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RESEARCH ARTICLE

Continuity of the Sacker-Sell spectrum on the half line

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The Sacker-Sell (also called dichotomy or dynamical) spectrum $\Sigma^+ \subseteq \mathbb{R}$ is an important notion in the stability theory of nonautonomous dynamical systems. For instance, when dealing with variational equations on the (nonnegative) half line, the set Σ^+ determines uniform asymptotic stability or instability of a solution and more general, it is crucial to construct invariant manifolds from the stable hierarchy. Compared to the spectrum associated to dichotomies on the entire line, Σ^+ has stronger and more flexible perturbation features.

In this paper, we study continuity properties of the Sacker-Sell spectrum by means of an operator-theoretical approach. We provide an explicit example that the generally upper-semicontinuous set Σ^+ can suddenly collapse under perturbation, establish continuity on the class of equations with discrete spectrum and identify system classes having a continuous spectrum. These results for instance allow to vindicate numerical approximation techniques.

Keywords: Dichotomy spectrum, Sacker-Sell spectrum, exponential dichotomy, nonautonomous hyperbolicity, shift operator, difference equation, robust stability

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1. Introduction

It is a classical observation, which can be traced back at least to Lyapunov that eigenvalues of a time-dependent linear system yield essentially no information on its stability behavior. Thus, a variety of alternative spectral notions to describe the long-term behavior of nonautonomous dynamical systems were developed and became relevant. This is largely due to the fact that stability theory for time-variant equations is more complex than in the autonomous situation and one has to carefully distinguish between uniform and nonuniform concepts — both coincide for time-invariant or periodic problems.

Without question, a distinctive property of a feasible stability notion is its robustness under a sufficiently wide and adequate class of perturbations. This has practical reasons (persistence under slightly modified data or numerical discretization), as well as solid theoretical motivations like the construction of invariant manifolds and an overall geometric theory of nonautonomous dynamical systems (cf. [1, 16, 32]). Throughout this whole area the concept of an exponential dichotomy (ED for short) is of crucial importance. An ED geometrically means that the extended state space of a linear differential or difference equation allows a splitting into two invariant bundles of subspaces: The *stable bundle* consists of solutions decaying exponentially to 0 in forward time and is uniquely determined on the nonnegative half line, while a complementary *unstable bundle* comprises of

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solutions allowing an exponential backward estimate.

As associate spectral notion one obtains the *Sacker-Sell spectrum*, in finite dimension a union of finitely many closed so-called *spectral intervals*, which is sometimes also named *dichotomy* or *dynamical spectrum.* It indeed dates back to [38], whose authors examine the situation of continuous time linear skew-product flows over a compact base. The discrete-time case of nonautonomous difference equations was tackled later in [2, 3], while [4, 6, 33] follow a more operator-theoretical approach. When dealing with dichotomies, it is a decisive observation whether this property is assumed on the whole time axis, or merely on a half line i.e. a semiaxis unbounded above. On the one hand, the full line situation is important for persistence, as well as bifurcation problems of bounded entire solutions (cf. [34] for a survey). On the other hand, EDs on the half line are central in stability theory — they indicate uniform asymptotic stability — and more general for the construction of invariant manifolds in the stable hierarchy (cf., for example, [32, p. 202, Cor. 4.2.12]). Thereby, a half line dichotomy is a significantly weaker assumption than an ED on the entire line. Accordingly, the related dichotomy spectrum $\Sigma^+ \subseteq \mathbb{R}$ is a smaller subset, has simpler fine structure (cf. [33]) and allows more flexible perturbation properties than in the full line case. For instance, Σ^+ does not change under perturbations decaying to 0 (see [6, 30]) and the dichotomy spectrum of triangular problems is fully determined by the diagonal elements (cf. [33]) — both properties fail on the full axis.

While our companion paper [35] is devoted to the full line dichotomy spectrum, we are currently interested in linear equations defined on the half line. Already in this context it is well-known that not even the stability radius behaves continuously under perturbations (cf. [36]). However, rather than studying merely the stability radius $\max \Sigma^+$ (see e.g. [19, 21, 36]), let us actually focus on the whole Sacker-Sell spectrum Σ^+. Although it generally persists in an upper-semicontinuous way too, concrete continuity results for Σ^+ are necessary due to the facts that

(1) one expects smooth changes in the exponential decay/growth of solutions,
(2) a certain robustness in the hierarchy of stable manifolds (precisely their dimensions) under perturbations is useful, and
(3) numerical approximation schemes (see [12, 22]) for the upper-semicontinuous dichotomy spectrum become validated for a certain class of problems.

Finally, there is the observation that Σ^+ generalizes the set of eigenvalues or Floquet multipliers, which depend at least continuously on parameters (see, e.g. [20, pp. 369ff]). Such a behavior is certainly also desirable in a nonautonomous framework.

1.1. *Main results and structure*

This paper focusses on nonautonomous linear difference equations

$$\boxed{x_{k+1} = A_k x_k} \qquad \text{for all } k \geq \kappa \tag{Δ_A}$$

in $\mathbb{R}^d$, where the coefficients A_k form a bounded sequence of invertible matrices. Notice that (Δ_A) for example occur when investigating the behavior of nonlinear difference equations $x_{k+1} = f_k(x_k)$ near given aperiodic forward solutions $(\phi_k^*)_{k \geq \kappa}$ in terms of their variational equation $x_{k+1} = Df_k(\phi_k^*)x_k$.

We are interested in the behavior of the corresponding Sacker-Sell spectrum

$$\Sigma^+(A) = \left\{ \gamma > 0 : \, x_{k+1} = \gamma^{-1} A_k x_k \text{ has no exponential dichotomy on } \mathbb{Z}_\kappa^+ \right\} \subset \mathbb{R}$$

on a half line $\mathbb{Z}_\kappa^+ := \{\kappa, \kappa+1, \dots\}$, $\kappa \in \mathbb{Z}$, with respect to the Hausdorff metric. Our main results can be summarized as follows:

- By adapting an abstract example of Kakutani, we present a scalar difference equation, whose Sacker-Sell spectrum abruptly shrinks under perturbation (see Ex. 4.2). To the author's knowledge, previous examples of equations with discontinuous spectrum were defined on the entire axis and were at least 2-dimensional.
- Continuity of the Sacker-Sell spectrum is a generic property. If one equips the set of equations (Δ_A) with the uniform topology, then Σ^+ is continuous on a dense G_δ-set and discontinuous on a meagre set (see Thm. 4.1).
- Thm. 4.4 establishes that the Sacker-Sell spectrum is continuous for equations with a discrete spectrum. This includes the classical autonomous and periodic cases, as well as problems under the frequently made assumption of a *full spectrum*, where all spectral intervals are singletons and their number coincides with the system dimension.
- Finally, in Thm. 4.5 and 4.6 we identify some classes of difference equations whose spectrum behaves continuously.

Addressing this paper's structure, we begin with remarks on Bohl exponents as a tool indicating uniform exponential stability and yielding sharp bounds for the dichotomy spectrum. The central idea for our overall approach preludes the following Sect. 3 and has its origin in [4–6]: The dichotomy spectrum Σ^+ can be characterized as Fredholm (or approximate point) spectrum of a unilateral matrix-weighted shift operator on an appropriate sequence space (ℓ^2 in our case). Based on abstract operator theoretical results from [9, 10], we then provide sufficient criteria for the continuity of the dichotomy spectrum in Sect. 4 — among them is the mentioned and geometrically clear condition of singletons as spectral intervals. The concluding Sect. 5 explains how the obtained results carry over to the Sacker-Sell spectrum on the nonpositive line, compares them to the full line situation discussed in [35] and illustrates their applicability in continuous time.

Throughout, our results are formulated in a strict dynamical systems language, although their proofs heavily rely on operator and spectral theory. For the convenience of the reader, we accordingly collected the required preliminaries in the appendices, which constitute almost one third of the paper. They contain several basics including a spectral picture for weighted unilateral shifts.

Finally, our confinement to a discrete time setting of difference equations in finite dimensions has three reasons: First, tools from operator theory apply instantly and it is not even necessary to leave the realm of bounded operators. Second, corresponding results on ordinary differential equations immediately follow via an application to their time-1-maps (cf. Sect. 5) and third, weighted shifts with matrix- rather than general operator-weights have simpler spectral pictures.

1.2. *Dynamics via operator theory*

The use of operator theory to study dynamical systems is not new, but turned out to be fruitful (cf., for instance the monograph [7] or the papers [4–6]). Indeed, such a connection has several technical merits, since the whole array of perturbation tools from operator theory becomes available. This allows quite elegant and short proofs of results like the robustness of EDs in [4, 5], their invariance under perturbations decaying to 0 from [6], the smooth dependence of associated invariant projectors on parameters, or the classification of nonautonomous bifurcations via [33]. From this perspective it suggested itself to tackle also continuity properties of the Sacker-Sell spectrum within such a framework.

Nevertheless, applying rather general operator-theoretical tools to quite particular situations (shift operators, in the present case) naturally leads to some losses. For instance, our approach might impede insights into the actual dynamical features of a problem. The use of dynamical systems methods could also lead to more concrete results applicable to wider classes of problems. In any case, the author is looking forward to further contributions on the continuous behavior of the Sacker-Sell spectrum, or even on a smooth dependence of its boundary points on parameters — applications are ubiquitous.

1.3. *Terminology*

A *discrete interval* $\mathbb{I}$ is the intersection of a real interval I with the integers $\mathbb{Z}$; we often write $\mathbb{I}' := \{k \in \mathbb{I} : k+1 \in \mathbb{I}\}$ and $I_{\mathbb{Z}} := I \cap \mathbb{Z}$. In our setting, $\mathbb{I}$ is typically a discrete half line of the form $\mathbb{Z}_\kappa^+ := [\kappa, \infty)_{\mathbb{Z}}$, $\mathbb{Z}_\kappa^- := (-\infty, \kappa]_{\mathbb{Z}}$, where $\kappa \in \mathbb{Z}$ is fixed from now on.

Suppose that X is a Banach space with norm $|\cdot|$ and $B_\rho(x)$ is the open ball in X with center x and radius $\rho > 0$. The Banach algebra of bounded linear operators on X is denoted by $L(X)$, id_X is the unit element, i.e. the identity mapping, and $GL(X)$ are the invertible elements. In concrete settings, X is the unitary space $\mathbb{C}^d$ with the inner product $\langle x, y\rangle := \sum_{j=1}^d x_j \bar{y}_j$ for all $x, y \in \mathbb{C}^d$ and the induced norm $|x| := \sqrt{\langle x, x\rangle}$, or the bounded sequences $\ell^\infty(\mathbb{C}^d)$ in $\mathbb{C}^d$ or the Hilbert space of square-summable sequences $\ell^2(\mathbb{C}^d)$ in $\mathbb{C}^d$ with the inner product

$$\langle\!\langle \phi, \psi \rangle\!\rangle := \sum_{k=\kappa}^{\infty} \langle \phi_k, \psi_k\rangle \quad \text{for all } \phi = (\phi_k)_{\kappa \le k}, \psi = (\psi_k)_{\kappa \le k}$$

inducing the norm $\|\phi\| := \sqrt{\langle\!\langle \phi, \phi \rangle\!\rangle}$. We write $\ell^2 = \ell^2(\mathbb{C}^d)$ throughout.

Given a subset $\Omega \subseteq X$, we denote its *interior* by $\operatorname{int}\Omega$ and its *closure* by $\overline{\Omega}$.

2. Dichotomies and Bohl exponents

When linearizing a nonlinear difference eqn. $x_{k+1} = f_k(x_k)$ along a non-constant solution $\phi^* = (\phi_k^*)_{k\in\mathbb{I}}$, the resulting variational eqn.

$$x_{k+1} = Df_k(\phi_k^*)x_k$$

is a linear nonautonomous difference equation, even if the right-hand side f_k does not depend on k. Spectral properties of this variational equations allow tangible conclusions on the dynamical behavior near ϕ^* (cf. for instance, [1, 32]). We take this as a motivation to abstractly study linearly homogeneous nonautonomous difference equations

$$\boxed{x_{k+1} = A_k x_k} \tag{Δ_A}$$

on an unbounded discrete interval $\mathbb{I}$. Here, we suppose that $A_k \in GL(\mathbb{C}^d)$, $k \in \mathbb{I}'$, are invertible matrices and uniformly bounded in k — this is legitimate when linearizing along a bounded solution ϕ^*. A difference eqn. (Δ_A) is uniquely determined by this sequence $(A_k)_{k\in\mathbb{I}'}$ and we consider the corresponding space

$$\mathcal{L}^\infty(\mathbb{C}^d) := \ell^\infty(L(\mathbb{C}^d)), \qquad\qquad \|A\| := \sup_{k\in\mathbb{I}'} |A_k|$$

carrying the topology of uniform convergence.

The solutions to (Δ_A) are given in terms of the *transition matrix*

$$\Phi : \mathbb{I} \times \mathbb{I} \to GL(\mathbb{C}^d), \qquad\qquad \Phi(k,l) := \begin{cases} A_{k-1} \cdots A_l, & l < k, \\ \mathrm{id}_{\mathbb{C}^d}, & k = l, \\ A_k^{-1} \cdots A_{l-1}^{-1}, & k < l \end{cases}$$

and in order to indicate the dependence on A we sometimes write $\Phi_A(k,l)$.

For local stability questions concerning ϕ^* or the stability of (Δ_A), the notion of an exponential dichotomy is central. One says (Δ_A) has an *exponential dichotomy* on a discrete interval $\mathbb{I}$ (abbreviated ED, cf. [16, p. 229, Def. 7.6.4] or [2, 6]), if there exists a sequence of projections $P_k \in L(\mathbb{C}^d)$, $k \in \mathbb{I}$, satisfying

$$P_{k+1}A_k = A_k P_k \quad \text{for all } k \in \mathbb{I}',$$

as well as reals $\alpha \in (0,1)$, $K \geq 1$ guaranteeing the hyperbolic splitting

$$|\Phi(k,l)P_l| \leq K\alpha^{k-l}, \qquad\qquad |\Phi(l,k)[\mathrm{id}_{\mathbb{C}^d} - P_k]| \leq K\alpha^{k-l} \quad \text{for all } l \leq k$$

and $k, l \in \mathbb{I}$. The *Sacker-Sell* of *dichotomy spectrum* of (Δ_A) is given as

$$\Sigma_{\mathbb{I}}(A) = \left\{\gamma > 0 : \, x_{k+1} = \gamma^{-1}A_k x_k \text{ does not have an ED on } \mathbb{I}\right\}.$$

Due to [4, Thm. 4] or [3, Thm. 3.4], this set $\Sigma_{\mathbb{I}}(A) \subseteq \mathbb{R}^+$ is empty or consists of up to d disjoint *spectral intervals*, i.e. there is an $1 \leq m \leq d$ with

$$\Sigma_{\mathbb{I}}(A) = \begin{cases} (0, \beta_m] \\ [\alpha_m, \beta_m] \end{cases} \cup \bigcup_{i=1}^{m-1} [\alpha_i, \beta_i] \tag{2.1}$$

and reals $0 < \alpha_m \leq \beta_m < \alpha_{m-1} \leq \ldots \leq \beta_1$. One speaks of a *discrete spectrum* (2.1), provided all intervals $[\alpha_i, \beta_i]$ are singletons, i.e. $\alpha_i = \beta_i$, $1 \leq i \leq m$; the eqn. (Δ_A) has *full spectrum*, if additionally $m = d$ holds. Autonomous difference eqns. $x_{k+1} = Ax_k$ with coefficients $A \in GL(\mathbb{C}^d)$ have the discrete Sacker-Sell spectrum

$$\Sigma_{\mathbb{I}}(A) = \{|\lambda| > 0 : \, \lambda \in \sigma(A)\} \tag{2.2}$$

and in case all moduli are pairwise different, one has a full spectrum.

In order to distinguish the three basic dichotomy spectra, let us fix $\kappa \in \mathbb{Z}$ and write

$$\Sigma^+(A) := \Sigma_{\mathbb{Z}_\kappa^+}(A), \qquad\qquad \Sigma^-(A) := \Sigma_{\mathbb{Z}_\kappa^-}(A), \qquad\qquad \Sigma(A) := \Sigma_{\mathbb{Z}}(A)$$

for the *forward*, the *backward* resp. the *all time spectrum*.

We first and foremost are interested in $\Sigma^+(A)$ here. The spectral interval of $\Sigma^+(A)$ with right boundary point β_1 is called *dominant*, because it determines stability properties:

- In case $\beta_1 = \max \Sigma^+(A) < 1$ the eqn. (Δ_A) is uniformly asymptotically stable.
- Each gap in $\Sigma^+(A)$ gives rise to a stable subspace, which persists locally under ambient nonlinear perturbations (cf. [32, p. 259, Thm. 4.6.4(a)]).

The assumption $A_k \in GL(\mathbb{C}^d)$, $k \in \mathbb{I}'$, ensures that a spectral interval $(0, \beta_m]$ can be avoided under the boundedness condition

$$\sup_{k\in\mathbb{I}'} \left|A_k^{-1}\right| < \infty \tag{2.3}$$

and that $\Sigma^+(A)$ is independent of $\kappa \in \mathbb{Z}$. This picture is softened for noninvertible matrices: Although EDs can be defined under such weaker conditions (cf. [2, 16]), spectral intervals with lower bound 0 might occur.

There exists a close relation between EDs, the corresponding Sacker-Sell spectrum and the notion of Bohl exponents (cf. [20]). Their convenient definition requires an abstract setting. Let us assume that $\mathcal{A}$ is a normed unital algebra over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with norm $|\cdot|$. Given this, we introduce the *upper* resp. *lower Bohl* exponent as

$$\overline{\beta}(a) := \limsup_{n\to\infty} \sup_{\kappa\le k} \sqrt[n]{\left|\prod_{j=k}^{k+n-1} a_j\right|}, \qquad \underline{\beta}(a) := \liminf_{n\to\infty} \inf_{\kappa\le k} \sqrt[n]{\left|\prod_{j=k}^{k+n-1} a_j\right|} \tag{2.4}$$

for a sequence $a = (a_k)_{\kappa\le k}$ in $\mathcal{A}$. These nonnegative real numbers satisfy

$$\underline{\beta}(a) \le \overline{\beta}(a) \le \limsup_{k\to\infty} |a_k|$$

(for this, cf. [36, Cor. to Thm. 5]), as well as the positive homogeneity $\overline{\beta}(\lambda a) = |\lambda|\, \overline{\beta}(a)$ and $\underline{\beta}(\lambda a) = |\lambda|\, \underline{\beta}(a)$ for every scalar $\lambda \in \mathbb{K}$. Moreover, the left-hand limit in (2.4) exists with the characterization (cf. [36, Thm. 2])

$$\overline{\beta}(a) = \lim_{n\to\infty} \sup_{\kappa\le k} \sqrt[n]{\left|\prod_{j=k}^{k+n-1} a_j\right|} = \inf_{n\ge 1} \sup_{\kappa\le k} \sqrt[n]{\left|\prod_{j=k}^{k+n-1} a_j\right|}.$$

In an algebra $\mathcal{A}$ with multiplicative norm (i.e. $|ab| = |a|\,|b|$ for all $a, b \in \mathcal{A}$) the relations $\overline{\beta}(|a|) = \overline{\beta}(a)$, $\underline{\beta}(|a|) = \underline{\beta}(a)$ hold and if every $a_k \in \mathcal{A}$ is invertible, then

$$\underline{\beta}(a) = \lim_{n\to\infty} \inf_{\kappa\le k} \sqrt[n]{\left|\prod_{j=k}^{k+n-1} a_j\right|} = \sup_{n\ge 1} \inf_{\kappa\le k} \sqrt[n]{\left|\prod_{j=k}^{k+n-1} a_j\right|}.$$

The relation between Bohl exponents and the Sacker-Sell spectrum is illuminated in

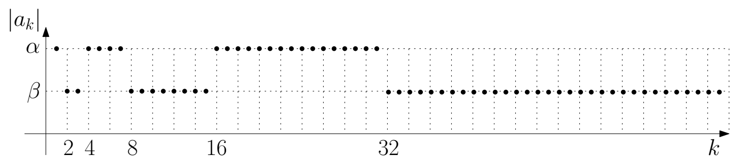

Figure 1. The sequence $(a_k)_{1\le k}$ from Ex. 2.1(1) with $\beta < \alpha$

Example 2.1: *Suppose that $\mathcal{A}$ is the Banach algebra $\mathbb{C}$ and $\mathbb{I} = \mathbb{Z}_1^+$. For sequences*

$a \in \ell^\infty(\mathbb{C})$ with $a_k \neq 0$ for all $1 \leq k$, let us consider scalar difference equations

$$\boxed{x_{k+1} = a_k x_k.} \tag{Δ_a}$$

Then, $\overline{\beta}(a) = \max(\Sigma^+(a) \cup \{0\})$ holds and in case $\sup_{1\leq k} \left|a_k^{-1}\right| < \infty$ one arrives at a compact interval $\Sigma^+(a) = [\underline{\beta}(a), \overline{\beta}(a)]$ as spectrum.

(1) More specifically, let us consider coefficient sequences fulfilling

$$|a_k| = \begin{cases} \alpha, & k \in [2^n, 2^{n+1})_{\mathbb{Z}} \text{ with even } n \in \mathbb{Z}_0^+, \\ \beta, & k \in [2^n, 2^{n+1})_{\mathbb{Z}} \text{ with odd } n \in \mathbb{Z}_1^+ \end{cases}$$

with reals $\alpha, \beta > 0$ (cf. Fig. 1). The values $|a_k|$ are alternately constant to α, β on increasingly larger discrete intervals and consequently

$$\Sigma^+(a) = [\min\{|\alpha|, |\beta|\}, \max\{|\alpha|, |\beta|\}].$$

(2) Following [18], if we recursively define

$$\gamma_1 := 1, \qquad \gamma_2 := \tfrac{1}{4}, \qquad \gamma_{n+1} := (\gamma_1\gamma_2\cdots\gamma_n)^{n+1} \quad \text{for all } n \geq 2,$$

then the real sequence $a = (\gamma_1, \ldots, \gamma_9, \gamma_1, \ldots, \gamma_{90}, \gamma_1, \ldots, \gamma_{900}, \ldots)$ satisfies the limit relation $\limsup_{k\to\infty} a_k = 1$ and has a vanishing upper Bohl exponent $\overline{\beta}(a) = 0$. This yields $\Sigma^+(a) = \emptyset$, because every forward solution to (Δ_a) decays super-exponentially.

3. Weighted shifts and system classes

Although dichotomy spectra are initially defined on a dynamical systems basis via (2.2), a close connection to the theory of shift operators holds (cf. [6]). The crucial observation is that $\Sigma^+(A)$ and the Fredholm spectrum of the unilateral matrix-weighted shift

$$T_A \in L(\ell^2), \qquad T_A\phi := (0, A_\kappa\phi_\kappa, A_{\kappa+1}\phi_{\kappa+1}, \ldots) \quad \text{for all } \phi \in \ell^2$$

are related by (cf. [6], [33, Thm. 3.22])

$$\Sigma^+(A) = \sigma_F(T_A) \cap \mathbb{R}^+. \tag{3.1}$$

This, for instance, allows to deduce the dichotomy spectra given in the above Ex. 2.1 from [37, Thm. 1] (cf. also (C.3)) resp. [18]. By Prop. C.5 and the spectral mapping theorem [26, p. 33, Thm. 1.11.1] a further immediate consequence of (3.1) are

$$\Sigma^+(A) \subseteq \left[\underline{\beta}(A), \overline{\beta}(A)\right], \qquad \Sigma^+(\lambda A) = |\lambda|\,\Sigma^+(A) \quad \text{for all } \lambda \in \mathbb{C} \setminus \{0\}.$$

Proposition 3.1: *The Fredholm spectrum $\sigma_F(T_A) \subseteq \mathbb{C}$ is rotationally invariant w.r.t. the origin. If $A, B \in \mathcal{L}^\infty(\mathbb{C}^d)$, then the following holds:*

$$\Sigma^+(A) = \Sigma^+(B) \quad \Leftrightarrow \quad \sigma_F(T_A) = \sigma_F(T_B).$$

Proof. Due to (3.1) it suffices to show that the Fredholm spectrum $\sigma_F(T_A) \subset \mathbb{C}$ is rotationally invariant w.r.t. 0, which was done in [33, Lemma 3.21]. □

We next identify difference eqns. (Δ_A) for which $\Sigma^+(A)$ behaves nicely under perturbation of $A \in \mathcal{L}^\infty(\mathbb{C}^d)$. This requires two preparations:

- First, the linear space of matrix sequences commuting asymptotically with A is

$$\mathcal{EC}(A) := \left\{ B \in \mathcal{L}^\infty(\mathbb{C}^d) : \lim_{k\to\infty} |A_{k+1}B_k - B_{k+1}A_k| = 0 \right\}$$

  and contains those $B \in \mathcal{L}^\infty(\mathbb{C}^d)$ fulfilling $\lim_{k\to\infty}|B_k - A_k| = 0$.
- Second, given a nonempty subset $\mathcal{X} \subseteq \mathcal{L}^\infty(\mathbb{C}^d)$, compact perturbations to matrix sequences in $\mathcal{X}$ are abbreviated as

$$\mathcal{E}\mathcal{X} := \left\{ X + K \in \mathcal{L}^\infty(\mathbb{C}^d) : X \in \mathcal{X},\, K \in \mathcal{L}^\infty(\mathbb{C}^d) \text{ with } \lim_{k\to\infty} K_k = 0 \right\}.$$

  Obviously $\mathcal{X} \subseteq \mathcal{E}\mathcal{X}$ holds and an element of $\mathcal{E}\mathcal{X}$ is said to be *essentially in* $\mathcal{X}$. E.g. the matrix sequences $X \in \mathcal{E}\{0\}$ satisfy $\overline{\beta}(X) = 0$, while the converse is not true (see Exs. 2.1(2) and C.4 for a drastic example). With subsets $\mathcal{X} \subseteq \mathcal{Y}$ of $\mathcal{L}^\infty(\mathbb{C}^d)$ it results $\mathcal{E}\mathcal{X} \subseteq \mathcal{E}\mathcal{Y}$.

For arbitrary $p \in \mathbb{Z}_1^+$, let us now introduce

$$\begin{aligned}\mathcal{H}_p(\mathbb{C}^d) := \Big\{ A \in \mathcal{L}^\infty(\mathbb{C}^d) : &\Phi(k+2p, k+p)^*\Phi(k+2p, k+p) \\ &\geq \Phi(k+p, k)\Phi(k+p, k)^* \text{ for all } \kappa \leq k \Big\}\end{aligned}$$

with the relation $B \geq A :\Leftrightarrow B - A$ is Hermitian positive semidefinite.

The sets $\mathcal{H}_p(\mathbb{C}^d) \subseteq \mathcal{L}^\infty(\mathbb{C}^d)$ are topologically closed and as in [35, Prop. 3.6(a)] one establishes that $a \in \mathcal{H}_p(\mathbb{C})$ implies $aA \in \mathcal{H}_p(\mathbb{C}^d)$ for each $A \in \mathcal{H}_p(\mathbb{C}^d)$.

Example 3.2: *(1) In the special case $d = 1$ of scalar eqns. (Δ_a) one has $L(\mathbb{C}^1) = \mathbb{C}$ and commutativity yields*

$$\mathcal{H}_p(\mathbb{C}) = \left\{ a \in \ell^\infty(\mathbb{C}) : \prod_{j=k}^{k+p-1} |a_j| \leq \prod_{j=k+p}^{k+2p-1} |a_j| \quad \text{for all } \kappa \leq k \right\}. \tag{3.2}$$

This means that sequences $a \in \mathcal{H}_p(\mathbb{C})$ are determined by the following: Consecutive geometric means over the terms $|a_k|, \ldots, |a_{k+p-1}|$ and $|a_{k+p}|, \ldots, |a_{k+2p-1}|$ do not decrease in $k \geq \kappa$. The characterization (3.2) *yields that $a \in \mathcal{H}_p(\mathbb{C}) \Leftrightarrow |a| \in \mathcal{H}_p(\mathbb{C})$. Moreover, one has $\mathcal{H}_1(\mathbb{C}^d) \subseteq \mathcal{H}_p(\mathbb{C}^d)$. Note that* (3.2) *extends to the diagonal elements in order to describe sequences of diagonal matrices in $\mathcal{H}_p(\mathbb{C}^d)$.*

(2) If (Δ_A) is a p-periodic difference equation with $A \in \mathcal{H}_p(\mathbb{C}^d)$, then its transition matrix satisfies the condition

$$\Phi(k+p, k)^*\Phi(k+p, k) \geq \Phi(k+p, k)\Phi(k+p, k)^* \quad \textit{for all } \kappa \leq k.$$

We remind the reader that linear difference eqns. (Δ_A) and (Δ_B) are called *kinematically similar*, provided there exists a sequence $(\Lambda_k)_{\kappa \le k}$ (called *Lyapunov transformation*) such that beyond $\Lambda_{k+1} B_k = A_k \Lambda_k$ also

$$\Lambda_k \in GL(\mathbb{C}^d) \quad \text{for all } \kappa \le k, \qquad\qquad \sup_{\kappa \le k} \max\left\{|\Lambda_k|, \left|\Lambda_k^{-1}\right|\right\} < \infty$$

holds. In this case we write $A \simeq_\Lambda B$ and point out that kinematic similarity defines an equivalence relation on the space of all difference eqns. (Δ_A). As in [35, Prop. 3.6(b)] it can be shown that the equivalence class of $A \in \mathcal{H}_p(\mathbb{C}^d)$ in $\mathcal{H}_p(\mathbb{C}^d)$ contains all $B \simeq_U A$ with unitary matrices $U_k \in L(\mathbb{C}^d)$, $k \in \mathbb{Z}_\kappa^+$.

Linear difference eqns. (Δ_A) with coefficients essentially in $\mathcal{H}_p(\mathbb{C}^d)$ and discrete Sacker-Sell spectrum allow the following result:

Proposition 3.3: *If (Δ_A) has discrete spectrum and $A \in \mathcal{E}\mathcal{H}_p(\mathbb{C}^d)$, then*

$$\lim_{k\to\infty} \left(\Phi(k+2p, k+p)^* \Phi(k+2p, k+p) - \Phi(k+p, k)\Phi(k+p, k)^*\right) = 0.$$

Proof. The assumptions imply that A is of the form $A = \bar{A} + K$, where $\bar{A} \in \mathcal{H}_p(\mathbb{C}^d)$ and $K \in \mathcal{L}^\infty(\mathbb{C}^d)$ with $\lim_{k\to\infty} K_k = 0$. For $A \in \mathcal{E}\mathcal{H}_p(\mathbb{C}^d)$ our Prop. C.9(a) shows that $S_{\bar{A}} := T_{\bar{A}}^p$ is hyponormal. Since the sequence $A = (A_k)_{\kappa \le k}$ is bounded, it is easy to see from Prop. C.3(a) that the commutators $S_A^* S_A - S_A S_A^*$ and $S_{\bar{A}}^* S_{\bar{A}} - S_{\bar{A}} S_{\bar{A}}^*$ only differ by a compact operator and so their essential norms satisfy

$$\|S_A^* S_A - S_A S_A^*\|_F = \left\|S_{\bar{A}}^* S_{\bar{A}} - S_{\bar{A}} S_{\bar{A}}^*\right\|_F \le \tfrac{1}{\pi}\lambda_2(\sigma_F(S_{\bar{A}})) \tag{3.3}$$

with the latter inequality valid due to [8, Theorem]. By the spectral mapping theorem [26, p. 33, Thm. 1.11.1] for σ_F, we obtain $\sigma_F(S_{\bar{A}}) = \sigma_F(T_{\bar{A}}^p) = \sigma_F(T_{\bar{A}})^p$ and because σ_F is invariant under compact perturbations, this implies

$$\sigma_F(S_{\bar{A}}) = \sigma_F(T_A)^p \overset{(3.1)}{=} \left\{e^{it}\lambda \in \mathbb{C} : t \in [0, 2\pi) \text{ and } \lambda \in \Sigma^+(A)\right\}^p.$$

As a discrete (in fact finite) set, $\Sigma^+(A)$ yields the measure $\lambda_2(\sigma_F(S_{\bar{A}})) = 0$ and (3.3) guarantees that the commutator $S_A^* S_A - S_A S_A^*$ is a compact operator. Then the claim follows from Props. C.1(b) and C.3(e). □

Elements of $\mathcal{H}_1(\mathbb{C}^d)$ have distinguished representatives under kinematic similarity, namely sequences increasing w.r.t. the relation $\ge$ and converging to a Hermitian positive semidefinite matrix. In particular, they have discrete dichotomy spectra:

Theorem 3.4: *If $A \in \mathcal{H}_1(\mathbb{C}^d)$, then $A \simeq_U B$ with an eqn. (Δ_B) satisfying:*

(a) The coefficient matrices $B_k \in GL(\mathbb{C}^d)$ are positive semidefinite Hermitian and

$$B_k \le B^+ := \lim_{n\to\infty} B_n \quad \textit{for all } \kappa \le k. \tag{3.4}$$

(b) The corresponding Lyapunov transformation U consists of unitary matrices.

(c) For $B^+ \in GL(\mathbb{C}^d)$ one has $\Sigma^+(A) = \{|\lambda| > 0 : \lambda \in \sigma(B^+)\}$.

Proof. (a) and (b) Due to Prop. C.8 the shift operator $T_A \in L(\ell^2)$ is unitarily equivalent to a weighted shift $T_B \in L(\ell^2)$ by means of a multiplication operator $M_U \in L(\ell^2)$ with unitary weights $U_k \in L(\mathbb{C}^d)$ and positive semidefinite Hermitian B_k. In particular, it is $B_k = U_{k+1}^* A_k U_k$ for all $\kappa \le k$ and (Δ_A) is kinematically similar to (Δ_B). One has

$$0 \le B_k \le \sup_{\kappa \le n} |B_n| \operatorname{id}_{\mathbb{C}^d} \quad \text{for all } \kappa \le k \tag{3.5}$$

and we show convergence for B_k: Our assumption $A \in \mathcal{H}_1(\mathbb{C}^d)$ yields

$$B_{k+1}^* B_{k+1} - B_k B_k^* = U_{k+1}^* \left(A_{k+1}^* A_{k+1} - A_k A_k^*\right) U_{k+1} \ge 0 \quad \text{for all } \kappa \le k$$

and thus $B_{k+1}^2 = B_{k+1}^* B_{k+1} \ge B_k B_k^* = B_k^2 \ge 0$ (thanks to (B.2)) holds. Because of (3.5) the Löwner-Heinz inequality (see [31]) applies and establishes that $(B_k)_{\kappa \le k}$ is bounded nondecreasing. Hence, the limit (3.4) exists (see [17]).

(c) We proved that the difference eqns. (Δ_A) and (Δ_B) are kinematically similar and therefore have the same dichotomy spectrum. Thanks to (3.4), the eqn. (Δ_B) is a compact perturbation of $x_{k+1} = B^+ x_k$ and the claim follows from [33, Cor. 3.26]. Here the assumption $A_k \in GL(\mathbb{C}^d)$ implies that also each B_k is invertible. □

Given any $p \in \mathbb{Z}_1^+$, we furthermore introduce the sets

$$\begin{aligned} \mathcal{P}_p(\mathbb{C}^d) := \Big\{ & A \in \mathcal{L}^\infty(\mathbb{C}^d) : \Phi(k+2p,k)^* \Phi(k+2p,k) \\ & - 2r\Phi(k+p,k)^*\Phi(k+p,k) + r^2 \operatorname{id}_{\mathbb{C}^d} \ge 0 \text{ for all } \kappa \le k,\, r > 0 \Big\}, \end{aligned}$$

which satisfy the following properties:

Proposition 3.5: *Let $p \in \mathbb{Z}_1^+$ and assume that $A \in \mathcal{P}_p(\mathbb{C}^d)$.*

(a) $\lambda A \in \mathcal{P}_p(\mathbb{C}^d)$ for all $\lambda \in \mathbb{C}$.
(b) If $A \simeq_U B$ and U consists of unitary matrices $U_k \in L(\mathbb{C}^d)$, then $B \in \mathcal{P}_p(\mathbb{C}^d)$.
(c) $\mathcal{H}_p(\mathbb{C}^d) \subseteq \mathcal{P}_p(\mathbb{C}^d) \subseteq \mathcal{P}_{np}(\mathbb{C}^d)$ for all $n \in \mathbb{Z}_1^+$.

Remark 3.6: As further class of coefficients $A \in \mathcal{L}^\infty(\mathbb{C}^d)$, which not only lay between $\mathcal{H}_p(\mathbb{C}^d)$ and $\mathcal{P}_p(\mathbb{C}^d)$, but are also easy to characterize, we mention those satisfying

$$\Phi(k+2p,k)^*\Phi(k+2p,k) \ge [\Phi(k+p,k)^*\Phi(k+p,k)]^2 \quad \text{for all } \kappa \le k.$$

Proof. Assertion (a) readily follows from the definition. In order to show (b) we observe that $\Phi_B(k,l) = U_k^* \Phi_A(k,l) U_l$ for all $k,l \in \mathbb{Z}_\kappa^+$ implies the relations

$$\begin{aligned} \Phi_B(k+2p,k)^*\Phi_B(k+2p,k) &= U_k^* \Phi_A(k+2p,k)^*\Phi_A(k+2p,k) U_k, \\ \Phi_B(k+p,k)^*\Phi_B(k+p,k) &= U_k^* \Phi_A(k+p,k)^*\Phi_A(k+p,k) U_k \end{aligned}$$

for all $\kappa \le k$. Thus, Prop. C.9(b) and (B.2) ensure the claim. Concerning (c), the first inclusion follows since Prop. C.9 guarantees

$$\begin{aligned} \mathcal{H}_p(\mathbb{C}^d) &= \left\{ A \in \mathcal{L}^\infty(\mathbb{C}^d) : T_A \in H_p(\ell^2) \right\}, \\ \mathcal{P}_p(\mathbb{C}^d) &= \left\{ A \in \mathcal{L}^\infty(\mathbb{C}^d) : T_A \in P_p(\ell^2) \right\} \end{aligned}$$

and that hyponormal operators are paranormal by [26, p. 74, (2.57)]. The second inclusion $\mathcal{P}_p(\mathbb{C}^d) \subseteq \mathcal{P}_{np}(\mathbb{C}^d)$, $n \geq 1$, is due to [14, Thm. 1]. $\square$

Example 3.7 (scalar equations)**:** *In case $d = 1$ it is $\mathcal{H}_p(\mathbb{C}) = \mathcal{P}_p(\mathbb{C})$ for every $p \in \mathbb{Z}_1^+$ and $\kappa \leq k$. In addition, the characterization* (3.2) *from Ex. 3.2(1) holds. To establish $\mathcal{P}_{p+1}(\mathbb{C}^d) \setminus \mathcal{P}_p(\mathbb{C}^d) \neq \emptyset$, consider a $p+1$-periodic complex sequence a with*

$$|a_k| = \begin{cases} \alpha, & k \mod (p+1) \neq 0, \\ \beta, & \text{else} \end{cases} \qquad \text{for all } \kappa \leq k$$

and reals $0 < \beta < \alpha$. This yields $\prod_{j=k}^{k+p} |a_j| = \alpha^p \beta = \prod_{j=k+p+1}^{k+2p+1} |a_j|$ for all $\kappa \leq k$. Therefore (3.2) *implies $a \in \mathcal{P}_{p+1}(\mathbb{C})$. On the other side, for $k = 1$ one obtains*

$$\prod_{j=k}^{k+p-1} |a_j| = \alpha^p > \alpha^{p-1}\beta = \prod_{j=k+p}^{k+2p-1} |a_j| \,,$$

from which $a \notin \mathcal{P}_p(\mathbb{C})$ follows.

Moreover, in dimensions $d > 1$ the set $\mathcal{P}_p(\mathbb{C}^d)$ is strictly larger than $\mathcal{H}_p(\mathbb{C}^d)$:

Example 3.8: *On the discrete interval $\mathbb{Z}_\kappa^+$ the coefficient sequence*

$$A_\kappa := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad\qquad A_k := \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{for all } \kappa < k$$

satisfies $\Phi(\kappa+2,\kappa+1)^*\Phi(\kappa+2,\kappa+1) - \Phi(\kappa+1,\kappa)\Phi(\kappa+1,\kappa)^* = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. This matrix has the eigenvalues ± 1 and is thus not Hermitian positive semidefinite, i.e. $A \notin \mathcal{H}_1(\mathbb{C}^2)$. However, $A \in \mathcal{P}_1(\mathbb{C}^2)$ holds because for every $r > 0$ the matrix

$$\begin{aligned} &\Phi(\kappa+p,\kappa)^*\Phi(\kappa+2,\kappa) - 2r\Phi(\kappa+p,\kappa)^*\Phi(\kappa+p,\kappa) + r^2 \operatorname{id}_{\mathbb{C}^2} \\ &\qquad = \begin{pmatrix} (r-1)^2 & 0 \\ 0 & r^2 \end{pmatrix} \quad \text{for all } r > 0 \end{aligned}$$

is obviously positive semidefinite.

4. Continuity of the Sacker-Sell spectrum

In this main section, we finally address continuity properties of the Sacker-Sell spectrum. To be specific, we interpret

$$\bar{\Sigma}^+(A) := \Sigma^+(A) \cup \{0\}$$

as a mapping from $\mathcal{L}^\infty(\mathbb{C}^d)$ (identified with the eqns. (Δ_A)) into the metric space $K(\mathbb{R})$ of nonempty compact subsets of $\mathbb{R}$; the union with $\{0\}$ enforces $\bar{\Sigma}^+(A)$ to be nonempty compact. It is known that $\bar{\Sigma}^+ : \mathcal{L}^\infty(\mathbb{C}^d) \to K(\mathbb{R})$ is upper-semicontinuous, i.e. given a sequence $(A^n)_{n\geq 1}$ with limit $A \in \mathcal{L}^\infty(\mathbb{C}^d)$ one has (cf. [33, Cor. 3.24])

$$\lim_{n\to\infty} d(\bar{\Sigma}^+(A^n), \bar{\Sigma}^+(A)) = 0,$$

where d denotes the Hausdorff semidistance (see Appendix A). Equivalently, if $\Sigma^+(A)$ is of the form (2.1), then for every $\varepsilon > 0$ there exists a $N \in \mathbb{Z}_1^+$ such that (see [6, Thm. 2.5])

$$\Sigma^+(A^n) \subseteq \begin{cases} (0, \beta_1 + \varepsilon] \\ [\alpha_1 - \varepsilon, \beta_1 + \varepsilon] \end{cases} \cup \bigcup_{i=2}^{m} [\alpha_i - \varepsilon, \beta_i + \varepsilon] \quad \text{for all } n \geq N.$$

From general properties of upper-semicontinuous functions we derive

Theorem 4.1: $\bar{\Sigma}^+ : \mathcal{L}^\infty(\mathbb{C}^d) \to K(\mathbb{R})$ *is continuous on a dense* G_δ*-set and discontinuous on a meagre set.*

Proof. First, as upper-semicontinuous function mapping the Banach space $\mathcal{L}^\infty(\mathbb{C}^d)$ into $K(\mathbb{R})$, we get from Fort's theorem (cf. e.g. [28]) that $\bar{\Sigma}^+$ is continuous on a dense G_δ-set. Second, [29, p. 57, Thm. 14] yields that the points of discontinuity for $\sigma_F : L(X) \to K(\mathbb{C})$ form a meagre set and due to (3.1) this property extends to $\bar{\Sigma}^+$. $\square$

The known examples of discontinuous spectra addressed EDs on the full axis and required at least 2-dimensional systems. Yet, on the half line and even for scalar difference eqns. (Δ_a) one cannot expect a continuous behavior of the Sacker-Sell spectrum. Inspired by an abstract example from Kakutani (see Ex. C.7) we have

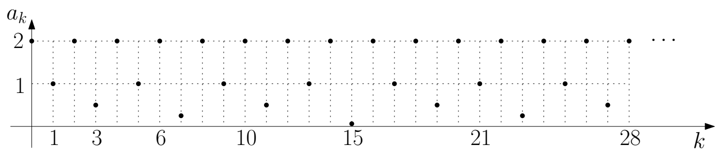

Figure 2. The sequence $(a_k)_{0\leq k}$ from Ex. C.7

Example 4.2: *Let* $\mathbb{I} = \mathbb{Z}_0^+$ *and suppose that* $(\gamma_k)_{k\in\mathbb{Z}_0^+}$ *is a real sequence with positive values and limit* 0*. Given this, we define further real sequences as follows:*

- *The bounded sequence* a *(see Fig. 2) is pointwise given as*

$$a_k := \begin{cases} \gamma_0, & k \in 2^0 - 1 + 2^1\mathbb{Z}_0^+, \\ \gamma_1, & k \in 2^1 - 1 + 2^2\mathbb{Z}_0^+, \\ \gamma_2, & k \in 2^2 - 1 + 2^3\mathbb{Z}_0^+, \ldots \end{cases}$$

and in general $a_k := \gamma_l$ *for nonnegative integers* $k \in 2^l - 1 + 2^{l+1}\mathbb{Z}_0^+$, $l \in \mathbb{Z}_0^+$, *i.e.*

$$a = (\gamma_0, \gamma_1, \gamma_0, \gamma_2, \gamma_0, \gamma_1, \gamma_0, \gamma_3, \gamma_0, \gamma_1, \gamma_0, \gamma_2, \gamma_0, \gamma_1, \gamma_0, \gamma_4, \gamma_0, \gamma_1, \gamma_0, \gamma_2, \gamma_0, \gamma_1, \gamma_0, \gamma_3, \gamma_0, \ldots).$$

- *For every* $n \in \mathbb{Z}_1^+$ *and* $k \in \mathbb{Z}_0^+$ *set*

$$a_k^n := \begin{cases} 0, & a_k = \gamma_n, \\ a_k, & \text{else} \end{cases} \quad \text{for all } k \in \mathbb{Z}_0^+$$

and this yields bounded real sequences a^n*, which explicitly read as*

$$a^1 = (\gamma_0, 0, \gamma_0, \gamma_2, \gamma_0, 0, \gamma_0, \gamma_3, \gamma_0, 0, \gamma_0, \gamma_2, \gamma_0, 0, \gamma_0, \gamma_4, \gamma_0, 0, \gamma_0, \gamma_2, \gamma_0, 0, \gamma_0, \gamma_3, \gamma_0, \ldots),$$

$a^2 = (\gamma_0, \gamma_1, \gamma_0, 0, \gamma_0, \gamma_1, \gamma_0, \gamma_3, \gamma_0, \gamma_1, \gamma_0, 0, \gamma_0, \gamma_1, \gamma_0, \gamma_4, \gamma_0, \gamma_1, \gamma_0, 0, \gamma_0, \gamma_1, \gamma_0, \gamma_3, \gamma_0, \ldots)$,
$a^3 = \ldots$

We next compare the upper Bohl exponents of a and a^n, $n \in \mathbb{Z}_1^+$. On the one hand, in order to show that a has a positive upper Bohl exponent, we observe that

$$\prod_{j=0}^{0} a_j = \gamma_0, \qquad \prod_{j=0}^{2} a_j = \gamma_1\gamma_0^2, \qquad \prod_{j=0}^{6} a_j = \gamma_2\gamma_1^2\gamma_0^4$$

and in general by mathematical induction

$$\prod_{j=0}^{2^p-2} a_j = \gamma_{p-1}\gamma_{p-2}^4 \cdots \gamma_0^{2^{p-1}} \quad \text{for all } p \in \mathbb{Z}_1^+.$$

For the upper Bohl exponent of a this implies

$$\begin{aligned}\overline{\beta}(a) &= \lim_{n\to\infty}\sup_{k\geq 0} \sqrt[n]{\prod_{j=k}^{k+n-1} a_j} = \lim_{p\to\infty}\sup_{k\geq 0} \sqrt[2^p-1]{\prod_{j=k}^{k+2^p-2} a_j} \geq \liminf_{p\to\infty} \sqrt[2^p-1]{\prod_{j=0}^{2^p-2} a_j} \\ &= \liminf_{p\to\infty} \sqrt[2^p-1]{\gamma_{p-1}\gamma_{p-2}^4\cdots\gamma_0^{2^{p-1}}}\end{aligned}$$

and it remains to show that the limit inferior on the right-hand side is positive for appropriate sequences γ. Indeed, setting $\gamma_k := e^{-k}$ and taking the logarithm yields

$$\begin{aligned}\ln \sqrt[2^p-1]{\gamma_{p-1}\gamma_{p-2}^4\cdots\gamma_0^{2^{p-1}}} &= \frac{2^p}{2^p-1}\sum_{j=0}^{p-1}\frac{\ln\gamma_j}{2^{j+1}} = -\frac{2^p}{2^p-1}\sum_{j=0}^{p-1}\frac{j}{2^{j+1}} \\ &= \frac{p-2^p-1}{2^p-1} \xrightarrow[p\to\infty]{} -1.\end{aligned}$$

Consequently, $e^{-1} > 0$ is a lower bound for $\overline{\beta}(a)$. On the other hand, for every $n \in \mathbb{Z}_1^+$ there exists a finite discrete subinterval of $\mathbb{Z}_0^+$ on which (at least) one entry of the sequence a^n vanishes, which guarantees $\overline{\beta}(a^n) = 0$.

Let us now consider the scalar difference eqns. (Δ_b) and (Δ_{b^n}), whose coefficients

$$b_k := a_k + \tfrac{1}{2^k}, \qquad b_k^n := a_k^n + \tfrac{1}{2^k} \quad \text{for all } k \in \mathbb{Z}_0^+,$$

are positive and bounded. This leads to:

- *Since upper Bohl exponents are invariant under perturbations with limit zero (see [36, Thm. 5]), one has $\overline{\beta}(b) = \overline{\beta}(a) > e^{-1}$ and $\overline{\beta}(b^n) = 0$ for all $n \in \mathbb{Z}_1^+$*
- *The relation $\sup_{k\geq 0}|b_k - b_k^n| = \sup_{k\geq 0}|a_k - a_k^n| = \gamma_n \xrightarrow[n\to\infty]{} 0$ ensures that every neighborhood of (Δ_b) contains an equation (Δ_{b^n}) with Bohl exponent 0.*

In conclusion, the difference eqn. (Δ_b) having nontrivial Sacker-Sell spectrum $\Sigma(b)$ is a uniform limit of eqns. (Δ_{b^n}) with empty spectrum, i.e. $\bar{\Sigma}(b^n) = \{0\}$ for all $n \in \mathbb{Z}_1^+$.

In order to obtain also the desired lower semicontinuity for Σ^+ and therefore convergence in the Hausdorff metric h, let us again rely on the relations (3.1) and employ the weighted shifts $T_A \in L(\ell^2)$ from (C.1), as well as

Proposition 4.3: *Keep $\alpha \in \{F, \pi\}$ fixed. If $\sigma_\alpha : L(\ell^2) \to K(\mathbb{C})$ is continuous at T_A, then $\bar{\Sigma}^+ : \mathcal{L}^\infty(\mathbb{C}^d) \to \mathbb{R}$ is continuous at A.*

Proof. The elementary argument for the Fredholm spectrum, although geometrically clear, can be found in [35, Prop. 5.3]. Thanks to Prop. C.5 one also has $\sigma_F = \sigma_\pi$ on the class of weighted unilateral shifts and the claim follows. □

Difference equations with discrete spectrum are always points of continuity for Σ^+; they particularly include (asymptotically) autonomous or periodic equations:

Theorem 4.4 (continuity of Σ^+)**:** *If $\Sigma^+(A)$ is discrete, then the Sacker-Sell spectrum Σ^+ is continuous at A.*

While our subsequent proof of Thm. 4.4 is purely operator-theoretical, a referee provided the following more intuitive (and shorter!) argument: The spectrum of a perturbed difference eqn. (Δ_B) is contained in a neighborhood of the discrete set $\Sigma^+(A)$ by upper-semicontinuity. W.l.o.g. this neighborhood is a union of open intervals containing the spectral points of (Δ_A). It remains to show that $\Sigma^+(B)$ has a nonempty intersection with each of these intervals. Crossing such an interval geometrically means for both (Δ_A) and (Δ_B) that the dimension of the stable subspace changes. Hence, each interval must contain some spectrum of (Δ_B).

Proof. We briefly write for the resolvent sets

$$\begin{aligned} \rho_{sF}^n(T_A) &:= \{\lambda : T_A - \lambda\,\mathrm{id} \text{ is semi-Fredholm with index } n\} \quad \text{for all } n \in \mathbb{Z} \cup \{\pm\infty\}, \\ \rho_\pm(T_A) &:= \bigcup_{1 \le |n| \le \infty} \rho_{sF}^n(T_A) \end{aligned}$$

and verify the following conditions:

(I) $\sigma_{F_0}(T_A) \setminus \overline{\rho_\pm(T_A)} = \emptyset$, because thanks to Prop. C.5 one has

$$\sigma_{F_0}(T_A) \stackrel{(C.3)}{=} \sigma(T_A) = \sigma_F(T_A) \dot\cup \bigcup_{n=1}^{d} \rho^{-n}(T_A)$$

with the open bounded annuli $\rho^n(T_A)$ introduced in Prop. C.5. Since $\sigma_F(T_A)$ has empty interior by assumption, we thus conclude

$$\sigma_{F_0}(T_A) = \overline{\bigcup_{n=1}^{d} \rho^{-n}(T_A)} \subseteq \overline{\rho_\pm(T_A)}.$$

(II) *For every $n \neq 0$, $\lambda \in \operatorname{int} \overline{\rho_{sF}^n(T_A)} \setminus \rho_{sF}^n(T_A)$ and $\varepsilon > 0$ the ball $B_\varepsilon(\lambda)$ contains a component of $\sigma_F(T_A)$*: In order to establish this, we observe from Prop. C.3(f) and the spectral picture given in Prop. C.5 that

$$\rho^n_{sF}(T_A) = \begin{cases} \rho^n(T_A), & -d \le n < 0, \\ \emptyset, & n < -d \text{ or } 0 < n. \end{cases}$$

These open sets clearly satisfy the inclusion $\operatorname{int} \overline{\rho^n(T_A)} \subseteq \rho^n(T_A)$ and therefore $\operatorname{int} \overline{\rho^n_{sF}(T_A)} \setminus \rho^n_{sF}(T_A) = \emptyset$.

Thanks to (I) we deduce from [9, Thm. 3.6] that σ_{F_0} is continuous at T_A. In combination with (II) this enables us to apply [10, Thm. 4.1]. Consequently, also the Fredholm spectrum σ_F is continuous at T_A. Eventually, Prop. 4.3 guarantees as desired that the Sacker-Sell spectrum Σ^+ is continuous at A. □

While Thm. 4.4 is a criterion that the dichotomy spectrum of a particular (Δ_A) is continuous, we now obtain whole classes of equations with this behavior:

Theorem 4.5: *The Sacker-Sell spectrum Σ^+ of (Δ_A) is continuous on $\mathcal{EC}(A)$ w.r.t. perturbations in $\mathcal{EC}(A)$.*

Proof. Let $(A^n)_{n\ge1}$ be a sequence in $\mathcal{EC}(A) \subseteq \mathcal{L}^\infty(\mathbb{C}^d)$ with limit A and thus

$$\lim_{k\to\infty} \left| A^n_{k+1} A_k - A_{k+1} A^n_k \right| = 0 \quad \text{for all } n \ge 1.$$

Combining (C.2) and Prop. C.3(a), implies that $T_{A^n} T_A - T_A T_{A^n} \in L(\ell^2)$, $n \ge 1$, are compact. Because Lemma C.2 yields $\lim_{n\to\infty} T_{A^n} = T_A$ we can deduce from [26, p. 53, Lemma 2.3.2] that the limit relation $\lim_{n\to\infty} \sigma_F(T_{A^n}) = \sigma_F(T_A)$ holds. Then the claim again follows from Prop. 4.3. □

Theorem 4.6: *The Sacker-Sell spectrum Σ^+ is continuous on the sets $\mathcal{EP}_p(\mathbb{C}^d)$, $p \in \mathbb{Z}^+_1$.*

Proof. Let $p \in \mathbb{Z}^+_1$. If $A \in \mathcal{EP}_p(\mathbb{C}^d)$ is limit of a sequence $(A^n)_{n\ge1}$ in $\mathcal{EP}_p(\mathbb{C}^d)$, then the representations $A = \bar{A} + K$, $A^n = \bar{A}^n + K^n$ with

$$\begin{aligned} &\bar{A} \in \mathcal{P}_p(\mathbb{C}^d), && \lim_{k\to\infty} K_k = 0, \\ &\bar{A}^n \in \mathcal{P}_p(\mathbb{C}^d), && \lim_{k\to\infty} K^n_k = 0 \quad \text{for all } n \ge 1 \end{aligned}$$

hold. Now Prop. C.9(b) shows that the powers $T^p_{\bar{A}}$ and $T^p_{\bar{A}^n}$ are paranormal, and both $T_A = T_{\bar{A}} + T_K$, as well as $T_{A^n} = T_{\bar{A}^n} + T_{K^n}$ are compact perturbations to pth roots of paranormal operators by Prop. C.3(a). Thus, [13, Thm. 2.5] guarantees the limit relation $\lim_{n\to\infty} \sigma_F(T_{A^n}) = \sigma_F(T_A)$ in the Hausdorff metric and anew Prop. 4.3 yields continuity on the set $\mathcal{EP}_p(\mathbb{C}^d)$. □

The Thm. 4.6 also guarantees continuity of Σ^+ on the subsets $\mathcal{EH}_p(\mathbb{C}^d) \subseteq \mathcal{EP}_p(\mathbb{C}^d)$ (cf. Prop. 3.5(c)), whose elements are easier to characterize than those of $\mathcal{EP}_p(\mathbb{C}^d)$.

5. Further remarks

5.1. *Backward spectrum*

While the forward spectrum $\Sigma^+(A)$ has immediate applications in stability theory, also the backward spectrum is relevant. Indeed, gaps in $\Sigma^-(A)$ lead to unstable subspaces persisting under small nonlinearities leading to unstable fiber bundles, i.e. nonautonomous unstable manifolds (cf. [32, p. 259, Thm. 4.6.4(b)]). Furthermore, $\Sigma^-(A)$ is independent of the initial time κ. The forward and backward spectra have analogous properties. Concerning a proof, it suffices to note that our obtained results carry over to the backward spectrum by means of the following spectral mapping theorem:

Proposition 5.1: *Under* (2.3) *the relation*

$$\Sigma^-(A) = 1/\Sigma^+(B) := \left\{\tfrac{1}{\gamma} > 0 : \gamma \in \Sigma^+(B)\right\}$$

holds for the coefficient sequence $A_k := B^{-1}_{2\kappa-k-1}$, $k \in \mathbb{Z}^-_{\kappa-1}$.

This ensures that the backward spectrum of (Δ_A) can be represented via the forward spectrum of the inverted equation (Δ_B).

Remark 5.2 (bounded growth)**:** (1) Throughout the paper, we restricted to bounded coefficient matrices. Technically, this had the enormous benefit to work with bounded shift operators T_A. Dynamically, this assumption prevented arbitrarily large growth. The additional assumption (2.3) avoids also arbitrarily small growth (as featured in Ex. 2.1(2)). Such systems, whose spectrum has positive distance from 0, are said to possess *bounded growth* (see, e.g., [3]). Prop. 5.1 applies to this class and consequently also the spectrum $\Sigma^+(B)$ stays away from 0.

(2) Note that the equations from Ex. 4.2 featuring a discontinuous Sacker-Sell spectrum fail to have bounded growth, since their coefficients violate (2.3).

Proof. W.l.o.g. we can assume an initial time $\kappa = 0$. Moreover, our assumption (2.3) guarantees that $(|A_k|)_{k<\kappa}$ is bounded away from 0. It is easy to show that (Δ_A) has an ED on $\mathbb{Z}^+_0$ with projector P_k, $k \in \mathbb{Z}^+_0$, if and only if

$$x_{k+1} = A_k x_k, \qquad\qquad A_k := B^{-1}_{-k-1}$$

possesses an ED on $\mathbb{Z}^-_{-1}$ with projector $\mathrm{id} - P_{-k}$. Therefore, the following equivalences

$$\begin{aligned}\gamma \in \Sigma^+(B) &\Leftrightarrow x_{k+1} = \gamma^{-1} B_k x_k \text{ has an ED on } \mathbb{Z}^+_0\\ &\Leftrightarrow x_{k+1} = \gamma A_k x_k \text{ has an ED on } \mathbb{Z}^-_{-1} \Leftrightarrow \tfrac{1}{\gamma} \in \Sigma^-(A)\end{aligned}$$

hold and imply the claim. □

5.2. *All time spectrum*

We point out that Thm. 4.4 has immediate consequences for difference eqns. (Δ_A) defined on the whole line $\mathbb{Z}$. In fact, their full line Fredholm dichotomy spectrum $\Sigma_F(A)$ allows the representation (see [33, Prop. 4.29(b)]) $\Sigma_F(A) = \Sigma^+(A) \cup \Sigma^-(A)$. This implies that continuity properties of the half line spectra Σ^+ and Σ^- extend to Σ_F.

Let us compare our current approach to the companion paper [35]: An exponential dichotomy on the full line is a significantly tighter assumption than merely having an ED on a semi axis. For this reason, $\Sigma^{\pm}(A)$ are not only smaller than $\Sigma(A)$, but also have stronger robustness properties. On a technical level, $\Sigma^{+}(A)$ is related to the essential spectrum, while $\Sigma(A)$ could be treated via the usual spectrum and we needed to employ different operator theoretical tools. Concretely, in [35, Thm. 5.3] we identified equations with a large Weyl and a small Fredholm spectrum to feature a continuous all time spectrum. This criterion applies to equations having spectral intervals of positive length and excludes examples with discontinuous spectrum (see [35, Ex. 5.6]). Our sufficient criterion Thm. 4.4 is limited to equations with discrete spectrum. Furthermore, the equation classes $\mathcal{P}_p(\mathbb{C}^d)$ featuring continuity from [35, Thm. 5.5] were smaller and more rigid than in the present case of Thm. 4.6

5.3. *Continuous time*

Our results imminently allow an application to ordinary differential equations. We actually can study the continuous time situation of linear Carathéodory differential equations

$$\boxed{\dot{x} = A(t)x} \quad \text{with } A \in L^{\infty}_{\text{loc}}([\tau, \infty), L(\mathbb{C}^d)) \tag{5.1}$$

on a half line $[\tau, \infty)$, $\tau \in \mathbb{R}$ (cf. [1]). Their Sacker-Sell spectrum $\hat{\Sigma}^{+}(A) \subseteq \mathbb{R}$ was investigated in [38, 40]. The corresponding transition matrix $U(t, s) \in GL(\mathbb{C}^d)$, $s, t \in \mathbb{R}$ is exponentially bounded (see [1, Lemma 2.9]) and we define the matrix sequence

$$A_k := U(k + 1, k) \in GL(\mathbb{C}^d) \quad \text{for all } k \in [\tau, \infty)_{\mathbb{Z}}. \tag{5.2}$$

Then [11, p. 111, Lemma 2.4] implies the estimates

$$\exp\left(-\int_k^{k+1} |A(s)|\, ds\right) \leq \left|A_k^{\pm 1}\right| \leq \exp\left(\int_k^{k+1} |A(s)|\, ds\right) \quad \text{for all } k \geq \tau$$

and therefore the assumption $\sup_{\tau \leq k} \int_k^{k+1} |A(s)|\, ds < \infty$ guarantees that a difference eqn. (Δ_A) has bounded growth.

The characterization [39, Thm. 5.1] implies that the dichotomy spectra of the difference eqn. (Δ_A) and of the ODE (5.1) allow the spectral mapping relations

$$\Sigma^{+}(A) = \exp \hat{\Sigma}^{+}(A), \qquad \hat{\Sigma}^{+}(A) = \ln \Sigma^{+}(A). \tag{5.3}$$

By means of these identities several previous results apply to the specific coefficient sequences (5.2) and yield information on the continuous time spectrum $\hat{\Sigma}^{+}(A)$. As a particular example including autonomous or periodic time dependence we obtain

Theorem 5.3 (continuous spectrum for (5.1))**:** *If $\hat{\Sigma}^{+}(A)$ is discrete, then the Sacker-Sell spectrum $\hat{\Sigma}^{+}$ is continuous at A.*

Proof. By means of the relations (5.3) this immediately follows from Thm. 4.4. □

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Appendix A. Hausdorff distance

Given a metric space X, let $K(X)$ denote the family of all compact nonempty subsets of X. Then $d(M_1, M_2) := \sup_{x \in M_1} \operatorname{dist}(x, M_2)$ is the *Hausdorff semidistance* of sets $M_1, M_2 \in K(X)$, while its symmetrized version defines the *Hausdorff distance*

$$h : K(X) \times K(X) \to [0, \infty), \qquad h(M_1, M_2) := \max \{d(M_1, M_2), d(M_2, M_1)\}.$$

The pair $(K(X), h)$ is a metric space (cf. [20, p. 374]).

Appendix B. Operators on Hilbert spaces

For an infinite-dimensional separable and complex Hilbert space X with inner product $\langle \cdot, \cdot \rangle$, let $L(X)$ denote the Banach algebra of bounded linear operators on X with identity id_X. Furthermore, $L_0(X)$ is the ideal of compact operators in $L(X)$.

Given an operator $T \in L(X)$, we denote its kernel by $N(T)$. Let us introduce the

$$\begin{aligned}
\textit{spectrum } \sigma(T) = \sigma_a(T) &:= \{\lambda \in \mathbb{C} : T - \lambda\,\mathrm{id} \text{ is not invertible}\},\\
\textit{point spectrum } \sigma_p(T) &:= \{\lambda \in \mathbb{C} : \dim N(T - \lambda\,\mathrm{id}) > 0\},\\
\textit{approximate point spectrum } \sigma_\pi(T) &:= \{\lambda \in \mathbb{C} : T - \lambda\,\mathrm{id} \text{ is not bounded below}\},\\
\textit{Fredholm spectrum } \sigma_F(T) &:= \{\lambda \in \mathbb{C} : T - \lambda\,\mathrm{id} \text{ is not Fredholm}\},\\
\textit{Weyl spectrum } \sigma_{F_0}(T) &:= \{\lambda \in \mathbb{C} : T - \lambda\,\mathrm{id} \text{ is not Weyl}\},
\end{aligned}$$

where a Fredholm operator with index 0 is called *Weyl operator*; one has

$$\sigma_F(T) \subseteq \sigma_{F_0}(T) \subseteq \sigma(T) \supseteq \sigma_\pi(T) \supseteq \partial\sigma(T). \tag{B.1}$$

We write $r(T) := \sup_{\lambda \in \sigma(T)} |\lambda|$ for the *spectral radius* and define the *essential spectral radius* resp. the *essential norm* of T by

$$r_F(T) := \sup_{\lambda \in \sigma_F(T)} |\lambda|, \qquad \|T\|_F := \inf_{K \in L_0(X)} \|T + K\|.$$

One speaks of a *quasi-nilpotent operator* T, if $r(T) = 0$ i.e. $\sigma(T) = \{0\}$.

The adjoint of $T \in L(X)$ is denoted by T^*. A self-adjoint operator $T = T^*$ is said to be *positive* (in symbols, $T \geq 0$), provided $\langle x, Tx \rangle \geq 0$ holds for all $x \in X$. In case the difference $T - S$ of two self-adjoint operators $S, T \in L(X)$ is positive, we abbreviate $T \geq S$ and obtain the cone-like conditions

$$\beta T \geq \alpha T \geq \alpha S, \qquad T + R \geq S + R \quad \text{for all } 0 \leq \alpha \leq \beta$$

and self-adjoint $R \in L(X)$. With a unitary operator $U \in L(X)$ one finally has

$$T \geq S \quad \Leftrightarrow \quad U^*TU \geq U^*SU. \tag{B.2}$$

An operator $T \in L(X)$ is called

$$\begin{aligned} &\textit{hyponormal} &:\Leftrightarrow \quad & T^*T \geq TT^*, \\ &\textit{paranormal} &:\Leftrightarrow \quad & T^{*2}T^2 - 2rT^*T + r^2 \operatorname{id}_X \geq 0 \quad \text{for all } r > 0 \end{aligned}$$

and for every $p \in \mathbb{Z}_1^+$ we abbreviate the operator classes

$$\begin{aligned} H_p(X) &:= \{S \in L(X) : \, S^p \text{ is hyponormal}\}, \\ P_p(X) &:= \{S \in L(X) : \, S^p \text{ is paranormal}\}. \end{aligned}$$

The elements of $H_p(X)$ (or $P_p(X)$) are called *pth roots* of a hyponormal (resp. paranormal) operator. The sets $H_p(X)$ are closed in the norm topology (cf. [24, Prop. 1.5]), while $H_1(X)$ is nowhere dense in $L(X)$ (cf. [27, Thm. 2.4]). Furthermore, the above operator classes are invariant under multiplication with a complex scalar.

Appendix C. Multiplication and weighted shift operators

Let us write ℓ^2 for the prototypical separable Hilbert space of square-summable sequences $\phi = (\phi_k)_{\kappa \leq k}$ in $\mathbb{C}^d$ equipped with the inner product

$$\langle\!\langle \phi, \psi \rangle\!\rangle := \sum_{k=\kappa}^{\infty} \langle \phi_k, \psi_k \rangle \quad \text{for all } \phi, \psi \in \ell^2$$

and the resulting norm $\|\phi\| = \sqrt{\langle\!\langle \phi, \phi \rangle\!\rangle}$.

Given a bounded *weight sequence* $\Lambda = (\Lambda_k)_{\kappa \leq k}$ of matrices $\Lambda_k \in L(\mathbb{C}^d)$ we denote

$$M_\Lambda : \ell^2 \to \ell^2, \qquad (M_\Lambda \phi)_k := \Lambda_k \phi_k \quad \text{for all } \kappa \leq k$$

as *multiplication operator*. It is bounded with $\|M_\Lambda\| \leq \sup_{\kappa \leq k} |\Lambda_k|$ and

$$\langle\!\langle M_\Lambda \phi, \psi \rangle\!\rangle = \sum_{k=\kappa}^{\infty} \langle \Lambda_k \phi_k, \psi_k \rangle = \sum_{k=\kappa}^{\infty} \langle \phi_k, \Lambda_k^* \psi_k \rangle \quad \text{for all } \phi, \psi \in \ell^2$$

yields the adjoint $(M_\Lambda^* \phi)_k = \Lambda_k^* \phi_k$ for all $\kappa \leq k$. Accordingly, a multiplication operator M_Λ is unitary, if and only if $\Lambda_k^{-1} = \Lambda_k^*$, $\kappa \leq k$, holds.

Proposition C.1 (properties of multiplication operators)**:** *Let* $\Lambda \in \mathcal{L}^\infty(\mathbb{C}^d)$.

(a) *If* $\Lambda_k \in GL(\mathbb{C}^d)$, $\kappa \leq k$, *then* $M_\Lambda \in GL(\ell^2) \Leftrightarrow \sup_{\kappa \leq k} \left|\Lambda_k^{-1}\right| < \infty$ *holds. In particular, it is* $(M_\Lambda^{-1}\phi)_k = \Lambda_k^{-1}\phi_k$ *for all* $\kappa \leq k$.

(b) $M_\Lambda \in L_0(\ell^2)$, *if and only if* $\lim_{k \to \infty} \Lambda_k = 0$.

Proof. Analogous to [35, Prop. B.1]. □

Given another *weight sequence* $A \in \mathcal{L}^\infty(\mathbb{C}^d)$, the *weighted unilateral left shift* reads as

$$T_A : \ell^2 \to \ell^2, \qquad T_A\phi := (0, A_\kappa\phi_\kappa, A_{\kappa+1}\phi_{\kappa+1}, \ldots). \tag{C.1}$$

Clearly, T_A is bounded with $\|T_A\| = \sup_{\kappa\le k} |A_k|$ and such shift operators form a closed subspace of $L(\ell^2)$. Given weight sequences $A, B \in \mathcal{L}^\infty(\mathbb{C}^d)$ one has

$$T_B T_A\phi = (0, 0, B_{\kappa+1}A_\kappa\phi_\kappa, B_{\kappa+2}A_{\kappa+1}\phi_{\kappa+1}, \ldots) \quad \text{for all } \phi \in \ell^2. \tag{C.2}$$

Lemma C.2: *The mapping $T. : \mathcal{L}^\infty(\mathbb{C}^d) \to L(\ell^2)$ is linear and continuous.*

Proof. The linearity of $T.$ is clear. For arbitrary $A \in \mathcal{L}^\infty(\mathbb{C}^d)$ it holds

$$\|T_A\phi\|^2 = \sum_{k=\kappa}^\infty |(T_A\phi)_k|^2 \overset{(C.1)}{=} \sum_{k=\kappa}^\infty |A_k\phi_k|^2 \le \big(\sup_{\kappa\le k} |A_k|\big)^2 \|\phi\|^2 \quad \text{for all } \phi \in \ell^2$$

and consequently $\|T_A\| \le \sup_{\kappa\le k} |A_k|$. □

Our analysis requires several basic properties of unilateral weighted shifts. In this context it is convenient to introduce the *discrete Heaviside function*

$$\theta_l : \mathbb{Z} \to \{0, 1\}, \qquad \theta_l(k) := \begin{cases} 1, & k \ge l, \\ 0, & \text{else} \end{cases} \quad \text{for all } l \in \mathbb{Z}.$$

Proposition C.3 (properties of shifts)**:** *Let $p \in \mathbb{Z}_0^+$, $A \in \mathcal{L}^\infty(\mathbb{C}^d)$ and $\lambda \in \mathbb{C}$.*

(a) $T_A \in L_0(\ell^2)$, if and only if $\lim_{k\to\infty} A_k = 0$.
(b) T_A is quasi-nilpotent, if and only if $\overline{\beta}(A) = 0$.
(c) The adjoint of T_A is given by $T_A^* \in L(\ell^2)$, $(T_A^*\phi)_k = A_k^*\phi_{k+1}$ for all $\kappa \le k$.
(d) T_A is an isometry, if and only if each weight $A_k \in L(\mathbb{C}^d)$, $\kappa \le k$, is unitary.
(e) For every $k \in \mathbb{Z}_\kappa^+$ one has $(T_A^{*p}\phi)_k = \Phi(k+p, k)^*\phi_{k+p}$ and

$$(T_A^p\phi)_k = \theta_{\kappa+p}(k)\Phi(k, k-p)\phi_{k-p} = \begin{cases} 0, & \kappa \le k < \kappa + p, \\ \Phi(k, k-p)\phi_{k-p}, & k \ge \kappa + p. \end{cases}$$

(f) $T_A - \lambda\,\mathrm{id}_{\ell^2}$ is Fredholm, if and only if it is semi-Fredholm. In this case the index satisfies $|\mathrm{ind}(T_A - \lambda\,\mathrm{id})| \le d$.

The above result shows that every compact shift operator is quasi-nilpotent.

Proof. (a) Thanks to the factorization $T_A = T_I M_A$ the claim follows from Prop. C.1(b); here I denotes the sequence of identity mappings on $\mathbb{C}^d$.

(b) This is immediate from $\overline{\beta}(A) = r(T_A)$ (cf. [5, Thm. 1.1(i)]).

(c) and (d) For arbitrary sequences $\phi, \psi \in \ell^2$ we obtain

$$\langle T_A\phi, \psi\rangle \overset{(C.1)}{=} \sum_{k=\kappa}^\infty \langle A_k\phi_k, \psi_{k+1}\rangle = \sum_{k=\kappa}^\infty \langle \phi_k, A_k^*\psi_{k+1}\rangle = \langle \phi, T_A^*\psi\rangle$$

with $(T_A^*\psi)_k := A_k^*\psi_{k+1}$ for all $\kappa \le k$. Furthermore, it is

$$\langle T_A\phi, T_A\psi\rangle \overset{(C.1)}{=} \sum_{k=\kappa}^{\infty}\langle A_k\phi_k, A_k\psi_k\rangle = \sum_{k=\kappa}^{\infty}\langle \phi_k, A_k^*A_k\psi_k\rangle$$

and thus also the assertion concerning T_A being an isometry follows.

(e) Proceeding inductively the claim holds for $p = 0$. As induction step $p \to p+1$,

$$\begin{aligned}(T_A^{p+1}\phi)_k = (T_A(T_A^p\phi))_k &= \begin{cases} 0, & \kappa \le k < \kappa + p, \\ A_{k-1}0, & k = \kappa + p, \\ A_{k-1}\Phi(k-1, k-1-p)\phi_{k-1-p}, & k > \kappa + p \end{cases} \\ &= \begin{cases} 0, & \kappa \le k \le \kappa + p, \\ \Phi(k, k-(p+1))\phi_{k-(p+1)}, & k > \kappa + p \end{cases}\end{aligned}$$

and for all $\kappa \le k$ it is

$$\begin{aligned}((T_A^*)^{p+1})\phi)_k &= (T_A^*T_A^{*p}\phi)_k = A_k^*(T_A^{*p}\phi)_{k+1} = A_k^*\Phi(k+1+p, k+1)^*\phi_{k+p+1} \\ &= (\Phi(k+1+p, k+1)A_k)^*\phi_{k+p+1} = \Phi(k+p+1, k)^*\phi_{k+p+1}.\end{aligned}$$

(f) Because of $\dim N(T_A - \lambda\,\mathrm{id}) \le d$ and $\dim N(T_A - \lambda\,\mathrm{id})^* \le d$ the operator $T_A - \lambda\,\mathrm{id}$ is semi-Fredholm if and only if it is Fredholm with index

$$|\mathrm{ind}(T_A - \lambda\,\mathrm{id})| \le |\dim N(T_A - \lambda\,\mathrm{id}) - \dim N(T_A - \lambda\,\mathrm{id})^*| \le d$$

and this implies the claim. $\square$

Example C.4: *Let us return to the sequence $(a_k)_{k\ge 1}$ defined in Ex. 2.1(2). It follows by Prop. C.3(b) that the weighted shift $T_a \in L(\ell^2)$ has spectral radius 0 and is therefore quasi-nilpotent. On the other hand, due to Prop. C.3(a) it is not compact. In fact, [18] even shows that no power T_a^p, $p \in \mathbb{Z}_1^+$, is compact.*

The spectral properties of unilateral shifts are summarized in

Proposition C.5 (spectra of shifts)**:** *For $A \in \mathcal{L}^\infty(\mathbb{C}^d)$ one has $\sigma_p(T_A) = \emptyset$ and*

$$\partial\sigma(T_A) \subseteq \sigma_\pi(T_A) = \sigma_F(T_A) \subseteq \sigma_{F_0}(T_A) = \sigma(T_A) = \bar{B}_{\overline{\beta}(A)}(0), \tag{C.3}$$

where the occurring spectra are rotationally invariant w.r.t. 0. The spectral picture

$$\sigma(T_A) = \sigma_F(T_A)\dot{\cup}\bigcup_{n=1}^{d}\rho^{-n}(T_A), \qquad \sigma_F(T_A) \subseteq \left\{\lambda \in \mathbb{C} : \underline{\beta}(A) \le |\lambda| \le \overline{\beta}(A)\right\}$$

holds for the pairwise disjoint bounded and open resolvent sets

$$\rho^n(T_A) := \{\lambda \in \mathbb{C} : T_A - \lambda\,\mathrm{id} \text{ is Fredholm with index } n\} \subseteq \sigma(T_A) \quad \text{for all } n \in \mathbb{Z}.$$

We refer to Fig. C1 giving an illustration of Prop. C.5.

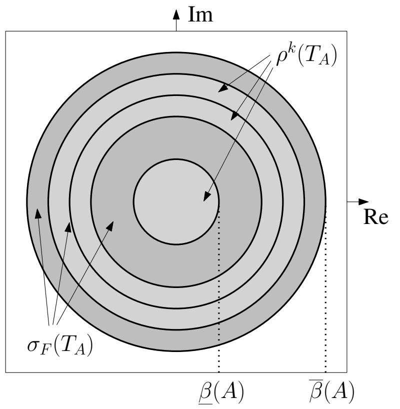

Figure C1 The spectral picture of unilateral weighted shift operators $T_A \in L(\ell^2)$, where the open and bounded annuli $\rho^n(T_A) \subseteq \sigma(T_A)$ with indices $-d \leq n < 0$ (shaded in light grey), are holes in the compact and rotationally invariant Fredholm spectrum $\sigma_F(T_A)$ (dark grey)

Proof. Combining [4, Thm. 5] and [5, Thm. 1.1(i)] yields that $\sigma(T_A)$ is precisely the closed disk $\bar{B}_{\overline{\beta}(A)}(0) \subseteq \mathbb{C}$. Furthermore, referring to [23, p. 127] unilateral weighted shifts have no eigenvalues, i.e. $\sigma_p(T_A) = \emptyset$ and so the inclusion (for this cf. [25, p. 151]) $\rho^0(T_A) \subseteq \sigma_p(T_A)$ implies $\rho^0(T_A) = \emptyset$. Hence, we can deduce from [25, p. 160] that $\sigma(T_A) = \sigma_{F_0}(T_A)$ holds. The identity $\sigma_F(T_A) = \sigma_\pi(T_A)$ also results from [23, p. 127], where $\sigma_F(T_A)$ is rotationally invariant due to [33, Lemma 3.21]. The remaining inclusions in (C.3) follow from (B.1).

Using [25, p. 148, Thm. 5.16(b)] and $\sigma_p(T_A) = \emptyset$ we moreover obtain

$$\rho^n(T_A) = \emptyset \quad \text{for all } n \geq 0 \tag{C.4}$$

(the case $n = 0$ was already shown above). Then a combination of [25, p. 152, Cor. 5.18] and Prop. C.3(f) implies

$$\sigma(T_A) = \sigma_F(T_A) \dot{\cup} \bigcup_{n\in\mathbb{Z}} \rho^n(T_A) = \sigma_F(T_A) \dot{\cup} \bigcup_{n=-d}^{d} \rho^n(T_A) \overset{\text{(C.4)}}{=} \sigma_F(T_A) \dot{\cup} \bigcup_{n=1}^{d} \rho^{-n}(T_A)$$

with mutually disjoint open and bounded annuli $\rho^n(T_A) \subseteq \mathbb{C}$. Finally, the claimed inclusion for the Fredholm spectrum $\sigma_F(T_A)$ can be found in [23, p. 127]. $\square$

Corollary C.6: $r_F(T_A) = r(T_A) = \overline{\beta}(A)$.

Proof. The relation $r(T_A) = \overline{\beta}(A)$ is clear from (C.3), and in case $\sigma(T_A) = \{0\}$ there is nothing to show. Otherwise, [25, p. 156, Cor. 5.21] yields $\partial\sigma(T_A) = \sigma_F(T_A)$ and thanks to (C.3) one concludes $\sup_{\lambda\in\sigma_F(T_A)} |\lambda| = r(T_A)$. $\square$

The next example due to Kakutani illustrates that the spectral radius $r : L(\ell^2) \to \mathbb{R}$ of an unilateral shift operator is an upper-semicontinuous function and forms the basis of Ex. 4.2. Thereby the peripheral spectrum is always contained in the Fredholm spectrum σ_F (cf. (C.3)), which shows that also σ_F can suddenly shrink. We quote from [15, p. 242]:

Example C.7: *Given a sequence $(\gamma_k)_{k\in\mathbb{Z}_0^+}$ we define $a, a^n \in \ell^\infty(\mathbb{C})$ as in Ex. 4.2. In [15, pp. 242–243] it is shown that $r(T_a) > 0$ and thanks to Cor. C.6 one obtains $r_F(T_a) > 0$. Moreover, our construction yields $\|T_{a^n} - T_a\| = \gamma_n$ for all $n \in \mathbb{Z}_1^+$. Due to Prop. C.3(e) one has $T_{a^n}^l = 0$ for $l > 2^{n+1}$, therefore T_{a^n} is nilpotent and*

$$\{0\} = \sigma(T_{a^n}) \overset{\text{(C.3)}}{=} \sigma_F(T_{a^n}) \quad \textit{for all } n \in \mathbb{Z}_1^+$$

holds. Hence, every neighborhood of T_a contains a nilpotent weighted shift.

Proposition C.8: *Every T_A is unitarily equivalent to a weighted left shift T_B with*

(a) *$B_k \in L(\mathbb{C}^d)$ is positive semidefinite Hermitian.*
(b) $\sup_{\kappa \le k} |B_k| < \infty$.

Proof. We set $U_\kappa := \mathrm{id}_{\mathbb{C}^d}$ and claim there exist unitary $U_k \in L(\mathbb{C}^d)$ such that

$$B_k = U_{k+1}^{-1} A_k U_k \quad \text{for all } \kappa \le k. \tag{C.5}$$

The matrix A_κ has a polar decomposition $A_\kappa = U_{\kappa+1} B_\kappa$ with unitary $U_{\kappa+1} \in L(\mathbb{C}^d)$ and positive semidefinite $B_\kappa \in L(\mathbb{C}^d)$. This yields (C.5) for $k = \kappa$. In the induction step $k-1 \to k$ we invest that $A_k U_k$ admits the polar decomposition $A_k U_k = U_{k+1} B_k$ with unitary U_{k+1} and a positive semidefinite B_k, with U_k is known by the hypothesis. Hence, (C.5) holds for $k \ge \kappa$. For the multiplication operator $M_U \in L(\ell^2)$ we obtain

$$(M_U^* T_A M_U \phi)_k = \begin{cases} 0, & k = \kappa, \\ U_k^* A_{k-1} U_{k-1} \phi_{k-1}, & k > \kappa \end{cases} \overset{\text{(C.5)}}{=} (T_B \phi)_k \quad \text{for all } \kappa \le k$$

and (C.5) immediately shows that the boundedness of A_k carries over to B_k. □

Proposition C.9: *Let $p \in \mathbb{Z}_0^+$ and $A \in \mathcal{L}^\infty(\mathbb{C}^d)$. The pth power T_A^p is*

(a) *hyponormal, if and only if for all $\kappa \le k$ one has*

$$\Phi(k+2p, k+p)^* \Phi(k+2p, k+p) \ge \Phi(k+p, k) \Phi(k+p, k)^*, \tag{C.6}$$

(b) *paranormal, if and only if for all $\kappa \le k$ and $r > 0$ one has*

$$\Phi(k+2p, k)^* \Phi(k+2p, k) - 2r \Phi(k+p, k)^* \Phi(k+p, k) + r^2 \mathrm{id}_{\mathbb{C}^d} \ge 0. \tag{C.7}$$

Proof. For $p = 0$ the claims are trivial. Given $p \in \mathbb{Z}_1^+$ let us abbreviate $S := T_A^p$ and choose any sequence $\phi \in \ell^2$. We obtain from Prop. C.3(e) the multiplication operators $(S^* S \phi)_k = \Phi(k+p, k)^* \Phi(k+p, k) \phi_k$,

$$\begin{aligned} (SS^*\phi)_k &= \begin{cases} 0, & \kappa \le k < \kappa + p, \\ \Phi(k, k-p) \Phi(k, k-p)^* \phi_k, & \kappa + p \le k, \end{cases} \\ (S^{*2} S^2 \phi)_k &= \Phi(k+2p, k)^* \Phi(k+2p, k) \phi_k, \\ ((S^* S)^2 \phi)_k &= (\Phi(k+p, k)^* \Phi(k+p, k))^2 \phi_k \quad \text{for all } \kappa \le k \end{aligned}$$

by means of the weighted shift operators $(S^{*2}\phi)_k = \Phi(k+2p, k)^* \phi_{k+2p}$ and

$$(S^2 \phi)_k = \begin{cases} 0, & \kappa \le k < \kappa + 2p, \\ \Phi(k, k-2p) \phi_{k-2p}, & \kappa + 2p \le k. \end{cases}$$

(a) Because of

$$\langle\!\langle S^*S\phi - SS^*\phi, \phi\rangle\!\rangle = \sum_{k=\kappa}^{\kappa+p-1} \underbrace{\langle \Phi(k+p,k)\phi_k, \Phi(k+p,k)\phi_k\rangle}_{\geq 0}$$
$$+ \sum_{k=\kappa+p}^{\infty} \langle (\Phi(k+p,k)^*\Phi(k+p,k) - \Phi(k,k-p)\Phi(k,k-p)^*)\phi_k, \phi_k\rangle$$

the relation $S^*S - SS^* \geq 0$ holds, if and only if (C.6) is satisfied.

(b) Finally, $S^{*2}S^2 - 2rS^*S\phi + r^2 \operatorname{id}_{\mathbb{C}^d} \geq 0$, $r > 0$, characterizes (C.7) due to

$$\langle\!\langle (S^{*2}S^2)\phi - 2rS^*S\phi + r^2\phi, \phi\rangle\!\rangle = r^2 \sum_{k=\kappa}^{\infty} \langle \phi_k, \phi_k\rangle$$
$$- 2r \sum_{k=\kappa}^{\infty} \langle \Phi(k+p,k)^*\Phi(k+p,k)\phi_k, \phi_k\rangle + \sum_{k=\kappa}^{\infty} \langle \Phi(k+2p,k)^*\Phi(k+2p,k)\phi_k, \phi_k\rangle$$

and this completes the proof. □

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