^2 w_2\Big) \end{aligned}$$

with the terms $w_j := D_2^j w(t,x,\lambda)$, $0 \leq j \leq 3$, which are uniformly bounded in a tubular set $\mathbb{R}\times U$, where $U \subseteq \mathbb{R}^2$ is a neighborhood of 0. Together with the boundedness of $S_a^\star$ this guarantees that the two limit relations assumed in Theorem C.1 are fulfilled.

Therefore, Theorem C.1 applies to the scalar equation (2.3).

(III) The attraction/repulsion properties of $(S_a^\star, 0)$ are a consequence of the reduction principle (see [3] or [13, Theorem 6.25]) guaranteeing that the behavior of (2.3) extends the the planar equation $(*_2)$.

It remains to show that the nonautonomous equilibrium $(S_a^\star(t), 0)$ is unstable for $\gamma = a+b$. This follows from the reduction property (d_3) of $\mathcal{W}$. Indeed, in case $\lambda = 0$ and referring to (2.3), the reduced differential equation becomes $\dot{y}_2 = -(a+b)y_2^2/S_a^\star(t) + O(y_2^3)$ uniformly in $t \in \mathbb{R}$, with unstable zero solution. Again, by the reduction principle, this stability property extends to $(*_2)$. $\square$

Such an observation corresponds to Propositions 2.1 and 2.2, where the asymptotic stability of the solution $(S_a^\star(t), 0)$ is transferred to $(S_\gamma^*(t), I_\gamma^*(t))$.

3. The SI equations with variable interaction

Consider the SI equations in $(*)$ for a constant driving q and thus with a constant limiting population

$$N(t) = S(t) + I(t) \equiv 1 \quad \text{on } J, \tag{3.1}$$

but now with a time-variable interaction term $\gamma(t)$. For the sake of a bifurcation analysis we suppose that $\gamma : J \to \mathbb{R}$ depends on a real parameter λ and write instead γ_λ. In particular, we assume that $\gamma_\lambda : J \to [\gamma^-, \gamma^+]$ is a continuous function, where $0 < \gamma^- \leq \gamma^+$; hence $(*)$ becomes

$$\begin{cases} \dot{S} = a - aS + bI - \gamma_\lambda(t)SI, \\ \dot{I} = -(a+b)I + \gamma_\lambda(t)SI. \end{cases} \tag{$*_3$}$$

This planar ODE $(*_3)$ has the disease-free steady state equilibrium solution

$$(S^\star(t), I^\star(t)) \equiv (1,0) \quad \text{on } J$$

for all parameters λ. On adding, the equations reduce to the autonomous ODE

$$\dot{N} = a - aN,$$

which has the globally asymptotically stable steady state solution $N^\star(t) \equiv 1$. This allows the analysis to be restricted to the compact 2-simplex

$$\Sigma_2 = \left\{(S,I) \in \mathbb{R}^2 : S, I \geq 0,\, S + I = 1\right\}.$$

Equation $(*_3)$ can be reduced to a Bernoulli differential equation

$$\dot{I} = (\gamma_\lambda(t) - a - b)I - \gamma_\lambda(t)I^2, \tag{$*_{3.1}$}$$

which has the explicit solution

$$I(t) = \frac{\exp\left(\int_{t_0}^t [\gamma_\lambda(s) - a - b]\, ds\right)}{1 + I_0 \int_{t_0}^t \gamma_\lambda(s) \exp\left(\int_{t_0}^s [\gamma_\lambda(r) - a - b]\, dr\right) ds}$$

satisfying $I(t_0) = I_0$. Taking the pullback limit $t_0 \to -\infty$, the planar system $(*_3)$ possesses the nonautonomous equilibrium $(I_\lambda^*(t), S_\lambda^*(t))$,

$$I_\lambda^*(t) = \left(\int_{-\infty}^t \gamma_\lambda(r) e^{-\int_r^t \gamma_\lambda(s) - a - b\, ds}\, dr\right)^{-1}, \quad S_\lambda^*(t) = 1 - I_\lambda^*(t). \tag{3.2}$$

For constant interaction rates $\gamma_\lambda(t) \equiv \lambda$ on J we obtain

$$I_\lambda^*(t) \equiv \frac{\lambda - a - b}{\lambda}, \qquad S_\lambda^*(t) \equiv \frac{a+b}{\lambda}.$$

This is the nontrivial equilibrium to what is then the autonomous equation $(*_3)$.

3.1. Disease-free equilibrium

In [14] it was shown that the disease-free steady state equilibrium $(S^\star, I^\star) = (1,0)$ is globally asymptotically stable w.r.t. Σ_2 when $\gamma^+ \leq a + b$; it is unstable when $\gamma^- > a + b$. In order to determine the stability of the trivial solution without such uniformity assumptions on the function γ_λ, we need the concepts of upper and lower *Bohl exponents*, that are defined in Appendix A.

Proposition 3.1. *The disease-free equilibrium* $(1,0)$ *to* $(*_3)$ *is uniformly asymptotically stable for* $\overline{\beta}_J(\gamma_\lambda) < a+b$ *and unstable for* $\underline{\beta}_J(\gamma_\lambda) > a+b$.

In the following we sometimes use the convenient abbreviation

$$\gamma + [\alpha, \beta] := [\alpha+\gamma, \beta+\gamma] \quad \text{for all } \alpha, \beta, \gamma \in \mathbb{R}.$$

Proof. Due to (3.1) we can restrict to the scalar equation $(*_{3.1})$. Since both a and b are constant in time, thanks to (A.1) the linear part of $(*_{3.1})$ has the dichotomy spectrum $-a-b+[\underline{\beta}_J(\gamma_\lambda), \overline{\beta}_J(\gamma_\lambda)]$. Propositions 3.9 and 3.10 of [24] then imply the claim. □

This result indicates a bifurcation in $(*_{3.1})$ (and thus $(*_3)$) for interaction rates γ_λ, when the Bohl exponents satisfy $\underline{\beta}_J(\gamma_\lambda) = a+b$ or $\overline{\beta}_J(\gamma_\lambda) = a+b$. A more detailed analysis in this direction will be given now.

The first result is based on a skew-product formulation of $(*_3)$ resp. $(*_{3.1})$, i.e., the nonautonomity in $(*)$ is induced by a driving-system on a compact metric space Ω. The required terminology is introduced in Section B.2 of the Appendix. More precisely, this means the interaction coefficient γ_λ becomes

$$\gamma_\lambda(t) = \Gamma(\phi^t(\omega), \lambda) \quad \text{for all } t \in J = \mathbb{R}$$

with a function $\Gamma : \Omega \times \mathbb{R} \to \mathbb{R}$ and a compact space Ω. The flow $\phi^t : \Omega \to \Omega$ is assumed to be minimal and uniquely ergodic w.r.t. a unique ergodic measure m_0 satisfying $m_0(\Omega) > 0$. Writing $I_\lambda(\cdot; \iota, \omega)$ for the solution to the corresponding initial value problem

$$\dot{I} = (\Gamma(\phi^t(\omega), \lambda) - a - b)I - \Gamma(\phi^t(\omega), \lambda)I^2, \qquad I(0) = \iota,$$

we see that $(*_{3.1})$ generates a local skew-product flow $\Phi_\lambda^t : \mathbb{R} \times \Omega \to \mathbb{R} \times \Omega$ with

$$\Phi_\lambda^t(\iota, \omega) := \begin{pmatrix} I_\lambda(t, \iota, \omega) \\ \phi^t(\omega) \end{pmatrix}$$

on $\mathbb{R} \times \Omega$. We use the terminology in Section B.2, in particular $\dot{B}_r(a)$ and $B_r(a)$ represent, respectively, the open and closed balls about a of radius r.

Theorem 3.2 (transcritical bifurcation in $(*_3)$ I)**.** *Let* $J = \mathbb{R}$. *If there exists a neighborhood* $V \subseteq \mathbb{R}$ *of* 0 *such that* Γ, $D_2\Gamma : \Omega \times V \to \mathbb{R}$ *are continuous and satisfy*

$$\Gamma(\omega, \lambda) > 0, \quad D_2\Gamma(\omega, \lambda) > 0 \quad \textit{for all } (\omega, \lambda) \in \Omega \times V,$$

then the disease-free equilibrium $(1,0)$ *to* $(*_3)$ *undergoes a transcritical bifurcation at the critical parameter* $\lambda^* = 0$ *as follows: There exists a* $\delta > 0$ *and a compact neighborhood* $U \subseteq \Sigma_2$ *of* $(1,0)$ *such that*

(a) The set $O := \{(1,0)\} \times \Omega$ *is the unique* Φ_0*-minimal set in* $U \times \Omega$ *and a nonhyperbolic copy of the base.*

(b) *For all* $\lambda \in \dot{B}_\delta(0)$ *the set* $U \times \Omega$ *contains exactly one* Φ_λ*-invariant set besides* O*, namely*

$$M_\lambda = \left\{ \left(1 - \tilde{I}_\lambda(\omega), \tilde{I}_\lambda(\omega), \omega\right) : \omega \in \Omega \right\}. \tag{3.3}$$

Here, M_λ *varies continuously in the parameter* $\lambda \in B_\delta(0)$*. Furthermore, both* O *and* M_λ *are hyperbolic copies of the base and have opposite-sign Lyapunov exponents*

$$\Lambda(0) = \int_\Omega \Gamma(\omega, \lambda)\, dm_0(\omega) - (a+b)m_0(\Omega),$$

$$\Lambda(\tilde{I}_\lambda) = \int_\Omega \Gamma(\omega, \lambda)(1 - 2\tilde{I}_\lambda(\omega))\, dm_0(\omega) - (a+b)m_0(\Omega).$$

(c) *The limit* $\lim_{\lambda\to 0} \sup_{\omega\in\Omega} \left|\tilde{I}_\lambda(\omega)\right| = 0$ *holds.*

Proof. (I) We mimic the notation of [19, Section 4.1] and write the right-hand side of the scalar ODE $(*_{3.1})$ accordingly in the form $\dot{I} = W(\phi^t\omega, I, \lambda)I$ with the affine function $W(\omega, I, \lambda) := \Gamma(\omega, \lambda) - a - b - \Gamma(\omega, \lambda)I$. Clearly, W is continuous as are its partial derivatives $D_2W(\omega, I, \lambda) = -\Gamma(\omega, \lambda)$,

$$D_3W(\omega, I, \lambda) = D_2\Gamma(\omega, \lambda)(1 - I), \qquad D_2^2W(\omega, I, \lambda) = 0.$$

The positivity assumption on $\Gamma(\omega, \lambda)$ ensures partial derivatives $D_2W(\omega, 0, 0) > 0$ and $D_3W(\omega, 0, 0) > 0$. Therefore, [19, Theorem 4.4] applies and yields the nonhyperbolic set $\{0\} \times \Omega$ as a unique Φ_0-minimal set in $U \times \Omega$, which is a copy of the base. In addition, for $\lambda \in \dot{B}_\delta(0)$ the set $U \times \Omega$ contains exactly two Φ_λ-invariant sets $M_\lambda^\pm = \left\{(I_\lambda^\pm(\omega), \omega) : \omega \in \Omega\right\}$, which, furthermore, are both hyperbolic copies of the base, vary continuously in the parameter $\lambda \in B_\delta(0)$ and have opposite-sign Lyapunov exponents

$$\Lambda(I_\lambda^\pm) = \int_\Omega \Gamma(\omega, \lambda)(1 - 2I_\lambda^\pm(\omega))\, dm_0(\omega) - (a+b)m_0(\Omega).$$

Moreover, $\lim_{\lambda\to 0} \sup_{\omega\in\Omega} \left|I_\lambda^\pm(\omega)\right| = 0$.
(II) Since $(*_{3.1})$ has the trivial solution, one of the Φ_λ-minimal sets $M_\lambda^\pm$ from step (I) is the graph of the function $I_\lambda^+(\omega) \equiv 0$ on Ω. In view of the identity (3.1), the assertion follows for $\tilde{I}_\lambda := I_\lambda^-$ and M_λ as defined in (3.3). □

We now discuss two further and different possible bifurcation scenarios:

- A transcritical bifurcation caused by the fact that the amplitude of the temporal fluctuation is increased.
- A shovel bifurcation due to a change in the range of the fluctuation.

3.1.1. ***Transcritical bifurcation***

We suppose that the interaction term γ_λ is given by $\gamma_\lambda(t) = a + b + \lambda\delta(t)$ for a continuous function $\delta : J \to \mathbb{R}$ having positive values in $[\delta_-, \delta_+]$, where

$0 < \delta_- \leq \delta_+$. With λ serving as bifurcation parameter controlling the amplitude of the interaction coefficient, $(*_{3.1})$ reduces to

$$\dot{I} = \lambda\delta(t)I - (a + b + \lambda\delta(t))I^2 \qquad (*^1_{3.1})$$

and we obtain

Theorem 3.3 (transcritical bifurcation in $(*_3)$ II)**.** *If J is unbounded, then there exists a neighborhood $\Lambda \subseteq \mathbb{R}$ of the critical parameter $\lambda^* = 0$ such that for all $\lambda \in \Lambda$ the disease-free equilibrium $(1, 0)$ to $(*_3)$*

(a) is J-attractive for $\lambda < 0$ and J-repulsive for $\lambda > 0$. In the case $\lambda = 0$ and $J = \mathbb{R}$, it is unstable.

(b) undergoes a bifurcation in the sense that the corresponding radii of J-attraction and J-repulsion satisfy

$$\lim_{\lambda \nearrow 0} \rho^+_\lambda(1, 0) = 0 = \lim_{\lambda \searrow 0} \rho^-_\lambda(1, 0).$$

An illustration of Theorem 3.3 for a concrete function δ can be seen in Fig. 3.

Remark 1. (1) Restricting to solutions with nonnegative initial values, the equilibrium $(1, 0)$ is asymptotically stable for $\lambda = 0$.
(2) In the biologically irrelevant situation that $\delta : J \to \mathbb{R}$ has negative values in an interval $[\delta_-, \delta_+]$ with $\delta_- \leq \delta_+ < 0$, a dual version to Theorem 3.3 holds with attraction/repulsion properties as claimed in Theorem C.1(b).

Proof. Since it is a scalar ODE with trivial solution, we aim to apply Theorem C.1 directly to $(*_{3.1})$ with the right-hand side $g(t, x, \lambda) := \lambda\delta(t)x - (a+b+\lambda\delta(t))x^2$, while the transfers to the full planar system $(*_3)$ follows using (3.1).

(i) First, since $D_2 g(t, 0, \lambda) = \lambda\delta(t)$, the transition matrix of the linear part is

$$\Phi_\lambda(t, s) = \exp\left(\lambda \int_s^t \delta(r)\, dr\right) \quad \text{for all } s, t \in J$$

and thus we choose $\gamma_\pm(\lambda) = \delta_\pm\lambda$ for $\lambda \in \mathbb{R}$. In particular, $\gamma_\pm$ are strictly increasing.

(ii) Second, the relation $D_2^2 g(t, 0, \lambda) = -2(a + b + \lambda\delta(t))$ together with the facts $a + b > 0$ and $\delta(t) \in [\delta_-, \delta_+] \subseteq (0, \infty)$ ensure that assumption (ii) of Theorem C.1 holds.

(iii) Finally, $D_2^3 g(t, x, \lambda) \equiv 0$ on $J \times \mathbb{R} \times \mathbb{R}$.

Therefore, Theorem C.1(a) applies to $(*^1_{3.1})$ and (3.1) then establishes the claim. □

On the other hand, equation $(*^1_{3.1})$ resembles the problems investigated in [17] and beyond Theorems 3.2 and 3.4 we obtain the additional information:

Theorem 3.4 (transcritical bifurcation in $(*_3)$ III)**.** *If $J = \mathbb{R}$, then there exist neighborhoods $U \subseteq \mathbb{R}$ and Λ of $\lambda = 0$ so that for all $\lambda \in \Lambda$ the disease-free steady state equilibrium $(1, 0)$ of $(*_3)$ undergoes a transcritical bifurcation at the critical parameter $\lambda^* = 0$ as follows:*

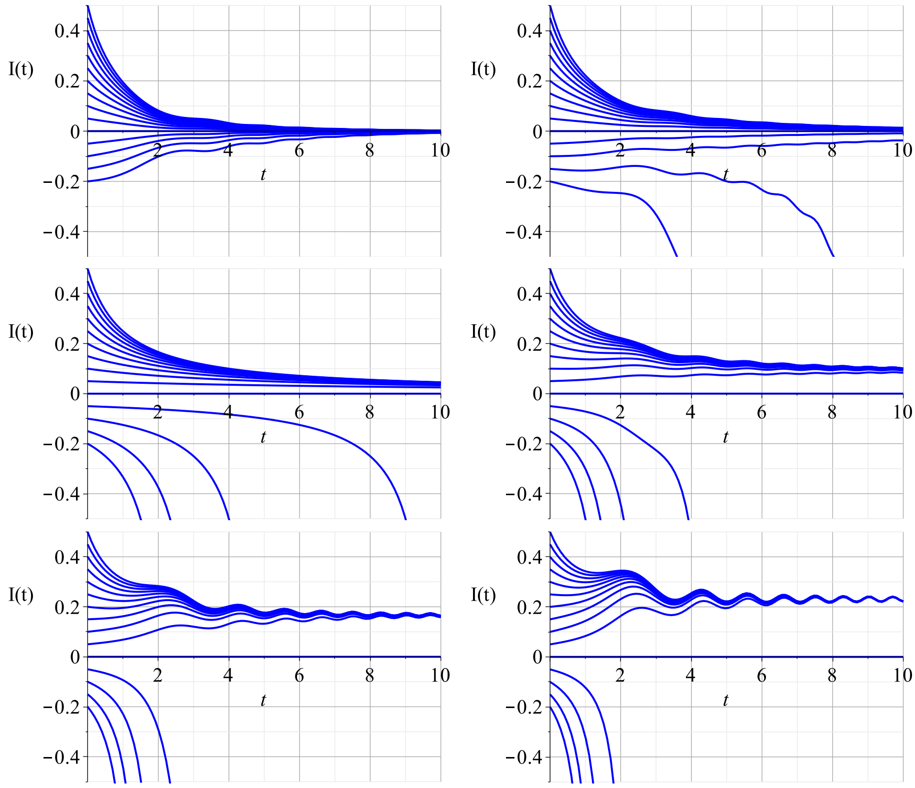

Figure 3: Nonautonomous transcritical bifurcation in $(*^1_{3.1})$: The trivial solution looses stability at the parameter value $\lambda = 0$ and a bounded entire solution becomes attractive (parameters $a = b = 1$, $\delta(t) = 4+3.5\sin(\frac{t^2}{2})$ and λ increases from the value -0.1 (top, left) to 0.15 (bottom, right) in steps of 0.05)

(a) For $\lambda < 0$ the equilibrium $(1, 0)$ is pullback attracting and there exists another entire solution $(S^*_\lambda(t), I^*_\lambda(t))$ with $I^*_\lambda(t) \in U \cap (-\infty, 0)$ for all $t \in \mathbb{R}$, which is asymptotically unstable and satisfies

$$\lim_{\lambda \nearrow 0} (S^*_\lambda(t), I^*_\lambda(t)) = (1, 0) \quad \text{for all } t \in \mathbb{R}.$$

(b) For $\lambda = 0$ the equilibrium $(1, 0)$ is (forwards) asymptotically unstable, but still pullback attracting within Σ_2.

(c) For $\lambda > 0$ the equilibrium $(1, 0)$ is asymptotically unstable and there exists another entire solution $(S^*_\lambda(t), I^*_\lambda(t))$ with $I^*_\lambda(t) \in U \cap (0, \infty)$ for all $t \in \mathbb{R}$, which is pullback attracting within Σ_2 and satisfies

$$\lim_{\lambda \searrow 0} (S^*_\lambda(t), I^*_\lambda(t)) = (1, 0) \quad \text{for all } t \in \mathbb{R}.$$

Proof. We mimic the notation of [17, Theorem 7] and denote the right-hand side of $(*^1_{3.1})$ by $G(t, x, \lambda) := \lambda\delta(t)x - (a + b + \lambda\delta(t))x^2$. Obviously, $G(t, 0, \lambda) \equiv 0$ and $D_2G(t, 0, 0) \equiv 0$ on $\mathbb{R} \times \mathbb{R}$. We introduce the abbreviations

$$f(t) := D_2D_3G(t, 0, 0) = \delta(t), \quad g(t) := -\tfrac{1}{2}D_2^2G(t, 0, 0) = a + b \quad \text{for all } t \in \mathbb{R},$$

as well as $\gamma(t,\lambda) := \lambda\delta(t)$. Hence, we can write $(*^1_{3.1})$ as

$$\dot{I} = \lambda f(t)I - [g(t) + \gamma(t,\lambda)]I^2.$$

Then all of the assumptions of [17, Theorem 7] are fulfilled. $\square$

3.1.2. ***Shovel bifurcation***

Another, somewhat rough, bifurcation scenario occurs when the interaction function γ_λ has the form

$$\gamma_\lambda(t) = a + b + \lambda + \delta(t)$$

with some continuous function δ. The real bifurcation parameter λ controls the fluctuation of the interaction coefficient. Here, $(*_{3.1})$ reads as

$$\dot{I} = (\lambda + \delta(t))I - (a + b + \lambda + \delta(t))I^2. \tag{$*^2_{3.1}$}$$

We understand a bifurcation now as a change in the number of bounded entire solutions to $(*^2_{3.1})$ under variation of λ.

Theorem 3.5 (supercritical shovel bifurcation in $(*_3)$)**.** *If $J = \mathbb{R}$, then there exist neighborhoods $U \subseteq \Sigma_2$ of the disease-free steady state equilibrium $(1,0)$ and Λ of the critical parameter $\lambda^* = -\overline{\beta}_{\mathbb{R}}(\delta)$ such that with $\lambda \in \Lambda$:*

- *(a) For $\lambda < -\overline{\beta}_{\mathbb{R}}(\delta)$ the unique entire bounded solution of $(*_3)$ in U is the equilibrium $(1,0)$; it is uniformly asymptotically stable w.r.t. Σ_2.*
- *(b) For $\lambda = -\overline{\beta}_{\mathbb{R}}(\delta)$ and $\overline{\beta}_{\mathbb{R}_+}(\delta) < \overline{\beta}_{\mathbb{R}}(\delta)$ the equilibrium $(1,0)$ is asymptotically stable w.r.t. Σ_2.*
- *(c) For $\lambda > -\overline{\beta}_{\mathbb{R}}(\delta)$ and*
  - *(c_1) if $\overline{\beta}_{\mathbb{R}_+}(\delta) < \overline{\beta}_{\mathbb{R}}(\delta)$, then the equilibrium $(1,0)$ is asymptotically stable, and embedded into a 1-parameter family of bounded entire solutions to $(*_3)$,*
  - *(c_2) if $\underline{\beta}_{\mathbb{R}_+}(\delta) = \overline{\beta}_{\mathbb{R}}(\delta)$, then the equilibrium $(1,0)$ is unstable.*

Proof. The linearization of the scalar ODE $(*^2_{3.1})$ reads as $\dot{I} = (\lambda + \delta(t))I$ and the corresponding dichotomy spectra are $\Sigma_J(\lambda) = \lambda + [\underline{\beta}_J(\delta), \overline{\beta}_J(\delta)]$. Hence, their maxima and minima are increasing with λ and the smaller critical parameter value is given for $\lambda^* = -\overline{\beta}_{\mathbb{R}}(\delta)$. The claim then follows from [24, Theorem 4.14(b)] in view of (3.1). $\square$

Example 3.6. Given strictly decreasing functions $\delta : \mathbb{R} \to \mathbb{R}$ with limits $\lim_{t\to\pm\infty} \delta(t) = \delta_\pm$ and reals $\delta_+ < \delta_-$, one deduces from Example A.1 that

$$\overline{\beta}_{\mathbb{R}}(\delta) = \max\{\delta_+, \delta_-\} = \delta_-, \qquad \overline{\beta}_{\mathbb{R}_+}(\delta) = \underline{\beta}_{\mathbb{R}_+}(\delta) = \delta_+.$$

Consequently the assertions of Theorem 3.5(a)–(c_1) hold, while the assumption of assertion (c_2) is violated. More precisely, for $\lambda < -\delta_-$ the trivial solution of $(*^2_{3.1})$ is uniformly asymptotically stable and the unique bounded entire solution to $(*^2_{3.1})$. In the critical case $\lambda = -\delta_-$ the trivial solution is asymptotically stable and for $\lambda > -\delta_-$ it is embedded into a family of bounded entire solutions. This is illustrated illustration in Fig. 4(left).

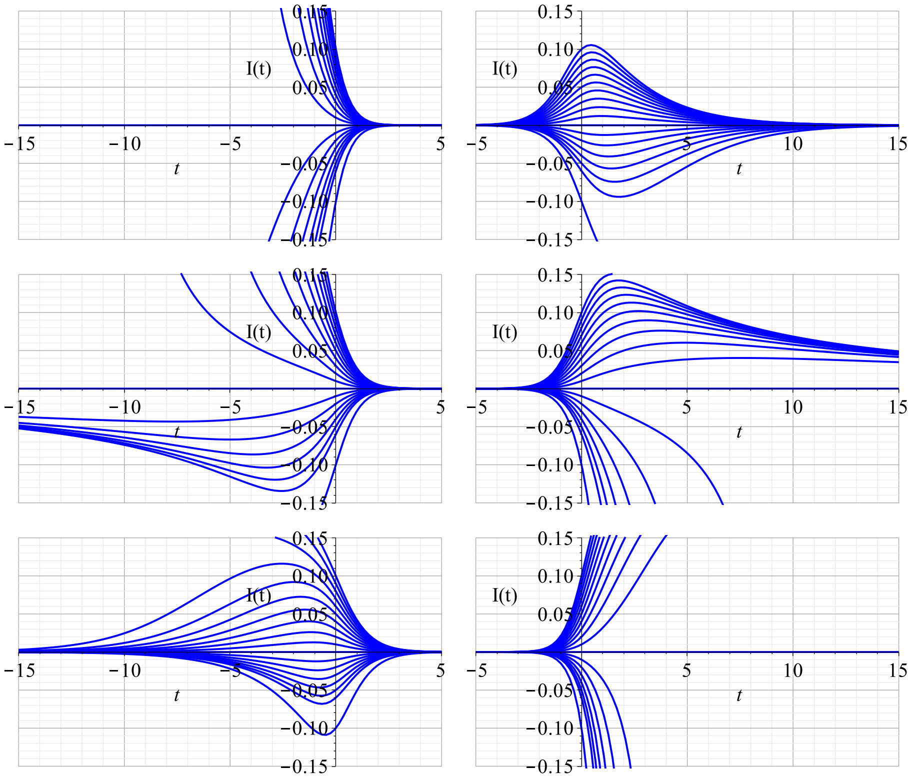

Figure 4: Shovel bifurcation in $(*^2_{3.1})$ with parameters $a = b = 1$ and the strictly decreasing function $\delta(t) := \frac{a+b}{\pi}\arctan(-t)$:
Left (supercritical situation from Ex. 3.6): The uniformly asymptotically stable trivial solution bifurcates into a 1-parameter family of asymptotically stable solutions as λ grows through $\lambda^* = -1$ (for parameters $\lambda \in \{-1.5, -1, -0.5\}$).
Right (subcritical situation from Ex. 3.8): The asymptotically stable trivial solution is embedded into a 1-parameter family of asymptotically stable solutions and becomes unstable as λ grows through the value $\lambda^* = 1$ (for parameters $\lambda \in \{0.5, 1, 1.5\}$).

Theorem 3.7 (subcritical shovel bifurcation in $(*_3)$)**.** *If $J = \mathbb{R}$, then there exist neighborhoods $U \subseteq \Sigma_2$ of disease-free steady state equilibrium $(1, 0)$ and Λ of the critical parameter $\lambda^* = -\underline{\beta}_{\mathbb{R}}(\delta)$ such that with $\lambda \in \Lambda$:*

(a) For $\lambda < -\underline{\beta}_{\mathbb{R}}(\delta)$

(a_1) if $\underline{\beta}_{\mathbb{R}}(\delta) = \overline{\beta}_{\mathbb{R}}(\delta)$, then the equilibrium $(1, 0)$ is uniformly asymptotically stable and the unique entire bounded solution in U of $(*^2_{3.1})$,

(a_2) if $\underline{\beta}_{\mathbb{R}}(\delta) = \overline{\beta}_{\mathbb{R}_+}(\delta)$, then the equilibrium $(1, 0)$ is asymptotically stable, and for $\underline{\beta}_{\mathbb{R}}(\delta) < \underline{\beta}_{\mathbb{R}_-}(\delta)$ it is embedded into an 1-parameter family of bounded entire solutions to $(*_3)$,

(a_3) if $\underline{\beta}_{\mathbb{R}}(\delta) < \underline{\beta}_{\mathbb{R}_+}(\delta)$, then the equilibrium $(1, 0)$ is unstable.

(b) For $\lambda = -\underline{\beta}_{\mathbb{R}}(\delta)$ and $\underline{\beta}_{\mathbb{R}}(\delta) < \underline{\beta}_{\mathbb{R}_+}(\delta)$ the equilibrium $(1, 0)$ is unstable.

(c) For $\lambda > -\underline{\beta}_{\mathbb{R}}(\delta)$ the equilibrium $(1, 0)$ is unstable and the unique entire bounded solution of $(*_3)$ in U.

Proof. For the critical value $\lambda^* = -\underline{\beta}_{\mathbb{R}}(\delta)$ the claims follow as in the above proof of Theorem 3.5 using [24, Theorem 4.15(a)]. $\square$

The following example illuminates that a subcritical shovel bifurcation in $(*^2_{3.1})$ can be interpreted as transcritical bifurcation on the nonnegative semi-axis $J = \mathbb{R}_+$ in the sense of Theorem C.1.

Example 3.8. With the same asymptotically constant function $\delta : \mathbb{R} \to \mathbb{R}$ as in Example 3.6 it follows from Example A.1 that

$$\overline{\beta}_{\mathbb{R}}(\delta) = \delta_-, \qquad \underline{\beta}_{\mathbb{R}}(\delta) = \delta_+,$$
$$\underline{\beta}_{\mathbb{R}_-}(\delta) = \delta_-, \qquad \overline{\beta}_{\mathbb{R}_+}(\delta) = \underline{\beta}_{\mathbb{R}_+}(\delta) = \delta_+.$$

Theorem 3.7 can then be applied to $(*^2_{3.1})$ with the critical parameter value $\lambda^* = -\delta_+$. In detail, for $\lambda < -\delta_+$ case (a_2) applies, the trivial solution is asymptotically stable and embedded into a 1-parameter family of bounded entire solutions. At the critical value $\lambda = -\delta_+$, Theorem 3.7(b) yields no stability information on the trivial solution, but for $\lambda > -\delta_+$ the trivial solution becomes unstable and isolated as a bounded entire solution to $(*^2_{3.1})$; see Fig. 4(right).

A further branching process occurs beyond this subcritical shovel bifurcation, which we will describe now. Since the function δ is asymptotically constant for $t \to \infty$ with limit δ_+, the ODE $(*^2_{3.1})$ is asymptotically autonomous with the limit system

$$\dot{I} = (\lambda + \delta_+)\, I - (a + b + \lambda + \delta_+)\, I^2.$$

It is clear that the trivial solution to this autonomous ODE transcritically bifurcates into the nontrivial branch $(\lambda + \delta_+)/(a + b + \lambda + \delta_+)$ at the critical parameter $\lambda^* = -\delta_+$. In addition, λ^* is also a critical value for a transcritical bifurcation of the zero solution to the nonautonomous ODE $(*^2_{3.1})$ in the sense of Theorem C.1 on the semi-axis $J = \mathbb{R}_+$. Indeed, writing the right-hand side of $(*^2_{3.1})$ as $g(t, x, \lambda) = (\lambda + \delta(t))x - (a + b + \lambda + \delta(t))x^2$ gives

(i) $D_2 g(t, 0, \lambda) = \lambda + \delta(t)$, thus $\lim_{t\to\infty} D_2 g(t, 0, \lambda) = \lambda + \delta_+$, so we can write the linearization of $(*^2_{3.1})$ along the zero solution as $\dot{x} = \alpha_\lambda x + b(t)x$ with

$$\alpha_\lambda := \lambda + \delta_+, \qquad b(t) := \delta(t) - \delta_+.$$

Under the summability assumption $\delta - \delta_+ \in L^1(J, \mathbb{R})$, Proposition A.3 applies and yields

$$\exp\left(\int_s^t D_2 g(r, 0, \lambda)\, dr\right) \le K e^{(\lambda+\delta_+)(t-s)} \quad \text{for all } s, t \ge 0$$

with $K := \exp\left(\int_0^\infty |\delta(r) - \delta_+|\, dr\right)$. Hence, we can choose the increasing functions $\gamma_\pm(\lambda) := \lambda + \delta_+$ which satisfy $\lim_{\lambda\to-\delta_+} \gamma_\pm(\lambda) = 0$.

(ii) For the partial derivative $D_2^2 g(t, 0, \lambda) = -2(a + b + \lambda + \delta(t))$, since

$$a + b + \lambda + \delta_+ \le a + b + \lambda + \delta(t) \le a + b + \lambda + \delta_- \quad \text{for all } t \ge 0,$$

we have

$$-2(a+b+\lambda+\delta_-) \le \inf_{t\ge0} D_2^2 g(t, 0, \lambda) \le \sup_{t\ge0} D_2^2 g(t, 0, \lambda) \le -2(a+b+\lambda+\delta_+)$$

and in the limit $\lambda \to -\delta_+$ we obtain

$$-2(a+b+\delta_- - \delta_+) \le \lim_{\lambda \to -\delta_+} \inf_{t \ge 0} D_2^2 g(t,0,\lambda) \\ \le \lim_{\lambda \to -\delta_+} \sup_{t \ge 0} D_2^2 g(t,0,\lambda) \le -2(a+b) < 0.$$

(iii) Since $D_2^3 g(t,x,\lambda) \equiv 0$ the assumption (iii) of Theorem C.1 is also satisfied.

Specifically, Theorem C.1(a) applies to $(*_{3.1}^2)$ on $J = \mathbb{R}_+$ and shows that the $\mathbb{R}_+$-attractive trivial solution becomes $\mathbb{R}_+$-repulsive as λ increases through the critical value $-\delta_+$ and, moreover, that the scalar ODE $(*_{3.1}^2)$ exhibits a $\mathbb{R}_+$-bifurcation with radii satisfying $\lim_{\lambda \nearrow -\delta_+} \rho_\lambda^+(0) = 0 = \lim_{\lambda \searrow -\delta_+} \rho_\lambda^-(0)$.

3.2. Endemic equilibrium

Consider now the possibility that a nontrivial equilibrium exists for the SI-system $(*_3)$, i.e., with $I \neq 0$. It cannot exist as a steady state, so a nonautonomous equilibrium $(S_\lambda^*(t), I_\lambda^*(t))$ will be sought. As above, its explicit form is given in (3.2). A sufficient criterion for its stability can be found in [14, Lemma 4]. In general, the dynamical behavior of the entire solution $I_\lambda^*(t)$ to $(*_{3.1})$ corresponds to the asymptotics of the trivial solution to the associated equation of perturbed motion

$$\dot{I} = \big[(1 - 2I_\lambda^*(t))\gamma_\lambda(t) - a - b\big] I - \gamma_\lambda(t) I^2. \tag{$*_{3.2}$}$$

Since both differential equations $(*_{3.1})$ and $(*_{3.2})$ share a very similar structure, the same techniques from [17, 19, 26, 24], as illustrated in the previous Section 3.1, apply and we will not discuss them explicitly here.

On the other hand, in the Section 3.1 we discussed various stability changes for the equilibrium $(1,0)$ to $(*_3)$, as a parameter crosses a critical value, and stability of $(1,0)$ is transferred to $(S_\lambda^*(t), I_\lambda^*(t))$. We therefore expect a dual behavior of the nontrivial equilibrium here.

4. The SIR model with variable population

In this section we return to the full SIR system $(*)$, specifically with a variable population, but constant interaction rates, and the asymptotically stable limiting population $N_a^\star(t)$ given in (1.3), i.e.,

$$\begin{cases} \dot{S} = aq(t) - aS + bI - \gamma \dfrac{SI}{N_a^\star(t)}, \\ \dot{I} = -(a+b+c)I + \gamma \dfrac{SI}{N_a^\star(t)}, \\ \dot{R} = cI - aR, \end{cases} \tag{$*_4$}$$

from which it is convenient to eliminate the S variable and just consider the equations for the I and R variables in $(*_4)$. Essentially, we replace S by the

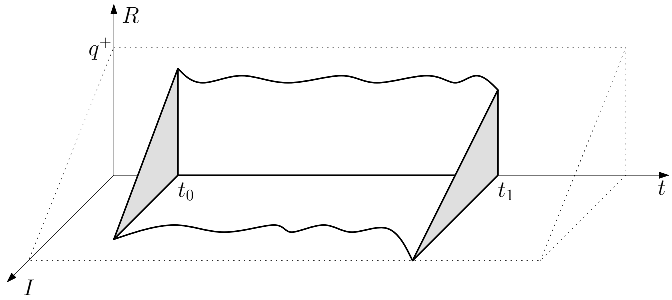

Figure 5: Time-dependent triangular sets T_2 with fibers $T_2(t_0)$ and $T_2(t_1)$

difference $S = N_a^\star(t) - I - R$ in the I equation of $(*_4)$ to obtain the IR-system

$$\begin{cases} \dot{I} = (\gamma - a - b - c)I - \gamma \dfrac{I(I+R)}{N_a^\star(t)}, \\ \dot{R} = cI - aR, \end{cases} \tag{$*_{4.1}$}$$

where $0 \leq I,\ R \leq N_a^\star(t)$, i.e., with solutions in time-dependent triangular sets

$$T_2(t) := \big\{(I,R)\ :\ I,R \geq 0,\ 0 \leq I+R \leq N_a^\star(t)\big\},$$

and hence in the common compact subset (see Fig. 5)

$$T_2(t) \subset \big\{(I,R) \in \mathbb{R}^2\ :\ I,R \geq 0,\ 0 \leq I+R \leq q^+\big\}.$$

Proposition 4.1. *[14, Lemma 5] The disease-free nonautonomous equilibrium $(S_a^\star(t),0,0)$ to $(*_4)$ with*

$$S_a^\star(t) = N_a^\star(t) = ae^{-at}\int_{-\infty}^t q(s)e^{as}\,ds,$$

is globally asymptotically stable w.r.t. $\Sigma_3^\pm$ for $\gamma \leq a+b+c$ and unstable for $\gamma > a+b+c$.

The nontrivial dynamics for $\gamma > a+b+c$ is complicated and could be chaotic. However, a nonautonomous bifurcation takes place as γ passes through $a+b+c$ and we might expect a simpler pullback attractor to exist for γ slightly larger than $a+b+c$. To specify the kind of nonautonomous bifurcation happening here, as well as the pullback attractor, we again apply a nonautonomous center manifold reduction. Nonetheless, differing from the proof of Theorem 2.3 we now apply the approach of [17] to the scalar reduced equation:

Theorem 4.2 (transcritical bifurcation in $(*_4)$)**.** *If $J = \mathbb{R}$, then there exist neighborhoods $U \subseteq \mathbb{R}^2$ and $\Gamma \subseteq \mathbb{R}$ of $a+b+c$ so that for all $\gamma \in \Gamma$ the disease-free nonautonomous equilibrium $(S_a^\star(t),0,0)$ of $(*_4)$ undergoes a transcritical bifurcation at the critical parameter $\gamma^* = a+b+c$ as follows:*

(a) For $\gamma < a+b+c$ the solution $(S_a^\star(t), 0, 0)$ is pullback attracting and there exists another entire solution $(S_a^\star(t) - I_\gamma^*(t) - R_\gamma^*(t), I_\gamma^*(t), R_\gamma^*(t))$ with $(I_\gamma^*(t), R_\gamma^*(t)) \in U$ for all $t \in \mathbb{R}$, which is asymptotically unstable and satisfies

$$\lim_{\gamma \nearrow a+b+c} (I_\gamma^*(t), R_\gamma^*(t)) = (0,0) \quad \textit{for all } t \in \mathbb{R}.$$

(b) For $\gamma = a+b+c$ the solution $(S_a^\star(t), 0, 0)$ is asymptotically unstable, but still pullback attracting.

(c) For $\gamma > a+b+c$ the solution $(S_a^\star(t), 0, 0)$ is asymptotically unstable and there exists another entire solution $(S_a^\star(t) - I_\gamma^*(t) - R_\gamma^*(t), I_\gamma^*(t), R_\gamma^*(t))$ with $(I_\gamma^*(t), R_\gamma^*(t)) \in U$ for all $t \in \mathbb{R}$, which is pullback attracting and satisfies

$$\lim_{\gamma \searrow a+b+c} (I_\gamma^*(t), R_\gamma^*(t)) = (0,0) \quad \textit{for all } t \in \mathbb{R}.$$

Proof. Mimicking the strategy from Theorem 2.3, we split the proof into three parts.
(I) To shorten notation, we substitute $\lambda := \gamma - a - b - c$ and attach the trivial equation $\dot\lambda = 0$ to $(*_{4.1})$, i.e., we consider the system

$$\begin{cases} \dot I = \lambda I - \gamma \dfrac{I(I+R)}{N_a^\star(t)}, \\ \dot R = cI - aR, \\ \dot\lambda = 0. \end{cases}$$

Applying the linear transformation

$$\begin{pmatrix} I \\ R \\ \lambda \end{pmatrix} := \begin{pmatrix} 0 & \frac{a+\lambda}{c} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

to this problem yields the 3-dimensional nonautonomous ODE

$$\dot y = \begin{pmatrix} -a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} y + F(t,y), \quad F(t,y) := y_2 \begin{pmatrix} \gamma \dfrac{y_2(a+y_3) + c(y_1+y_2)}{cN_a^\star(t)} \\ y_3 - \gamma \dfrac{y_2(a+y_3) + c(y_1+y_2)}{cN_a^\star(t)} \\ 0 \end{pmatrix}.$$

As explained in Appendix D this system has a 2-dimensional center integral manifold $\mathcal{W}$, which is the graph of a smooth function $y_1 = w(t, y_2, y_3)$. In particular, the first component of the reduced equation (D.3) becomes

$$\dot y_2 = y_2 \left(y_3 - \gamma \frac{(a+y_3)y_2 + c(w(t,y_2,y_3) + y_2)}{cN_a^\star(t)} \right)$$

and the fact $y_3 = \lambda$ finally yields the scalar nonautonomous ODE

$$\dot y = y \left(\lambda - \gamma \frac{(a+\lambda)y + c(w(t,y,\lambda) + y)}{cN_a^\star(t)} \right) =: G(t,y,\lambda). \tag{4.1}$$

(II) Our next goal is to apply [17, Theorem 7] to (4.1). Indeed, for the right-hand side G we have the identities

$$G(t,0,\lambda) = 0, \qquad D_2 G(t,0,0) \overset{(d_1)}{=} 0,$$

$$D_2D_3G(t,0,0) \overset{(\mathrm{D.8})}{=} 1 =: f(t), \qquad D_2^2G(t,0,0) \overset{(d_1)}{=} -\frac{2\gamma(a+c)}{cN_a^\star(t)}.$$

With $g(t) := \frac{\gamma(a+c)}{cN_a^\star(t)}$ it is

$$\frac{\gamma(a+c)}{cq^+} \le g(t), \qquad \frac{cq^-}{\gamma(a+b)} \le \frac{f(t)}{g(t)} \le \frac{cq^+}{\gamma(a+b)} \quad \text{for all } t \in \mathbb{R}.$$

To verify the remaining assumptions of [17, Theorem 7], we apply Taylor's theorem (see [15, p. 349]) together with the above relations to arrive at

$$\begin{aligned}
G(t,y,\lambda) &= D_2G(t,0,\lambda)y + \frac{1}{2}\left(D_2^2G(t,0,\lambda) + \int_0^1 (1-h)^2 D_2^3G(t,hy,\lambda)\,dh\,y\right)y^2 \\
&= \lambda\left(D_2D_3G(t,0,0) + \int_0^1 (1-h)D_2D_3^2G(t,0,h\lambda)\,dh\,\lambda\right)y \\
&\quad + \frac{1}{2}\left(D_2^2G(t,0,\lambda) + \int_0^1 (1-h)^2 D_2^3G(t,hy,\lambda)\,dh\,y\right)y^2 \\
&= \lambda\left(D_2D_3G(t,0,0) + \int_0^1 (1-h)D_2D_3^2G(t,0,h\lambda)\,dh\lambda\right)y \\
&\quad + \frac{1}{2}\left(D_2^2G(t,0,0) + \int_0^1 D_2^2D_3G(t,0,h\lambda)\,dh\,\lambda\right)y^2 \\
&\quad + \frac{1}{2}\int_0^1 (1-h)^2 D_2^3G(t,hy,\lambda)\,dh\,y^3 \\
&= \lambda\left(f(t) + \lambda\phi(t,\lambda)\right)y - \left(g(t) + \hat{\gamma}(t,y,\lambda)\right)y^2
\end{aligned}$$

with the functions

$$\begin{aligned}
\phi(t,\lambda) &:= -\frac{\gamma}{N_a^\star(t)}\int_0^1 (1-h)D_3^2w(t,0,h\lambda)\,dh \overset{(\mathrm{D.8})}{=} 0, \\
\hat{\gamma}(t,y,\lambda) &:= \frac{\gamma}{cN_a^\star(t)}\int_0^1 \left(1 + cD_2D_3w(t,0,h\lambda)\right)\,dh\,\lambda \\
&\quad + \frac{\gamma}{2N_a^\star(t)}\int_0^1 (1-h)^2\left(3D_2^2w(t,hy,\lambda) + hyD_2^3(t,hy,\lambda)\right)\,dh\,y.
\end{aligned}$$

Since the partial derivatives $D^j_{(2,3)}w : \mathbb{R} \times U_1 \times U_2 \to \mathbb{R}$ are bounded, we conclude that the derivatives $D_2\hat{\gamma}, D_3\hat{\gamma}$ are also bounded $\mathbb{R} \times U_1 \times U_2$. This ensures that also the quotients $D_2\hat{\gamma}(t,y,\lambda)/g(t)$, $D_3\hat{\gamma}(t,y,\lambda)/g(t)$ are bounded above. Hence, the assumptions of [17, Theorem 7] are fulfilled and imply a transcritical bifurcation in the reduced equation (4.1) at the critical parameter value $\lambda = 0$. The trivial solution bifurcates into an entire solution $x^*_\lambda : \mathbb{R} \to \mathbb{R}$ of (4.1) and exchanges its stability properties as λ crosses 0.

(III) Now we transfer this result back to the IR system $(*_{4.1})$. Recalling the transformations of step (I), we define

$$I^*_\gamma(t) := \frac{\gamma - b - c}{c}x^*_{\gamma-a-b-c}(t),$$

$$R_\gamma^*(t) := x_{\gamma-a-b-c}^*(t) + w(t, x_{\gamma-a-b-c}^*(t), \gamma - a - b - c)$$

and conclude from the reduction principle of [3] or [13, p. 133, Theorem 6.25] that the stability properties of the trivial solution 0 and x_λ^* provided in [17, Theorem 7] are transferred to $(I_\gamma^*(t), R_\gamma^*(t))$. $\square$

5. The SIR model with a variable interaction coefficient

We finally consider the SIR equations $(*)$ with variable interaction coefficient $\gamma(t)$ and constant population $N(t) \equiv 1$ on an interval J unbounded below, i.e.,

$$\begin{cases} \dot{S} = a - aS + bI - \gamma(t)SI, \\ \dot{I} = -(a+b+c)I + \gamma(t)SI, \\ \dot{R} = cI - aR. \end{cases} \tag{$*_5$}$$

By identity (3.2) the S equation can be eliminated to give

$$\begin{cases} \dot{I} = (\gamma(t) - a - b - c)I - \gamma(t)I(I+R), \\ \dot{R} = cI - aR. \end{cases} \tag{$*_{5.1}$}$$

Stability properties of the trivial solution to this planar ODE with nonautonomous linear part are given in the next proposition, and carry over to $(*_5)$.

Proposition 5.1. *The disease-free steady state equilibrium* $(1,0,0)$ *of* $(*_5)$ *is uniformly asymptotically stable w.r.t.* Σ_3 *for* $\overline{\beta}_J(\gamma) < a+b+c$ *and unstable for* $\underline{\beta}_J(\gamma) > a+b+c$.

Proof. Linearizing $(*_{5.1})$ along the trivial solution yields the variational equation

$$\begin{cases} \dot{I} = (\gamma(t) - a - b - c)I, \\ \dot{R} = cI - aR. \end{cases} \tag{5.1}$$

In view of its lower triangular structure, the corresponding dichotomy spectrum computes as

$$\Sigma_J = \{-a\} \cup [\underline{\beta}_J(\gamma) - a - b - c, \overline{\beta}_J(\gamma) - a - b - c].$$

The stability assertion then follows from [24, Propositions 4.9 and 4.10]. $\square$

In the rest of the section we describe the essential ingredients for a bifurcation analysis of the trivial solution to $(*_{5.1})$ and the disease-free equilibrium to $(*_5)$. For this we consider $c > 0$ as bifurcation parameter and are interested in the two critical cases

$$c = c^* \in \left\{\underline{\beta}_J(\gamma) - a - b, \overline{\beta}_J(\gamma) - a - b\right\}.$$

To perform a center manifold reduction as described in Appendix D, we need to find a Lyapunov transformation $L : J \to \mathbb{R}^{2\times 2}$ decoupling the linear part of $(*_{5.1})$, which turns out to be possible under the assumption

$$b < \underline{\beta}_J(\gamma). \tag{5.2}$$

We make the ansatz

$$L(t) := \begin{pmatrix} 1 & 0 \\ \ell(t) & 0 \end{pmatrix} \tag{5.3}$$

where $\ell : J \to \mathbb{R}$ is a C^1-function that is to be determined. Applying this transformation to (5.1) yields the lower triangular system

$$\dot{x} = \begin{pmatrix} \gamma(t) - a - b - c & 0 \\ \dot{\ell}(t) + (\gamma(t) - b - c)\ell(t) + c & -a \end{pmatrix} x,$$

which becomes diagonal, if and only if the function ℓ satisfies the scalar linear inhomogeneous ODE $\dot{\ell} = (b + c - \gamma(t))\ell - c$. Assumption (5.2) allows us to find positive parameters $c > 0$ such that $\underline{\beta}_J(\gamma) > b + c$. This linear ODE thus possesses a unique bounded solution $\ell_c : J \to \mathbb{R}$ given by

$$\ell_c(t) := -c \int_{-\infty}^{t} e^{(b+c)(t-s)} \exp\left(- \int_s^t \gamma(r)\, dr \right) ds.$$

Since $\gamma : J \to \mathbb{R}$ is bounded, we see that $\dot{\ell}_c$ is also bounded. Moreover,

$$L(t)^{-1} = \begin{pmatrix} 1 & 0 \\ -\ell(t) & 0 \end{pmatrix},$$

so $L : J \to \mathbb{R}^{2\times 2}$ is a Lyapunov transformation for this choice of $\ell(t) = \ell_c(t)$.

Next we apply the transformation $\binom{y_2}{y_1} = L(t)\binom{I}{R}$ to (5.1) and arrive at

$$\dot{x} = \begin{pmatrix} -a & 0 \\ 0 & \gamma(t) - a - b - c \end{pmatrix} x - \gamma(t) y_2 \begin{pmatrix} 2(1 - \ell_c(t))\ell_c(t) y_2 + \ell_c(t) y_1 \\ 2(1 - \ell_c(t)) y_2 + y_1 \end{pmatrix}.$$

Denoting the bifurcation parameter c in $(*_5)$, respectively $(*_{5.1})$, by $\lambda := c$, we attach the trivial equation $\dot{\lambda} = 0$. This gives us the 3-dimensional ODE

$$\begin{aligned} \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\lambda} \end{pmatrix} &= \begin{pmatrix} -a & 0 & 0 \\ 0 & \gamma(t) - a - b & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \lambda \end{pmatrix} \\ &\quad - \begin{pmatrix} \gamma(t) y_2 \left[2(1 - \ell_\lambda(t))\ell_\lambda(t) y_2 + \ell_\lambda(t) y_1 \right] \\ \lambda y_2 + \gamma(t) y_2 \left[2(1 - \ell_\lambda(t)) y_2 + y_1 \right] \\ 0 \end{pmatrix}, \end{aligned} \tag{5.4}$$

the linear part of which has the dichotomy spectrum

$$\Sigma_J = \{-a, 0\} \cup \left[\underline{\beta}_J(\gamma) - a - b, \overline{\beta}_J(\gamma) - a - b \right].$$

Hence, assumption (5.2) implies that $-a < \underline{\beta}_J(\gamma) - a - b$. Consequently there exist real numbers $\alpha_s < \alpha_c$, $\alpha_s < 0$ such that $\Sigma_J \cap (\alpha_s, \alpha_s) = \emptyset$. Appendix D can then be used to obtain a 2-dimensional center integral manifold $y_1 = w(t, y_2, \lambda)$ for (5.4). Renaming λ as c again, this yields the reduced equation

$$\dot{y}_2 = (\gamma(t) - a - b - c)y_2 - \gamma(t)y_2 \left[2(1 - \ell_c(t))y_2 + w(t, y_2, c)\right]$$

which has a very similar structure to $(*_{3.1})$ or the previously considered reduced equations (2.3) and (4.1). Hence, the transcritical bifurcation results for scalar nonautonomous equations from [17, 19, 26] apply.

6. Conclusion

We have essentially discussed three different approaches to describe nonautonomous counterparts [17, 26, 19] to what can be interpreted as a transcritical bifurcation. Due to its "normal form" $\dot{x} = \lambda a(t)x + b(t)x^2 + o(x^3)$ this kind of bifurcation appears to be even more typical in a time-variant framework than in the classical autonomous setting: An identically vanishing coefficient $b(t) \equiv 0$ represents a higher degeneracy than merely a vanishing real number.

- In the sense of [26], a bifurcation is understood in a dynamical way, i.e. as loss of attractivity (gain of repulsivity) for the trivial solution as the parameter crosses a critical value. Here, a certain flexibility concerning the temporal interval $J \in \{\mathbb{R}_-, \mathbb{R}_+, \mathbb{R}\}$ is included and allows to understand the shovel bifurcation of a family of bounded entire solutions from [24] as transcritical bifurcation on the semiaxis $\mathbb{R}_+$.
- The approach of [19] also provides the mentioned continuous exchange from stability to instability by means of opposite sign Lyapunov exponents. In addition, it guarantees the existence of two (unique) invariant sets which generalize the equilibria of the autonomous situation and, being a copy of the base, additionally reflect the particular time-dependence.
- Likewise, the scenario of [17] yields a plausible analogy to the autonomous situation: Steady state equilibria are generalized to bounded entire solutions and one observes a change in their stability properties (if needed in a pullback sense).

Moreover, in order to deal with higher-dimensional ODEs we have exemplified a nonautonomous center manifold reduction. Technically, dealing with only transcritical bifurcation does not require to know Taylor coefficients of the center manifolds explicitly (for this, see [25] however). Our overall analysis was greatly simplified by the fact that the solutions along which bifurcations occur where explicitly known.

Appendices

A. Linear scalar equations

The widely known Lyapunov exponents are a tool to measure the exponential growth of functions. In nonautonomous stability theory, however, it is advantageous to have information on the *uniform* exponential growth described by Bohl exponents.

Let $J \subseteq \mathbb{R}$ denote an interval. The *upper Bohl exponent* of a locally integrable function $a : J \to \mathbb{R}$ is defined by

$$\overline{\beta}_J(a) := \inf \left\{ \omega \in \mathbb{R} : \sup_{s \leq t,\, s,t \in J} \frac{1}{t-s} \int_s^t (a(r) - \omega)\, dr < \infty \right\}$$

and the *lower Bohl exponent* by

$$\underline{\beta}_J(a) := \sup \left\{ \omega \in \mathbb{R} : \sup_{t \leq s,\, s,t \in J} \frac{1}{t-s} \int_s^t (a(r) - \omega)\, dr < \infty \right\}.$$

It is not difficult to see that this is equivalent to the definition in [6, p. 118] and

$$\overline{\beta}_J(\lambda + a) = \lambda + \overline{\beta}_J(a), \qquad \underline{\beta}_J(\lambda + a) = \lambda + \underline{\beta}_J(a) \quad \text{for all } \lambda \in \mathbb{R} \tag{A.1}$$

holds. The Bohl exponents are finite when a is integrally bounded, i.e., if

$$\sup_{0 \leq t-s \leq 1} \exp\left(\int_s^t a(r)\, dr \right) < \infty, \qquad \sup_{0 \leq s-t \leq 1} \exp\left(\int_s^t a(r)\, dr \right) < \infty$$

(cf. [6, p. 119, Theorem 4.2]) hold, respectively. In addition,

$$\underline{\beta}_J(a) \leq \underline{\beta}_{J'}(a) \leq \overline{\beta}_{J'}(a) \leq \overline{\beta}_J(a)$$

for a subinterval $J' \subseteq J$:

Example A.1. The functions $a_i : \mathbb{R} \to \mathbb{R}$,

$$a_0(t) :\equiv \alpha \in \mathbb{R}, \quad a_1(t) = \sin t, \quad a_2(t) := \arctan(-t), \quad a_3(t) = \sin \ln(1 + |t|)$$

have the Bohl exponents on $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively:

i	$\underline{\beta}_{\mathbb{R}}(a_i)$	$\overline{\beta}_{\mathbb{R}}(a_i)$	$\underline{\beta}_{\mathbb{R}_+}(a_i)$	$\overline{\beta}_{\mathbb{R}_+}(a_i)$	$\underline{\beta}_{\mathbb{R}_-}(a_i)$	$\overline{\beta}_{\mathbb{R}_-}(a_i)$
0	α	α	α	α	α	α
1	0	0	0	0	0	0
2	$-\pi/2$	$\pi/2$	$-\pi/2$	$-\pi/2$	$\pi/2$	$\pi/2$
3	-1	1	-1	1	-1	1

Generalizing the asymptotically constant example a_2 to general continuous functions $a : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{t \to \pm\infty} a(t) = \alpha_\pm \in \mathbb{R}$ gives

$$\overline{\beta}_{\mathbb{R}}(a) = \max\{\alpha_-, \alpha_+\}, \qquad \underline{\beta}_{\mathbb{R}}(a) = \min\{\alpha_-, \alpha_+\},$$
$$\overline{\beta}_{\mathbb{R}_+}(a) = \underline{\beta}_{\mathbb{R}_+}(a) = \alpha_+, \qquad \overline{\beta}_{\mathbb{R}_-}(a) = \underline{\beta}_{\mathbb{R}_-}(a) = \alpha_-.$$

There is a strong relation between Bohl exponents, exponential dichotomies (see [4]) and the dichotomy spectrum (see [27]). The last notion can be considered as the appropriate counterpart to eigenvalue real parts in a nonautonomous framework. In the simplest situation of the scalar ODE

$$\dot{x} = a(t)x \tag{A.2}$$

one has:

Proposition A.2. *If $a : J \to \mathbb{R}$ is a continuous bounded function, then the dichotomy spectrum of* (A.2) *is given by $\Sigma_J(a) = [\underline{\beta}_J(a), \overline{\beta}_J(a)]$.*

Proof. Let $\gamma \in \mathbb{R}$, s, $t \in J$ and suppose $\Phi(t,s) = \exp\left(\int_s^t a(r)\,dr\right)$ is the corresponding transition operator of (A.2). Then the scalar equation

$$\dot{x} = [a(t) - \gamma]x \tag{A.3}$$

has an exponential dichotomy on J if and only if there exist constants $K \geq 1$ and $\alpha > 0$ such that

$$\Phi(t,s) \leq Ke^{(\gamma-\alpha)(t-s)} \quad \text{for all } s \leq t \text{ or } \Phi(t,s) \leq Ke^{(\gamma+\alpha)(t-s)} \quad \text{for all } t \leq s,$$

which is equivalent to the existence of a constant $\alpha > 0$ such that (cf. (A.1))

$$\overline{\beta}_J(c) + \alpha \leq \gamma \quad \text{or} \quad \gamma \leq \underline{\beta}_J(c) - \alpha.$$

Thus, contrapositively, γ is in the dichotomy spectrum of (A.3) if and only if

$$\gamma \in \left(\underline{\beta}_J(c) - \alpha, \overline{\beta}_J(c) + \alpha\right) \quad \text{for all } \alpha > 0,$$

which yields the assertion. □

We close this section with an elementary perturbation result.

Proposition A.3. *If $\alpha \in \mathbb{R}$ and $b \in C(J, \mathbb{R})$, then the transition operator Φ of*

$$\dot{x} = (\alpha + b(t))x \tag{A.4}$$

satisfies $\Phi(t,s) \leq \exp\left(\alpha(t-s) + \left|\int_s^t b(r)\,dr\right|\right)$ for all s, $t \in J$.

Proof. By the variation of constants the transition operator of (A.4) satisfies

$$\Phi(t,s) = e^{\alpha(t-s)} + \int_s^t e^{\alpha(t-r)} b(r)\Phi(r,s)\,dr \quad \text{for all } s,t \in J.$$

Consequently for all s, $t \in J$

$$e^{\alpha(s-t)}\Phi(t,s) = 1 + \int_s^t b(r)e^{\alpha(s-r)}\Phi(r,s)\,dr \leq 1 + \left|\int_s^t b(r)e^{\alpha(s-r)}\Phi(r,s)\,dr\right|.$$

The result then follows by the Gronwall lemma (cf. [1, p. 90, (6.2)]). □

B. Nonautonomous dynamics

There are two formulations of nonautonomous dynamical systems, processes and skew-product flows, each of which offers different advantages, see [13].

B.1. Processes

Let $\Omega \subseteq \mathbb{R} \times \mathbb{R}^d$ be an open and connected set with the *t-fibers*

$$\Omega(t) := \left\{x \in \mathbb{R}^d : (t, x) \in \Omega\right\}.$$

Given $f : \Omega \to \mathbb{R}^d$, we consider a nonautonomous ODE

$$\dot{x} = f(t, x) \tag{B.1}$$

for which solutions exist and are unique. The maximal solutions $\varphi(\cdot; t_0, x_0)$ to (B.1) satisfying the initial condition $x(t_0) = x_0$, where $(t_0, x_0) \in \Omega$, thus generate a *local process*, i.e., a mapping φ satisfying

$$\varphi(t_0; t_0, x_0) = x_0, \qquad \varphi(t; t_0, x_0) = \varphi(t; s, \varphi(s; t_0, x_0))$$

for all t, $s \in \mathbb{R}$ in the respective maximal solutions intervals. These intervals may be finite or bounded from above or below, so one speaks of a local process.

In the following, h denotes the Hausdorff semidistance between nonempty compact subsets of $\mathbb{R}^d$ [13, p. 257]. Using the terminology of [17], a solution $\phi : J \to U$ of (B.1) on an interval J which is unbounded below is said to be

- *pullback attractive*, if there exists a $\rho > 0$ with
$$\lim_{t_0 \to -\infty} h\left(\varphi\big(t; t_0, B_\rho(\phi(t_0))\big), \{\phi(t)\}\right) = 0 \quad \text{for all } t \in J,$$
- *asymptotically unstable*, if there exists an instant $t_0 \in J$ such that
$$\left\{x_0 \in \Omega(t_0) : \lim_{t \to -\infty} h(\varphi(t; t_0, x_0), \{\phi(t)\}) = 0\right\} \neq \{\phi(t_0)\}.$$

On the other hand, modifying the terminology of [13, pp. 33ff] slightly, a solution $\phi : J \to \mathbb{R}^d$ of (B.1) is said to be

- *J-attractive*, if there exists a $\rho > 0$ with
$$\lim_{t \to \infty} h\left(\varphi\big(t; t_0, B_\rho(\phi(t_0))\big), \{\phi(t)\}\right) = 0 \quad \text{for all } t_0 \in J \text{ and } J \text{ unbounded above},$$
$$\lim_{t_0 \to -\infty} h\left(\varphi\big(t; t_0, B_\rho(\phi(t_0))\big), \{\phi(t)\}\right) = 0 \quad \text{for all } t \in J \text{ and } J \text{ unbounded below},$$
$$\lim_{t \to \infty} \sup_{t_0 \in \mathbb{R}} h\left(\varphi\big(t_0 + t; t_0, B_\rho(\phi(t_0))\big), \phi(t_0 + t)\right) = 0 \quad \text{for } J = \mathbb{R}.$$

  Note that $\mathbb{R}$-attractivity means both $\mathbb{R}_+$- and $\mathbb{R}_-$-attractive uniformly in $t_0 \in \mathbb{R}$ and $t \in \mathbb{R}$, respectively. The supremum over all these $\rho > 0$ is called the *J-attraction radius* and denoted by $\rho^+(\phi)$.

- *J-repulsive*, if there exists a $\rho > 0$ with

$$\lim_{t\to-\infty} h\left(\varphi\big(t;t_0,B_\rho(\phi(t_0))\big),\{\phi(t)\}\right)=0 \quad \text{for all } t_0\in J \text{ and } J \text{ unbounded below,}$$
$$\lim_{t_0\to\infty} h\left(\varphi\big(t;t_0,B_\rho(\phi(t_0))\big),\{\phi(t)\}\right)=0 \quad \text{for all } t\in J \text{ and } J \text{ unbounded above,}$$
$$\lim_{t\to\infty}\sup_{t_0\in\mathbb{R}} h\left(\varphi\big(t_0-t;t_0,B_\rho(\phi(t_0))\big),\phi(t_0-t)\right)=0 \quad \text{for } J=\mathbb{R}.$$

Similarly, $\mathbb{R}$-repulsivity is equivalent to being $\mathbb{R}_-$- and $\mathbb{R}_+$-repulsive uniformly in $t_0\in\mathbb{R}$ and $t\in\mathbb{R}$, respectively. The supremum over all these $\rho>0$ is called *J-repulsion radius* and denoted by $\rho^-(\phi)$.

B.2. Skew-product flows

Let X be a nonempty set. A mapping $\phi^t : X\to X$, $t\in\mathbb{R}$, is called a *flow* on the *state space* X if

$$\phi^0(x)=x, \qquad \phi^{t+s}=\phi^t\circ\phi^s \quad \text{for all } s,t\in\mathbb{R},\, x\in X \tag{B.2}$$

holds. A subset $A\subseteq X$ is said to be *ϕ-invariant* if $\phi^t(A)=A$ for all $t\in\mathbb{R}$.

In the subsequent considerations and terminology we follow [19] closely. The Borel-σ-algebra on a metric space X is denoted by $\mathcal{B}(X)$, a Borel-measure μ said to be *ϕ-invariant* if $\mu(\phi^t(B))=\mu(B)$ holds for all $t\in\mathbb{R}$ and $B\in\mathcal{B}(X)$, and a normalized ϕ-invariant measure is called *ϕ-ergodic* if any ϕ-invariant set has measure 0 or 1. In addition, a compact, ϕ-invariant subset $A\subseteq X$ is called *ϕ-minimal* if it does not properly contain any other compact ϕ-invariant set. A *minimal flow* ϕ has a minimal state space X. A flow ϕ^t on a compact metric space X is said to be *uniquely ergodic* if there exists a unique ϕ-ergodic measure m_0 defined on a complete σ-algebra of X containing the Borel sets $\mathcal{B}(X)$.

Suppose Ω is a compact metric space and that the mapping $(t,\omega)\mapsto\phi^t(\omega)$ is continuous. Consider the nonautonomous ODE

$$\dot{x}=f(\phi^t(\omega),x) \tag{B.3}$$

with continuous right-hand side $f:\Omega\times D\to\mathbb{R}^d$, where D is an open subset of $\mathbb{R}^d$, that guarantees existence and uniqueness of solutions. For fixed $\omega\in\Omega$, denote the maximal solution to (B.3) satisfying the initial condition $x(0)=x_0$ by $\varphi(\cdot;x_0,\omega)$. Then the mapping Φ^t given by

$$\Phi^t\begin{pmatrix}x\\ \omega\end{pmatrix}:=\begin{pmatrix}\varphi(t;x,\omega)\\ \phi^t(\omega)\end{pmatrix}$$

defines a local flow on $D\times\Omega$, the *(local) skew-product flow* generated by (B.3). In this context, ϕ^t is called the *base flow* and one speaks of a local skew-product flow, since $\varphi(\cdot;x,\omega)$ might be defined on an interval being bounded above or below, with the consequence that the flow properties (B.2) for Φ^t hold only for times t in the corresponding maximal existence intervals.

We say a Φ-minimal set $\mathcal{M}$ is a *copy of the base* if each ω*-fiber*

$$\mathcal{M}(\omega) := \{x \in D : (x, \omega) \in \mathcal{M}\} \quad \text{for all } \omega \in \Omega$$

consists of a unique point, in which case it coincides with the graph of a continuous Φ*-invariant curve* $\tilde{x} : \Omega \to \mathbb{R}$. The latter is a bounded measurable map satisfying

$$\varphi(t, \tilde{x}(\omega), \omega) = \tilde{x}(\phi^t(\omega)) \quad \text{for all } (t, \omega) \in \mathbb{R} \times \Omega.$$

The existence of a Φ-minimal set turns out to be equivalent to the existence of a pair $(x_0, \omega) \in D \times \Omega$ such that $\varphi(\cdot; x_0, \omega)$ is defined on the whole axis $\mathbb{R}$ as a bounded function, in which case one speaks of a *globally bounded* Φ*-orbit* $\{(\varphi(t; x_0, \omega), \phi^t(\omega))\}_{t \in \mathbb{R}}$. This, in turn, holds if and only if the set

$$\mathcal{B} := \left\{ (x_0, \omega) \in D \times \Omega : \sup_{t \in \mathbb{R}} \|\varphi(t, x_0, \omega)\| < \infty \right\}$$

is nonempty.

Finally, we restrict to scalar ODEs (B.3), i.e., the situation where $D \subseteq \mathbb{R}$ is an interval, the partial derivative $D_2 f$ exists as a continuous function and the mapping $(t, \omega) \mapsto \phi^t(\omega)$ is continuous. Provided $\mathcal{B}$ is bounded, it then makes sense to define the real numbers

$$\phi^+(\omega) := \sup\{x \ : \ (x, \omega) \in \mathcal{B}\}, \qquad \phi^-(\omega) := \inf\{x \ : \ (x, \omega) \in \mathcal{B}\},$$

as well as the corresponding sets $\mathcal{M}^\pm := \{(\phi^\pm(\omega), \omega) \in D \times \Omega : \omega \in \Omega\}$, which are copies of the base. The *Lyapunov exponent* of a Φ*-invariant curve* $\tilde{x} : \Omega \to \mathbb{R}$ is defined by

$$\Lambda(\tilde{x}) := \int_\Omega D_2 f(\omega, \tilde{x}(\omega)) dm_0(\omega).$$

The curve $\tilde{x}$ is said to be *hyperbolic* when $\mathrm{d}\Lambda(\tilde{x}) \neq 0$.

C. Nonautonomous bifurcations

Consider a scalar nonautonomous ODE

$$\dot{x} = g(t, x, \lambda) \tag{C.1}$$

with a continuous right-hand side $g : J \times U \times \Lambda \to \mathbb{R}$ defined on an unbounded interval J, a neighborhood $U \subseteq \mathbb{R}$ of 0, and an interval $\Lambda \subseteq \mathbb{R}$. Also assume

$$g(t, 0, \lambda) \equiv 0 \quad \text{on } J \times \Lambda,$$

i.e., (C.1) has the trivial solution for all $\lambda \in \Lambda$. The following theorem is a slight formal modification of [26, Theorem 5.1] or [13, Theorem 8.1, pp. 155–156]:

Theorem C.1 (nonautonomous transcritical bifurcation). *Let $\lambda^* \in \Lambda$, $K \geq 1$ and suppose g is of class C^3 in the second variable. In addition, suppose that* *(i) There exist functions* γ_-, $\gamma_+ : \Lambda \to \mathbb{R}$*, that are both either increasing or decreasing, with* $\lim_{\lambda\to\lambda^*} \gamma_-(\lambda) = \lim_{\lambda\to\lambda^*} \gamma_+(\lambda) = 0$*, such that*

$$\exp\left(\int_s^t D_2 g(r,0,\lambda)\,dr\right) \le K e^{\gamma_+(\lambda)(t-s)} \quad \text{for all } s \le t,\ \lambda \in \Lambda,$$

$$\exp\left(\int_s^t D_2 g(r,0,\lambda)\,dr\right) \le K e^{\gamma_-(\lambda)(t-s)} \quad \text{for all } t \le s,\ \lambda \in \Lambda.$$

(ii) One of the following conditions is fulfilled:

$$0 < \liminf_{\lambda\to\lambda^*} \inf_{t\in J} D_2^2 g(t,0,\lambda) \le \limsup_{\lambda\to\lambda^*} \sup_{t\in J} D_2^2 g(t,0,\lambda) < \infty,$$

$$-\infty < \liminf_{\lambda\to\lambda^*} \inf_{t\in J} D_2^2 g(t,0,\lambda) \le \limsup_{\lambda\to\lambda^*} \sup_{t\in J} D_2^2 g(t,0,\lambda) < 0.$$

(iii) The following limit relations hold:

$$\lim_{x\to 0} \sup_{\lambda\in(\lambda^*-|x|,\lambda^*+|x|)} \sup_{t\in J} x \int_0^1 (1-h)^2 D_2^3 g(t,hx,\lambda)\,dh = 0,$$

$$K \limsup_{\lambda\to\lambda^*} \limsup_{x\to 0} \sup_{t\in J} \frac{x^2}{\max\{\gamma_+(\lambda), -\gamma_-(\lambda)\}} \int_0^1 (1-h)^2 D_2^3 g(t,hx,\lambda)\,dh < 1.$$

Then there exists a neighborhood $\Lambda_0 \subseteq \Lambda$ *of* λ^* *such that for* $\lambda \in \Lambda_0$*:*

(a) For increasing functions γ_-, γ_+ *the zero solution to* (C.1) *is* J*-attractive for* $\lambda < \lambda^*$ *and* J*-repulsive for* $\lambda > \lambda^*$*. The scalar ODE* (C.1) *admits a* J*-bifurcation in the sense that the corresponding radii of* J*-attraction and -repulsion satisfy*

$$\lim_{\lambda\nearrow\lambda^*} \rho_\lambda^+(0) = 0 = \lim_{\lambda\searrow\lambda^*} \rho_\lambda^-(0).$$

(b) For decreasing functions γ_-, γ_+ *the zero solution to* (C.1) *is* J*-repulsive for* $\lambda < \lambda^*$ *and* J*-attractive for* $\lambda > \lambda^*$*. The scalar ODE* (C.1) *admits a* J*-bifurcation in the sense that the corresponding radii of* J*-repulsion and -attraction satisfy*

$$\lim_{\lambda\nearrow\lambda^*} \rho_\lambda^-(0) = 0 = \lim_{\lambda\searrow\lambda^*} \rho_\lambda^+(0).$$

Proof. Due to the smoothness assumption on g we can write

$$\dot{x} = a(t,\lambda)x + b(t,\lambda)x^2 + r(t,x,\lambda), \tag{C.2}$$

where Taylor's theorem as in [15, p. 349] guarantees $a(t,\lambda) = D_2 g(t,0,\lambda)$,

$$b(t,\lambda) = \tfrac{1}{2} D_2^2 g(t,0,\lambda), \qquad r(t,x,\lambda) = \int_0^1 \frac{(1-h)^2}{2} D_2^3 g(t,hx,\lambda)\,dh\,x^3.$$

Our assumptions guarantee that [26, Theorem 5.1] can be applied to (C.2). □

D. Nonautonomous center manifolds

D.1. Basic ideas

Here we consider a d-dimensional ODE

$$\dot{y} = A(t)y + F(t, y) \tag{D.1}$$

with continuous functions $A : J \to \mathbb{R}^{d\times d}$, $F : J \times U \to \mathbb{R}^d$ defined on an interval $J \subseteq \mathbb{R}$ unbounded below and an open neighborhood $U \subseteq \mathbb{R}^d$ of 0. For simplicity, we assume that A is in block-diagonal form $A = \operatorname{diag}(A_s, A_c)$ and that the transition matrices Φ_s and Φ_c associated with the diagonal blocks $A_s : J \to \mathbb{R}^{s\times s}$ and $A_c : J \to \mathbb{R}^{c\times c}$, respectively fulfill

$$\|\Phi_s(t, s)\| \le Ke^{\alpha_s(t-s)}, \qquad \|\Phi_c(s, t)\| \le Ke^{\alpha_c(s-t)} \quad \text{for all } s \le t \tag{D.2}$$

with real numbers $K \ge 1$ and $\alpha_s < \alpha_c$, where $\alpha_s < 0$. We also introduce a corresponding splitting $F = (F_s, F_c)$ with functions $F_s : J \times U \to \mathbb{R}^s$ and $F_c : J \times U \to \mathbb{R}^c$ of the nonlinearity and suppose that F is of class C^m for some $m \in \mathbb{N}$ in the second variable with

$$F(t, 0) \equiv 0 \quad \text{on } J, \qquad \lim_{x\to 0} D_2F(t, x) = 0 \quad \text{uniformly in } t \in J.$$

Under the spectral gap condition $\alpha_s < m\alpha_c$ there is a neighborhood $U_0 \subseteq \mathbb{R}^c$ of 0 and a continuous mapping $w : J \times U \to \mathbb{R}^s$ of class C^m in the second variable, which satisfies (cf. [3, 21, 25])

(d_1) $w(t, 0) \equiv 0$ and $D_2w(t, 0) \equiv 0$ on J

(d_2) $\mathcal{W} := \{(t, w(t, x), x) \in J \times \mathbb{R}^d : x \in U_0\}$ is *locally invariant* w.r.t. (D.1), i.e., a solution $\phi = (\phi_s, \phi_c)$ to (D.1) satisfies the implication

$$(t_0, \phi(t_0)) \in \mathcal{W} \quad \Rightarrow \quad (t, \phi(t)) \in \mathcal{W}$$

for all $t \in J$ as long as $\phi_c(t) \in U_0$

(d_3) the trivial solution of (D.1) has the same stability properties as the zero solution of the c-dimensional *equation reduced to* $\mathcal{W}$

$$\dot{x} = A_c(t)x + F_c(t, w(t, x), x). \tag{D.3}$$

D.2. Parameter-dependent equations

In applications to bifurcation theory, the ODEs under investigation depend on a parameter λ in an open set $\Lambda \subseteq \mathbb{R}^p$, i.e., are of the form

$$\dot{x} = f(t, x, \lambda) \tag{D.4}$$

with a continuous right-hand side $f : \Omega \times \Lambda \to \mathbb{R}^d$ and an open set $\Omega \subseteq \mathbb{R} \times \mathbb{R}^d$. We assume that f is of class C^m in (x, λ) with continuous partial derivatives, that Λ is a neighborhood of 0 and that

- (D.4) has the trivial solution, i.e., $\mathbb{R}\times U \subseteq \Omega$ holds for some neighborhood $U \subseteq \mathbb{R}^d$ of 0 and
$$f(t,0,\lambda) \equiv 0 \quad \text{on } \mathbb{R}\times\Lambda \tag{D.5}$$
- $D_2f(t,0,0)$ has the block diagonal form $\begin{pmatrix} A_s(t) & \\ & A_c(t)\end{pmatrix}$ with diagonal blocks satisfying (D.2).

If we extend (D.4) with the trivial equation $\dot\lambda = 0$, then the resulting equation
$$\begin{cases} \dot x = f(t,x,\lambda), \\ \dot\lambda = 0 \end{cases} \tag{D.6}$$
can be written as a $(d+p)$-dimensional ODE of the form (D.1). Since (D.5) implies $D_3f(t,0,0) \equiv 0$ on $\mathbb{R}$, we obtain
$$A(t) := \begin{pmatrix} D_2f(t,0,0) & 0 \\ 0 & 0 \end{pmatrix}, \qquad F(t,x,\lambda) := \begin{pmatrix} f(t,x,\lambda) - D_2f(t,0,0)x \\ 0 \end{pmatrix}$$
and due to our assumptions, the center manifold theory from Subsection D.1 applies with $y = (x,\lambda)$. In particular, there exists a $(c+p)$-dimensional center integral manifold $\mathcal{W}$ of (D.6), which is graph of a C^m-mapping $w : \mathbb{R}\times U_0 \to \mathbb{R}^s$ with $U_0 \subseteq \mathbb{R}^c \times \Lambda$ being a neighborhood of 0. From (D.5) we see that $(0,\lambda)$ are constant solutions of (D.6), hence the invariance of $\mathcal{W}$ implies
$$w(t,0,\lambda) \equiv 0 \quad \text{on } \Lambda, \tag{D.7}$$
as well as the identities
$$D_3^j w(t,0,\lambda) \equiv 0 \quad \text{on } \mathbb{R}\times\Lambda,\ 0 \le j \le m. \tag{D.8}$$

Acknowledgements: We thank the referees for various remarks improving our presentation.

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