SMOOTH ROUGHNESS OF EXPONENTIAL DICHOTOMIES, REVISITED

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ABSTRACT. In their recent paper [Proc. Am. Math. Soc. 139 (2011), no. 3, 999-1012], Barreira and Valls show that the invariant projectors of exponential dichotomies, and therefore the associated stable and unstable invariant vector bundles, depend continuously differentiable on parameters, provided the perturbation is small in the C^1-topology.

We give a direct and independent proof of this result and moreover enhance it in various aspects.

An exponential dichotomy (ED for short) is a hyperbolic splitting of the extended state space for linear nonautonomous differential or difference equations into two bundles of linear subspaces: The so-called stable vector bundle consists of all solutions decaying exponentially in forward time, while the complementary unstable vector bundle consists of all solutions which exist and decay in backward time. For various reasons, EDs turned out to be an ambient hyperbolicity concept when dealing with nonautonomous dynamical systems: In stability theory the associated dichotomy spectrum of a linearization yields the appropriate uniform asymptotic stability, while gaps in this spectrum give rise to invariant manifolds or fiber bundles (cf. [KR11]). Moreover, EDs allow to deduce continuation results and thus provide a necessary condition for bifurcations of entire solutions (cf. [Pöt11, Pöt10b]).

Beyond this, in nonautonomous bifurcation theory it plays a crucial role, how invariant projectors or the associated invariant vector bundles for EDs behave under parameter-variation. First, corresponding results might be instrumental to obtain a feasible perturbation theory for the dichotomy spectrum. Second, the vector bundles induced by EDs serve as domain for invariant manifolds, whose intersection in turn gives rise to bifurcating objects, namely entire bounded solutions. Further applications of parameter-dependent dichotomic equations can be found, for instance, in [Pal86, San93, Yi93].

Now [BV11] recently proved that dichotomy projectors vary continuously differentiable, provided a dichotomic difference equation is subject to small perturbations w.r.t. the C^1-norm. However, in the light of versatile applications, particularly the ones sketched above, this C^1-closeness assumption is too restrictive and appears to be of solely technical nature.

In this short note, we therefore demonstrate that the smooth dependence of dichotomy projectors on the difference equation is merely a corollary from meanwhile established proof techniques (cf., for example, [San93, AM96]). Beyond that our basic Thm. 2 improves [BV11] in various regards:

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- Rather than C^1-closeness, which involves a smallness condition on the derivative, we essentially require continuity of the coefficient mapping and its derivatives in the parameter (see relation (2) below).
- At once general C^m-dependence for $0 \le m \le \infty$ is obtained.
- Finally, we do not assume invertibility of the coefficient operators. Especially in an infinite-dimensional setting, invertibility is too restrictive, since discrete-time equations in Banach spaces typically occur as discretizations of compact semigroups (cf. [Pöt10a]), or as integro-difference equations.

Our mathematical set-up is as follows: Let the state space X be a Banach space with norm $|\cdot|$ and Λ be an open convex subset of a further Banach space Y. The Banach algebra of bounded linear operators on X is denoted by $L(X)$ and I_X stands for its unit element, i.e. the identity mapping. A *discrete interval* $\mathbb{I}$ is the intersection of a real interval with the integers $\mathbb{Z}$ and $\mathbb{I}' := \{k \in \mathbb{Z} : k+1 \in \mathbb{I}\}$. Finally, $\delta_{k,m}$ is the Kronecker symbol.

This paper is centered around linear difference equations

$$\boxed{x_{k+1} = A_k(\lambda)x_k} \tag{L_λ}$$

depending on a parameter $\lambda \in \Lambda$ and with coefficients $A_k : \Lambda \to L(X)$, $k \in \mathbb{I}'$. Throughout, we impose the global boundedness assumption

$$\sup_{k\in\mathbb{I}'} |A_k(\lambda)| < \infty \quad \text{for all } \lambda \in \Lambda,$$

which is justifiable since nonautonomous difference eqns. (L_λ) typically occur as variational equations along bounded solutions.

The (forward) solutions to (L_λ) can be expressed in terms of the *transition operator* given by $\Phi_\lambda : \{(k,l) \in \mathbb{I}\times\mathbb{I} : l \le k\} \to L(X)$,

$$\Phi_\lambda(k,l) := \begin{cases} A_{k-1}(\lambda)\cdots A_l(\lambda), & l < k, \\ I_X, & k = l \end{cases} \quad \text{for all } \lambda \in \Lambda.$$

Without invertibility assumptions on $A_k(\lambda) \in L(X)$, $k \in \mathbb{I}'$, neither backward solutions to (L_λ) nor the transition operator $\Phi_\lambda(k,l)$ exists for $k < l$.

Now keep the parameter $\lambda \in \Lambda$ fixed. A linear difference eqn. (L_λ) is said to possess an *exponential dichotomy* on $\mathbb{I}$ (cf. [Hen81, p. 229, Def. 7.6.4]), provided there exists a sequence of projections $P_k(\lambda) \in L(X)$, $k \in \mathbb{I}$, satisfying

$$\begin{aligned} P_{k+1}(\lambda)A_k(\lambda) &= A_k(\lambda)P_k(\lambda) \quad \text{for all } k \in \mathbb{I}', \\ A_k(\lambda) : N(P_k(\lambda)) &\to N(P_{k+1}(\lambda)) \quad \text{is invertible for all } k \in \mathbb{I}', \end{aligned} \tag{1}$$

as well as reals $\alpha \in (0,1)$, $K \ge 1$ guaranteeing the hyperbolic splitting

$$|\Phi_\lambda(k,l)P_l(\lambda)| \le K\alpha^{k-l}, \qquad \left|\bar\Phi_\lambda(l,k)[I_X - P_k(\lambda)]\right| \le K\alpha^{k-l} \quad \text{for all } l \le k$$

and $k,l \in \mathbb{I}$. Here, $\bar\Phi_\lambda(k,l) : N(P_l(\lambda)) \to N(P_k(\lambda))$ denotes the *extended transition operator* given by

$$\bar\Phi_\lambda(k,l) := A_k(\lambda)|_{N(P_k(\lambda))}^{-1} \cdots A_{l-1}(\lambda)|_{N(P_{l-1}(\lambda))}^{-1} \quad \text{for all } k < l,$$

which is well-defined due to the *regularity condition* (1) and bounded by the open mapping theorem (cf., e.g., [Yos80, p. 77, Corollary]). For later use it is convenient to additionally introduce *Green's function*

$$G_\lambda(k,l) := \begin{cases} \Phi_\lambda(k,l)P_l(\lambda), & l \le k, \\ -\bar{\Phi}_\lambda(k,l)[I_X - P_l(\lambda)], & k < l. \end{cases}$$

Remark 1 (Uniqueness of invariant projectors). The sequence $(P_k(\lambda))_{k\in\mathbb{I}}$ in $L(X)$ is denoted as *invariant projector* for (L_λ). It is well-known (see [Pal88] for the invertible finite-dimensional situation, or [Pöt10a, pp. 137–138, Rems. 3.4.17–3.4.18] for our setting) that invariant projectors are uniquely determined in case $\mathbb{I} = \mathbb{Z}$, whereas only the *stable vector bundle*

$$\mathcal{V}_\lambda^+ = \{(k,x) \in \mathbb{I} \times X : x \in R(P_k(\lambda))\}$$

is unique for $\mathbb{I}$ unbounded above and merely the *unstable vector bundle*

$$\mathcal{V}_\lambda^- = \{(k,x) \in \mathbb{I} \times X : x \in N(P_k(\lambda))\}$$

being unique for $\mathbb{I}$ unbounded below.

The coefficient operators $A_k : \Lambda \to L(X)$ in (L_λ) are said to be of class C^m *uniformly in* $k \in \mathbb{I}'$, if they are m-times continuously Fréchet-differentiable and the derivatives satisfy

$$\lim_{\lambda\to\lambda_0} \sup_{k\in\mathbb{I}'} |D^n A_k(\lambda) - D^n A_k(\lambda_0)| = 0 \quad \text{for all } 0 \le n < m+1,\ \lambda_0 \in \Lambda. \tag{2}$$

This canonical assumption allows us to formulate our main result:

Theorem 2 (smooth roughness of EDs)**.** *Let $m \in \mathbb{N}_0 \cup \{\infty\}$, suppose that $\mathbb{I}$ is an unbounded discrete interval and that the difference eqn. (L_{λ^*}) admits an ED on $\mathbb{I}$ for a parameter $\lambda^* \in \Lambda$. If $A_k : \Lambda \to L(X)$ is of class C^m uniformly in $k \in \mathbb{I}'$, then there exists an open neighborhood $\Lambda_0 \subseteq \Lambda$ of λ^*, such that the following holds true:*

(a) *(L_λ) has an ED on $\mathbb{I}$ for every $\lambda \in \Lambda_0$,*
(b) *the associate invariant projectors $P_k : \Lambda_0 \to L(X)$, $k \in \mathbb{I}$, are of class C^m.*

Remark 3. (1) Based on the quantitative nature of the roughness properties [Hen81, p. 232, Thm. 7.6.7, Remark] or [Pöt10a, p. 165, Thm. 3.6.5], it is not difficult to obtain information on the size of the perturbation neighborhood Λ_0.

(2) A closer inspection of the following proofs shows that similar persistence results hold under a $C^{m,\mathrm{lip}}$-dependence (uniformly in $k \in \mathbb{I}'$) of (L_λ) on λ, as well.

(3) Due to the uniformity in assumption (2) (for $n = 0$), assertion (a) of Thm. 2 is commonly denoted as ℓ^∞-roughness. However, it can be shown that EDs persist under a significantly wider class of perturbations (cf. [Hen81, Pal87, Sak94] in continuous time).

(4) Rather than (L_λ), difference equations of the form $x_{k+1} = A_k x_k + B_k(\lambda)x_k$ are considered in [BV11] with $x_{k+1} = A_k x_k$ being exponentially dichotomic. The interested reader might verify that our subsequent proofs remain essentially unaffected in such a framework, if the global bound $\sup_{\lambda\in\Lambda_0} \sup_{k\in\mathbb{I}'} |B_k(\lambda)|$ is sufficiently small and the perturbation $B_k : \Lambda \to L(X)$ is of class C^m uniformly in $k \in \mathbb{I}'$.

We will verify Thm. 2 below using two different techniques. Since its assertion (a) is well-known with an abundant literature (see [Pöt10a, p. 185] for a short survey), we give a precise reference and only sketch the basic ingredients, such that the detailed arguments required to establish claim (b) become transparent: Our first proof demands an ED on a semiaxis $\mathbb{I}$, while the second one is valid in case $\mathbb{I} = \mathbb{Z}$.

Dichotomies on semiaxes. The dissertation [San93] contains a constructive approach to the roughness for EDs (of parabolic evolutionary equations), which received a certain popularity since then. It constructs invariant projectors of some perturbed linear system using a contraction mapping argument. Hence, parameter-dependent problems can be tackled on basis of the classical uniform contraction principle (cf., for example, [CH96, p. 25, Thm. 2.2]). Under this premise, in [Pöt10a, p. 165, Thm. 3.6.5] we derived Thm. 2(a) using a discrete time version of Sandstede's approach from [San93, p. 8, Lemma 1.1].

Proof. We restrict to the case that the discrete interval $\mathbb{I}$ is unbounded above with minimum $\kappa \in \mathbb{Z}$. The dual situation of $\mathbb{I}$ being unbounded below, can be done analogously. Above all, we set $\mathbb{I}^2_+ := \{(k,l) \in \mathbb{I}^2 : l \le k\}$, $\mathbb{I}^2_- := \{(k,l) \in \mathbb{I}^2 : k \le l\}$ and write (L_λ) as

$$x_{k+1} = [A^*_k + B_k(\lambda)]x_k$$

with $A^*_k := A_k(\lambda^*)$ and $B_k(\lambda) := A_k(\lambda) - A_k(\lambda^*)$.[1] Our assumption implies an ED of the difference eqn. $x_{k+1} = A^*_k x_k$ on $\mathbb{I}$ with invariant projector $P^*_k \in L(X)$.

(a) Choose reals γ, δ such that $\alpha < \gamma < \delta < \alpha^{-1}$, define the canonically normed Banach spaces

$$\begin{aligned} L_0 &:= \left\{ \Xi : \mathbb{I} \to L(X) \middle| \sup_{k\in\mathbb{I}} \|\Xi(k)\| < \infty \right\}, \\ L^+_\gamma &:= \left\{ \Xi : \mathbb{I}^2_+ \to L(X) \middle| \sup_{l\le k} \|\Xi(k,l)\| \gamma^{l-k} < \infty \right\}, \\ L^-_\delta &:= \left\{ \Xi : \mathbb{I}^2_- \to L(X) \middle| \sup_{k\le l} \|\Xi(k,l)\| \delta^{l-k} < \infty \right\}, \end{aligned}$$

as well as the affine-linear operators $T^0 : L^+_\gamma \times L^-_\delta \times L_0 \to L_0$, $T^+ : L^+_\gamma \times L^2_0 \to L^+_\gamma$ and $T^- : L^-_\delta \times L^2_0 \to L^-_\delta$ given by

$$\begin{aligned} T^0(\Xi^+, \Xi^-, B)(k) := P^*_k &- \sum_{n=\kappa}^{k-1} \Phi_{\lambda^*}(k, n+1) P^*_{n+1} B_n \Xi^-(n,k) \\ &- \sum_{n=k}^{\infty} \bar\Phi_{\lambda^*}(k, n+1)[I_X - P^*_{n+1}] B_n \Xi^+(n,k) \text{ for all } k \in \mathbb{I}, \end{aligned}$$

$$T^+(\Xi^+, \Xi, B)(k,l) := \Phi_{\lambda^*}(k,l) P^*_l \Xi(l) + \sum_{n=\kappa}^{\infty} G_{\lambda^*}(k, n+1) B_n \Xi^+(n,l) \text{ for all } l \le k,$$

$$T^-(\Xi^-, \Xi, B)(k,l) := \bar\Phi_{\lambda^*}(k,l)[I_X - P^*_l][I_X - \Xi(l)] + \sum_{n=\kappa}^{\infty} G_{\lambda^*}(k, n+1) B_n \Xi^-(n,l)$$

for all $k \le l$. Then $B : \Lambda \to L_0$, $B(\lambda) := (B_k(\lambda))_{k\in\mathbb{I}}$ is well-defined, of class C^m and furthermore satisfies $B(\lambda^*) = 0$. Using [Pöt10a, p. 165, Thm. 3.6.5] we guarantee a neighborhood $\Lambda_0 \subseteq \Lambda$ of λ^* such that the mapping $T : L^+_\gamma \times L^-_\delta \times \Lambda_0 \to L^+_\gamma \times L^-_\delta$,

$$T(\Xi^+, \Xi^-, \lambda) := \begin{pmatrix} T^+(\Xi^+, T^0(\Xi^+, \Xi^-, B(\lambda)), B(\lambda)) \\ T^-(\Xi^-, T^0(\Xi^+, \Xi^-, B(\lambda)), B(\lambda)) \end{pmatrix} \tag{3}$$

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[1] This representation resembles the setting of Rem. 3(4) or [BV11]. We obtain the uniform smallness of $|B_k(\lambda)|$ from continuity, but alternatively a smallness condition can be imposed in advance without consequences for the on-going proof.

becomes a uniform contraction in $\lambda \in \Lambda_0$. Thus, for every parameter $\lambda \in \Lambda_0$ the mapping T possesses a unique fixed point $(\Xi_\kappa^+(\lambda), \Xi_\kappa^-(\lambda)) \in L_\gamma^+ \times L_\delta^-$. Proceeding as in the proof of [Pöt10a, p. 165, Thm. 3.6.5], one sets

$$P_k(\lambda) := T^0(\Xi_\kappa^+(\lambda), \Xi_\kappa^-(\lambda), \lambda)(k) \quad \text{for all } k \in \mathbb{I} \tag{4}$$

and obtains that $P_k(\lambda) \in L(X)$ is actually an invariant projector for an ED to (L_λ).

(b) Using [Pöt10a, pp. 163ff, Lemma 3.6.2–3.6.4] one can verify that the above mappings T^+, T^- and T^0 are affine-linear and moreover continuous. Accordingly, with $B \in C^m(\Lambda, L_0)$ also the composition $T : L_\gamma^+ \times L_\delta^- \times \Lambda_0 \to L_\gamma^+ \times L_\delta^-$ in (3) is of class C^m. Therefore, the uniform contraction principle [CH96, p. 25, Thm. 2.2] applies and guarantees that the fixed-point mapping $(\Xi_\kappa^+, \Xi_\kappa^-) : \Lambda_0 \to L_\gamma^+ \times L_\delta^-$ of T is m-times continuously differentiable. From relation (4) we see that P_k is a composition of C^m-mappings with a linear bounded evaluation map $\Lambda_0 \to L(X)$, consequently also of class C^m. □

A similar justification involving Lyapunov-Perron sums appears to be possible in the $\mathbb{I} = \mathbb{Z}$ situation, as well as for evolutionary equations with continuous time. Nonetheless, it is both advantageous and illustrative to follow a different path:

Dichotomies on $\mathbb{Z}$. The following proof is suitable for EDs on the whole integer axis $\mathbb{Z}$ and relies on an elegant operator theoretical characterization due to [AM96]: On the Banach space ℓ^∞ of bounded sequences $(\phi_k)_{k\in\mathbb{Z}}$ in X we define the weighted shift operators $T(\lambda) \in L(\ell^\infty)$,

$$(T(\lambda)\phi)_k := A_{k-1}(\lambda)\phi_{k-1} \quad \text{for all } k \in \mathbb{Z},\ \lambda \in \Lambda.$$

Proof. Suppose that $\mathbb{I} = \mathbb{Z}$ and keep $\lambda^* \in \Lambda$ fixed. We begin with a technical preparation: Since the derivatives $D^n A_k : \Lambda \to L_n(Y, L(X))$ fulfill the continuity assumption (2), it is straight-forward to show that the mapping $T : \Lambda \to L(\ell^\infty)$ is m-times continuously differentiable (see [Pöt11, Prop. 2.4] for a related proof).

(a) By assumption, the difference eqn. (L_{λ^*}) admits an ED on $\mathbb{Z}$ and thanks to [AM96, Thm. 2] this implies that the spectrum of $T(\lambda^*) \in L(\ell^\infty)$ is disjoint from the unit circle $\mathbb{S}^1$ in $\mathbb{C}$, i.e. $\sigma(T(\lambda^*)) \cap \mathbb{S}^1 = \emptyset$. The upper-semicontinuity of the spectrum (see [Kat80, p. 208, Rem. 3.3]) shows that there exists a neighborhood $\Lambda_0 \subseteq \Lambda$ of λ^* such that

$$\sigma(T(\lambda)) \cap \mathbb{S}^1 = \emptyset \quad \text{for all } \lambda \in \Lambda_0 \tag{5}$$

and therefore again the characterization from [AM96, Thm. 2] implies that also eqn. (L_λ) is exponentially dichotomic for $\lambda \in \Lambda_0$.

(b) Relation (5) ensures that the complex unit circle $\mathbb{S}^1$ can be used to decompose the spectrum of $T(\lambda)$, where the associate Riesz projection (cf. [Kat80, p. 178]) is given by

$$P(\lambda) := -\frac{1}{2\pi i} \int_{\mathbb{S}^1} [zI_X - T(\lambda)]^{-1}\, dz \quad \text{for all } \lambda \in \Lambda_0.$$

Thanks to our preparation and [AMR88, p. 104, 2.5.5 Lemma], $P : \Lambda_0 \to L(\ell^\infty)$ turns out to be a composition of smooth mappings and is hence of class C^m. Owing to [AM96, p. 255] the invariant projectors associated to the ED of (L_λ) are given by $P_k(\lambda)x := (P(\lambda)x^k)_k$ for all $k \in \mathbb{Z}$ and $x \in X$, where the sequence $x^k \in \ell^\infty$ is defined by $x_m^k := \delta_{k,m}x$. Accordingly, each $P_k : \Lambda \to L(X)$, $k \in \mathbb{Z}$, is a composition of $P \in C^m(\Lambda_0, L(\ell^\infty))$ with linear bounded evaluation maps and thus m-times continuously differentiable. □

Corollary 4. *For all $k \in \mathbb{Z}$, $\lambda \in \Lambda_0$ and $y \in Y$ has the representation*

$$DP_k(\lambda)y = -\frac{1}{2\pi i}\left(\int_{\mathbb{S}^1}[zI_X - T(\lambda)]^{-1}DT(\lambda)y[zI_X - T(\lambda)]^{-1}\,dz(\delta_{k,m}\cdot)_{m\in\mathbb{Z}}\right)_k$$

with $(DT(\lambda)y\phi)_k = DA_{k-1}(\lambda)y\phi_{k-1}$ and sequences $\phi \in \ell^\infty$.

Proof. Referring to [AMR88, p. 104, 2.5.5 Lemma] and the chain rule, we obtain

$$DP(\lambda)y = -\frac{1}{2\pi i}\int_{\mathbb{S}^1}[zI_X - T(\lambda)]^{-1}DT(\lambda)y[zI_X - T(\lambda)]^{-1}\,dz \quad \text{for all } y \in Y.$$

Then the claim follows from the definition of $P_k(\lambda)$. □

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