Fine Structure of the Dichotomy Spectrum

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Abstract. The dichotomy spectrum is a crucial notion in the theory of dynamical systems, since it contains information on stability and robustness properties. However, recent applications in nonautonomous bifurcation theory showed that a detailed insight into the fine structure of this spectral notion is necessary. On this basis, we explore a helpful connection between the dichotomy spectrum and operator theory. It relates the asymptotic behavior of linear nonautonomous difference equations to the point, surjectivity and Fredholm spectra of weighted shifts. This link yields several dynamically meaningful subsets of the dichotomy spectrum, which not only allows to classify and detect bifurcations, but also simplifies proofs for results on the long term behavior of difference equations with explicitly time-dependent right-hand side.

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1. Nonautonomous hyperbolicity

Depending on the purpose of inquiry, the spectrum of a bounded linear operator can be decomposed into subsets in various different ways. First, this is due to the fact that sometimes the spectrum is too coarse and only appropriate subsets yield desirable information. Second, when it comes to a classification of the solution behavior for nonlinear equations (in Banach spaces) on a local basis, Fredholm, surjectivity and point spectra of Fréchet derivatives become relevant in order to apply miscellaneous analytical tools. The goal of this paper is to illustrate a significant application of operator theory to the field of dynamical systems. Indeed, we study certain subsets of the so-called dichotomy spectrum using corresponding results on weighted shift operators.

Weighted shift operators. In operator and spectral theory, weighted shift operators on sequence spaces are frequently encountered for illustrative and didactical reasons. They are sometimes even called the "Building Block" of operator theory and often serve as valuable source for (counter-) examples. This consequently led to a rich theory and a quite comprehensive understanding of their spectral properties; for an introduction see [Lam71, Shi74].

Beyond that it turned out that weighted shifts feature useful applications in systems theory. They include a characterization of stability, observability and detectability properties for linear time-varying difference equations (cf. [KKP85, PI97, Wir98] and the references therein) in terms of spectral properties for matrix- or operator-weighted shifts. For instance, [KKP85, Thm. 4.5] show that a nonautonomous linear difference equation

$$\boxed{x_{k+1} = A_k x_k} \tag{Δ}$$

is uniformly asymptotically stable, if and only if the weighted shift operator

$$(T_A\phi)_k := A_{k-1}\phi_{k-1} \quad \text{for all } k \in \mathbb{Z}$$

has spectral radius $r(T_A) < 1$. This suggests a strong connection between the asymptotic behavior of linear dynamical systems (Δ) and operator theoretical concepts, i.e. the spectrum of a shift operator on certain ambient sequence spaces. We point out that such a bridge was probably first observed by Mather in [Mat68] (see also the monographs [CL99, HL05]). Related ideas were later taken up by [BAG91, AM96], who characterize the spectrum $\sigma(T_A)$ for difference eqns. (Δ) defined on the whole integer axis $\mathbb{Z}$, as the union of up to d concentric annuli, where d is the finite dimension of the state space for (Δ). The radii of these annuli are so-called *Bohl exponents* and [BAG91] obtain explicit forms of the spectrum $\sigma(T_A)$ for constant, periodic, asymptotically constant and scalar sequences A_k. An identical description of the set $\sigma(T_A)$ using local spectral theory can be found in [Bou06, Thm. 2.4] and a characterization of weighted operator-shifts satisfying the single value extension property (SVEP for short) is given in [BC04]. Moreover, in [GKK96] the similarity of weighted shifts T_A is discussed for several sequences A_k.

The previously described shape of the spectrum changes when dealing with unilateral shifts. In this situation $\sigma(T_A)$ becomes a solid disc, whereas the essential spectrum $\sigma_F(T_A)$ inherits the above structure as union of rings and is therefore invariant under compact perturbations. This is shown in [BAG93] or [LJS01] by means of Fredholm properties of the shift operator

$$T_A^+\phi = (0, A_0\phi_0, A_1\phi_1, \dots);$$

note that also [BAG92] tackle these problems and relate them to Stein equations. Unitary equivalence for T_A^+ is studied in [Lam71], local spectral theory for matrix-weighted shifts was investigated in [Li94] and finally, [LJS01] characterize Banach reducibility.

Exponential dichotomies. A further fruitful field of applications for weighted shifts is the recent theory of nonautonomous dynamical systems. Indeed, gaps in the above spectra are dynamically important, since they induce a hyperbolic splitting for difference eqns. (Δ). This means there exist two complementary bundles of subspaces containing solutions to (Δ) with a particular exponential growth behavior in forward resp. backward time (cf. [AM96, Lemma 1]). In a dynamical systems language this splitting means that (Δ) has an exponential dichotomy (ED for short). The spectral notion correlated to an ED is called *dichotomy spectrum* $\Sigma_{ED}(A) \subset \mathbb{R}$ associated to a linear nonautonomous differential or difference equation (cf. [SS78] resp. [BAG91, AS01]) — equivalently one also speaks of the *dynamical* or *Sacker-Sell spectrum*. In stability theory the dichotomy spectrum extends the role of the classical spectrum $\sigma(A)$ from an autonomous setting $x_{k+1} = Ax_k$ to a general time-dependent set-up of general eqns. (Δ). Indeed, owing to the rotational invariance of spectra for shift operators, [Pöt09] and [BAG93] observed the crucial relations

$$\Sigma_{ED}(A) = \sigma(T_A) \cap (0,\infty), \qquad \Sigma_{ED}(A) = \sigma_F(T_A^+) \cap (0,\infty) \tag{1.1}$$

relating dichotomy spectra of a linear difference eqn. (Δ) to the spectra of a weighted shift on an appropriate sequence space.

Continuation and bifurcation. This concept of an ED is of crucial relevance to understand the long-term behavior of dynamical systems. For instance, it ensures that entire solutions $\phi^* = (\phi_k^*)_{k\in\mathbb{Z}}$ to *nonlinear* nonautonomous difference equations

$$x_{k+1} = f_k(x_k, \lambda) \tag{1.2}$$

can be continued in parameters λ and it yields (uniform) stability concepts. In addition, it guarantees the persistence of invariant subspaces under a sufficiently large and germane class of nonlinear perturbations giving rise to invariant manifold theorems. Therefore, an ED represents the correct hyperbolicity notion for an effective geometric theory in a nonautonomous framework and we legitimately denote (Δ) as *hyperbolic*, if it admits an ED. When dealing with nonlinear difference eqns. (1.2), linear problems (Δ) typically occur as variational equations with $A_k = D_1 f_k(\phi_k^*, \lambda^*)$ along ϕ^*. One speaks of a *hyperbolic* solution ϕ^*, if the associated operator

$$(S_1\phi)_k := \phi_{k+1} - A_k\phi_k \quad \text{for all } k \in \mathbb{Z}$$

is invertible or equivalently, $1 \notin \sigma(T_A)$. It was first observed in [Hen81, p. 239, Thm. 7.6.5] that this assertion is tantamount to an ED of the variational equation on $\mathbb{Z}$. Thus, by means of the relations (1.1), the long term behavior of (Δ), as well as (1.2) can be studied using classical spectral theory for T_A.

On the other hand, when interested in the behavior of (1.2) under varying parameters λ, nonhyperbolicity $1 \in \Sigma_{ED}(A)$ becomes a necessary condition for bifurcation of bounded entire solutions. Nevertheless, for sufficient conditions it turned out in this context that a more detailed insight into the fine structure of the dichotomy spectrum is required. First, this observation is due to the fact that hyperbolicity of (Δ) is not a generic property anymore, like it is in the classical case of finite-dimensional autonomous or periodic eqns. (Δ). While the set of hyperbolic equations is open (see e.g. [Hen81, p. 232, Thm. 7.6.7]), it is not dense — one can actually construct whole neighborhoods of nonhyperbolic systems (Δ) (cf. [Pöt10a, p. 149, Exam. 3.4.34] or Exam. 5.3). Second, appropriate subsets of the dichotomy spectrum allow to classify nonautonomous bifurcations. This can be understood when S_1 is the Fréchet derivative of an operator formulation for (1.2) evaluated in ϕ^*:

- In case S_1 is an index 0 Fredholm operator, a whole array of tools from classical branching theory becomes available via Lyapunov-Schmidt reduction (cf. [Zei93, Chapt. 8], [Zei95, Sect. 5.12] or [Pöt10b]).
- Provided S_1 looses its invertibility but remains onto, then the surjective implicit function theorem (cf. [Zei93, p. 177, Thm. 4.H]) applies. This yields sufficient conditions for the branching of whole families of entire solutions and the so-called *shovel bifurcation pattern* (see [Pöt11a]).
- Finally, if S_1 is not Fredholm, more specific tools from dynamical systems theory rather than comparatively crude functional analysis become important (for a survey we refer to [Ras07]).

These applications underline the importance of appropriate subsets of the (dichotomy) spectrum, namely essential, surjectivity and point spectra. To our knowledge, this is the first paper suggesting and investigating such a decomposition of $\Sigma_{ED}(A)$. In this endeavor we largely benefit from known results for the corresponding spectra of weighted shifts here.

Our paper is therefore structured into five sections. After introducing our terminology, Sect. 2 contains some basic facts on EDs and the weaker trichotomies, as well as the associated spectra; in particular, the trichotomy spectrum has not been investigated before. Difference equations on semiaxes are studied in Sect. 3. We tackle the question when dichotomies can be extended to larger intervals (a new result in discrete time), and characterize Fredholm properties of weighted shifts in terms of EDs. As a result, we observe that the dichotomy spectrum is related to the Fredholm and surjectivity spectra of T_A^+. This yields certain properties of the dichotomy spectrum (upper-semicontinuity, ℓ_0-roughness), where proofs are drastically simplified, and an explicit formula for $\Sigma_{ED}(A)$ when the coefficient matrices A_k are upper-triangular. Our subsequent Sect. 4 addresses similar questions for difference eqns. (Δ) defined on the whole integer axis. Due to the occurrence of point spectrum, the resulting spectral theory is richer than in the semiaxis situation. We give a short proof relating EDs and Fredholm properties (the stronger general result [LT05, Thm. 1.6] has a much more involved proof and is given for slightly different sequence spaces), which is new in discrete time. The connection to operator theory not only simplifies proofs of known results (e.g. the ℓ_∞-roughness of exponential trichotomies), but also provides novel insights as, e.g., the meagerness of discontinuity points for the dichotomy spectra. We relate all the studied dichotomy spectra in Cor. 4.31. Finally, we address difference equations with an almost periodic coefficient sequence and show that most of the dichotomy spectra for such problems coincide. This drastically simplifies the spectral theory for these (and in particular autonomous or periodic) problems. The concluding Sect. 5 demonstrates our results using various examples, including noninvertible and infinite-dimensional ones. For instance, we illustrate the upper-semicontinuity of dichotomy spectra or show that the dichotomy spectrum for equations defined on the whole axis needs not to be invariant under compact perturbations (cf. Exam. 5.6).

Let us close this motivation with the following remarks and perspectives:

- In the operator theoretical literature [Lam71, BAG91, BAG92, BAG93, Li94, LJS01, BC04, Bou06] the weighted shift operators T_A and T_A^+ are typically studied on the Hilbert space of square-summable sequences ℓ_2. To a large extend, we instead focus on the Banach space ℓ_∞ of bounded sequences. This choice is well-motivated from a dynamical systems point of view, where bounded solutions are of obvious interest, rather than somehow artificial ℓ_2-solutions or -perturbations.
- While we are exclusively dealing with the discrete time case of difference equations, an extension of our theory to a large class of evolutionary differential equations is possible. The key to such an endeavor are results like [Hen81, p. 229, Exam. 10], [HM01, Thm. 4.8] or [LT05, Thm. 1.4 and Lemma 1.5].

Notation. Throughout the paper, we suppose that X, Y are Banach spaces, whose norm and induced operator norm on the space of bounded operators $L(X,Y)$ are denoted by $|\cdot|$. The dual space to X is X', $\langle \cdot, \cdot \rangle$ stands for the duality product and furthermore $X_0^\perp$ for the annihilator of a subset $X_0 \subset X$. We write $N(T) \subset X$ for the kernel and $R(T) \subset Y$ for the range of a linear operator $T \in L(X,Y)$ and $T' \in L(Y', X')$ for its dual operator. Abbreviating $L(X) := L(X,X)$ and $GL(X,Y) := \{T \in L(X,Y) : T \text{ is invertible}\}$ we define the following subsets

$$\begin{aligned} GL(X) &:= GL(X,X),\\ L_s(X) &:= \{S\in L(X):\ S \text{ is onto}\},\\ L_p(X) &:= \{S\in L(X):\ S \text{ is one-to-one}\},\\ L_F(X) &:= \{S\in L(X):\ S \text{ is Fredholm}\},\\ L_{F_0}(X) &:= \{S\in L(X):\ S \text{ is Fredholm with index } 0\};\end{aligned}$$

note that the first four of them are regularities in the sense of [Mül07, pp. 51ff], while the set $L_{F_0}(X)$ merely defines an upper-semiregularity (cf. [Mül00] and [Mül07, pp. 211ff]). Given a bounded operator $S \in L(X)$ we introduce the *reduced minimum modulus* (cf. [Mül07, pp. 97ff])

$$\Gamma(S) := \inf\{c>0:\ \|Sx\| \ge c\,\mathrm{dist}(x,N(S)) \quad \text{for all } x\in X\}$$

and the *surjectivity modulus* (cf. [Mül07, p. 86])

$$k(S) := \sup\{r\ge 0:\ rB_X \subset SB_X\},$$

where B_X is the closed unit ball in X centered around 0. We make use of the following induced spectra:

$$\begin{aligned} \sigma(S) &:= \{\lambda\in\mathbb{C}:\ S-\lambda\,\mathrm{id}\notin GL(X)\} \text{ for the } \textit{spectrum},\\ \sigma_s(S) &:= \{\lambda\in\mathbb{C}:\ S-\lambda\,\mathrm{id}\notin L_s(X)\} \text{ for the } \textit{surjectivity spectrum},\\ \sigma_p(S) &:= \{\lambda\in\mathbb{C}:\ S-\lambda\,\mathrm{id}\notin L_p(X)\} \text{ for the } \textit{point spectrum} \text{ and}\\ \sigma_F(S) &:= \{\lambda\in\mathbb{C}:\ S-\lambda\,\mathrm{id}\notin L_F(X)\},\\ \sigma_{F_0}(S) &:= \{\lambda\in\mathbb{C}:\ S-\lambda\,\mathrm{id}\notin L_{F_0}(X)\} \text{ for the } \textit{Fredholm spectra};\end{aligned}$$

note that $\sigma_s(S)$ is also denoted as *(approximate) defect spectrum*, $\sigma_{F_0}(S)$ as *Weyl spectrum* and $\sigma_F(S), \sigma_{F_0}(S)$ as *essential spectra* of S.

Let $\mathbb{I}$ be a discrete interval, i.e. the intersection of a real interval with the integers $\mathbb{Z}$, and $\mathbb{I}' := \{k\in\mathbb{I}:\ k+1\in\mathbb{I}\}$. In particular, we define the discrete intervals $\mathbb{I}_\kappa^+ := [\kappa,\infty)\cap\mathbb{I}$ and $\mathbb{I}_\kappa^- := (-\infty,\kappa]\cap\mathbb{I}$ for $\kappa\in\mathbb{I}$. Often it is convenient to write $\mathbb{I}_\kappa^\pm$ for either $\mathbb{I}_\kappa^+$ or $\mathbb{I}_\kappa^-$ and we proceed similarly with our further notation. Typically, $\mathbb{I}$ will be unbounded to avoid trivialities.

We denote by $\ell_\infty(\mathbb{I},X)$, or for brevity ℓ_∞, the space of bounded sequences $\phi = (\phi_k)_{k\in\mathbb{I}}$ in X, ℓ_0 is the subspace of limit zero sequences (in case $\mathbb{I}=\mathbb{Z}$ we mean the two-sided limit) and ℓ_{00} the space of sequences with only finitely many nonzero elements. One has the inclusions $\ell_{00}\subset\ell_0\subset\ell_\infty$ and ℓ_0 is a closed subspace of ℓ_∞, in which ℓ_{00} is dense. Moreover, $\ell_p\subset\ell_0$ is the space of p-summable sequences, $p\ge 1$. Finally as convention, the norm on spaces of sequences with values in X is denoted as $\|\cdot\|$.

2. Exponential dichotomy and trichotomy

We consider a nonautonomous linear difference eqn. (Δ) in a Banach space X with coefficient operators $A_k \in L(X)$, $k\in\mathbb{I}'$. Such discrete equations generate a *transition operator* $\Phi : \{(k,l)\in\mathbb{I}\times\mathbb{I}:\ l\le k\} \to L(X)$, i.e. for each instant $\kappa \in \mathbb{I}$ the sequence $\Phi(\cdot,\kappa) : \mathbb{I}_\kappa^+ \to L(X)$ is the unique forward solution to the initial value problem $X_{k+1} = A_k X_k$, $X_\kappa = \mathrm{id}$ in $L(X)$ and thus given by

$$\Phi(k,l) := \begin{cases} A_{k-1} \cdots A_l, & l < k, \\ \mathrm{id}, & k = l; \end{cases}$$

in the invertible case

$$A_k \in GL(X) \quad \text{for all } k \in \mathbb{I}' \tag{2.1}$$

we supplement this definition by setting

$$\Phi(k,l) := A_k^{-1} \cdots A_{l-1}^{-1} \quad \text{for all } k < l.$$

Remark 2.1. (1) We call a difference eqn. (Δ) or its transition operator Φ *eventually compact*, if there exists an $m \in \mathbb{N}$ such that $\Phi(k,l)$ is compact for $k, l \in \mathbb{I}$ satisfying $k - l \geq m$. This situation typically occurs when dealing with temporal discretization of diffusion equations (cf. [Hen81, p. 196, Exam. 10*]) or retarded functional differential equations (cf. [HVL93, p. 91, Cor. 6.2]).

(2) One says a difference equation

$$y_{k+1} = B_k y_k \tag{2.2}$$

with coefficient operators $B_k \in L(Y)$, $k \in \mathbb{I}'$, is *kinematically similar* to (Δ), if there exists a sequence $(C_k)_{k\in\mathbb{I}}$ of operators $C_k \in GL(Y,X)$ satisfying

$$C_{k+1} B_k = A_k C_k \quad \text{for all } k \in \mathbb{I}' \tag{2.3}$$

and also the boundedness condition $\sup_{k\in\mathbb{I}} \max\left\{|C_k|, \left|C_k^{-1}\right|\right\} < \infty$ holds. Such sequences are denoted as *Lyapunov transformations*.

(3) The solution space $\Lambda := \{(\phi_k)_{k\in\mathbb{I}} : \phi_{k+1} \equiv A_k \phi_k \text{ on } \mathbb{I}'\}$ for (Δ) is linear. If one of the conditions

- $\mathbb{I} = \mathbb{Z}_\kappa^+$,
- $\mathbb{I}$ is unbounded below and (2.1)

holds, then $\xi \mapsto \Phi(\cdot,\kappa)\xi$ is a vector space isomorphism from X onto Λ.

The following notions aim to capture the geometrical behavior of (Δ). For a discrete interval $\mathbb{I}$ unbounded above we define the *stable bundle*

$$\mathcal{V}^+ := \left\{(\kappa,\xi) \in \mathbb{I} \times X : \sup_{\kappa \leq k} |\Phi(k,\kappa)\xi| < \infty\right\}$$

consisting of forward bounded solutions to (Δ). Dually, for discrete intervals $\mathbb{I}$ unbounded below the *unstable bundle* is given by

$$\mathcal{V}^- := \left\{(\kappa,\xi) \in \mathbb{I} \times X : \begin{array}{l} \text{there exists a solution } \phi = (\phi_k)_{k\in\mathbb{I}} \text{ of} \\ (\Delta) \text{ with } \phi_\kappa = \xi \text{ and } \sup_{k\leq\kappa} |\phi_k| < \infty \end{array}\right\}$$

and consists of solutions which exist and are bounded in backward time; in the invertible case (2.1) one has $\mathcal{V}^- = \left\{(\kappa,\xi) \in \mathbb{I} \times X : \sup_{k\leq\kappa} |\Phi(k,\kappa)\xi| < \infty\right\}$. Finally, for $\mathbb{I} = \mathbb{Z}$ we can introduce the *center bundle*

$$\mathcal{V}^0 := \mathcal{V}^- \cap \mathcal{V}^+$$

consisting of entire bounded solutions to (Δ). Since we frequently deal with subsets $\mathcal{V} \subseteq \mathbb{I} \times X$, it is convenient use the convenient notation for its *κ-fiber*

$$\mathcal{V}(k) := \{x \in X : (k, x) \in \mathcal{V}\} \quad \text{for all } k \in \mathbb{I}.$$

An exponential dichotomy or trichotomy is a concept to provide a convenient characterization of the vector bundles $\mathcal{V}^+$, $\mathcal{V}^-$ and $\mathcal{V}^0$. More precisely, a sequence of projections $P_k \in L(X)$, $k \in \mathbb{I}$, is denoted as *invariant projector* of (Δ), provided $P_{k+1}A_k = A_kP_k$ for all $k \in \mathbb{I}'$ holds and a *regular* invariant projector additionally satisfies the *regularity condition*

$$A_k|_{R(P_k)} : R(P_k) \to R(P_{k+1}) \text{ is bijective for all } k \in \mathbb{I}'. \tag{2.4}$$

Using mathematical induction and [Yos80, p. 77, Corollary] this yields

$$\bar{\Phi}(k, l) := \Phi(k, l)|_{R(P_l)} \in GL(R(P_l), R(P_k)) \quad \text{for all } l \leq k,$$

we denote the bounded inverse of $\bar{\Phi}(k, l)$ by $\bar{\Phi}(l, k)$, $l \leq k$, and point out that the ranges $R(P_k)$ have constant dimension for all $k \in \mathbb{I}$.

Definition 2.2. A difference eqn. (Δ) or the associated transition operator Φ is said to have an *exponential trichotomy* (ET for short) on $\mathbb{I}$, if

(i) there exist invariant projectors P_k, Q_k satisfying $P_kQ_k = Q_kP_k = 0$ such that $Q_k, \mathrm{id} - P_k - Q_k$ are regular,
(ii) there exist reals $K \geq 1$, $\alpha \in (0, 1)$ and a $\kappa \in \mathbb{I}$ such that (with $k, l \in \mathbb{I}$)

$$\begin{aligned} |\Phi(k, l)P_l| &\leq K\alpha^{k-l} && \text{for all } l \leq k, \\ \left|\bar{\Phi}(k, l)Q_l\right| &\leq K\alpha^{|k-l|} && \text{for all } \kappa \leq l \leq k \text{ or } k \leq l \leq \kappa, \\ \left|\bar{\Phi}(k, l)[\mathrm{id} - P_l - Q_l]\right| &\leq K\alpha^{l-k} && \text{for all } k \leq l. \end{aligned} \tag{2.5}$$

In case $Q_k \equiv 0$ one speaks of an *exponential dichotomy* (ED for short) and the *Morse index* ι of an ED is the (constant) dimension of $N(P_k)$.

Remark 2.3. (1) The above ET notion is stronger than the frequently used trichotomy concept from [SS76], where the estimate (2.5) is replaced by

$$\left|\bar{\Phi}(k, l)Q_l\right| \leq K \quad \text{for all } k, l \in \mathbb{I}. \tag{2.6}$$

While (2.6) captures the classical situation of a constant (or periodic) coefficient sequence A_k in (Δ) with semi-simple eigenvalues (Floquet multipliers) on the complex unit circle, the notion in Def. 2.2 is due to [EJ98, Pap91]. It is intrinsically nonautonomous and strongly related to EDs on semiaxes.

Indeed, an ET on $\mathbb{I}$ is essentially equivalent to EDs on both semiaxes (see [EJ98, Lemma 2] for finite-dimensional invertible equations):

Proposition 2.4. *If a difference eqn.* (Δ) *admits*

(a) *an ET on $\mathbb{I}$ with projectors P_k, Q_k, then it has an ED on $\mathbb{I}_\kappa^+$ with projector $P_k^+ = P_k + Q_k$, an ED on $\mathbb{I}_\kappa^-$ with projector $P_k^- = P_k$ and*

$$P_\kappa^- = P_\kappa^- P_\kappa^+ = P_\kappa^+ P_\kappa^-, \tag{2.7}$$

(b) *an ED on* $\mathbb{I}_\kappa^+$ *with projector* P_k^+, *an ED on* $\mathbb{I}_\kappa^-$ *with* P_k^- *satisfying* (2.7) *and fulfills* (2.1), *then it admits an ET on* $\mathbb{I}$ *with*

$$P_k = \Phi(k,\kappa)P_\kappa^-\Phi(\kappa,k), \qquad Q_k = \Phi(k,\kappa)[P_\kappa^+ - P_\kappa^-]\Phi(\kappa,k). \tag{2.8}$$

Remark 2.5. The condition (2.7) is equivalent to $X = R(P_\kappa^+) + N(P_\kappa^-)$.

Proof. (a) It is easy to see that the projections P_κ^+, P_κ^- defined above fulfill the relation (2.7). We show the backward time estimate for the ED on $\mathbb{I}_\kappa^-$,

$$\begin{aligned} \left|\bar\Phi(k,l)[\mathrm{id} - P_l^-]\right| &= \left|\bar\Phi(k,l)[Q_l + \mathrm{id} - P_l - Q_l]\right| \\ &\le \left|\bar\Phi(k,l)Q_l\right| + \left|\bar\Phi(k,l)[\mathrm{id} - P_l - Q_l]\right| \overset{(2.5)}{\le} 2K\alpha^{l-k} \end{aligned}$$

for all $k \le l \le \kappa$ and the other estimates can be established similarly.

(b) Due to (2.1) we can extend the projectors $P_k^+ := \Phi(k,\kappa)P_\kappa^+\Phi(\kappa,k)$ for $k < \kappa$ and $P_k^- := \Phi(k,\kappa)P_\kappa^-\Phi(\kappa,k)$ for $k > \kappa$ on the whole axis $\mathbb{I}$. Thus, for arbitrary $k, l \in \mathbb{I}$ we obtain

$$\begin{aligned} |\Phi(k,l)P_l| &\overset{(2.8)}{=} \left|\Phi(k,l)\Phi(l,\kappa)P_\kappa^-\Phi(\kappa,l)\right| \\ &\overset{(2.7)}{=} \left|\Phi(k,\kappa)P_\kappa^+\Phi(\kappa,l)\Phi(l,\kappa)P_\kappa^-\Phi(\kappa,l)\right| \\ &\overset{(2.8)}{\le} \left|\Phi(k,l)P_l^+\right| |P_l| \le K^2\alpha^{k-l} \quad \text{for all } \kappa \le l \le k, \\ |\Phi(k,l)P_l| &\overset{(2.7)}{=} \left|\Phi(k,\kappa)P_\kappa^+P_\kappa^-\Phi(\kappa,l)\right| \\ &\overset{(2.8)}{\le} \left|\Phi(k,\kappa)P_\kappa^+\right| \left|\Phi(\kappa,l)P_l^-\right| \le K^2\alpha^{k-l} \quad \text{for all } l \le \kappa \le k, \\ |\Phi(k,l)P_l| &\overset{(2.8)}{=} \left|\Phi(k,\kappa)P_\kappa^+\Phi(\kappa,k)\Phi(k,\kappa)P_\kappa^-\Phi(\kappa,l)\right| \\ &\overset{(2.8)}{\le} \left|P_k^+\right| \left|\Phi(k,l)P_l^-\right| \le K^2\alpha^{k-l} \quad \text{for all } l \le k \le \kappa \end{aligned}$$

and consequently the first trichotomy inequality. The remaining two estimates in (2.5) follow along the same lines. □

Corollary 2.6. *Under* (2.1) *one can replace* $\kappa \in \mathbb{I}$ *in* (2.5) *by any other* $\bar\kappa \in \mathbb{I}$.

Proof. Let $\bar\kappa \in \mathbb{I}$. Since the eqn. (Δ) has an ET, using Prop. 2.4(a) we conclude EDs on $\mathbb{I}_\kappa^+$ with $P_k^+ = P_k + Q_k$, as well as on $\mathbb{I}_\kappa^-$ with $P_k^- = P_k$ satisfying (2.7). Due to (2.1) these dichotomies can be extended to the discrete intervals $\mathbb{I}_{\bar\kappa}^+$ and $\mathbb{I}_{\bar\kappa}^-$ by means of the projectors

$$P_k^+ := \Phi(k,\kappa)[P_\kappa + Q_\kappa]\Phi(\kappa,k), \qquad P_k^- := \Phi(k,\kappa)P_\kappa\Phi(\kappa,k)$$

satisfying $P_{\bar\kappa}^- P_{\bar\kappa}^+ = \Phi(\bar\kappa,\kappa)P_\kappa[P_\kappa + Q_\kappa]\Phi(\kappa,\bar\kappa) = \Phi(\bar\kappa,\kappa)P_\kappa\Phi(\kappa,\bar\kappa) = P_{\bar\kappa}^-$ and $P_{\bar\kappa}^+ P_{\bar\kappa}^- = P_{\bar\kappa}^-$. Thus, Prop. 2.4(b) ensures an ET on $\mathbb{I}$ with $\bar\kappa$ instead of κ. □

Lemma 2.7. *Let* $\lambda \in \mathbb{C} \setminus \{0\}$. *A scaled difference equation* $x_{k+1} = \lambda A_k x_k$ *admits an ET on* $\mathbb{I}$, *if and only if* $x_{k+1} = |\lambda| A_k x_k$ *is exponentially trichotomic on* $\mathbb{I}$ *with the same data* P_k, Q_k, K, α *and* κ.

Proof. Given a nonzero $\lambda \in \mathbb{C}$, the transition operator of $x_{k+1} = \lambda A_k x_k$ reads as $\Phi_\lambda(k,l) = \lambda^{k-l}\Phi(k,l)$ for all $l \le k$. Then it is straight forward to show that the estimates (2.5) are inherited between the equations in question. $\square$

Along with (Δ) we consider the *scaled difference equation*

$$\boxed{x_{k+1} = \tfrac{1}{\gamma} A_k x_k} \quad \text{for reals } \gamma > 0, \tag{Δ_γ}$$

with transition operator $\Phi_\gamma(k,l) := \gamma^{l-k}\Phi(k,l)$ and invariant vector bundles

$$\mathcal{V}_\gamma^+ := \left\{(\kappa,\xi) \in \mathbb{I} \times X : \sup_{\kappa \le k} |\Phi(k,\kappa)\xi|\gamma^{\kappa-k} < \infty\right\},$$

$$\mathcal{V}_\gamma^- := \left\{(\kappa,\xi) \in \mathbb{I} \times X : \begin{array}{l} \text{there exists a solution } \phi = (\phi_k)_{k\in\mathbb{I}} \text{ of } (\Delta) \\ \text{with } \phi_\kappa = \xi \text{ and } \sup_{k \le \kappa} |\phi_k|\gamma^{\kappa-k} < \infty \end{array}\right\}.$$

We clearly have $\mathcal{V}^\pm = \mathcal{V}_1^\pm$ and define

- the *dichotomy spectrum*
$$\Sigma_{ED}(A) := \{\gamma > 0 : \Phi_\gamma \text{ does not have an ED on } \mathbb{I}\},$$
- the *trichotomy spectrum*
$$\Sigma_{ET}(A) := \{\gamma > 0 : \Phi_\gamma \text{ does not have an ET on } \mathbb{I}\},$$
- the *dichotomy resolvent* $\rho_{ED}(A) := (0,\infty) \setminus \Sigma_{ED}(A)$

of a linear eqn. (Δ). In case $1 \notin \Sigma_{ED}(A)$ one calls (Δ) *hyperbolic* on $\mathbb{I}$. One also introduces the *forward* resp. *backward dichotomy spectrum*

$$\Sigma_\kappa^\pm(A) = \left\{\gamma > 0 : \Phi_\gamma \text{ does not have an ED on } \mathbb{I}_\kappa^\pm\right\} \subset \Sigma_{ED}(A)$$

for $\kappa \in \mathbb{I}$ and deduces the obvious inclusions

$$\Sigma_k^+(A) \subset \Sigma_\kappa^+(A), \qquad \Sigma_\kappa^-(A) \subset \Sigma_k^-(A) \quad \text{for all } \kappa \le k. \tag{2.9}$$

Proposition 2.8. *One has* $\Sigma_{ET}(A) \subset \Sigma_{ED}(A)$ *and for every* $k \in \mathbb{I}$ *it is*

(a) $\Sigma_k^+(A) \cup \Sigma_k^-(A) \subset \Sigma_{ED}(A)$,
(b) $\Sigma_k^+(A) \cup \Sigma_k^-(A) \subset \Sigma_{ET}(A)$ *under* (2.1).

Proof. First, one clearly has the inclusion $\Sigma_{ET}(A) \subset \Sigma_{ED}(A)$ and our assertion (a) is evident from the above. Concerning (b), due to Prop. 2.4(a) we obtain the implications

$$\gamma \notin \Sigma_{ET}(A) \Leftrightarrow \Phi_\gamma \text{ has an ET on } \mathbb{I} \Rightarrow \Phi_\gamma \text{ has EDs on } \mathbb{I}_\kappa^+ \text{ and } \mathbb{I}_\kappa^-,$$

where $\kappa \in \mathbb{I}$ can depend on γ. However, Cor. 2.6 allows us to choose κ equal to any given $k \in \mathbb{I}$ and we conclude

$$\gamma \notin \Sigma_{ET}(A) \Leftrightarrow \gamma \notin \Sigma_k^+(A) \text{ and } \gamma \notin \Sigma_k^-(A),$$

which is equivalent to assertion (b). $\square$

The structure of the above dichotomy spectra resembles the autonomous case, where eigenvalue moduli correspond to spectral intervals and generalized eigenspaces become invariant vector bundles. Precisely, one has:

Theorem 2.9 (spectral theorem). *The dichotomy spectrum of* (Δ) *is closed in* $(0,\infty)$. *If* $\mathbb{I}$ *is unbounded, then* $\Sigma_{ED}(A)$ *is either empty or in case* $\dim X<\infty$ *the disjoint union of* $1\leq n\leq \dim X$ *closed* spectral intervals. *Precisely, one either has* $\Sigma_{ED}(A)=\emptyset$, $\Sigma_{ED}(A)=(0,\infty)$ *or one of the four cases*

$$\Sigma_{ED}(A)=\begin{cases}[\alpha_1,\beta_1]\\ or\\ (0,\beta_1]\end{cases}\cup[\alpha_2,\beta_2]\cup\ldots\cup[\alpha_{n-1},\beta_{n-1}]\cup\begin{cases}[\alpha_n,\beta_n]\\ or\\ [\alpha_n,\infty)\end{cases}$$

with reals $0<\alpha_j\leq\beta_j<\alpha_{j+1}$. *Finally, for* $\mathbb{I}=\mathbb{Z}$ *one has the Whitney sum*

$$\bigoplus_{j=1}^{n}\mathcal{U}_j=\mathbb{Z}\times X$$

with nontrivial invariant vector bundles $\mathcal{U}_j:=\mathcal{V}^+_{\gamma_j}\cap\mathcal{V}^-_{\gamma_{j-1}}$. *The growth rates are chosen according to* $\gamma_j\in(\beta_j,\alpha_{j+1})$ *and* $\gamma_0\in\rho_{ED}(A)$ *such that* $(0,\gamma_0)\subset\rho_{ED}(A)$*; if this is not possible, define* $\mathcal{U}_1:=\mathcal{V}^+_{\gamma_1}$.

Remark 2.10. (1) If (Δ) has *bounded forward growth*, i.e. the estimate

$$\omega_+:=\sup_{k\in\mathbb{I}}|A_k|<\infty \tag{2.10}$$

holds, then $\Sigma_{ED}(A)\subset(0,\omega_+]$. In applications, this is a legitimate assumption, since nonautonomous linear problems (Δ) often occur as variational eqn. $x_{k+1}=D_1f_k(\phi_k^*,\lambda)x_k$ of a nonlinear problem (1.2) along a bounded reference solution $\phi=(\phi_k^*)_{k\in\mathbb{I}}$. Then (2.10) holds under natural assumptions on the nonlinearity f_k, i.e. $D_1f_k(\cdot,\lambda)$ maps bounded sets into bounded sets uniformly in $k\in\mathbb{I}$. This justifies that we often assume (2.10) in the following. On the other hand, provided a difference eqn. (Δ) has *bounded backward growth*, i.e. beyond (2.1) the estimate

$$\omega_-:=\sup_{k\in\mathbb{I}}\left|A_k^{-1}\right|<\infty \tag{2.11}$$

holds, then $(0,\omega_-)\cap\Sigma(A)=\emptyset$ (see [AS02, Thm. 2.1]). Hence, under both conditions (2.10) and (2.11) the dichotomy spectrum is compact.

(2) If (Δ) has an ET with $Q_k\neq 0$, then $[\alpha,\alpha^{-1}]\subset\Sigma_{ED}(A)$.

(3) A different dichotomy spectrum was introduced in [AS01] based on the weaker dichotomy concept from [AK01]. Here, a difference eqn. (Δ) or Φ is said to have an *exponential forward dichotomy* (EFD for short), provided there exist reals $\alpha\in(0,1)$, $K\geq 1$ and an invariant projector $(P_k)_{k\in\mathbb{I}}$, which is additionally a bounded sequence, such that

$$|\Phi(k,l)P_lx|\leq K\alpha^{k-l}|P_lx|,\qquad K^{-1}\alpha^{l-k}|[\mathrm{id}-P_l]x|\leq|\Phi(k,l)[\mathrm{id}-P_l]x|$$

holds for all $x\in X$ and $l\leq k$. An EFD does not even require the regularity condition (2.4), but as shown in [AK01, Exam. 2.7], it has the disadvantage of not being robust w.r.t. ℓ_∞-perturbations. The resulting spectrum

$$\hat{\Sigma}_{ED}(A):=\{\gamma>0:\ \Phi_\gamma\text{ does not have an EFD on }\mathbb{I}\},$$

satisfies $\hat{\Sigma}_{ED}(A) \subset \Sigma_{ED}(A)$ and this inclusion can be strict in the sense that $\hat{\Sigma}_{ED}(A)$ might consist of up to $1+\dim X$ disjoint spectral intervals (cf. [AS01, Thm. 3.4] for details). Under (2.1) however, one has $\hat{\Sigma}_{ED}(A) = \Sigma_{ED}(A)$.

Proof. Due to [AS02, Lemma 2.1] the dichotomy spectrum is closed. For the remaining assertions, see [AS01, Thm. 3.4] and [AS02, Spectral theorem]. □

3. Equations on semiaxes

Throughout the section, we fix an instant $\kappa \in \mathbb{Z}$ and focus on linear difference eqns. (Δ) where the time axis $\mathbb{I}$ is of the form $\mathbb{Z}_\kappa^+$ or $\mathbb{Z}_\kappa^-$. As mentioned above, an ET on such an unbounded discrete interval provides a geometrical characterization of the invariant vector bundles $\mathcal{V}^+, \mathcal{V}^-$.

Proposition 3.1. *Suppose a difference eqn.* (Δ) *admits an ET on* $\mathbb{I}$.

(a) *If* $\mathbb{I}$ *is unbounded above, then*

$$\mathcal{V}^+(\kappa) = \left\{\xi \in X : \lim_{k\to\infty} \Phi(k,\kappa)\xi = 0\right\} = R(P_\kappa + Q_\kappa),$$

(b) *if* $\mathbb{I}$ *is unbounded below, then*

$$\mathcal{V}^-(\kappa) = \left\{\xi \in X : \begin{array}{l} \textit{there exists a solution } \phi = (\phi_k)_{k\in\mathbb{Z}_\kappa^-} \textit{ to} \\ (\Delta) \textit{ with } \phi_\kappa = \xi \textit{ and } \lim_{k\to-\infty} \phi_k = 0 \end{array}\right\} = N(P_\kappa),$$

and the above convergence assertions are even exponential.

Remark 3.2. (1) If a difference eqn. (Δ) is eventually compact and the discrete interval $\mathbb{I}$ is unbounded below, then all the fibers $\mathcal{V}^-(k)$, $k \in \mathbb{Z}_\kappa^-$, share the same finite dimension (cf. [Pöt10a, p. 142, Prop. 3.4.24]).

(2) Under (2.1) the above characterizations hold for all $\kappa \in \mathbb{I}$.

Proof. We only prove assertion (b) since (a) can be shown analogously. Due to Prop. 2.4(a) our difference eqn. (Δ) admits an ED on $\mathbb{Z}_\kappa^-$ with projector P_k and by [Pöt10a, Rem. 3.4.18] we have $N(P_\kappa) = \mathcal{V}_\gamma^-(\kappa)$, if $\gamma \in (\alpha, \alpha^{-1})$. For $\gamma \in (\alpha, 1)$ one has exponential convergence to 0 and our claim results. □

Our following result gives criteria that an ED on a semiaxis $\mathbb{I}$ can be extended to a larger discrete interval $\mathbb{J}$ as long as $\mathbb{J} \setminus \mathbb{I}$ remains finite. Under (2.1) this is possible by simply extending the invariant projector using

$$P_k := \Phi(k,\kappa)P_\kappa\Phi(\kappa,k) \quad \text{for all } k \in \mathbb{J} \setminus \mathbb{I}.$$

Without the strict invertibility assumption (2.1) we obtain the following discrete-time counterpart to [Lin86, Lemma 2.3, 2.4]:

Proposition 3.3. *Let* $\overline{\kappa}, \underline{\kappa} \in \mathbb{Z}$ *with* $\underline{\kappa} < \kappa < \overline{\kappa}$ *and suppose that a difference eqn.* (Δ) *admits an ED on* $\mathbb{I}$ *with projector* P_k.

(a) *If* $\mathbb{I} = \mathbb{Z}_\kappa^-$, $N(\Phi(\overline{\kappa},\kappa)|_{N(P_\kappa)}) = \{0\}$, *the subspaces* $\Phi(k,\kappa)N(P_\kappa) \subset X$ *are closed for* $\kappa < k \le \overline{\kappa}$ *and* $\Phi(\overline{\kappa},\kappa)N(P_\kappa) \subset X$ *is complemented, then* (Δ) *has an ED on the extended interval* $\mathbb{Z}_{\overline{\kappa}}^-$.

(b) *If* $\mathbb{I} = \mathbb{Z}_\kappa^+$ *and the subspaces* $\Phi(\kappa, \underline{\kappa})^{-1} R(P_\kappa), R(P_\kappa) \subset X$ *have the same finite codimension, then* (Δ) *has an ED on the extended interval* $\mathbb{Z}_{\underline{\kappa}}^+$.

Remark 3.4. (1) In Hilbert spaces X the subspaces $\Phi(\overline{\kappa}, \kappa) N(P_\kappa)$ are complemented, provided they are closed.

(2) For an eventually compact difference eqn. (Δ) we know that the images $\Phi(\overline{\kappa}, \kappa) N(P_\kappa)$ are finite-dimensional by Rem. 3.2(1) and thus complemented. Moreover, the spaces $\Phi(k, \kappa) N(P_\kappa)$ are closed for all $k \geq \kappa$.

Proof. (a) Let $\kappa \leq \overline{\kappa}$. Above all, we define the subspaces

$$X_k^+ := \begin{cases} N(P_k), & k \leq \kappa, \\ \Phi(k, \kappa) N(P_\kappa), & \kappa \leq k \leq \overline{\kappa} \end{cases}$$

which are closed due to our assumption, and the regularity of $\mathrm{id} - P_k$ shows that $\dim X_k^+$, $k \leq \overline{\kappa}$, is constant. Moreover, $\Phi(k, l) : X_l^+ \to X_k^+$ is a topological isomorphism for $l \leq k \leq \overline{\kappa}$. By assumption, $X_{\overline{\kappa}}^+$ is complemented and we denote by $X_{\overline{\kappa}}^-$ the corresponding closed subspace with $X_{\overline{\kappa}}^+ \oplus X_{\overline{\kappa}}^- = X$. Since $\Phi(\overline{\kappa}, \kappa)$ is continuous, the preimages $X_k^- := \Phi(\overline{\kappa}, k)^{-1} X_{\overline{\kappa}}^-$, $k \leq \overline{\kappa}$, are closed subspaces of X with the following properties:

- Given $\xi \in X_k^+ \cap X_k^-$ we obtain $\Phi(\overline{\kappa}, k)\xi \in X_{\overline{\kappa}}^+ \cap X_{\overline{\kappa}}^-$, thus $\Phi(\overline{\kappa}, k)\xi = 0$ and so $\xi = 0$, since $\Phi(\overline{\kappa}, k)$ is an isomorphism between X_k^+ and $X_{\overline{\kappa}}^+$.
- Let $\xi \in X$ with a representation $\Phi(\overline{\kappa}, k)\xi = \eta_1 + \eta_2$ for $\eta_1 \in X_{\overline{\kappa}}^+$, $\eta_2 \in X_{\overline{\kappa}}^-$. Then there exists a $\xi_1 \in X_k^+$ with $\eta_1 = \Phi(\overline{\kappa}, k)\xi_1$ and therefore $\Phi(\overline{\kappa}, k)[\xi - \xi_1] = \eta_2 \in X_{\overline{\kappa}}^-$, which guarantees $\xi - \xi_1 \in X_k^-$.

From this we conclude the relation

$$X = X_k^+ \oplus X_k^- \quad \text{for all } k \leq \overline{\kappa}$$

and define the complementary projectors $\bar{P}_k, \bar{Q}_k \in L(X)$ by virtue of the relations $R(\bar{P}_k) = X_k^+$ and $R(\bar{Q}_k) = X_k^-$ for all $k \leq \overline{\kappa}$. Both, $\bar{P}_k, \bar{Q}_k$ are invariant projectors of (Δ) for $k \leq \overline{\kappa}$. Since $R(\bar{Q}_k) = R(Q_k)$ implies

$$Q_\kappa \bar{Q}_\kappa = \bar{Q}_\kappa, \qquad \bar{Q}_k Q_k = Q_k,$$

we deduce the equation

$$\begin{aligned} \bar{Q}_k &= \bar{Q}_k(Q_k + P_k) = Q_k + \bar{Q}_k P_k = Q_k + \Phi(k, \kappa)\bar{Q}_\kappa \Phi(\kappa, k) P_k \\ &= Q_k + \Phi(k, \kappa) Q_\kappa \bar{Q}_\kappa \Phi(\kappa, k) P_k \quad \text{for all } k \leq \kappa. \end{aligned}$$

In connection with the exponential limit relations

$$\lim_{k \to -\infty} \Phi(\kappa, k) P_k = 0, \qquad \lim_{k \to -\infty} \bar{\Phi}(k, \kappa) Q_\kappa = 0$$

this implies that the difference $\bar{Q}_k - Q_k$ decays to 0 exponentially as $k \to -\infty$. Due to the relation $\bar{P}_k - P_k = \mathrm{id} - \bar{Q}_k - (\mathrm{id} - Q_k) = Q_k - \bar{Q}_k$ the same holds for the difference $\bar{P}_k - P_k$ and there exists a constant $M \geq 0$ such that

$$\max\left\{ \left|\bar{P}_k\right|, \left|\bar{Q}_k\right| \right\} \leq M \quad \text{for all } k \leq \overline{\kappa}.$$

First, we get an ED of (Δ) on the finite set $\{\kappa,\dots,\overline{\kappa}\}$. Second, due to $\bar{Q}_kQ_k = Q_k$ we have $\bar{P}_k = \bar{P}_k(P_k+Q_k) = \bar{P}_kP_k + \bar{P}_k\bar{Q}_kQ_k = \bar{P}_kP_k$ and obtain

$$\left|\Phi(k,l)\bar{P}_l\right| = \left|\bar{P}_k\Phi(k,l)\right| = \left|\bar{P}_kP_k\Phi(k,l)\right| \le MK\alpha^{k-l}$$

for all $l \le k \le \kappa$, while we derive the remaining dichotomy inequality

$$\left|\Phi(l,k)\bar{Q}_k\right| = \left|\Phi(l,k)Q_k\bar{Q}_k\right| \le \left|\Phi(l,k)Q_k\right|\left|\bar{Q}_k\right| \le KM\alpha^{k-l}$$

for all $l \le k \le \kappa$. This guarantees that (Δ) also has an ED on $\mathbb{Z}_\kappa^+$ w.r.t. the projector $\bar{P}_k$. Combining the results on the two intervals implies (a).

(b) Let $\underline{\kappa} \le \kappa$. We define the subspaces

$$X_k^+ := \begin{cases} R(P_k), & k \le \kappa, \\ \Phi(\kappa,k)^{-1}R(P_\kappa), & \underline{\kappa} \le k \le \kappa \end{cases}$$

and our assumption yields $m := \operatorname{codim} X_\kappa^+ = \operatorname{codim} X_{\underline{\kappa}}^+ < \infty$. By [Mül07, p. 398, Thm. A.1.25(ii)] we can choose a closed subspace $X_{\underline{\kappa}}^-$ with $X_{\underline{\kappa}}^+ \oplus X_{\underline{\kappa}}^- = X$ and define $X_k^- := \Phi(k,\underline{\kappa})X_{\underline{\kappa}}^-$ for $k \ge \underline{\kappa}$. In order to verify the direct sum

$$X_k^+ \oplus X_k^- = X \quad \text{for all } \underline{\kappa} \le k \tag{3.1}$$

we proceed in two steps:

- We first verify $X_k^+ \cap X_k^- = \{0\}$ for $k \ge \underline{\kappa}$: If $\xi_k \in X_k^+ \cap X_k^-$, then there exists a $\xi \in X_{\underline{\kappa}}^-$ such that $\xi_k = \Phi(k,\underline{\kappa})\xi$.
  For $k \le \kappa$, by definition of X_k^+ it is $\Phi(\kappa,k)\xi_k \in X_\kappa^+$ and this implies $\xi \in X_{\underline{\kappa}}^+$. Thus, $\xi = 0$ and $\xi_k = 0$.
  On the other hand, for $\kappa < k$ it is
  $$\Phi(k,\kappa)\Phi(\kappa,\underline{\kappa})\xi = \Phi(k,\underline{\kappa})\xi = \xi_k \in X_k^+ = R(P_k),$$
  then $\Phi(k,\underline{\kappa})\xi \in R(P_k)$, which yields $\xi \in X_{\underline{\kappa}}^+$. We conclude $\xi = \xi_k = 0$.
- We proceed to prove (3.1). Given $\xi \in X_{\underline{\kappa}}^- \setminus \{0\}$ we get $\Phi(k,\underline{\kappa})\xi \neq 0$ for $\underline{\kappa} \le k$, since otherwise $\xi \in X_{\underline{\kappa}}^+$ and also $X_{\underline{\kappa}}^+ \cap X_{\underline{\kappa}}^- = \{0\}$, which is a contradiction. Furthermore, $\Phi(k,\underline{\kappa})X_{\underline{\kappa}}^- = X_k^-$, $\underline{\kappa} \le k$, is an m-dimensional subspace of X. For $\kappa \le k$ one has $\operatorname{codim} X_k^+ = \operatorname{codim} R(P_k) = m$ and hence, (3.1) holds for $\kappa \le k$. Then the claim $X_k^+ + X_k^- = X$ for $\underline{\kappa} \le k < \kappa$ follows as in part (a) of the proof.

By virtue of the decomposition (3.1) we can define complementary projections $\bar{P}_k, \bar{Q}_k \in L(X)$ such that $R(\bar{P}_k) = X_k^+$ and $R(\bar{Q}_k) = X_k^-$ for $\underline{\kappa} \le k$. The invariance of X_k^+, X_k^- ensures that also $\bar{P}_k$ and $\bar{Q}_k$ are invariant w.r.t. (Δ). For $\kappa \le k$ it is $R(P_k) = R(\bar{P}_k)$, thus $P_k\bar{P}_k = \bar{P}_k$, $\bar{P}_kP_k = P_k$ and consequently

$$\begin{aligned}\bar{P}_k &= \bar{P}_kP_k + \bar{P}_kQ_k = P_k + \bar{P}_k\Phi(k,\kappa)\bar{\Phi}(\kappa,k)Q_k \\ &= P_k + \Phi(k,\kappa)\bar{P}_\kappa\bar{\Phi}(\kappa,k)Q_k = P_k + \Phi(k,\kappa)P_\kappa\bar{P}_\kappa\bar{\Phi}(\kappa,k)Q_k.\end{aligned}$$

Since our dichotomy estimates imply the exponential limit relations

$$\lim_{k\to\infty}\Phi(k,\kappa)P_\kappa = 0, \qquad \lim_{k\to\infty}\bar{\Phi}(\kappa,k)Q_k = 0,$$

also the differences $\bar{P}_k - P_k$ and $Q_k - \bar{Q}_k$ decay to 0 exponentially as $k \to \infty$. Thus, there exists a real $M \geq 0$ such that

$$\max\left\{\left|\bar{P}_k\right|, \left|\bar{Q}_k\right|\right\} \leq M \quad \text{for all } \kappa \leq k.$$

This enables us to establish the first dichotomy estimate

$$\left|\Phi(k,l)\bar{P}_l\right| = \left|\Phi(k,l)P_l\bar{P}_l\right| \leq KM\alpha^{k-l} \quad \text{for all } \kappa \leq l \leq k,$$

while the second one is more involved: Due to $\bar{Q}_k = \bar{Q}_k Q_k + \bar{Q}_k P_k = \bar{Q}_k Q_k$ we have

$$\begin{aligned}\Phi(k,l)\left[\bar{Q}_l\bar{\Phi}(l,k)Q_k\bar{Q}_k\right] &= \bar{Q}_k\Phi(k,l)\bar{\Phi}(l,k)Q_k\bar{Q}_k = \bar{Q}_kQ_k\bar{Q}_k \\ &= \bar{Q}_k = \Phi(k,l)\bar{\Phi}(l,k)\bar{Q}_k \quad \text{for all } \kappa \leq l \leq k.\end{aligned}$$

Since $\Phi(k,l)$ is an isomorphism from $R(\bar{Q}_l)$ onto $R(\bar{Q}_k)$, we can deduce

$$\bar{\Phi}(l,k)\bar{Q}_k = \bar{Q}_k\bar{\Phi}(l,k)Q_k\bar{Q}_k$$

and obtain the second dichotomy estimate

$$\left|\Phi(l,k)\bar{Q}_k\right| = \left|\bar{Q}_l\Phi(l,k)Q_k\bar{Q}_k\right| \leq M^2K\alpha^{k-l} \quad \text{for all } \kappa \leq l \leq k.$$

It remains to verify an ED on the finite set $\{\underline{\kappa}, \dots, \kappa\}$. This, however, follows since $\bar{Q}_k$ is regular on the above interval and the claim results by combining the results on both intervals. □

The subsequent corollary corresponds to [Pöt10a, p. 141, Cor. 3.4.23].

Corollary 3.5. *The dichotomy projector P_k of* (Δ) *on* $\mathbb{I}$ *and the dichotomy projector $\bar{P}_k$ on the extended discrete intervals from Prop.* 3.3 *converge to each other exponentially in the corresponding time direction.*

Proof. See the above proof of Prop. 3.3. □

The following result provides sufficient conditions such that the dichotomy spectrum is preserved when passing over to a larger interval:

Proposition 3.6. *Let $\overline{\kappa}, \underline{\kappa} \in \mathbb{I}$ with $\underline{\kappa} < \kappa < \overline{\kappa}$.*

(a) *If $\mathbb{I} = \mathbb{Z}_\kappa^-$ and (Δ) is eventually compact with $A_{\overline{\kappa}-1}, \dots, A_\kappa \in L(X)$ being one-to-one, then $\Sigma_{\overline{\kappa}}^-(A) = \Sigma_\kappa^-(A)$.*

(b) *If $\mathbb{I} = \mathbb{Z}_\kappa^+$ and for all closed subspaces $Y \subset X$ with $m = \operatorname{codim} Y < \infty$ also $\operatorname{codim}\Phi(\kappa, \bar{\kappa})^{-1}Y = m$ holds, then $\Sigma_{\underline{\kappa}}^+(A) = \Sigma_\kappa^+(A)$.*

Proof. (a) Let $\kappa < \overline{\kappa}$. For every fixed $\gamma \notin \Sigma_\kappa^-(A)$ we obtain that (Δ_γ) has an ED on $\mathbb{Z}_\kappa^-$ with projector P_k. By Rem. 3.2(1) we have $\dim N(P_\kappa) < \infty$ and thus $\Phi(\overline{\kappa}, \kappa)N(P_\kappa)$ is complemented with $\Phi(k,\kappa)N(P_\kappa)$ being closed. Our injectivity assumption ensures $N(\Phi(\overline{\kappa}, \kappa)) = \{0\}$ and so Prop. 3.3(a) can be employed, which yields an ED of (Δ_γ) on $\mathbb{Z}_{\overline{\kappa}}^-$, i.e. $\gamma \notin \Sigma_{\overline{\kappa}}^-(A)$. Therefore, we have the inclusion $\Sigma_{\overline{\kappa}}^-(A) \subset \Sigma_\kappa^-(A)$ and the claim follows with (2.9).

(b) Our assumption yields that $\Phi(\kappa, \underline{\kappa})^{-1}R(P_\kappa)$ and $R(P_\kappa)$ have the same finite codimension. Hence, Prop. 3.3(b) ensures an ED of (Δ_γ) on $\mathbb{Z}_{\underline{\kappa}}^+$, which means $\gamma \notin \Sigma_{\underline{\kappa}}^+(A)$. As above, the assertion results with (2.9). □

3.1. Multiplication operators

We suppose that $(C_k)_{k\in\mathbb{I}}$ is a sequence of bounded operators $C_k \in L(Y,X)$ satisfying $\sup_{k\in\mathbb{I}} |C_k| < \infty$. This ensures that the multiplication operator

$$(M_C\phi)_k := C_k\phi_k \quad \text{for all } k \in \mathbb{I}$$

satisfies the inclusion $M_C \in L(\ell(\mathbb{I},Y),\ell(\mathbb{I},X))$, where ℓ stands for one of the symbols ℓ_p, $1 \le p \le \infty$. Under the additional assumptions $C_k \in GL(X,Y)$ with $\sup_{k\in\mathbb{I}} \left|C_k^{-1}\right| < \infty$ one even has the inclusion $M_C \in GL(\ell(\mathbb{I},Y),\ell(\mathbb{I},X))$.

Lemma 3.7. *Let $\mathbb{I}$ be unbounded and $(C_k)_{k\in\mathbb{I}}$ be a sequence of compact operators $C_k \in L(Y,X)$. If*

$$\lim_{\substack{|k|\to\infty \\ k\in\mathbb{I}}} |C_k| = 0,$$

then the multiplication operator $M_C \in L(\ell_\infty(\mathbb{I},Y),\ell_\infty(\mathbb{I},X))$ is compact.

Proof. We define the operators $M_C^n \in L(\ell_\infty(\mathbb{I},Y),\ell_\infty(\mathbb{I},X))$ by

$$(M_C^n\phi)_k := \begin{cases} C_k\phi_k, & |k| \le n, \\ 0, & |k| > n \end{cases}$$

and obtain that each M_C^n is compact. Due to the limit relation

$$\|M_C - M_C^n\| \le \sup_{|k|>n} |C_k| \xrightarrow[n\to\infty]{} 0$$

also M_C is compact (cf. [Yos80, p. 278, Thm. (iii)]). □

3.2. Shift operators

For the discrete half line $\mathbb{I} = \mathbb{Z}_\kappa^+$, we can introduce the weighed shift

$$(T_A^+\phi)_k := \begin{cases} 0, & k = \kappa, \\ A_{k-1}\phi_{k-1}, & k > \kappa, \end{cases}$$

while in case $\mathbb{I} = \mathbb{Z}_\kappa^-$ we work with the unilateral weighted shift operator

$$(T_A^-\phi)_k := A_{k-1}\phi_{k-1}.$$

Both are defined on one of the sequence spaces $\ell = \ell_p(\mathbb{I},X)$, $1 \le p \le \infty$; for bounded forward growth (2.10) they fulfill the inclusion $T_A^\pm \in L(\ell)$. In particular, the spectrum $\sigma(T_A^\pm) \subset \mathbb{C}$ is rotationally invariant w.r.t. 0 and independent of the choice for ℓ. This is shown in [AM96, Thm. 1] for $\mathbb{I} = \mathbb{Z}$, but the proof carries over to arbitrary unbounded discrete intervals $\mathbb{I}$. Indeed, thanks to [AM96, Thm. 5] and [AMZ94, Thm. 1(i)] the spectrum and spectral radius r of T_A^+ are related to the *upper Bohl exponent* of (Δ) as follows

$$r(T_A^+) = \lim_{n\to\infty} \sup_{k\in\mathbb{Z}_\kappa^+} \sqrt[n]{|\Phi(k+n,k)|}.$$

Let us introduce further linear operators $S_\lambda^+, S_\lambda^- : \ell \to \ell$,

$$(S_\lambda^+\phi)_k := \phi_{k+1} - \tfrac{1}{\lambda}A_k\phi_k, \qquad \text{if } \mathbb{I} = \mathbb{Z}_\kappa^+,$$
$$(S_\lambda^-\phi)_k := \phi_k - \tfrac{1}{\lambda}A_{k-1}\phi_{k-1}, \qquad \text{if } \mathbb{I} = \mathbb{Z}_\kappa^-$$

for complex $\lambda \neq 0$. Clearly, S_λ^+ and S_λ^- are well-defined and bounded under our bounded forward growth assumption (2.10). The operators S_1^+ and S_1^- frequently occur in the literature (see, e.g. [Bas00, Lemma 2, 3]), since their left resp. right inverse is the solution operator to the inhomogeneous equation

$$\boxed{x_{k+1} = A_k x_k + b_k,} \tag{3.2}$$

where $(b_k)_{k\in\mathbb{I}'}$ is an X-valued sequence in ℓ. This is motivated by

Proposition 3.8. *Suppose that* (2.10) *holds. A difference eqn.* (Δ) *has an ED on* $\mathbb{Z}_\kappa^\pm$, *if and only if* $S_1^\pm \in L(\ell_\infty)$ *is onto and* $\mathcal{V}^\pm(\kappa)$ *is complemented.*

Remark 3.9. (1) In a Hilbert space X it is sufficient to assume the closedness of $\mathcal{V}^\pm(\kappa)$, a void assumption in finite-dimensional spaces X.

(2) The same characterization holds, if $S_1^\pm$ is defined on the spaces ℓ_p, $p \geq 1$ (see [HH06, Thm. 3.2]), and $\mathcal{V}^\pm(\kappa)$ is replaced by the ℓ_p-stable space

$$\left\{ x \in X : \sum_{k\in\mathbb{I}} |\Phi(k,\kappa)x|^p < \infty \right\}.$$

Under (2.11), the special case $p = 2$, $\dim X < \infty$ is due to [BAG93, Thm. 3.1].

Proof. We apply [HM01, Thm. 3.2] for the time axis $\mathbb{Z}_\kappa^+$, whereas the claim in case $\mathbb{Z}_\kappa^-$ follows analogously. □

Next we show that $S_1^\pm$ being Fredholm implies an ED on the corresponding semiaxis $\mathbb{Z}_\kappa^\pm$. This requires some preliminaries:

Lemma 3.10. *If* $b \in \ell_{00}$ *and* (2.1) *hold, then* (3.2) *has a solution in* ℓ_{00}.

Proof. First of all, we consider the case $\mathbb{I} = \mathbb{Z}_\kappa^+$. Using the variation of constants formula (cf. [Pöt10a, p. 100, Thm. 3.1.16(a)]) every forward solution ϕ of eqn. (3.2) allows the representation

$$\phi_k = \Phi(k,\kappa)\phi_\kappa + \sum_{n=\kappa}^{k-1} \Phi(k,n+1)b_n \quad \text{for all } \kappa \leq k.$$

Since the inhomogeneity b has finite support, there exists an integer $K \geq \kappa$ with $b_k = 0$ for all $k \geq K$ and we see

$$\phi_k = \Phi(k,\kappa)\phi_\kappa + \sum_{n=\kappa}^{\infty} \Phi(k,n+1)b_n \quad \text{for all } K \leq k.$$

Thus, ϕ is eventually zero if and only if $\phi_\kappa = -\sum_{n=\kappa}^{\infty} \Phi(\kappa,n+1)b_n$.

In the dual situation $\mathbb{I} = \mathbb{Z}_\kappa^-$ one employs [Pöt10a, p. 100, Rem. 3.1.17] in order to see that a backward solution ϕ of (3.2) is eventually zero if and only if $\phi_\kappa = \sum_{n=-\infty}^{\kappa-1} \Phi(\kappa,n+1)b_n$. □

For our subsequent considerations we need the *dual difference equation*

$$\boxed{x_k = A'_{k+1}x_{k+1}} \tag{Δ'}$$

with variables x_k in the dual space X'. It has uniquely determined backward solutions and its corresponding transition operator Φ' satisfies

$$\Phi'(k,l) = \Phi(l+1,k+1)' \quad \text{for all } k \le l.$$

Since $S_1^\pm : \ell_\infty \to \ell_\infty$ is continuous, the pre-image $E := (S_1^\pm)^{-1}\ell_0$ is a closed subspace of ℓ_∞ and we define $T := S_1^\pm|_E \in L(E,\ell_0)$. The following result allows us to characterize the kernel of the dual operator $T' : \ell_0' \to E'$.

Lemma 3.11. *If* (2.1) *holds, then* $N(T') = \{0\}$.

Proof. Given an arbitrary $b \in \ell_{00}$, our Lemma 3.10 implies $b \in R(S_1^\pm)$ and thus $b \in R(T)$. This means there exists a $\phi \in E$ with $b = T\phi$ and therefore

$$\mu(b) = \mu(T\phi) = (T'\mu)(\phi) = 0 \quad \text{for all } \mu \in N(T'). \tag{3.3}$$

We deduce $\mu(b) \equiv 0$ on ℓ_{00} and since $\ell_{00} \subset \ell_0$ is dense, $\mu(b) \equiv 0$ on ℓ_0. □

At this point we arrive at a discrete version of the results from [Pal88].

Proposition 3.12. *Suppose* (2.1) *and* (2.10) *hold. If* $S_1^\pm \in L(\ell_\infty)$ *is semi-Fredholm, then a difference eqn.* (Δ) *has an ED on the corresponding semiaxis.*

Remark 3.13. (1) In combination with Prop. 3.8, a semi-Fredholm operator $S_1^\pm$ is in fact onto and therefore satisfies $\operatorname{codim} R(S_1^\pm) = 0$.

(2) For $\dim N(S_1^\pm) < \infty$ the closed subspace $N(S_1^\pm)$ is complemented (see [Mül07, p. 398, Thm. A.1.25(i)]). Indeed, if $\Pi \in L(X)$ denotes a projection onto $\mathcal{V}^\pm(\kappa)$, then $\{\phi \in \ell_\infty : \Pi\phi_\kappa = 0\}$ is a complement. From (1) we can conclude that $S_1^\pm$ is semi-Fredholm, if and only if it is Fredholm.

(3) Let P_k denote an invariant projector associated to the ED from Prop. 3.12. By definition, the kernel $N(S_1^\pm)$ consists of the bounded solutions to eqn. (Δ) and in case $\dim X < \infty$ we obtain from Prop. 3.1:

- For $\mathbb{I} = \mathbb{Z}_\kappa^+$ one has $N(S_1^+) \cong \mathcal{V}^+(\kappa) = R(P_\kappa)$ and $\dim N(S_1^+) < \infty$.
- For $\mathbb{I} = \mathbb{Z}_\kappa^-$ it is $N(S_1^-) \cong \mathcal{V}^-(\kappa) = N(P_\kappa)$ and $\dim N(S_1^-) < \infty$.

Proof. For $\mathbb{I} = \mathbb{Z}_\kappa^+$ the semi-Fredholm property of S_1^+ shows that $R(S_1^+)$ is closed. Hence, also the range $R(T)$ is closed and [Kat80, p. 24, (3.37)] gives

$$R(T)^\perp = \{\mu \in \ell_0' : \mu(\phi) = 0 \quad \text{for all } \phi \in R(T)\} = N(T').$$

Thanks to Lemma 3.11(a) it is $N(T') = \{0\}$ and by the Hahn-Banach theorem we obtain $R(T) = \ell_0$. This means that for every inhomogeneity $b \in \ell_0$ there exists a solution $\phi \in \ell_\infty$ of (3.2) and [HM01, Thm. 3.2] (this reference gives the proof only for ℓ_∞, but the interested reader might verify that the arguments also hold for the space ℓ_0) implies that (Δ) has an ED on $\mathbb{Z}_\kappa^+$. The proof for the negative semiaxis $\mathbb{I} = \mathbb{Z}_\kappa^-$ is similar. □

3.3. Dichotomy spectra

Up to this point we collected technical preliminaries to analyze the dichotomy spectrum for difference eqns. (Δ) on semiaxes. Now the main observation of this section is the fact that the dichotomy spectrum on semiaxes is essential spectrum; we will see that the (point) spectrum of $T_A^\pm$ is of minor importance. In order to show this, the concept of a regularity (cf. [Mül07, pp. 51ff]) allows a unified treatment of the spectra $\sigma_s(T_A^\pm)$ and $\sigma_F(T_A^\pm)$. Here, we often (and particularly in proofs) abbreviate $\sigma = \sigma(T_A^\pm)$ and proceed accordingly with the other (dichotomy) spectra.

First, the mapping $\gamma \mapsto S_\gamma^-$ between $(0,\infty)$ and $L(\ell)$ is analytic. As in case of the usual spectrum, for $\mathbb{I} = \mathbb{Z}_\kappa^-$ and $\sigma(T_A^-)$ having a positive distance from the origin in $\mathbb{C}$, one can apply an ambient spectral mapping theorem (see [Kim02, Thm. 1.2]) to deduce

$$\sigma_s(S_\gamma^-) = 1 - \tfrac{1}{\gamma}\sigma_s(T_A^-), \qquad \sigma_s(T_A^-) = \gamma - \gamma\sigma_s(S_\gamma^-) \quad \text{for all } \gamma > 0.$$

The *surjectivity spectrum* of (Δ) read as

$$\Sigma_s^\pm(A) := \left\{\gamma > 0 : \, S_\gamma^\pm \text{ is not onto}\right\}$$

and we will study its properties. Thereto, for later reference, we observe

Lemma 3.14. *The set $L_s(X)$ is open in $L(X)$.*

Proof. If a mapping $T \in L(X)$ is onto, then its surjectivity modulus satisfies $k(T) > 0$ (cf. [Mül07, p. 86, Thm. 9.4]). Thus, since $k : L(X) \to \mathbb{R}$ is continuous (cf. [Mül07, p. 88, Prop. 9.9]), there is a neighborhood $U \subset L(X)$ of T on which k remains positive and so every $S \in U$ is onto. □

Lemma 3.15. *Suppose that* (2.10) *holds. If $\mathbb{I} = \mathbb{Z}_\kappa^-$ and* $\dim X < \infty$*, then $\sigma_s(T_A^-)$ is rotationally invariant w.r.t.* 0 *and compact.*

Proof. Choose $\ell = \ell_\infty(\mathbb{Z}_\kappa^-, X)$. Because subspaces of finite-dimensional spaces are complemented (cf. [Mül07, p. 398, Thm. A.1.25(i)]), we can obtain from Prop. 3.8 for $\lambda \neq 0$ that

$$\begin{aligned}
\lambda \notin \sigma_s \quad &\Leftrightarrow \quad T_A^- - \lambda\,\mathrm{id} \in L(\ell) \text{ is onto}\\
&\Leftrightarrow \quad \forall \psi \in \ell : \, \exists \phi \in \ell : \, T_A^-\phi - \lambda\phi = \psi\\
&\Leftrightarrow \quad \forall \psi \in \ell : \, x_k = \tfrac{1}{\lambda}A_{k-1}x_{k-1} + \tfrac{1}{\lambda}\psi_k \text{ has a solution in } \ell\\
&\Leftrightarrow \quad x_{k+1} = \tfrac{1}{\lambda}A_k x_k \text{ has an ED on } \mathbb{Z}_\kappa^-\\
&\Leftrightarrow \quad x_{k+1} = \tfrac{e^{-i\mu}}{\lambda}A_k x_k \text{ has an ED on } \mathbb{Z}_\kappa^- \quad \Leftrightarrow \quad e^{i\mu}\lambda \notin \sigma_s
\end{aligned}$$

for $\mu \in \mathbb{R}$, and so $\sigma_s = e^{i\mu}\sigma_s$. Since $L_s(\ell)$ is a regularity, due to Lemma 3.14 the compactness of σ_s follows from [Mül07, p. 55, Prop. 6.9]. □

Theorem 3.16. *Suppose that* (2.10) *holds and* $\dim X < \infty$*.*

(a) *If $\mathbb{I} = \mathbb{Z}_\kappa^+$, then $\Sigma_s^+(A) = \Sigma_\kappa^+(A)$.*
(b) *If $\mathbb{I} = \mathbb{Z}_\kappa^-$, then $\Sigma_s^-(A) = \Sigma_\kappa^-(A) = \sigma_s(T_A^-) \cap (0,\infty)$.*

Proof. First, in a finite-dimensional space X the fibers $\mathcal{V}^{\pm}(\kappa)$ are complemented. Given $\gamma > 0$ thanks to Prop. 3.8 the following is equivalent,

$$\gamma \notin \Sigma_s^{\pm} \Leftrightarrow S_\gamma^{\pm} \in L(\ell) \text{ is onto} \Leftrightarrow \Phi_\gamma \text{ has an ED on } \mathbb{Z}_\kappa^{\pm} \Leftrightarrow \gamma \notin \Sigma_\kappa^{\pm},$$

which yields the claimed relations $\Sigma_s^{\pm} = \Sigma_\kappa^{\pm}$. For the remaining assertion in (b), the proof of Lemma 3.15 together with Prop. 3.8 and Lemma 2.7 guarantees for $\lambda \neq 0$ that

$$\begin{aligned} \lambda \notin \sigma_s \quad &\Leftrightarrow \quad x_{k+1} = \tfrac{1}{\lambda} A_k x_k \text{ has an ED on } \mathbb{Z}_\kappa^- \\ &\Leftrightarrow \quad x_{k+1} = |\lambda|^{-1} A_k x_k \text{ has an ED on } \mathbb{Z}_\kappa^- \quad \Leftrightarrow \quad |\lambda| \notin \Sigma_\kappa^-. \end{aligned}$$

Hence, $\lambda \in \sigma_s$ if and only if $|\lambda| \in \Sigma_\kappa^-$. From Lemma 3.15 we see that the surjectivity spectrum σ_s is rotationally symmetric; the assertion holds. □

Next we introduce the *Fredholm spectrum*

$$\Sigma_F^{\pm}(A) := \left\{\gamma > 0 : S_\gamma^{\pm} \text{ is not Fredholm}\right\}$$

and obtain the following useful relation. It states that the dichotomy spectrum on semiaxes is actually essential spectrum:

Proposition 3.17. *If* (2.10) *holds and* $\dim X < \infty$*, then*

(a) $\Sigma_F^{\pm}(A) \subset \Sigma_s^{\pm}(A) = \Sigma_\kappa^{\pm}(A)$,
(b) $\Sigma_F^{\pm}(A) = \Sigma_s^{\pm}(A) = \Sigma_\kappa^{\pm}(A)$ *under* (2.1).

Remark 3.18. In [BAG93, Cor. 3.4] it is shown that $\Sigma_\kappa^{+}(A) = \sigma_F(T_A^+) \cap (0, \infty)$ holds for T_A^+ defined on ℓ_2 and $\dim X < \infty$.

Proof. Referring to Thm. 3.16 one has $\Sigma_s^{\pm} = \Sigma_\kappa^{\pm}$.

(a) For $\gamma \notin \Sigma_s^{\pm}$ the operator $S_\gamma^{\pm}$ is onto. Due to $\dim X < \infty$ its kernel must be finite-dimensional and thus $S_\gamma^{\pm}$ is Fredholm, i.e. $\gamma \notin \Sigma_F^{\pm}$.

(b) Let $\gamma \notin \Sigma_F^{\pm}$. Thanks to (2.1) and Prop. 3.12 the scaled difference eqn. (Δ_γ) admits an ED on $\mathbb{Z}_\kappa^{\pm}$. Therefore, Prop. 3.8 ensures that $S_\gamma^{\pm}$ is onto and this means $\gamma \notin \Sigma_s^{\pm}$. The claim follows with assertion (a). □

The assumption (2.10) allows us to introduce the concept of upper-semicontinuity for various dynamical spectral notions $\Sigma(A)$. This is understood in terms of a limit relation $\lim_{B \to A} \operatorname{dist}(\Sigma(B), \Sigma(A)) = 0$, where

- on the space of all linear eqns. (Δ) satisfying (2.10) we use the norm
$$\|A\| := \sup_{k \in \mathbb{I}} |A_k|$$
- $\operatorname{dist}(\Sigma_1, \Sigma_2) := \sup_{x_1 \in \Sigma_1} \inf_{x_2 \in \Sigma_2} |x_1 - x_2|$ for subsets $\Sigma_1, \Sigma_2 \subset \mathbb{R}$ denotes their *Hausdorff-semidistance.*

Corollary 3.19. *For* $\mathbb{I} = \mathbb{Z}_\kappa^-$ *the spectra* $\Sigma_s^-(A) = \Sigma_\kappa^-(A)$ *fulfill*

(a) $\Sigma_\kappa^-(A) \cup \{0\}$ *is compact,*
(b) $\Sigma_\kappa^-(A)$ *is upper-semicontinuous.*

Proof. Referring to [Mül07, p. 233, C.22.2] we know that $\sigma_s \neq \emptyset$ is compact and satisfies the inclusions $\partial\sigma \subset \sigma_s \subset \sigma$. Moreover, the upper-semicontinuity of σ_s follows from Lemma 3.14 and [Mül07, p. 55, Prop. 6.9]. These properties carry over to Σ_s^-, since Thm. 3.16(b) shows $\Sigma_s^- = \sigma_s \cap (0,\infty)$ and therefore Prop. 3.17(a) implies the assertions. ☐

Using the Fredholm spectrum $\Sigma_F^{\pm}(A)$ it is possible to obtain a counterpart to Cor. 3.19 for the forward dichotomy spectrum. It requires

Lemma 3.20. *The sets $L_F(X)$ and $L_{F_0}(X)$ are open in $L(X)$.*

Proof. This follows from [Mül07, p. 158, Thm. 16.11]. ☐

Lemma 3.21. *Suppose that* (2.1) *and* (2.10) *hold. If $\mathbb{I} = \mathbb{Z}_\kappa^+$ and* $\dim X < \infty$*, then $\sigma_F(T_A^+)$ is rotationally invariant w.r.t.* 0 *and compact.*

Proof. We choose $\ell = \ell_2(\mathbb{Z}_\kappa^+, X)$. This allows us to apply [BAG92, Thm. 1.1] in order to deduce the equivalence

$$\begin{aligned} \lambda \notin \sigma_F \quad &\Leftrightarrow \quad T_A^+ - \lambda\,\mathrm{id} \in L(\ell) \text{ is Fredholm} \\ &\Leftrightarrow \quad x_{k+1} = \tfrac{1}{\lambda} A_k x_k \text{ has an ED on } \mathbb{Z}_\kappa^+ \\ &\Leftrightarrow \quad x_{k+1} = \tfrac{e^{-i\mu}}{\lambda} A_k x_k \text{ has an ED on } \mathbb{Z}_\kappa^+ \quad \Leftrightarrow \quad e^{i\mu}\lambda \notin \sigma_F \end{aligned}$$

for all $\mu \in \mathbb{R}$, which implies $\sigma_F = e^{i\mu}\sigma_F$. Moreover, $L_F(\ell)$ is a regularity and Lemma 3.20 with [Mül07, p. 55, Prop. 6.9] ensures that σ_F is compact. ☐

Theorem 3.22. *Suppose that* (2.1), (2.10) *hold and* $\dim X < \infty$*. If $\mathbb{I} = \mathbb{Z}_\kappa^+$, then $\Sigma_\kappa^+(A) = \sigma_F(T_A^+) \cap (0,\infty)$.*

Remark 3.23 (kinematic similarity). If two difference eqns. (Δ) and (2.2) are kinematically similar by virtue of the Lyapunov transform $(C_k)_{k\in\mathbb{Z}_\kappa^\pm}$, then

$$M_C T_B^{\pm} = T_A^{\pm} M_C$$

with an invertible multiplication operator M_C. Hence, $T_A^{\pm}$ and $T_B^{\pm}$ have the same surjectivity spectrum σ_s, as well as Fredholm spectrum σ_F. From Thm. 3.22 and Thm. 3.16(b) we can therefore conclude that the dichotomy spectra $\Sigma_\kappa^{\pm}$ are invariant under kinematic similarity.

Proof. Again, suppose T_A^+ is defined on $\ell_2 = \ell_2(\mathbb{Z}_\kappa^+, X)$. As in the proof of Lemma 3.21 we obtain using Lemma 2.7 that

$$\begin{aligned} \lambda \notin \sigma_F \quad &\Leftrightarrow \quad x_{k+1} = \tfrac{1}{\lambda} A_k x_k \text{ has an ED on } \mathbb{Z}_\kappa^+ \\ &\Leftrightarrow \quad x_{k+1} = |\lambda|^{-1} A_k x_k \text{ has an ED on } \mathbb{Z}_\kappa^+ \quad \Leftrightarrow \quad |\lambda| \notin \Sigma_\kappa^+ \end{aligned}$$

holds for all $\lambda \neq 0$. Hence, $\lambda \in \sigma_F$ if and only if $|\lambda| \in \Sigma_\kappa^+$. Refereeing to Lemma 3.21, σ_F is rotationally symmetric and the assertion holds. ☐

Corollary 3.24. *For $\mathbb{I} = \mathbb{Z}_\kappa^+$ the spectra $\Sigma_s^+(A) = \Sigma_\kappa^+(A)$ fulfill*

(a) $\Sigma_\kappa^+(A) \cup \{0\}$ *is compact,*
(b) $\Sigma_\kappa^+(A)$ *is upper-semicontinuous.*

Proof. Thanks to [Mül07, p. 55, Prop. 6.9] and Lemma 3.20 the Fredholm spectrum $\sigma_F(T_A^+)$ is compact and upper-semicontinuous. By Thm. 3.22 we have $\Sigma_F^+ = \sigma(T_A^+) \cap (0,\infty)$ and then these properties carry over to Σ_κ^+. $\square$

For scalar difference equations

$$\boxed{x_{k+1} = a_k x_k} \tag{S}$$

with a coefficient sequence $(a_k)_{k\in\mathbb{I}}$ of nonzero complex numbers it is possible to compute dichotomy spectra explicitly. Provided both $(a_k)_{k\in\mathbb{I}}$ and $(a_k^{-1})_{k\in\mathbb{I}}$ are bounded, then due to [BAG91, p. 660] one has

$$\Sigma_\kappa^\pm(a) = \left[\beta^-_{\mathbb{Z}_\kappa^\pm}(a), \beta^+_{\mathbb{Z}_\kappa^\pm}(a)\right] \quad \text{for all } \kappa \in \mathbb{Z}. \tag{3.4}$$

The boundary points of this interval are *Bohl exponents* for $(a_k)_{k\in\mathbb{I}}$ given by

$$\beta_{\mathbb{I}}^-(a) := \lim_{n\to\infty} \inf_{k\in\mathbb{I}} \sqrt[n]{\prod_{i=k}^{k+n-1} |a_i|}, \quad \beta_{\mathbb{I}}^+(a) := \lim_{n\to\infty} \sup_{k\in\mathbb{I}} \sqrt[n]{\prod_{i=k}^{k+n-1} |a_i|}. \tag{3.5}$$

Then the dichotomy spectrum of an upper-triangular system (Δ) allows an explicit representation given by the diagonal elements:

Corollary 3.25. *If every $A_k \in L(\mathbb{K}^d)$ is upper-triangular with bounded diagonal sequences $(a_k^n)_{k\in\mathbb{Z}_\kappa^+}$, $n = 1,\dots,d$, in $\mathbb{C}\setminus\{0\}$, then*

$$\Sigma_\kappa^+(A) = \bigcup_{n=1}^d \Sigma_\kappa^+(a^n).$$

If also each $\left(\frac{1}{a_k^n}\right)_{k\in\mathbb{Z}_\kappa^+}$ is bounded, then $\Sigma_\kappa^+(a^n) = \left[\beta^-_{\mathbb{Z}_\kappa^+}(a^n), \beta^+_{\mathbb{Z}_\kappa^+}(a^n)\right]$.

Proof. We refer to [LJS01, Thm. 2.2] for the relation $\sigma_F(T_A^+) = \bigcup_{n=1}^d \sigma_F(T_{a^n}^+)$ and the claim follows with Thm. 3.22. $\square$

The following conclusion addresses perturbed difference equations

$$\boxed{x_{k+1} = [A_k + B_k]x_k} \tag{P}$$

with an operator sequence $B_k \in L(X)$, $k \in \mathbb{I}$.

Corollary 3.26 (ℓ_0-roughness of EDs). *Let (P) fulfill (2.1). If all the operators $B_k \in L(X)$, $k \in \mathbb{Z}_\kappa^\pm$, are compact and satisfy $\lim_{k\to\pm\infty}|B_k| = 0$ in the respective time direction, then $\Sigma_\kappa^\pm(A) = \Sigma_\kappa^\pm(A+B)$.*

Proof. Due to Lemma 3.7 the operator $M_B \in L(\ell_\infty)$ is compact.

(I) We begin with the case $\mathbb{I} = \mathbb{Z}_\kappa^+$ and show two inclusions:

($\subset$) Let $\gamma \notin \Sigma_F^+(A)$. This means $S_\gamma^+ \in L(\ell_\infty)$ is Fredholm and by [Mül07, p. 158, Thm. 16.9(iii)] also $S_\gamma^+ + \frac{1}{\gamma}M_B$ is Fredholm, i.e. $\gamma \notin \Sigma_F^+(A+B)$.

($\supset$) Conversely, for $\gamma \notin \Sigma_F^+(A+B)$ the operator $S_\gamma^+ + \frac{1}{\gamma}M_B$ is Fredholm, and the compactness of $-M_B$ ensures that also $S_\gamma^+ = (S_\gamma^+ + \frac{1}{\gamma}M_B) - \frac{1}{\gamma}M_B$ is Fredholm, i.e. $\gamma \notin \Sigma_F^+(A)$.

(II) For $\mathbb{I} = \mathbb{Z}_\kappa^-$ we define the backward shift operator $S \in L(\ell_\infty)$ given by $(S\phi)_k := \phi_{k-1}$ and observe that also the composition $T_B = SM_B$ is compact (cf. [Mül07, p. 149, Thm. 15.2(iii)]). Then $\Sigma_F^-(A) = \Sigma_F^-(A+B)$ can be deduced as above with M_B replaced by T_B.

Since (Δ) and (P) fulfill (2.1) the claim follows from Prop. 3.17(b). □

4. Equations on the whole axis

In this section, we investigate difference eqns. (Δ) defined on the whole discrete axis $\mathbb{I} = \mathbb{Z}$. This ensures that various results of the previous Sect. 3 apply, since an ED on $\mathbb{Z}$ trivially implies an ED on a semiaxis. For instance, Prop. 3.1 guarantees the Whitney sum decomposition $\mathbb{Z} \times X = \mathcal{V}^+ \oplus \mathcal{V}^-$.

Yet, the present situation is different, since an ED on $\mathbb{Z}$ is a significantly stronger assumption than an ED on a semiaxis. Consequently the dichotomy spectrum $\Sigma_{ED}(A)$ on $\mathbb{I} = \mathbb{Z}$ is typically larger than $\Sigma_\kappa^\pm(A)$. In particular, it does not need to coincide with the surjectivity or Fredholm spectra, which ensures a richer dynamical spectral theory.

We begin with the observation that for an ET on $\mathbb{Z}$ we can derive a geometrical characterization of the center bundle $\mathcal{V}^0$ (cf. Prop. 3.1):

Proposition 4.1. *If a difference eqn.* (Δ) *admits an ET on* $\mathbb{Z}$*, then*

$$\mathcal{V}^0(\kappa) = \left\{ \xi \in X : \begin{array}{l} \text{there exists a solution } \phi = (\phi_k)_{k\in\mathbb{Z}} \text{ to} \\ (\Delta) \text{ with } \phi_\kappa = \xi \text{ and } \lim_{k\to\pm\infty} \phi_k = 0 \end{array} \right\} = R(Q_\kappa)$$

and the convergence assertions are even exponential.

Remark 4.2. (1) If a difference eqn. (Δ) admits an ED on $\mathbb{Z}$, then the center bundle $\mathcal{V}^0$ becomes trivial.

(2) The center bundle $\mathcal{V}^0$ has finite dimension, if (Δ) is eventually compact (cf. [Pöt10a, p. 142, Prop. 3.4.24]).

Proof. Due to $\mathcal{V}^0(\kappa) = \mathcal{V}^+(\kappa) \cap \mathcal{V}^-(\kappa)$ the claim results from Prop. 3.1. □

The following result captures a situation dual to the above Prop. 2.4 and is a discrete infinite-dimensional counterpart to [Pal06, Lemma 1]:

Proposition 4.3. *Let* $\kappa \in \mathbb{Z}$ *and suppose* (Δ) *admits an ED on* $\mathbb{Z}_\kappa^+$ *with projector* P_k^+ *and an ED on* $\mathbb{Z}_\kappa^-$ *with* P_k^-.

(a) *If* (Δ) *has no nontrivial bounded solution on* $\mathbb{Z}$ *and* $R(P_\kappa^+) + N(P_\kappa^-)$ *is complemented, then*

$$P_\kappa^+ = P_\kappa^- P_\kappa^+ = P_\kappa^+ P_\kappa^-. \tag{4.1}$$

(b) *If* (4.1) *holds, then* (Δ) *has no nontrivial solution bounded on* $\mathbb{Z}$.

Remark 4.4. The condition (4.1) guarantees $R(P_\kappa^+) \cap N(P_\kappa^-) = \{0\}$. On the other hand, the subspace $R(P_\kappa^+) + N(P_\kappa^-) \subset X$ is closed and thus in a Hilbert space X always complemented.

Proof. Thanks to Prop. 3.1 we have the dynamical characterizations

$$R(P_\kappa^+) = \left\{\xi \in X : \sup_{k\geq\kappa} |\Phi(k,\kappa)\xi| < \infty\right\},$$

$$N(P_\kappa^-) = \left\{\xi \in X : \begin{array}{l} \text{there exists a bounded solution} \\ \phi = (\phi_k)_{k\in\mathbb{Z}_\kappa^-} \text{ of } (\Delta) \text{ with } \phi_\kappa = \xi \end{array}\right\}.$$

(a) Since eqn. (Δ) has no nontrivial bounded solution on $\mathbb{Z}$, we obtain $R(P_\kappa^+) \cap N(P_\kappa^-) = \{0\}$. Thus, by assumption $R(P_\kappa^+) \oplus N(P_\kappa^-)$ is complemented and one has $X = R(P_\kappa^+) \oplus N(P_\kappa^-) \oplus Y$ for a closed subspace $Y \subset X$. Because the kernel $N(P_\kappa^+)$ (resp. the range $R(P_\kappa^-)$) can be any closed subspace complementary to $R(P_\kappa^+)$ (resp. $N(P_\kappa^-)$), we may choose

$$N(P_\kappa^+) = Y \oplus N(P_\kappa^-), \qquad R(P_\kappa^-) = R(P_\kappa^+) \oplus Y.$$

This yields $N(P_\kappa^-) \subset N(P_\kappa^+)$, $R(P_\kappa^+) \subset R(P_\kappa^-)$ and hence (4.1) holds.

(b) If $\phi = (\phi_k)_{k\in\mathbb{Z}}$ is an entire bounded solution of (Δ), then we have $\phi_\kappa \in R(P_\kappa^+) \cap N(P_\kappa^-)$ and consequently

$$\phi_\kappa = P_\kappa^+\phi_\kappa \stackrel{(4.1)}{=} P_\kappa^+ P_\kappa^- \phi_\kappa = P_\kappa^+ 0 = 0.$$

Since ϕ is contained in $\mathcal{V}^-$, i.e. $\phi_k \in N(P_k^-)$ for all $k \leq \kappa$, one gets $\phi = 0$. ☐

4.1. The operator T_A

For a sequence space $\ell = \ell_p(\mathbb{Z}, X)$, $1 \leq p \leq \infty$, we introduce the bilateral weighted shift operator

$$T_A : \ell \to \ell, \qquad (T_A\phi)_k := A_{k-1}\phi_{k-1},$$

which fulfills the inclusion $T_A \in L(\ell)$ under bounded forward growth (2.10). Its spectrum $\sigma(T_A) \subset \mathbb{C}$ is rotationally invariant w.r.t. the origin in $\mathbb{C}$ and independent of the sequence space ℓ (cf. [AM96, Thm. 1]). For the spectrum and spectral radius r one again obtains the *upper Bohl exponent* (for this, see [AM96, Thm. 5] and [AMZ94, Thm. 1(i)])

$$r(T_A) = \lim_{n\to\infty} \sup_{k\in\mathbb{Z}} \sqrt[n]{|\Phi(k+n,k)|}.$$

Asymptotic properties of a difference eqn. (Δ) are closely connected to the shift operator T_A (see [BAG91, AM96, Pöt09]). Indeed, (1.1) relates its spectrum $\sigma(T_A)$ to the dichotomy spectrum $\Sigma_{ED}(A)$ of (Δ) (cf. [Pöt09, Thm. 1]). This enables us study the dichotomy spectrum by means of perturbation theory for linear operators (see, e.g. [Kat80]) applied to $T_A \in L(\ell)$. For instance, we obtain its upper-semicontinuity and rediscover Thm. 2.9 in

Theorem 4.5. *Suppose* $d := \dim X < \infty$ *and that* (2.10) *holds.*

(a) *If* A_k^{-1} *exists for all* $k \in \mathbb{Z}$ *with* (2.11), *then there exists an* $1 \leq n \leq d$ *and reals* $0 < \beta_{j-1} < \alpha_j \leq \beta_j$ *such that*

$$\sigma(T_A) = \bigcup_{j=1}^n \{z \in \mathbb{C} : \alpha_j \leq |z| \leq \beta_j\}.$$

(b) *If* A_k^{-1} *does not exist for some* $k \in \mathbb{Z}$ *or* (2.11) *is violated, then there exists an* $1 \leq n \leq d$ *and reals* $0 \leq \beta_{j-1} < \alpha_j \leq \beta_j$ *such that*

$$\sigma(T_A) = \{z \in \mathbb{C} : |z| \leq \beta_1\} \cup \bigcup_{j=2}^{n} \{z \in \mathbb{C} : \alpha_j \leq |z| \leq \beta_j\}.$$

Remark 4.6 (Bohl exponents). (1) The statement (a) in Thm. 4.5 is also shown in [BAG91, Thm. 0.3.1], with T_A being defined on ℓ_2.

(2) A counterpart to Thm. 4.5(a) for the Fredholm spectrum of T_A^+ (as an operator on ℓ_2) is due to [BAG91, Thm. 2.1.6].
In both cases, the moduli of boundary points to the intersection $\sigma(T_A) \cap (0, \infty)$ resp. $\sigma_F(T_A^+) \cap (0, \infty)$ are *Bohl exponents* for (Δ).

Proof. See [AM96, Thm. 4]. □

4.2. The operator S_γ

For $\ell = \ell_\infty(\mathbb{Z}, X)$ let us introduce another linear operator

$$S_\lambda : \ell \to \ell, \qquad (S_\lambda \phi)_k := \phi_{k+1} - \tfrac{1}{\lambda} A_k \phi_k \tag{4.2}$$

with a complex scaling parameter $\lambda \neq 0$. Clearly, S_λ is well-defined and continuous under our bounded forward growth assumption (2.10).

In particular, also the linear operator S_1 is often encountered in the literature (see e.g. [Hen81, p. 230, Thm. 7.6.5]), since its inverse is the solution operator to the inhomogeneous eqn. (3.2), provided $(b_k)_{k \in \mathbb{Z}}$ is a bounded sequence in X. This is motivated by

Proposition 4.7. *Suppose that* (2.10) *holds. A difference eqn.* (Δ) *has an*

(a) *ED on* $\mathbb{Z}$, *if and only if* $S_1 \in GL(\ell_\infty)$,
(b) *ET on* $\mathbb{Z}$, *if and only if* $S_1 \in L(\ell_\infty)$ *is onto.*

Proof. (a) See [Hen81, p. 230, Thm. 7.6.5].

(b) A proof can be found in [EJ98, Thm. 4] (or [Pap91, Prop. 1]) for finite-dimensional spaces X and invertible operators A_k. However, the interested reader might check that the arguments apply as well for general Banach spaces X and under the regularity condition on Q_k, $\mathrm{id} - P_k - Q_k$. □

Corollary 4.8. *Let* $\kappa \in \mathbb{Z}$. *A difference eqn.* (Δ) *has an ED on* $\mathbb{Z}$, *if and only if it admits EDs on* $\mathbb{Z}_\kappa^+$ *with* P_k^+ *and on* $\mathbb{Z}_\kappa^-$ *with* P_k^- *satisfying*

$$R(P_\kappa^+) \oplus N(P_\kappa^-) = X.$$

Proof. The direction $(\Rightarrow)$ is trivial. For the converse, see [Bas00, Cor. 2]. □

For a finite dimensional center bundle $\mathcal{V}^0$ the above Prop. 4.7(b) provides conditions that the operator S_1 is Fredholm (with index $\dim \mathcal{V}_0$). More general, sufficient criteria for S_1 to be Fredholm are given in

Proposition 4.9. *Let* $\kappa \in \mathbb{Z}$. *If* (Δ) *has EDs on* $\mathbb{Z}_\kappa^+$ *and* $\mathbb{Z}_\kappa^-$ *with respective Morse indices* ι_+ *and* ι_-, *then* $S_1 \in L(\ell_\infty)$ *is Fredholm with index* $\iota_- - \iota_+$.

Proof. See [Bas00, Thm. 8]. □

Our next objective is to show that also the converse of Prop. 4.9 holds, i.e. the fact that S_1 being Fredholm implies EDs on both semiaxes. This requires some preliminaries:

Lemma 4.10. *Let* $b \in \ell_{00}$ *and suppose* (2.1) *holds. Then an inhomogeneous eqn.* (3.2) *has a solution in* ℓ_{00}*, if and only if* $\sum_{n\in\mathbb{Z}} \Phi(\kappa, n+1)b_n = 0$.

Proof. On the axis $\mathbb{Z}$ we combine the results of Lemma 3.10 for $\mathbb{I} = \mathbb{Z}_\kappa^+$ and $\mathbb{I} = \mathbb{Z}_\kappa^-$ in order to see that an entire solution to (3.2) is in ℓ_{00}, if and only if

$$\sum_{n=-\infty}^{\kappa-1} \Phi(\kappa, n+1)b_n = \phi_\kappa = -\sum_{n=\kappa}^{\infty} \Phi(\kappa, n+1)b_n,$$

which means $\sum_{n\in\mathbb{Z}} \Phi(\kappa, n+1)b_n = 0$. □

With the continuous linear operator S_1 now given in (4.2), we introduce the closed subspace $E := S_1^{-1}\ell_0$ and the operator $T := S_1|_E \in L(E, \ell_0)$ in order to characterize the kernel of the dual operator $T' : \ell_0' \to E'$.

Lemma 4.11. *Let* $\dim X < \infty$*. If* (2.1) *holds, then the kernel* $N(T')$ *consists of all* $\mu \in \ell_0'$ *such there exists an entire solution* $\psi' \in \ell_1$ *to* (Δ') *such that*

$$\mu(b) = \sum_{n\in\mathbb{N}} \langle \psi_n', b_n \rangle \quad \text{for all } b \in \ell_0. \tag{4.3}$$

Proof. Since the state space X is assumed to be finite-dimensional, we identify it with $\mathbb{R}^d$ (for real X) or $\mathbb{C}^d$ (for complex spaces X) equipped with the canonical basis $e_1, \ldots, e_d$. We have to establish two inclusions:
($\subset$) Let $\mu \in N(T')$ and suppose $(\theta_k)_{k\in\mathbb{Z}}$ is a real sequence in ℓ_{00} with positive values and $\sum_{k\in\mathbb{Z}} \theta_k = 1$. Given a sequence $b \in \ell_{00}$ we define

$$\tilde{b}_k := b_k - \theta_k \sum_{n\in\mathbb{N}} \Phi(k+1, n+1)b_n \quad \text{for all } k \in \mathbb{Z}. \tag{4.4}$$

At once we observe $\tilde{b} \in \ell_{00}$ and due to

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \Phi(\kappa, k+1)\tilde{b}_k \quad &\overset{(4.4)}{=} \quad \sum_{k\in\mathbb{Z}} \Phi(\kappa, k+1)b_k \\
&\qquad\qquad - \sum_{k\in\mathbb{Z}} \Phi(\kappa, k+1)\theta_k \sum_{n\in\mathbb{Z}} \Phi(k+1, n+1)b_n \\
&= \quad \sum_{k\in\mathbb{Z}} \Phi(\kappa, k+1)b_k - \sum_{k\in\mathbb{Z}} \theta_k \sum_{n\in\mathbb{Z}} \Phi(\kappa, n+1)b_n \\
&= \quad \sum_{n\in\mathbb{Z}} \Phi(\kappa, n+1)b_n \left(1 - \sum_{k\in\mathbb{Z}} \theta_k\right) = 0
\end{aligned}$$

our Lemma 4.10 implies $\tilde{b} \in R(S_1)$ and $\tilde{b} \in R(T)$. As in (3.3) one gets

$$\mu(\tilde{b}) = 0 \quad \text{for all } \mu \in N(T'). \tag{4.5}$$

We apply the functional $\mu : \ell_0 \to \mathbb{K}$ to (4.4) and obtain using (4.5) that

$$\mu(b) = \mu\left(\theta.\sum_{n\in\mathbb{Z}} \Phi(\cdot+1, n+1)b_n\right). \tag{4.6}$$

Now we define the sequence $\psi' = (\psi'_k)_{k\in\mathbb{Z}}$ by

$$\psi'_k := \sum_{i=1}^{d} \mu(\theta.\Phi(\cdot+1,\kappa)e_i)\Phi(\kappa, k+1)'e_i \quad \text{for all } k \in \mathbb{Z},$$

which, as linear combination of solutions to (Δ'), also solves the dual difference eqn. (Δ') on $\mathbb{Z}$. In addition, we deduce the desired identity

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \langle \psi'_n, b_n\rangle &= \sum_{n\in\mathbb{Z}} \left\langle \sum_{i=1}^{d} \mu(\theta.\Phi(\cdot+1,\kappa)e_i)\Phi(\kappa, n+1)'e_i, b_n \right\rangle \\
&= \sum_{n\in\mathbb{Z}} \left\langle \sum_{i=1}^{d} \mu(\theta.\Phi(\cdot+1,\kappa)e_i)e_i, \Phi(\kappa, n+1)b_n \right\rangle \\
&= \left\langle \sum_{i=1}^{d} \mu(\theta.\Phi(\cdot+1,\kappa)e_i)e_i, \sum_{n\in\mathbb{Z}} \Phi(\kappa, n+1)b_n \right\rangle \\
&= \left\langle \begin{pmatrix} \mu(\theta.\Phi(\cdot+1,\kappa)e_1) \\ \vdots \\ \mu(\theta.\Phi(\cdot+1,\kappa)e_d) \end{pmatrix}, \sum_{n\in\mathbb{Z}} \Phi(\kappa, n+1)b_n \right\rangle \\
&= \sum_{i=1}^{d} \mu(\theta.\Phi(\cdot+1,\kappa)e_i)e'_i \sum_{n\in\mathbb{Z}} \Phi(\kappa, n+1)b_n \\
&= \sum_{i=1}^{d} \mu\left(\theta.\Phi(\cdot+1,\kappa)e_i e'_i \sum_{n\in\mathbb{Z}} \Phi(\kappa, n+1)b_n\right) \\
&= \mu\left(\theta.\Phi(\cdot+1,\kappa)\sum_{i=1}^{d} e_i e'_i \sum_{n\in\mathbb{Z}} \Phi(\kappa, n+1)b_n\right) \\
&= \mu\left(\theta.\sum_{n\in\mathbb{Z}} \Phi(\cdot+1, n+1)b_n\right) \stackrel{(4.6)}{=} \mu(b) \quad \text{for all } b \in \ell_{00}.
\end{aligned}$$

Using (4.3) we moreover have $\left|\sum_{n\in\mathbb{Z}} \langle \psi'_n, b_n\rangle\right| = \mu(b) \le |\mu| \, \|b\|$ for all $b \in \ell_{00}$ and therefore $\sum_{n\in\mathbb{Z}} |\psi'_n| \le |\mu|$. Both the mappings μ and $b \mapsto \sum_{n\in\mathbb{Z}} \langle \psi'_n, b_n\rangle$ are bounded linear functionals on ℓ_0, which coincide on the dense subset ℓ_{00}. Thus, the relation (4.3) holds on the whole space ℓ_0.

($\supset$) Conversely, let $\psi' \in \ell_1$ be a solution of (Δ'). One has $\psi' \in \ell_0$ and $\mu : \ell_0 \to \mathbb{K}$ defined via (4.3) satisfies $\mu \in \ell'_0$. Given $\phi \in E$ we obtain

$$(T'\mu)(\phi) = \mu(T\phi) = \sum_{n\in\mathbb{Z}} \langle \psi'_n, \phi_{n+1} - A_n\phi_n\rangle$$

$$= \sum_{n\in\mathbb{Z}} \left(\langle\psi_n', \phi_{n+1}\rangle - \left\langle\psi_{n-1}', \phi_n\right\rangle\right) + \sum_{n\in\mathbb{Z}} \left\langle\psi_{n-1}' - A_n'\psi_n', \phi_n\right\rangle$$
$$= \lim_{n\to\infty} \langle\psi_n', \phi_{n+1}\rangle - \lim_{n\to-\infty} \langle\psi_n', \phi_{n+1}\rangle + \sum_{n\in\mathbb{Z}} \left\langle\psi_{n-1}' - A_n'\psi_n', \phi_n\right\rangle = 0$$

and our claim follows. □

We arrive at the converse to Prop. 4.9. It is a discrete counterpart of the continuous time results from [Pal88]. See [LT05, Thm. 1.6] for a stronger version valid for difference equations in reflexive infinite-dimensional Banach spaces and the sequence spaces ℓ_0 and ℓ_p, requiring a more involved proof.

Proposition 4.12. *Let* $\kappa \in \mathbb{Z}$*, suppose* (2.1)*,* (2.10) *hold and* $\dim X < \infty$*. If* $S_1 \in L(\ell_\infty)$ *is semi-Fredholm, then a difference eqn.* (Δ) *has EDs on both half lines* $\mathbb{Z}_\kappa^+$ *and* $\mathbb{Z}_\kappa^-$ *with respective Morse indices* ι_+ *and* ι_-*. Furthermore, for the Fredholm index it is* $\operatorname{ind} S_1 = \iota_- - \iota_+$*.*

Remark 4.13. Let P_k^- and P_k^+ denote the invariant projectors associated to the ED on $\mathbb{Z}_\kappa^-$ resp. $\mathbb{Z}_\kappa^+$. Then [Pöt10b, Cor. 2.5] implies the relations:

(1) $\dim N(S_1) = \dim(R(P_\kappa^+) \cap N(P_\kappa^-))$, which yields $\dim N(S_1) < \infty$ for eventually compact Φ (see Rem. 3.2(1)) or for $\dim X < \infty$.

(2) $\operatorname{codim} R(S_1) = \dim(R(P_\kappa^+) + N(P_\kappa^-))^\perp$ and consequently the relation $\operatorname{codim} R(S_1) < \infty$ in case $\dim X < \infty$. Hence, in finite-dimensional state spaces X the operator S_1 is Fredholm, if and only if it is semi-Fredholm.

Proof. We define $d := \dim X < \infty$. Above all, $R(S_1)$ is closed, since the operator S_1 is semi-Fredholm (cf. [Zei93, p. 366, Prop. 8.14(2)]). Hence, also the range $R(T)$ is closed and by [Kat80, p. 24, (3.37)] we have

$$R(T)^\perp = \{\mu \in \ell_0' : \mu(\phi) = 0 \quad \text{for all } \phi \in R(T)\} = N(T').$$

Due to [Zei95, p. 294, Prop. 6(ii)] it is

$$R(T) = {}^\perp N(T') = \{\phi \in \ell_0 : \mu(\phi) = 0 \quad \text{for all } \mu \in N(T')\}.$$

Referring to the duality Lemma 4.11 this means that $b \in R(T)$ holds if and only if (4.3) is satisfied for all solutions $\phi' \in \ell_1$ of (Δ').

Let $\psi^1, \dots, \psi^m$ be a basis for the space $\left\{\psi \in \ell_1 : \psi_k \equiv A_{k+1}'\psi_{k+1} \text{ on } \mathbb{Z}\right\}$ of entire solutions to (Δ') in ℓ_1. We define the linear functionals

$$\alpha_i : \ell_0(\mathbb{Z}_\kappa^-) \to \mathbb{K}, \qquad \alpha_i(\phi) := \sum_{n=-\infty}^{\kappa-1} \left\langle\psi_n^i, \phi_n\right\rangle \quad \text{for all } 1 \le i \le m$$
$$\beta_j : \ell_0(\mathbb{Z}_\kappa^-) \to \mathbb{K}, \qquad \beta_j(\phi) := \phi_\kappa^j \quad \text{for all } 1 \le j \le d,$$

where ϕ^j is the jth component of ϕ. Suppose that we have the representation $\sum_{i=1}^m \gamma_i\alpha_i = \sum_{j=1}^d \delta_j\beta_j$ with scalars $\gamma_i, \delta_j \in \mathbb{K}$. Setting $\psi := \sum_{i=1}^m \gamma_i\psi^i$ yields

$$\sum_{n=-\infty}^{\kappa-1} \langle\psi_n, g_n\rangle = \sum_{i=1}^m \gamma_i\alpha_i(g) = \sum_{j=1}^d \delta_j b_\kappa^{-,j} \quad \text{for all } b^- \in \ell_0(\mathbb{Z}_\kappa^-). \tag{4.7}$$

Under the assumption $\psi \neq 0$ we can choose $b^- \in \ell_0(\mathbb{Z}_\kappa^-)$ such that $b_\kappa^- = 0$, but $\langle \psi_n, b_n^- \rangle > 0$ for all $n < \kappa$. Consequently, (4.7) implies the contradiction $0 < \sum_{n=-\infty}^{\kappa-1} \langle \psi_n, b_n^- \rangle = \sum_{j=1}^{d} \delta_j b_\kappa^{-,j} = 0$ and thus $\psi = 0$. Moreover, the linear independence of $\psi^1, \dots, \psi^m$ shows $\gamma_1 = \ldots = \gamma_m = 0$ and we have

$$\sum_{j=1}^{d} \delta_j b_\kappa^{-,j} = 0 \quad \text{for all } b^- \in \ell_0(\mathbb{Z}_\kappa^-).$$

For every $1 \leq j \leq d$ we choose $b_\kappa^{-,l} := \delta_{lj}$ and conclude that $\delta_j = 0$. Hence, the linear functionals $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_d$ are linearly independent. This guarantees that for each given $b^+ \in \ell_0(\mathbb{Z}_\kappa^+)$ we can construct a $b^- \in \ell_0(\mathbb{Z}_\kappa^-)$ satisfying $b_\kappa^+ = b_\kappa^-$ and $\sum_{n=-\infty}^{\kappa-1} \left\langle \psi_n^i, b_n^- \right\rangle = -\sum_{n=\kappa}^{\infty} \left\langle \psi_n^i, b_n^+ \right\rangle$ for $1 \leq i \leq m$.

We define the sequence $b = (b_k)_{k \in \mathbb{Z}}$,

$$b_k := \begin{cases} b_k^+, & k \geq \kappa, \\ b_k^-, & k < \kappa \end{cases}$$

and clearly obtain the relations $b \in \ell_0$ and $\sum_{n \in \mathbb{Z}} \left\langle \psi_n^i, b_n \right\rangle = 0$ for $1 \leq i \leq m$. Therefore, $b \in R(T)$, which means that (3.2) possesses a bounded entire solution. Restricting to $\mathbb{Z}_\kappa^+$ yields that $x_{k+1} = A_k x_k + b_k^+$ has a bounded entire solution. Since this holds for all $b \in \ell_0(\mathbb{Z}_\kappa^+)$, we conclude from [HM01, Thm. 3.2] (this result is formulated only for ℓ_∞, but the proof extends to the required case ℓ_0) that (Δ) admits an ED on $\mathbb{Z}_\kappa^+$. A dual argument shows that eqn. (Δ) has also an ED on $\mathbb{Z}_\kappa^-$ and the proof is complete. □

4.3. Dichotomy spectra

We now analyze various subsets of the dichotomy spectrum which are important in nonautonomous bifurcation theory (cf. [Ras07, Pöt10b, Pöt11a]). In doing so, we observe that the spectral theory for equations on the whole integer axis is richer than its previous counterpart dealing with semiaxes.

Anew, the concept of a regularity (cf. [Mül07, pp. 51ff]) allows a unified treatment, since the spectra $\sigma(T_A)$, $\sigma_s(T_A)$, $\sigma_F(T_A)$, $\sigma_p(T_A)$ fit into the related axiomatic spectral theory. Similarly, $\sigma_{F_0}(T_A)$ is induced by an upper-semiregularity (cf. [Mül07, pp. 211ff]).

4.3.1. The spectrum. We define the *spectrum* of a difference eqn. (Δ) by

$$\Sigma(A) := \{\gamma > 0 : \, S_\gamma \text{ is not invertible}\}$$

and obtain that it coincides with the dichotomy spectrum $\Sigma_{ED}(A)$. This relates the dynamical property of an ED for (Δ) to the invertibility of a bounded linear operator S_γ and motivates the given terminology "spectrum":

Theorem 4.14. *If* (2.10) *holds, then* $\Sigma_{ED}(A) = \Sigma(A) = \sigma(T_A) \cap (0, \infty)$.

Remark 4.15. One speaks of *discrete spectrum*, provided the set $\Sigma(A)$ consists of isolated points. This situation typically occurs for autonomous or periodic difference eqns. (Δ) (cf. [BAG91, Thm. 4.1]).

Proof. Due to Prop. 4.7(a) one has the chain of equivalences

$$\gamma \notin \Sigma \quad \Leftrightarrow \quad S_\gamma \text{ is invertible} \quad \Leftrightarrow \quad \Phi_\gamma \text{ has an ED on } \mathbb{Z} \quad \Leftrightarrow \quad \gamma \notin \Sigma_{ED}$$

and the remaining assertion is simply our relation (1.1). □

Throughout the remaining section, we will frequently assume (2.10) and therefore the previous Thm. 4.14 allows us to identify the spectrum $\Sigma(A)$ with the dichotomy spectrum $\Sigma_{ED}(A)$.

4.3.2. The point spectrum. We first introduce the *point spectrum*

$$\Sigma_p(A) := \{\gamma > 0 : \dim N(S_\gamma) > 0\}$$

of a difference eqn. (Δ). As dynamical interpretation, $\gamma \in \Sigma_p(A)$ means that a scaled eqn. (Δ_γ) possesses nontrivial bounded entire solutions.

Lemma 4.16. *If* (2.10) *holds, then* $\sigma_p(T_A)$ *is rotationally invariant w.r.t.* 0.

Proof. Let $\ell = \ell_p(\mathbb{Z}, X)$, $1 \leq p \leq \infty$. For $\lambda \neq 0$ this yields

$$\begin{aligned} \lambda \in \sigma_p \quad &\Leftrightarrow \quad \exists \phi \in \ell \setminus \{0\} : T_A\phi = \lambda\phi \\ &\Leftrightarrow \quad x_{k+1} = \tfrac{1}{\lambda} A_k x_k \text{ has a nontrivial solution } \phi \in \ell \\ &\Leftrightarrow \quad x_{k+1} = \tfrac{e^{-i\mu}}{\lambda} A_k x_k \text{ has a nontrivial solution } \phi \in \ell \\ &\Leftrightarrow \quad e^{i\mu}\lambda \in \sigma_p \quad \text{for all } \mu \in \mathbb{R} \end{aligned}$$

and thus $\sigma_p = e^{i\mu}\sigma_p$. □

Theorem 4.17. *If* (2.10) *holds, then* $\Sigma_p(A) = \sigma_p(T_A) \cap (0, \infty)$.

Proof. As in the proof of Lemma 4.16 we deduce with Lemma 2.7 that

$$\begin{aligned} \lambda \in \sigma_p \quad &\Leftrightarrow \quad x_{k+1} = \tfrac{1}{\lambda} A_k x_k \text{ has a nontrivial solution in } \ell \\ &\Leftrightarrow \quad x_{k+1} = |\lambda|^{-1} A_k x_k \text{ has a nontrivial solution in } \ell \\ &\Leftrightarrow \quad |\lambda| \in \Sigma_p(A) \quad \text{for all } \lambda \neq 0; \end{aligned}$$

consequently the claim follows. □

The following result ensures that in absence of point spectrum, the dichotomy spectrum of (Δ) is invariant under compact perturbations. This distinguishes $\Sigma(A)$ from the dichotomy spectra on semiaxes, which are always invariant under compact perturbations (cf. Cor. 3.26). A further related result can be found in [Pöt11b, Thm. 4]:

Proposition 4.18. *Suppose* (2.10) *holds. If the operators* $B_k \in L(X)$, $k \in \mathbb{Z}$, *are compact and satisfy* $\lim_{k\to\pm\infty} |B_k| = 0$, *then*

$$\Sigma(A + B) \setminus \Sigma_p(A + B) \subset \Sigma(A), \qquad \Sigma(A) \setminus \Sigma_p(A) \subset \Sigma(A + B).$$

Proof. (I) Let $\gamma \notin \Sigma(A)$, i.e. eqn. (Δ_γ) admits an ED. Then [Hen81, p. 235, Thm. 7.6.9] implies that either the scaled perturbed equation

$$x_{k+1} = \gamma^{-1}[A_k + B_k]x_k \qquad (P_\gamma)$$

admits an ED or (P_γ) has a nontrivial bounded solution. This, in turn, means either $\gamma \notin \Sigma(A+B)$ or $\gamma \in \Sigma_p(A+B)$. The first claimed inclusion is just the logical contraposition to this.

(II) Now we perturb (P) with $-B_k$ and obtain the original system (Δ). Indeed, if $\gamma \notin \Sigma(A+B)$, then again [Hen81, p. 235, Thm. 7.6.9] guarantees that either (Δ_γ) has an ED or nontrivial bounded solutions exist. Equivalently, either $\gamma \notin \Sigma(A)$ or $\gamma \in \Sigma_p(A)$. This shows the second inclusion. □

4.3.3. The surjectivity spectrum. The *surjectivity spectrum* of (Δ) reads as

$$\Sigma_s(A) := \{\gamma > 0 : S_\gamma \text{ is not onto}\}$$

and one trivially obtains $\Sigma(A) = \Sigma_p(A) \cup \Sigma_s(A)$.

Lemma 4.19. *If* (2.10) *holds, then $\sigma_s(T_A)$ is rotationally invariant w.r.t.* 0 *and compact.*

Proof. Let $\ell = \ell_\infty(\mathbb{Z}, X)$. We obtain from Prop. 4.7(b) with $\lambda \neq 0$ that

$$\begin{aligned}
\lambda \notin \sigma_s \quad &\Leftrightarrow \quad T_A - \lambda\,\mathrm{id} \in L(\ell) \text{ is onto} \\
&\Leftrightarrow \quad \forall \psi \in \ell : \exists \phi \in \ell : T_A\phi - \lambda\phi = \psi \\
&\Leftrightarrow \quad \forall \psi \in \ell : x_k = \tfrac{1}{\lambda}A_{k-1}x_{k-1} + \tfrac{1}{\lambda}\psi_k \text{ has a solution in } \ell \\
&\Leftrightarrow \quad x_{k+1} = \tfrac{1}{\lambda}A_k x_k \text{ has an ET on } \mathbb{Z} \\
&\Leftrightarrow \quad x_{k+1} = \tfrac{e^{-i\mu}}{\lambda}A_k x_k \text{ has an ET on } \mathbb{Z} \quad \Leftrightarrow \quad e^{i\mu}\lambda \notin \sigma_s
\end{aligned}$$

for $\mu \in \mathbb{R}$, and consequently $\sigma_s = e^{i\mu}\sigma_s$. It remains to show the compactness of σ_s: For this, note that $L_s(\ell)$ is a regularity and due to Lemma 3.14 the claim follows from [Mül07, p. 55, Prop. 6.9]. □

Theorem 4.20. *If* (2.10) *holds, then* $\Sigma_{ET}(A) = \Sigma_s(A) = \sigma_s(T_A) \cap (0, \infty)$.

Proof. The first relation $\Sigma_{ET} = \Sigma_s$ follows from Prop. 4.7(b). From the proof of the above Lemma 4.19 we obtain with Lemma 2.7 that

$$\begin{aligned}
\lambda \notin \sigma_s \quad &\Leftrightarrow \quad x_{k+1} = \tfrac{1}{\lambda}A_k x_k \text{ has an ET on } \mathbb{Z} \\
&\Leftrightarrow \quad x_{k+1} = |\lambda|^{-1} A_k x_k \text{ has an ET on } \mathbb{Z} \quad \Leftrightarrow \quad |\lambda| \notin \Sigma_s
\end{aligned}$$

for all $\lambda \neq 0$, and consequently the claim. □

Corollary 4.21. *The surjectivity spectrum $\Sigma_s(A)$ is nonempty and fulfills:*

(a) $\partial\Sigma(A) \subset \Sigma_s(A) \subset \Sigma(A)$,
(b) $\Sigma_s(A) \cup \{0\}$ *is compact,*
(c) $\Sigma_s(A)$ *is upper-semicontinuous.*

Remark 4.22 (ℓ_∞-roughness of ETs). The upper-semicontinuity of the spectra $\Sigma_{ET} = \Sigma_s$ implies that the weak hyperbolicity condition $1 \notin \Sigma_{ET}(A)$ persists when perturbing (Δ) with an operator sequence $B_k \in L(X)$, $k \in \mathbb{Z}$, where $\sup_{k\in\mathbb{Z}} |B_k|$ is sufficiently small. Equivalently, if (Δ) has an ET, than also (P) has an ET; this is a quick proof of [Pap91, Prop. 2].

Proof. Using Thm. 4.20 the proof follows as in Cor. 3.19. □

Next we determine difference eqns. (Δ) for which dichotomy and trichotomy spectra coincide. First, this is the case for discrete spectra $\Sigma(A)$. For the sake of a second sufficient condition, we introduce the *Lyapunov exponents*

$$\lambda_l^-(A,x) := \liminf_{n\to-\infty} \frac{1}{\sqrt[n]{|\Phi(n,\kappa)x|}}, \qquad \lambda_u^+(A,x) := \limsup_{n\to\infty} \sqrt[n]{|\Phi(n,\kappa)x|}$$

with some given $x \in X \setminus \{0\}$, $\kappa \in \mathbb{Z}$.

Corollary 4.23. *Suppose that* (2.1) *holds. If one of the conditions*

- *the cardinality of the point spectrum fulfills* $\#\Sigma_p(A) \le \dim X < \infty$,
- $\lambda_l^-(A,x) \le \lambda_u^+(A,x)$ *for all* $x \in X \setminus \{0\}$

is satisfied, then $\Sigma(A) = \Sigma_{ET}(A) = \Sigma_s(A)$.

Proof. Note that T_A has the SVEP, if and only if

- the cardinality of $\Sigma_p(A) = \sigma_p(T_A) \cap (0,\infty)$ (cf. Thm. 4.17) does not exceed $\dim X$ (cf. [Li94]), or
- $\lambda_l^-(A,x) \le \lambda_u^+(A,x)$ for all $x \in X \setminus \{0\}$ (cf. [BC04, Thm. 2.1]).

In both cases, [Mül07, p. 142, Cor. 14.17] implies $\sigma(T_A) = \sigma_s(T_A)$ and the claim follows with Thm. 4.20. □

4.3.4. Fredholm spectra. We finally introduce the *Fredholm spectra*

$$\Sigma_{F_0}(A) := \{\gamma > 0 : S_\gamma \text{ is not Fredholm or of index} \ne 0\},$$
$$\Sigma_F(A) := \{\gamma > 0 : S_\gamma \text{ is not Fredholm}\} \subset \Sigma_{F_0}(A),$$

which are closely related to EDs on semiaxes (cf. Sects. 3.2 and 4.2).

Lemma 4.24. *Let* $\dim X < \infty$ *and suppose* (2.1) *holds. If* (2.10) *is satisfied, then the Fredholm spectra* $\sigma_F(T_A)$ *and* $\sigma_{F_0}(T_A)$ *are rotationally invariant w.r.t.* 0 *and compact.*

Proof. Let $\lambda \ne 0$ and $\ell = \ell_\infty(\mathbb{Z}, X)$. We define the backward shift operator $(S\phi)_k := \phi_{k-1}$ and obtain the relation $SS_\lambda = \lambda\,\mathrm{id} - T_A$. From $S \in GL(\ell)$ we conclude that SS_λ is Fredholm if and only if S_λ has this property; in addition $\operatorname{ind} S_\lambda = \operatorname{ind}(SS_\lambda)$ holds true. Hence, using Props. 4.9 and 4.12 we deduce

$$\begin{aligned}
\lambda \notin \sigma_{F_0} \quad &\Leftrightarrow \quad T_A - \lambda\,\mathrm{id} \in L(\ell) \text{ is Fredholm with index } 0 \\
&\Leftrightarrow \quad S_\lambda \in L(\ell) \text{ is Fredholm with index } 0 \\
&\Leftrightarrow \quad x_{k+1} = \tfrac{1}{\lambda} A_k x_k \text{ has an ED on } \mathbb{Z}_\kappa^+ \text{ and } \mathbb{Z}_\kappa^- \\
&\qquad\ \text{with the same Morse indices} \\
&\Leftrightarrow \quad x_{k+1} = \tfrac{e^{i\mu}}{\lambda} A_k x_k \text{ has an ED on } \mathbb{Z}_\kappa^+ \text{ and } \mathbb{Z}_\kappa^-
\end{aligned}$$

$$\begin{aligned} &\text{with the same Morse indices} \\ \Leftrightarrow\quad & e^{i\mu}\lambda \notin \sigma_{F_0} \quad \text{for all } \mu \in \mathbb{R}. \end{aligned}$$

Thus, $\sigma_{F_0} = e^{i\mu}\sigma_{F_0}$ and similarly $\sigma_F = e^{i\mu}\sigma_F$. Because $L_F(\ell)$ is a regularity, σ_F must be compact by Lemma 3.20, as well as [Mül07, p. 55, Prop. 6.9]. The compactness of σ_{F_0} follows from [Mül07, p. 218, Exam. 23.21(ii)]. □

Theorem 4.25. *Let* $\dim X < \infty$ *and suppose* (2.1) *holds. If* (2.10) *is satisfied, then* $\Sigma_F(A) = \sigma_F(T_A) \cap (0, \infty)$ *and* $\Sigma_{F_0}(A) = \sigma_{F_0}(T_A) \cap (0, \infty)$.

Proof. As in the proof of Lemma 4.24 one shows with Lemma 2.7 that

$$\begin{aligned} \lambda \notin \sigma_{F_0} \quad \Leftrightarrow\quad & x_{k+1} = \tfrac{1}{\lambda} A_k x_k \text{ has an ED on } \mathbb{Z}_\kappa^+ \text{ and } \mathbb{Z}_\kappa^- \\ & \text{with the same Morse indices} \\ \Leftrightarrow\quad & x_{k+1} = |\lambda|^{-1} A_k x_k \text{ has an ET on } \mathbb{Z} \\ & \text{with the same Morse indices} \quad \Leftrightarrow \quad |\lambda| \notin \Sigma_{F_0} \end{aligned}$$

for all $\lambda \neq 0$. By Lemma 4.24 the set σ_{F_0} is rotationally symmetric and we get the claim for σ_{F_0}. The assertion for σ_F results analogously. □

Corollary 4.26. *The Fredholm spectra* $\Sigma_F(A)$ *and* $\Sigma_{F_0}(A)$ *are nonempty and*

(a) $\Sigma_F(A) \subset \Sigma_{F_0}(A) \subset \Sigma(A)$ *and* $\max \Sigma_F(A) = \max \Sigma_{F_0}(A)$,
(b) *both* $\Sigma_F(A) \cup \{0\}$, $\Sigma_{F_0}(A) \cup \{0\}$ *are compact,*
(c) $\Sigma_F(A)$ *and* $\Sigma_{F_0}(A)$ *are upper-semicontinuous,*
(d) *if* $0 \in \sigma(T_A)$ *is not an isolated point or* $0 \notin \sigma(T_A)$, *then* $\partial\Sigma(A) \subset \Sigma_F(A)$.

Remark 4.27. (1) We equip the compact sets $\mathcal{C}(\mathbb{C}) := \{S \subset \mathbb{C} : S \text{ is compact}\}$ of $\mathbb{C}$ with the Hausdorff metric. It follows from [Mül07, p. 57, Thm. 6.14] that the set of discontinuity points for the set-valued mapping $\sigma_F : L(Y) \to \mathcal{C}(\mathbb{C})$ is of first category in $L(Y)$, Y being a Banach space. From this we conclude that also the set of discontinuity points for Σ_F is meager in the space of eqns. (Δ) satisfying (2.10). A similar statement holds for the dichotomy spectra $\Sigma_\kappa^\pm$, Σ_s and Σ. This indicates that numerical methods to approximate $\Sigma(A)$ (cf. [Hül10]) remain reliable despite the upper-semicontinuity of Σ.

(2) E.g. the assumption of Thm. 4.5(a) yields the conclusion in (d).

Proof. Let $\ell = \ell_\infty(\mathbb{Z}, X)$. First of all, thanks to [Aie04, p. 133] one immediately has $\emptyset \neq \sigma_F$ and the inclusions $\sigma_F \subset \sigma_{F_0} \subset \sigma$ are clear. Using [EE87, p. 44, Cor. 4.11] the essential spectral radii fulfill $\max_{\lambda\in\sigma_F} |\lambda| = \max_{\lambda\in\sigma_{F_0}} |\lambda|$. The set $L_F(\ell)$ is a regularity and Lemma 3.20 with [Mül07, p. 55, Prop. 6.9] yield that σ_F is upper-semicontinuous; since L_{F_0} is an upper-semiregularity and Lemma 3.20 this property of σ_{F_0} is shown in [Mül07, p. 215]. Due to the punctured neighborhood theorem (cf. [Mül07, p. 171, Thm. 18.7]) one knows that the difference $\partial\sigma \setminus \sigma_F$ consists of isolated points in σ. However, thanks to the rotational invariance of σ, this is only possible for the origin. By assumption, 0 is not isolated and $\partial\sigma \setminus \sigma_F = \emptyset$ implies $\partial\sigma \subset \sigma_F$. Now the assertions follow from Lemma 4.24 and Thm. 4.25. □

Corollary 4.28. *Let* (P) *fulfill* (2.1)*. If all the operators* $B_k \in L(X)$*,* $k \in \mathbb{Z}$*, are compact and satisfy* $\lim_{k\to\pm\infty} |B_k| = 0$*, then* $\Sigma_F(A) = \Sigma_F(A + B)$ *and* $\Sigma_{F_0}(A) = \Sigma_{F_0}(A + B)$*.*

Proof. The proof for the Fredholm spectrum Σ_F follows as in Cor. 3.26. Furthermore, the claim also holds for Σ_{F_0} since compact perturbations do not affect the index of a Fredholm operator (cf. [Zei93, p. 366, Prop. 8.14(3)]). □

The following result shows that the dichotomy spectrum $\Sigma(A)$ can be significantly larger than the union of the dichotomy spectra $\Sigma^+_\kappa(A) \cup \Sigma^-_\kappa(A)$. This is due to the occurrence of point and surjectivity spectrum.

Proposition 4.29. *Let* $\kappa \in \mathbb{Z}$*. If* (2.10) *is satisfied, then*
(a) $\Sigma^+_\kappa(A) \cup \Sigma^-_\kappa(A) \supset \Sigma_F(A) \subset \Sigma_{F_0}(A)$*,*
(b) $\Sigma^+_\kappa(A) \cup \Sigma^-_\kappa(A) = \Sigma_F(A)$*, provided* $\dim X < \infty$ *and* (2.1) *holds.*

Proof. (a) The inclusion $\Sigma_F \subset \Sigma_{F_0}$ is trivial. Moreover, $\gamma \notin \Sigma^+_\kappa$ and $\gamma \notin \Sigma^-_\kappa$ means that (Δ_γ) has EDs on both half lines $\mathbb{Z}^+_\kappa$ and $\mathbb{Z}^-_\kappa$. Thus, thanks to Prop. 4.9 the operator S_γ is Fredholm, i.e. $\gamma \notin \Sigma_F$; this shows $\Sigma_F \subset \Sigma^+_\kappa \cup \Sigma^-_\kappa$.

(b) Conversely, for $\gamma \notin \Sigma_F$ the operator S_γ is Fredholm. By Prop. 4.12 the scaled eqn. (Δ_γ) has EDs on both semiaxes $\mathbb{Z}^+_\kappa$ and $\mathbb{Z}^-_\kappa$, i.e. $\gamma \notin \Sigma^+_\kappa$ and $\gamma \notin \Sigma^-_\kappa$; this establishes $\Sigma^+_\kappa \cup \Sigma^-_\kappa \subset \Sigma_F$ and the claim follows with (a). □

Corollary 4.30. *If every* $A_k \in L(\mathbb{K}^d)$ *is upper-triangular with bounded diagonal sequences* $(a^n_k)_{k\in\mathbb{Z}}$*,* $n = 1, \dots, d$*, in* $\mathbb{C} \setminus \{0\}$*, then*

$$\Sigma_F(A) = \bigcup_{n=1}^{d} \Sigma^+_\kappa(a^n) \cup \bigcup_{n=1}^{d} \Sigma^-_\kappa(a^n).$$

If also each $\left(\frac{1}{a^n_k}\right)_{k\in\mathbb{Z}^+_\kappa}$ *is bounded, then* $\Sigma^+_\kappa(a^n) = \left[\beta^-_{\mathbb{Z}^+_\kappa}(a^n), \beta^+_{\mathbb{Z}^+_\kappa}(a^n)\right]$*.*

Proof. Since the diagonal elements a^n_k of A_k do not vanish, (2.1) is satisfied. With the aid of [LJS01, Thm. 2.2] we deduce $\sigma_F(T_A) = \bigcup_{n=1}^d \sigma_F(T_{a^n})$ and then the assertion is a combination of Thm. 4.25 and Prop. 4.29(b). Finally, the explicit form for $\Sigma^+_\kappa(a^n)$ follows from (3.4). □

Our subsequent result relates the dichotomy spectra introduced so far:

Corollary 4.31. *In case* $\dim X < \infty$ *and* (2.1) *one has the inclusions*

$$\begin{array}{ccccccccc}
 & & & & \Sigma_p & \subset & \Sigma_p \cup \Sigma_s & & \\
 & & & & & & \parallel & & \\
\Sigma^+_\kappa \cup \Sigma^-_\kappa & = & \Sigma_F & \subset & \Sigma_{F_0} & \subset & \Sigma_{ED} & = & \Sigma_s \cup \Sigma_{F_0} \\
 & & \cap & & & & \parallel & & \\
\partial\Sigma_{ED} & \subset & \Sigma_s & = & \Sigma_{ET} & \subset & \Sigma_p \cup \Sigma_{F_0} & &
\end{array}$$

Remark 4.32. We abbreviate $L_a := GL$, $\sigma_a := \sigma$, $\Sigma_a := \Sigma$.

(1) The different dichotomy spectra defined above satisfy

$$\left\{\gamma > 0 : S_\gamma \notin \bigcap_{\alpha\in A} L_\alpha(\ell_\infty)\right\} = \bigcup_{\alpha\in A} \Sigma_\alpha \quad \text{for all } A \subset \{a, s, p, F, F_0\}.$$

This follows from [Mül07, p. 51, Thm. 6.3(i) and p. 215], since $L_\alpha(\ell_\infty)$ is a regularity for $\alpha \in \{a, s, p, F\}$ and an upper semi-regularity for $\alpha = F_0$.

(2) Under the assumption $\operatorname{dist}(0, \sigma(T_A)) > 0$ guaranteed e.g. by (2.11), there holds a spectral mapping theorem for the spectra $\sigma_\alpha(T_A)$, $\alpha \in \{a, s, F\}$. This means

$$\sigma_\alpha(S_\lambda) = 1 - \tfrac{1}{\lambda}\sigma_\alpha(T_A), \qquad \sigma_\alpha(T_A) = \lambda - \lambda\sigma_\alpha(S_\lambda) \quad \text{for all } \lambda \neq 0$$

are satisfied with $\alpha \in \{a, s, F\}$; concerning a proof we refer to the standard spectral mapping theorem [Yos80, p. 227, Cor. 1] for $\alpha = a$, [Kim02, Thm. 1.2] for $\alpha = s$ and [Aie04, pp. 148–148, Cor. 3.61(ii)] for $\alpha = F$.

(3) For discrete dichotomy spectra, $\partial\Sigma(A) = \Sigma(A)$ is satisfied and the dichotomy spectra $\Sigma_s(A)$, $\Sigma_{ET}(A)$ coincide with $\Sigma(A)$. Moreover, under the additional assumption of Cor. 4.26(d) one also has $\Sigma_F(A) = \Sigma_{F_0}(A) = \Sigma(A)$.

Remark 4.33 (kinematic similarity). (1) Note that for kinematically similar difference eqns. (Δ) and (2.2) it is

$$M_C T_B = T_A M_C,$$

with the corresponding Lyapunov transformation $(C_k)_{k\in\mathbb{Z}}$ and the invertible multiplication operator M_C. Thus, $T_A^\pm$ and $T_B^\pm$ have the same spectra $\sigma, \sigma_p, \sigma_s, \sigma_F$ and σ_{F_0}. For this reason all the induced dichotomy spectra $\Sigma, \Sigma_p, \Sigma_s, \Sigma_F$ and Σ_{F_0} are invariant under kinematic similarity.

(2) Kinematic similarity defines an equivalence relation on the class of all difference eqns. (Δ); its equivalence classes were characterized in [GKK96].

Proof. Thanks to Thm. 4.20, Cors. 4.21(a), 4.26(a) and Prop. 4.29 it only remains to show the following nontrivial relations:

($\Sigma_F \subset \Sigma_s$) If $\gamma \notin \Sigma_s$ the operator S_γ is onto. Thus, $\operatorname{codim} R(S_\gamma) = 0$ and $\dim X < \infty$ guarantees $\dim N(S_\gamma) < \infty$. Hence, S_γ is Fredholm.

($\Sigma = \Sigma_s \cup \Sigma_{F_0}$) The relation $\gamma \notin \Sigma$ means that S_γ is invertible, i.e. the operator S_γ is onto and Fredholm with index 0, i.e. $\gamma \notin \Sigma_s$ and $\gamma \notin \Sigma_{F_0}$.

($\Sigma = \Sigma_p \cup \Sigma_{F_0}$) Similarly, $\gamma \notin \Sigma$ means that S_γ is invertible, i.e. S_γ is one-to-one and Fredholm with index 0, i.e. $\gamma \notin \Sigma_p$ and $\gamma \notin \Sigma_{F_0}$. □

4.4. Almost periodic equations

A frequently studied situation is the case of difference eqns. (Δ) with an *almost periodic* coefficient sequence $(A_k)_{k\in\mathbb{Z}}$ in $L(X)$. This means that for all $\varepsilon > 0$ there exists an inclusion length $l_\varepsilon \in \mathbb{N}$ such that for every $m \in \mathbb{Z}$ there is a $\kappa \in \{m, \dots, m + l_\varepsilon\}$ such that $\sup_{k\in\mathbb{Z}} |A_{k+\kappa} - A_k| < \varepsilon$. An almost periodic difference eqn. (Δ) satisfies the boundedness assumption (2.10). Clearly, this class contains autonomous, as well as periodic equations, and features a simplified spectral theory. This is due to the essential

Theorem 4.34. *Let* $\dim X < \infty$. *An almost periodic difference eqn.* (Δ) *admits an ED on* $\mathbb{Z}$ *under one of the conditions*

- (2.1) *is satisfied and* (Δ) *has an ET,*
- (Δ) *has an ED*

on a sufficiently large discrete (but finite) interval.

Proof. We only show the ET case, since otherwise the claim follows directly from [Tka96, Thm. 2]. Let a difference eqn. (Δ) have an ET on the finite interval $\mathbb{I}$. Due to assumption (2.1) and Cor. 2.6 we can choose $\kappa = \min \mathbb{I}$ in Def. 2.2. Then Prop. 2.4(a) ensures that (Δ) admits an ED on $\mathbb{I}$ and thanks to [Tka96, Thm. 2] this ED extends to the whole axis $\mathbb{Z}$. $\square$

Corollary 4.35. *Under* (2.1) *an almost periodic difference eqn.* (Δ) *fulfills*

$$\Sigma(A) = \Sigma_\kappa^+(A) = \Sigma_\kappa^-(A) = \Sigma_{ET}(A) = \Sigma_s(A) = \Sigma_F(A) = \Sigma_{F_0}(A).$$

Proof. Let $\gamma \in \Sigma_\kappa^\pm$, i.e. (Δ_γ) does have an ED on a semiaxis $\mathbb{Z}_\kappa^\pm$. Referring to Thm. 4.34 we deduce that (Δ_γ) admits an ED on $\mathbb{Z}$ and consequently $\gamma \notin \Sigma$. This yields the inclusion $\Sigma \subset \Sigma_\kappa^\pm$ and thanks to $\Sigma_\kappa^\pm \subset \Sigma_F, \Sigma_{F_0}, \Sigma_s \subset \Sigma$ the claim follows from Cor. 4.31. $\square$

5. Examples

In general, the concrete form of dichotomy spectra can be obtained only on a numerical level (cf. [Hül10]). However, for scalar, autonomous, periodic and asymptotically autonomous equations on $\mathbb{Z}$, explicit formulas for $\Sigma_{ED}(A)$ have been derived in [BAG91, Sect. 0.4] — provided (2.1) holds.

In this section, we illustrate our above theoretical results and focus on a finer insight. Our initial examples concern scalar difference equations

$$\boxed{x_{k+1} = a_k x_k} \tag{S}$$

with a real sequence $(a_k)_{k\in\mathbb{Z}}$. We first tackle a noninvertible situation:

Example 5.1. Let $\alpha_-, \alpha_+ \in \mathbb{R}$ and consider (S) with the particular sequence

$$a_k := \begin{cases} \alpha_-, & k < 0, \\ 0, & k = 0, \\ \alpha_+, & k > 0. \end{cases}$$

In case $\alpha_+ = \alpha_- = 0$ it is not hard to see that $\Sigma(a) = \Sigma_\kappa^\pm(a) = \emptyset$. Otherwise, it is clear that we have $\Sigma_\kappa^\pm(a) = \{|\alpha_\pm|\}$ for $\kappa \in \mathbb{Z}_1^\pm$. More interesting is it to obtain, e.g., $\Sigma_\kappa^+(a)$ for $\kappa < 1$. Thereto, we suppose one-sided time $\mathbb{I} = \mathbb{Z}_\kappa^+$ and, for simplicity, $|\alpha_-| \leq |\alpha_+|$. For the spectral notion from Rem. 2.10(3), as in [AS01, Exam. 4.1] we obtain $\hat{\Sigma}_{ED}(a) = (0, |\alpha_-|] \cup \{\alpha_+\}$ for $\kappa < 1$ sufficiently small. Using Rem. 2.10(1) and (3) we see $(0, |\alpha_-|] \cup \{\alpha_+\} \subset \Sigma_\kappa^+(a) \subset (0, |\alpha_+|]$ and since $\Sigma_\kappa^+(a)$ consist of at most one interval (cf. Thm. 2.9), this yields

$$\Sigma_\kappa^+(a) = (0, |\alpha_+|]\,.$$

As claimed in (2.9), we see that one-sided dichotomy spectra can grow when the dichotomy interval is increased. Moreover, the ℓ_0-roughness from Cor. 3.26 does not hold without invertibility (2.1), since $\{|\alpha_+|\} = \Sigma_\kappa^+(a+b) \subsetneq \Sigma_\kappa^+(a)$ with the perturbation sequence

$$b_k := \begin{cases} \alpha_+, & \kappa \leq k \leq 0, \\ 0, & 0 < k. \end{cases}$$

Now we focus on two-sided time $\mathbb{I} = \mathbb{Z}$ and [Bas00, Thm. 1] yields

$$\Sigma(a) = (0, \max\{|\alpha_-|, |\alpha_+|\}].$$

Since every entire solution ϕ to $x_{k+1} = \gamma^{-1} a_k x_k$ fulfills $\phi_k = 0$ for $k \in \mathbb{Z}_1^+$, it is bounded in forward time. Moreover, the backward bounded solutions can be identified with Prop. 3.1(b) (provided an ED on $\mathbb{Z}_0^-$) and this sums up to

$$\Sigma_p(a) = (0, |\alpha_-|].$$

Referring to [Bas00, Thm. 8] the Fredholm spectra become

$$\Sigma_F(a) = \{|\alpha_-|, |\alpha_+|\}, \quad \Sigma_{F_0}(a) = [\min\{|\alpha_-|, |\alpha_+|\}, \max\{|\alpha_-|, |\alpha_+|\}]$$

and an analysis of the Fredholm and surjectivity properties for S_γ yields

$$\Sigma_{ET}(a) = \Sigma_s(a) = (0, |\alpha_+|] \cup \{\max\{|\alpha_-|, |\alpha_+|\}\},$$

where we have used Thm. 4.20.

All our following examples will be invertible:

Example 5.2. Let $(a_k)_{k\in\mathbb{Z}}$ be a bounded sequence of nonzero reals such that also $(a_k^{-1})_{k\in\mathbb{Z}}$ is bounded, i.e. (2.11) holds. We introduce *Lyapunov exponents*

$$\lambda_l^-(a) := \liminf_{n\to\infty} \sqrt[n]{\prod_{i=-n}^{-1} |a_i|}, \qquad \lambda_u^+(a) := \limsup_{n\to\infty} \sqrt[n]{\prod_{i=0}^{n-1} |a_i|},$$

as well as Bohl exponents $\beta_{\mathbb{Z}}^{\pm}(a)$ defined in (3.5) (cf. also [BAG91]). First of all, thanks to [BAG91, Thm. 0.4.14] we have $\Sigma(a) = \left[\beta_{\mathbb{Z}}^-(a), \beta_{\mathbb{Z}}^+(a)\right]$ and due to Cor. 4.23 it is $\Sigma(a) = \Sigma_s(a) = \Sigma_{ET}(a) = \left[\beta_{\mathbb{Z}}^-(a), \beta_{\mathbb{Z}}^+(a)\right]$, if $\lambda_l^-(a) \leq \lambda_u^+(a)$.

Using Rem. 4.33(1) also the scaled difference equation

$$x_{k+1} = c_k a_k x; \tag{5.1}$$

has the same dichotomy spectra, provided $(c_k)_{k\in\mathbb{Z}}$ satisfies

$$\tfrac{1}{M} \leq \prod_{i=l}^{k} |c_i| \leq M \quad \text{for all } l \leq k$$

with some real $M > 0$; this is due to [GKK96, Thm. 3.1] guaranteeing that the scalar eqns. (S) and (5.1) are kinematically similar (cf. Rem. 4.33(1)).

The following example shows that the Fredholm spectrum $\Sigma_F(A)$ can be smaller than $\Sigma_{F_0}(A)$ and illustrates a breakdown of the point spectrum.

Example 5.3. Given reals $\alpha_-, \alpha_+ \neq 0$, we determine the various dichotomy spectra for asymptotically constant scalar difference eqn. (S) with

$$a_k := \begin{cases} \alpha_-, & k < 0, \\ \alpha_+, & k \geq 0. \end{cases} \tag{5.2}$$

- The dichotomy spectra (for all $\kappa \in \mathbb{Z}$, cf. [BAG91, Thm. 0.4.6])

  $$\Sigma_\kappa^{\pm}(a) = \{|\alpha_\pm|\}, \qquad \Sigma(a) = [\min\{|\alpha_-|, |\alpha_+|\}, \max\{|\alpha_-|, |\alpha_+|\}].$$

- The point spectrum

$$\Sigma_p(a) = \begin{cases} [|\alpha_+|, |\alpha_-|], & |\alpha_+| \le |\alpha_-|, \\ \emptyset, & \text{else.} \end{cases}$$

- The surjectivity and trichotomy spectrum (cf. Thm. 4.20)

$$\Sigma_s(a) = \Sigma_{ET}(a) = \begin{cases} \{|\alpha_-|, |\alpha_+|\}, & |\alpha_+| < |\alpha_-|, \\ [|\alpha_-|, |\alpha_+|], & |\alpha_-| \le |\alpha_+|. \end{cases}$$

- The Fredholm spectra

$$\Sigma_F(a) = \{|\alpha_-|, |\alpha_+|\}, \quad \Sigma_{F_0}(a) = [\min\{|\alpha_-|, |\alpha_+|\}, \max\{|\alpha_-|, |\alpha_+|\}].$$

The constellation $|\alpha_+| < |\alpha_-|$ typically occurs for supercritical shovel bifurcations (see [Pöt11a]), which are caused by the fact that 1 is an isolated point of the surjectivity spectrum.

Next we deal with 2-dimensional difference equations

$$\boxed{x_{k+1} = A_k(\lambda)x_k} \tag{5.3}$$

depending on a real parameter λ and write $\Sigma(\lambda)$ etc. for its dichotomy spectra.

Example 5.4. Let α, β be fixed nonzero reals and we investigate (5.3) with

$$A_k(\lambda) := \begin{pmatrix} \alpha & \lambda_k \\ 0 & \beta \end{pmatrix}, \qquad \lambda_k := \begin{cases} \lambda, & k \ge 0, \\ 0, & k < 0. \end{cases}$$

Due to its upper triangular form one has constant one-sided dichotomy spectra $\Sigma_0^+(\lambda) = \Sigma_0^-(\lambda) = \{|\alpha|, |\beta|\}$ and Prop. 4.29 yields $\Sigma_F(\lambda) = \{|\alpha|, |\beta|\}$ for every $\lambda \in \mathbb{R}$. In order to derive the other spectra, we compute the scaled transition matrix Φ_γ of (5.3) for several cases:

- $|\alpha| < |\beta|$: One has

$$\Phi_\gamma(k,0) = \gamma^{-k} \begin{cases} \begin{pmatrix} \alpha^k & \frac{\lambda\gamma}{\alpha-\beta}(\alpha^k - \beta^k) \\ 0 & \beta^k \end{pmatrix}, & k \ge 0, \\ \begin{pmatrix} \alpha^k & 0 \\ 0 & \beta^k \end{pmatrix}, & k \le 0 \end{cases} \tag{5.4}$$

and the resulting stable resp. unstable fibers

$$\mathcal{V}_\gamma^+(0) = \begin{cases} \mathbb{R}^2, & |\beta| \le \gamma, \\ \mathbb{R} \times \{0\}, & |\alpha| \le \gamma < |\beta|, \\ \{0\}, & \gamma < |\alpha|, \end{cases} \quad \mathcal{V}_\gamma^-(0) = \begin{cases} \{0\}, & |\beta| < \gamma, \\ \{0\} \times \mathbb{R}, & |\alpha| < \gamma \le |\beta|, \\ \mathbb{R}^2, & \gamma \le |\alpha|. \end{cases}$$

- $|\beta| < |\alpha|$: The scaled transition operator is given in (5.4) and we obtain

$$\mathcal{V}_\gamma^+(0) = \begin{cases} \mathbb{R}^2, & |\alpha| \le \gamma, \\ \mathbb{R}\begin{pmatrix} \lambda\gamma \\ \beta-\alpha \end{pmatrix}, & |\beta| \le \gamma < |\alpha|, \\ \{0\}, & \gamma < |\beta|, \end{cases}$$

$$\mathcal{V}_\gamma^-(0) = \begin{cases} \{0\}, & |\alpha| < \gamma, \\ \mathbb{R} \times \{0\}, & |\beta| < \gamma \leq |\alpha|, \\ \mathbb{R}^2, & \gamma \leq |\beta|. \end{cases}$$

- $\alpha = \beta$: Here, the scaled transition operator reads as

$$\Phi_\gamma(k,0) = \gamma^{-k} \begin{cases} \begin{pmatrix} \alpha^k & k\frac{\lambda\gamma}{\alpha}\alpha^k \\ 0 & \alpha^k \end{pmatrix}, & k \geq 0, \\ \begin{pmatrix} \alpha^k & 0 \\ 0 & \alpha^k \end{pmatrix}, & k \leq 0 \end{cases}$$

  and this gives us

$$\mathcal{V}_\gamma^+(0) = \begin{cases} \mathbb{R}^2, & |\alpha| < \gamma, \\ \mathbb{R} \times \{0\}, & |\alpha| = \gamma, \\ \{0\}, & \gamma < |\alpha|, \end{cases} \qquad \mathcal{V}_\gamma^-(0) = \begin{cases} \{0\}, & |\alpha| < \gamma, \\ \mathbb{R}^2, & \gamma \leq |\alpha|. \end{cases}$$

- $\alpha = -\beta$: The scaled transition operator becomes

$$\Phi_\gamma(k,0) = \gamma^{-k} \begin{cases} \begin{pmatrix} \alpha^k & \frac{\lambda\gamma}{\alpha}\alpha^k \begin{cases} 0, & k \text{ is even} \\ -1, & k \text{ is odd} \end{cases} \\ 0 & \alpha^k \end{pmatrix}, & k \geq 0, \\ \begin{pmatrix} \alpha^k & 0 \\ 0 & \alpha^k \end{pmatrix}, & k \leq 0 \end{cases}$$

  and this leads to

$$\mathcal{V}_\gamma^+(0) = \begin{cases} \mathbb{R}^2, & |\alpha| \leq \gamma, \\ \{0\}, & \gamma < |\alpha|, \end{cases} \qquad \mathcal{V}_\gamma^-(0) = \begin{cases} \{0\}, & |\alpha| < \gamma, \\ \mathbb{R}^2, & \gamma \leq |\alpha|. \end{cases}$$

In all of the above cases one has $\mathcal{V}_\gamma^+(0) \oplus \mathcal{V}_\gamma^-(0) = \mathbb{R}^2$ for $\gamma \notin \{|\alpha|, |\beta|\}$. Referring to the above Cor. 4.8, this immediately yields a discrete dichotomy spectrum $\Sigma(\lambda) = \{|\alpha|, |\beta|\}$ and Cor. 4.31 ensures

$$\Sigma_{F_0}(\lambda) = \Sigma_s(\lambda) = \Sigma_{ET}(\lambda) = \{|\alpha|, |\beta|\};$$

moreover, (Δ_γ) has nontrivial bounded entire solutions for $\gamma \in \{|\alpha|, |\beta|\}$ and consequently $\Sigma_p(\lambda) = \{|\alpha|, |\beta|\}$ for all $\lambda \in \mathbb{R}$.

In the previous example the dichotomy spectra turned out to be independent of the parameter $\lambda \in \mathbb{R}$. In general this needs not to be true, even for triangular difference equations and parameter dependence only in the upper triangular entries. Beyond that we now illustrate that the Fredholm spectra $\Sigma_F(\lambda)$, $\Sigma_{F_0}(\lambda)$ can be strictly smaller than $\Sigma_s(\lambda)$ or $\Sigma(\lambda)$.

Example 5.5. Let δ be a nonzero real number and consider (5.3) with

$$A_k(\lambda) := \begin{pmatrix} a_k & \lambda_k \\ 0 & a_k^{-1} \end{pmatrix}, \qquad a_k := \begin{cases} \delta, & k \geq 0, \\ \frac{1}{\delta}, & k < 0, \end{cases} \qquad \lambda_k := \begin{cases} \lambda, & k \geq 0, \\ 0, & k < 0. \end{cases}$$

The matrices A_k, $k \geq 0$, are the ones of Exam. 5.4 with $\alpha = \delta$, $\beta = \frac{1}{\delta}$, while A_k, $k < 0$, coincide with the ones discussed in Exam. 5.4 with $\alpha = \frac{1}{\delta}$, $\beta = \delta$.

Therefore, we obtain constant spectra $\Sigma_0^+(\lambda) = \Sigma_0^-(\lambda) = \left\{|\delta|, \frac{1}{|\delta|}\right\}$ and using Prop. 4.29 also $\Sigma_F(\lambda) = \left\{|\delta|, \frac{1}{|\delta|}\right\}$ for all $\lambda \in \mathbb{R}$.

- $|\delta| < 1$: One has the fibers

$$\mathcal{V}_\gamma^+(0) = \begin{cases} \mathbb{R}^2, & \frac{1}{|\delta|} \leq \gamma, \\ \mathbb{R} \times \{0\}, & |\delta| \leq \gamma < \frac{1}{|\delta|}, \\ \{0\}, & \gamma < |\delta|, \end{cases} \quad \mathcal{V}_\gamma^-(0) = \begin{cases} \{0\}, & \frac{1}{|\delta|} < \gamma, \\ \mathbb{R} \times \{0\}, & |\delta| < \gamma \leq \frac{1}{|\delta|}, \\ \mathbb{R}^2, & \gamma \leq |\delta| \end{cases}$$

and the relation $\mathcal{V}_\gamma^+(0) \oplus \mathcal{V}_\gamma^-(0) = \mathbb{R}^2$ holds for $\gamma \notin \left[|\delta|, \frac{1}{|\delta|}\right]$. Hence, we conclude the spectrum $\Sigma(\lambda) = \left[|\delta|, \frac{1}{|\delta|}\right]$ for all $\lambda \in \mathbb{R}$. With $\gamma \in \Sigma(\lambda)$ it is $\dim(\mathcal{V}_\gamma^+(0) \cap \mathcal{V}_\gamma^-(0)) = 1$ and since the elements of $\mathcal{V}_\gamma^+(0) \cap \mathcal{V}_\gamma^-(0)$ yield bounded entire solutions to (Δ_γ), one has $\Sigma_p(\lambda) = \left[|\delta|, \frac{1}{|\delta|}\right]$. For rates $\gamma \notin \left\{|\delta|, \frac{1}{|\delta|}\right\}$ the operator S_γ is Fredholm and Prop. 4.9 yields $\operatorname{ind} S_\gamma = 0$ for $\gamma \in \left(|\delta|, \frac{1}{|\delta|}\right)$ which implies $\Sigma_{F_0}(\lambda) = \left\{|\delta|, \frac{1}{|\delta|}\right\}$. Finally, the criterion from Prop. 4.7(b) and Cor. 4.31 ensure

$$\Sigma_{ET}(\lambda) = \Sigma_s(\lambda) = \left[|\delta|, \frac{1}{|\delta|}\right] \quad \text{for all } \lambda \in \mathbb{R}.$$

- $1 < |\delta|$: In this situation the fibers read as

$$\mathcal{V}_\gamma^+(0) = \begin{cases} \mathbb{R}^2, & |\delta| \leq \gamma, \\ \mathbb{R}\binom{\lambda\gamma\delta}{1-\delta^2}, & \frac{1}{|\delta|} \leq \gamma < |\delta|, \\ \{0\}, & \gamma < \frac{1}{|\delta|}, \end{cases} \quad \mathcal{V}_\gamma^-(0) = \begin{cases} \{0\}, & |\delta| < \gamma, \\ \{0\} \times \mathbb{R}, & \frac{1}{|\delta|} < \gamma \leq |\delta|, \\ \mathbb{R}^2, & \gamma \leq \frac{1}{|\delta|} \end{cases}$$

and for $\gamma \notin \left\{|\delta|, \frac{1}{|\delta|}\right\}$ we have

$$\mathcal{V}_\gamma^+(0) + \mathcal{V}_\gamma^-(0) = \begin{cases} \mathbb{R}^2, & \lambda \neq 0, \\ \{0\} \times \mathbb{R}, & \lambda = 0, \end{cases}$$

$$\mathcal{V}_\gamma^+(0) \cap \mathcal{V}_\gamma^-(0) = \begin{cases} \{0\}, & \lambda \neq 0, \\ \{0\} \times \mathbb{R}, & \lambda = 0. \end{cases}$$

Therefore, the dichotomy and point spectrum become

$$\Sigma(\lambda) = \Sigma_p(\lambda) = \begin{cases} \left\{\frac{1}{|\delta|}, |\delta|\right\}, & \lambda \neq 0, \\ \left[\frac{1}{|\delta|}, |\delta|\right], & \lambda = 0, \end{cases}$$

which exemplifies their upper-semicontinuity. Moreover, as opposed to EDs on semiaxes, for $\lambda \neq 0$ the dichotomy spectrum is not the union of the spectra associated to the diagonal elements. Due to the discrete dichotomy spectrum, Cor. 4.31 implies

$$\Sigma_{F_0}(\lambda) = \Sigma_s(\lambda) = \Sigma_{ET}(\lambda) = \left\{\frac{1}{|\delta|}, |\delta|\right\} \quad \text{for all } \lambda \neq 0.$$

For the parameter $\lambda = 0$ we obtain as above that $\Sigma_{F_0}(0) = \left\{|\delta|, \frac{1}{|\delta|}\right\}$, whereas using Prop. 4.7(b) and Cor. 4.31 one sees

$$\Sigma_s(0) = \Sigma_{ET}(0) = \left[\frac{1}{|\delta|}, |\delta|\right].$$

- $\delta = 1$: This gives us

$$\mathcal{V}_\gamma^+(0) = \begin{cases} \mathbb{R}^2, & 1 < \gamma, \\ \mathbb{R} \times \{0\}, & \gamma = 1, \\ \{0\}, & \gamma < 1, \end{cases} \qquad \mathcal{V}_\gamma^-(0) = \begin{cases} \{0\}, & 1 < \gamma, \\ \mathbb{R}^2, & \gamma \leq 1 \end{cases}$$

and for $\gamma \neq 1$ one has $\mathcal{V}_\gamma^+(0) \oplus \mathcal{V}_\gamma^-(0) = \mathbb{R}^2$; hence $\Sigma(\lambda) = \{1\}$ is discrete and accordingly Cor. 4.31 guarantees

$$\Sigma_s(\lambda) = \Sigma_{ET}(\lambda) = \Sigma_{F_0}(\lambda) = \{1\} \quad \text{for all } \lambda \in \mathbb{R}.$$

The difference eqn. (Δ) has nonzero bounded entire solutions, the operator S_1 possesses a nontrivial kernel and we finally get $\Sigma_p(\lambda) = \{1\}$.

- $\delta = -1$: This leads to

$$\mathcal{V}_\gamma^+(0) = \begin{cases} \mathbb{R}^2, & 1 \leq \gamma, \\ \{0\}, & \gamma < 1, \end{cases} \qquad \mathcal{V}_\gamma^-(0) = \begin{cases} \{0\}, & 1 < \gamma, \\ \mathbb{R}^2, & \gamma \leq 1. \end{cases}$$

As above we obtain the relation

$$\Sigma_s(\lambda) = \Sigma_{ET}(\lambda) = \Sigma_{F_0}(\lambda) = \Sigma_p(\lambda) = \{1\} \quad \text{for all } \lambda \in \mathbb{R}.$$

Our next example shows that in contrast to one-sided dichotomy spectra (cf. Cor. 3.26), the dichotomy spectrum on the whole integer axes is not necessarily invariant under compact perturbations.

Example 5.6 (compact perturbation). Suppose that δ, ε are fixed nonzero reals satisfying $|\varepsilon| < 1 < |\delta|$. We consider (5.3) with

$$A_k(\lambda) := \begin{pmatrix} a_k & \lambda_k \\ 0 & a_k^{-1} \end{pmatrix}, \quad a_k := \begin{cases} \delta, & k \geq 0, \\ \frac{1}{\delta}, & k < 0 \end{cases}, \quad \lambda_k := \lambda \begin{cases} \varepsilon^k, & k \geq 0, \\ 0, & k < 0. \end{cases}$$

In the above Exam. 5.5 we obtained the dichotomy spectrum

$$\Sigma(0) = \left[\frac{1}{|\delta|}, |\delta|\right]$$

and now we consider the matrix sequence $\left(\begin{smallmatrix} 0 & \lambda_k \\ 0 & 0 \end{smallmatrix}\right)$, $k \in \mathbb{Z}$, as compact perturbation of $x_{k+1} = A_k(0)x_k$ (cf. Lemma 3.7). Due to Cors. 3.26 and 4.28 such perturbations do neither affect the one-sided dichotomy spectra $\Sigma_0^\pm(\lambda)$, nor the Fredholm spectra $\Sigma_F(\lambda), \Sigma_{F_0}(\lambda)$. Nevertheless, they do have an effect on the dichotomy spectrum $\Sigma(\lambda)$. This can be seen as follows: For $\gamma > 0$ the scaled transition matrix of the perturbed equation with $\lambda \neq 0$ reads as

$$\Phi_\gamma(k, 0) = \gamma^{-k} \begin{pmatrix} \delta^k & \frac{\lambda\delta}{\delta^2 - \varepsilon}\left(\delta^k - \left(\frac{\varepsilon}{\delta}\right)^k\right) \\ 0 & \delta^{-k} \end{pmatrix} \quad \text{for all } k \geq 0$$

yielding the fibers (see the case $|\delta| > 1$ in Exam. 5.5)

$$\mathcal{V}_\gamma^+(0) = \begin{cases} \mathbb{R}^2, & |\delta| \le \gamma, \\ \mathbb{R}\begin{pmatrix} \delta\lambda \\ \varepsilon-\delta^2 \end{pmatrix}, & \frac{1}{|\delta|} \le \gamma < |\delta|, \\ \{0\}, & \gamma < \frac{1}{|\delta|}, \end{cases} \quad \mathcal{V}_\gamma^-(0) = \begin{cases} \{0\}, & |\delta| < \gamma, \\ \{0\} \times \mathbb{R}, & \frac{1}{|\delta|} < \gamma \le |\delta|, \\ \mathbb{R}^2, & \gamma \le \frac{1}{|\delta|}. \end{cases}$$

Hence, for scaling values $\gamma \notin \left\{|\delta|, \frac{1}{|\delta|}\right\}$ we have $\mathcal{V}_\gamma^+(0) \oplus \mathcal{V}_\gamma^-(0) = \mathbb{R}^2$ and Cor. 4.8 shows that Φ_γ admits an ED on $\mathbb{Z}$. This manifests a shrinking in the dichotomy spectrum under compact perturbations, since we can conclude

$$\Sigma(\lambda) = \begin{cases} \left\{\frac{1}{|\delta|}, |\delta|\right\}, & \lambda \neq 0, \\ \left[\frac{1}{|\delta|}, |\delta|\right], & \lambda = 0. \end{cases}$$

Much of this paper dealt with difference eqns. (Δ) in finite-dimensional spaces X. Actually, to our knowledge, there are no studies on the dichotomy spectrum in discrete time with $\dim X = \infty$. In order to illuminate this situation somewhat, we close with remarks and examples illustrating two extreme cases in the spectral theory for (Δ) with an infinite-dimensional state space:

- In general one cannot expect a nice characterization of $\Sigma(A)$ like in the Thms. 2.9 or 4.5. Indeed, for $\dim X = \infty$ it is possible to obtain any compact subset of $(0, \infty)$ as dichotomy spectrum, where corresponding examples can be constructed as follows: Suppose that $S \subset (0, \infty)$ is an arbitrary nonempty compact set and define $\Omega := \{\lambda \in \mathbb{C} : |\lambda| \in S\}$. On the Banach space $X := C_0(\Omega)$ of continuous complex-valued functions vanishing at the boundary of Ω, equipped with the sup-norm, we define the multiplication operator $(A\phi)(t) := t\phi(t)$. It is clearly bounded $A \in L(X)$ and in [EN00, p. 241, 1.5 Examples(ii)] it is shown that $\sigma(A) = \Omega$. Since the dichotomy spectrum of the autonomous difference equation $x_{k+1} = Ax_k$ is given by the moduli of the spectral points to this particular coefficient operator A, our construction yields $\Sigma(A) = S$.
- On the other hand, e.g. compactness properties of the coefficient operators yields a more regular spectrum; for continuous time linear skew-product semiflows this was illustrated in [CL94, Lei00]. The latter reference deals with scalar parabolic equations
$$u_t = a(t)u_{xx} + b(t)u \tag{5.5}$$
on the unit interval $(0, 1)$, where the coefficient functions $a : \mathbb{R} \to (0, \infty)$, $b : \mathbb{R} \to \mathbb{R}$ are assumed to be continuous with limits $\lim_{t\to\pm\infty} a(t) = \alpha^\pm$, $\lim_{t\to\pm\infty} b(t) = \beta^\pm$ and $\alpha^- \le \alpha^+$, $\beta^- \le \beta^+$. Equipped with ambient boundary conditions $u_x(t, 0) = u_x(t, 1) = 0$ (Neumann type) resp. $u(t, 0) = u(t, 1)$ (Dirichlet type), the PDE (5.5) can be formulated as abstract evolution equation in the Hilbert space $X = L^2[0, 1]$ generating an exponentially bounded and instantly compact evolution operator $U(t, s) \in L(X)$, $s \le t$. We associate a difference eqn. (Δ) in $L^2[0, 1]$ with coefficients $A_k := U(k + 1, k)$, $k \in \mathbb{Z}$. Based on the precise information on the dichotomy spectrum for the continuous time problem

from [Lei00, Lemma 1.1], the corresponding dichotomy spectrum $\Sigma(A)$ consists of countably many compact intervals accumulating at 0. This resembles the situation of compact operators, where eigenvalues rather than spectral intervals accumulate at 0.

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References

[Aie04] P. Aiena, *Fredholm and local spectral theory, with applications to multiplieres*, Kluwer, Dordrecht, 2004

[AK01] B. Aulbach & J. Kalkbrenner, *Exponential forward splitting for noninvertible difference equations*, Comput. Math. Appl. **42** (2001), 743–754

[AS01] B. Aulbach & S. Siegmund, *The dichotomy spectrum for noninvertible systems of linear difference equations*, J. Difference Equ. Appl. **7** (2001), no. 6, 895–913

[AS02] ______, *A spectral theory for nonautonomous difference equations*, Proceedings of the 5th Intern. Conference of Difference Eqns. and Application (Temuco, Chile, 2000) (ed. López-Fenner, J., et al), Taylor & Francis, London, 2002, pp. 45–55

[AM96] B. Aulbach & N. Van Minh, *The concept of spectral dichotomy for linear difference equations II*, J. Difference Equ. Appl. **2** (1996), 251–262

[AMZ94] B. Aulbach, N. Van Minh & P.P. Zabreiko, *The concept of spectral dichotomy for linear difference equations*, J. Math. Anal. Appl. **185** (1994), 275–287

[BAG91] A. Ben-Artzi & I. Gohberg, *Dichotomy, discrete Bohl exponents, and spectrum of block weighted shifts*, Integral Equations Oper. Theory **14** (1991), no. 5, 613–677

[BAG92] ______, *Inertial theorems for block weighted shifts and applications*, Time-variant systems and interpolation (ed. I. Gohberg), Operator Theory: Advances and Applications 56, Birkhäuser, Basel, 1992, pp. 120–152

[BAG93] ______, *Dichotomies of perturbed time varying systems and the power method*, Indiana Univ. Math. J. **42** (1993), no. 3, 699–720

[Bas00] A.G. Baskakov, *Invertibility and the Fredholm property of difference operators*, Mathematical Notes **67** (2000), no. 6, 690–698

[BC04] A. Bourhim & C.E. Chidume, *The single-valued extension property for bilateral operator weighted shifts*, Proc. Am. Math. Soc. **133** (2004), no. 2, 485–491

[Bou06] A. Bourhim, *Spectrum of bilateral shifts with operator-valued weights*, Proc. Am. Math. Soc. **134** (2006), no. 7, 2131–2137

[CL94] S.N. Chow & H. Leiva, *Dynamical spectrum for time-dependent linear systems in Banach spaces*, Japan J. of Industrial and Applied Math. **11** (1994), 379–415

[CL99] C. Chicone & Y. Latushkin, *Evolution semigroups in dynamical systems and differential equations*, Mathematical Surveys and Monographs 70, AMS, Providence RI, 1999

[EE87] D.E. Edmunds & W.D. Evans, *Spectral theory and differential operators*, Oxford Science Publications, Oxford, 1987

[EJ98] S. Elaydi & K. Janglajew, *Dichotomy and trichotomy of difference equations*, J. Difference Equ. Appl. **3** (1998), 417–448

[EN00] K.J. Engel & R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics 194, Springer, Berlin etc., 2000

[GKK96] I. Gohberg, M.A. Kaashoek & J. Kos, *Classification of linear time-varying difference equations under kinematic similarity*, Integral Equations Oper. Theory **25** (1996), no. 4, 445–480

[Hen81] D. Henry, *Geometric theory of semilinear parabolic equations*, Lect. Notes Math. 840, Springer, Berlin etc., 1981

[HH06] N.T. Huy & V.T.N. Ha, *Exponential dichotomy of difference equations in l_p-phase spaces on the half-line*, Adv. Difference Equ. **2006**, Article ID 58453, 1–14

[HL05] Á. Haro & R. de La Llave, *Spectral theory and dynamical systems*, Manuscript, ftp://ftp.ma.utexas.edu/pub/papers/llave/spectral.pdf, 2005

[HVL93] J.K. Hale & S.M. Verduyn Lunel, *Introduction to functional differential equations*, Applied Mathematical Sciences 99, Springer, Berlin etc., 1993

[Hül10] T. Hüls, *Computing Sacker-Sell spectra in discrete time dynamical systems*, SIAM Journal on Numerical Analysis **48** (2010), no. 6, 2043–2064

[HM01] N.T. Huy & N. Van Minh, *Exponential dichotomy of difference equations and applications to evolution equations on the half-line*, Comput. Math. Appl. **42** (2001), 301–311

[Kat80] T. Kato, *Perturbation theory for linear operators*, corrected 2nd ed., Grundlehren der mathematischen Wissenschaften 132, Springer, Berlin etc., 1980

[Kim02] F. Kimura, *Spectral mapping theorem for approximate spectra and applications*, Nihonkai Math. J. **13** (2002), 183–189

[KKP85] E.W. Kamen, P.P. Khargonekar & K.R. Poolla, *A transfer-function approach to linear time-varying discrete-time systems*, SIAM Journal of Control and Optimization **23** (1985), no. 4, 550–564

[Lam71] A. Lambert, *Unitary equivalence and reducibility of invertibly weighted shifts*, Bull. Austral. Math. Soc. **5** (1971), 157–173

[Lei00] H. Leiva, *Dynamical spectrum for scalar parabolic equations*, Appl. Anal. **76** (2000), no. 1–2, 9–28

[Li94] J.X. Li, *The single valued extension property for operator weighted shifts*, Northeastern Mathematical Journal **1** (1994), 99–103

[Lin86] X.-B. Lin, *Exponential dichotomies and homoclinic orbits in functional differential equations*, J. Differ. Equations **63** (1986), 227–254

[LJS01] J.X. Li, Y.Q. Ji & S.L. Sun, *The essential spectrum and Banach reducibility of operator weighted shifts*, Acta Mathematica Sinica **17** (2001), no. 3, 413–424

[LT05] Y. Latushkin and Y. Tomilov, *Fredholm differential operators with unbounded coefficients*, J. Differ. Equations **208** (2005), no. 2, 388–429

[Mat68] J.N. Mather, *Characterization of Anosov diffeomorphisms*, Proc. Koninklijke Nederlande Akad. van Wetenschappen, Series A Indag. Math. **30** (1968), 479–483

[Mül00] V. Müller, *Axiomatic theory of spectrum III: Semiregularities*, Studia Mathematica **14** (2000), no. 2, 159–169

[Mül07] ———, *Spectral theory of linear operators and spectral systems in Banach algebras*, Operator Theory: Advances and Applications 139, Birkhäuser, Basel etc., 2007

[Pal88] K.J. Palmer, *Exponential dichotomies and Fredholm operators*, Proc. Am. Math. Soc. **104** (1988), no. 1, 149–156

[Pal06] ———, *Exponential dichotomy and expansivity*, Annali di Matematica **185** (2006), 171–185

[Pap91] G. Papaschinopoulos, *On exponential trichotomy of linear difference equations*, Appl. Anal. **40** (1991), 89–109

[PI97] M.A. Peters & P. Iglesias, *A spectral test of observability and detectability of discrete linear time-varying systems*, Proceedings of the 36th IEEE Conference on Decision and Control, vol. 4, 1997, pp. 3920–3925

[Pöt09] C. Pötzsche, *A note on the dichotomy spectrum*, J. Difference Equ. Appl. **15** (2009), no. 10, 1021–1025

[Pöt10a] ———, *Geometric theory of discrete nonautonomous dynamical systems*, Lect. Notes Math. 2002, Springer, Berlin etc., 2010

[Pöt10b] ———, *Nonautonomous bifurcation of bounded solutions I: A Lyapunov-Schmidt approach*, Discrete Contin. Dyn. Syst. (Series B) **14** (2010), no. 2, 739–776

[Pöt11a] ———, *Nonautonomous bifurcation of bounded solutions II: A shovel bifurcation pattern*, Discrete Contin. Dyn. Syst. (Series A) **31** (2011), no. 3, 941–973

[Pöt11b] ———, *Corrigendum to "A note on the dichotomy spectrum"*, J. Difference Equ. Appl., DOI:10.1080/10236198.2011.573790 (2011)

[Ras07] M. Rasmussen, *Attractivity and bifurcation for nonautonomous dynamical systems*, Lect. Notes Math. 1907, Springer, Berlin etc., 2007

[Shi74] A.L. Shields, *Weighted shift operators and analytic function theory*, Topics in Operator Theory (Carl Pearcy, ed.), Mathematical Surveys and Monographs 13, AMS, Providence, RI, 1974, pp. 49–128

[SS76] R.J. Sacker & G.R. Sell, *Existence of dichotomies and invariant splittings for linear differential equations III*, J. Differ. Equations **22** (1976), 497–522

[SS78] ———, *A spectral theory for linear differential systems*, J. Differ. Equations **27** (1978), 320–358

[Tka96] V.I. Tkachenko, *On the exponential dichotomy of linear difference equations*, Ukr. Math. J. **48** (1996), no. 10, 1600–1608

[Wir98] F. Wirth, *On stability of infinite-dimensional discrete inclusions*, J. Math. Syst. Estim. Control **8** (1998), no. 4, 1–21

[Yos80] K. Yosida, *Functional analysis*, Grundlehren der mathematischen Wissenschaften 123, Springer, Berlin etc., 1980

[Zei93] E. Zeidler, *Nonlinear functional analysis and its applications I (Fixed-points theorems)*, Springer, Berlin etc., 1993

[Zei95] ———, *Applied functional analysis: Main principles and their applications*, Applied Mathematical Sciences 109, Springer, Heidelberg, 1995

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