A NOTE ON THE DICHOTOMY SPECTRUM

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ABSTRACT. In many ways, exponential dichotomies are an appropriate hyperbolicity notion for nonautonomous linear differential or difference equations. The corresponding dichotomy spectrum generalizes the classical set of eigenvalues or Floquet multipliers and is therefore of eminent importance in a stability theory for explicitly time-dependent systems, as well as to establish a geometric theory of nonautonomous problems with ingredients like invariant manifolds and normal forms, or to deduce continuation and bifurcation techniques.

In this note, we derive some invariance and perturbation properties of the dichotomy spectrum for nonautonomous linear difference equations in Banach spaces. They easily follow from the observation that the dichotomy spectrum is strongly related to a weighted shift operator on an ambient sequence space.

When interested in a robust hyperbolicity notion for nonautonomous equations, two spectral notations are commonly used: The *dichotomy spectrum*, which has been introduced in [11] for linear ODEs, is dynamically defined using asymptotic properties of transition matrices. Corresponding results for difference equations date back to [5], and for noninvertible equations to [1]. Less known is the concept of *spectral dichotomy* due to [2, 3] (see also [4]), who deduce dynamic properties using the spectrum of a weighted shift operator. Consequently, the motivation to write the present note is to combine both approaches. We relate the dynamically characterized dichotomy spectrum to the spectrum of a linear bounded operator. In doing so, it is accessible to a well-established perturbation theory (cf. [9]) and, as opposed to classical proofs, properties as the ℓ^∞- or ℓ_0-robustness of dichotomies are easily derived.

Corresponding perturbation results for the dichotomy spectrum of linear ODEs have been deduced in [11, Theorem 6] and [10, Section 4] in the flexible framework of linear skew-product flows. Yet, our approach is based on a discrete counterpart to the evolution semigroups considered in [6, pp. 62ff, Section 3.2] — a monograph devoted to the asymptotic behavior of nonautonomous evolutionary equations. Related discrete results

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(cf. [6, pp. 87ff, Section 4.1]) are used to obtain admissibility results for linear-inhomogeneously perturbed abstract differential equations

Throughout, let X be a Banach space over the field $\mathbb{K}$ of real or complex numbers, let $\mathbb{I}$ be a discrete interval and set $\mathbb{I}' := \{k \in \mathbb{I} : k+1 \in \mathbb{I}\}$. We suppose $A_k \in L(X)$, $k \in \mathbb{I}'$, is a sequence of bounded linear operators. In the center of our interest are explicit linear nonautonomous difference equations

$$(L) \qquad \boxed{x_{k+1} = A_k x_k}$$

with *transition operator* $\Phi(k,l) \in L(X)$, $k,l \in \mathbb{I}$, associated to A_k defined by

$$\Phi(k,l) := \begin{cases} I_X & \text{for } k = l, \\ A_{k-1}\cdots A_l & \text{for } k > l, \\ A_k^{-1}\cdots A_{l-1}^{-1} & \text{for } k < l, \text{ if } A_k,\ldots,A_{l-1} \text{ are invertible.} \end{cases}$$

We say (L) admits an *exponential dichotomy* (ED for short) on $\mathbb{I}$, if there exists a sequence of projections $P_k \in L(X)$ such that $A_k P_k = P_{k+1} A_k$, A_k is an isomorphism between the kernels $N(P_k)$ and $N(P_{k+1})$ for $k \in \mathbb{I}'$ and there exist reals $\alpha \in (0,1)$, $K \geq 1$ with

$$\|\Phi(k,l)P_l\| \leq K\alpha^{k-l}, \quad \|\Phi(l,k)[I-P_k]\| \leq K\alpha^{k-l} \quad \text{for all } l \leq k,\ k,l \in \mathbb{I}.$$

From now on we retreat to the situation of two-sided time, i.e., $\mathbb{I} = \mathbb{Z}$ (see [5, 1] for the one-sided case). Given reals $\gamma > 0$, we introduce the scaled difference equation

$$(L)_\gamma \qquad \boxed{x_{k+1} = \gamma^{-1} A_k x_k}$$

with transition operator $\Phi_\gamma(k,\kappa) = \gamma^{\kappa-k}\Phi(k,\kappa)$. Then the *dichotomy resolvent* and *dichotomy spectrum* of (L) are given as

$$\rho(A) := \{\gamma > 0 : (L)_\gamma \text{ admits an ED}\}, \qquad \Sigma(A) := (0,\infty) \setminus \rho(A),$$

respectively. As we intend to demonstrate, operator theoretical tools are convenient to investigate properties of the dichotomy spectrum. Thereto, let ℓ be one of the canonically normed spaces $\ell^p(X)$ with $p \in [1,\infty]$, or $\ell_0(X)$. Throughout this note, we suppose

$$(1) \qquad \omega_+(A) := \sup_{k\in\mathbb{Z}} \|A_k\| < \infty$$

and then it is easy to see that the linear operators $L_A, T_A : \ell \to \ell$ are well-defined

$$(L_A\phi)_k := \phi_{k+1} - A_k\phi_k, \qquad (T_A\phi)_k = A_{k-1}\phi_{k-1} \quad \text{for all } k \in \mathbb{Z}$$

and the iterates of the shift operator T_A read as $(T_A^n\phi)_k = \Phi(k,k-n)\phi_{k-n}$ for all $n \in \mathbb{N}_0$. Then the following properties are well-known (cf. [2, 3, 8]):

(P_1) $\sigma(T_A) \neq \emptyset$ does not depend on the particular choice of ℓ,

(P_2) $\sigma(T_A)$ is rotationally symmetric w.r.t. $0 \in \mathbb{C}$,
(P_3) $1 \notin \sigma(T_A)$ if and only L_A is invertible if and only (L) has an ED on $\mathbb{Z}$,
(P_4) for the spectral radius one has $r(T_A) \leq \omega_+(A)$, and if there exist reals $\omega_- > 0$, $K > 0$ such that $\|\Phi(k,l)\| \geq K\omega_-^{k-l}$ for all $k \geq l$, then $\sigma(T_A) \cap B_{\omega_-}(0) = \emptyset$.

Our first result is of basic importance for all the following. It relates the spectral dichotomy from [3] to the dichotomy spectrum, and thus the dynamically characterized set $\Sigma(A)$ to the usual spectrum of a bounded weighted shift operator on ℓ.

Theorem 1. $\Sigma(A) = \sigma(T_A) \cap (0, \infty)$

Proof. Due to Property (P_3) the following assertions are equivalent for values $\lambda \neq 0$,

$$\begin{aligned} \lambda \in \rho(T_A) \quad &\Leftrightarrow \quad T_A - \lambda I \in L(\ell) \text{ is invertible} \\ &\Leftrightarrow \quad \forall \psi \in \ell : \exists^1 \phi \in \ell : T_A\phi - \lambda\phi = \psi \\ &\Leftrightarrow \quad \forall \psi \in \ell : x_{k+1} = \lambda^{-1} A_k x_k + \lambda^{-1}\psi_{k+1} \\ &\qquad \text{has a unique solution in } \ell \\ &\Leftrightarrow \quad x_{k+1} = \lambda^{-1} A_k x_k \text{ has an ED} \\ &\Leftrightarrow \quad x_{k+1} = |\lambda|^{-1} A_k x_k \text{ has an ED} \quad \Leftrightarrow \quad |\lambda| \in \rho(A) \end{aligned}$$

and we obtain $\lambda \in \sigma(T_A)$ if and only if $|\lambda| \in \Sigma(A)$. Finally, from (P_2) we know that $\sigma(T_A)$ is rotationally symmetric and therefore the assertion holds. □

Corollary 2. *In case $d := \dim X < \infty$ the dichotomy spectrum $\Sigma(A)$ is the disjoint union of $n \leq d$ nonempty so-called* spectral intervals $\sigma_1 \ldots, \sigma_n \subseteq (0, \infty)$ *with*

$$\Sigma(A) = \sigma_1 \cup \ldots \cup \sigma_n, \qquad \sup \sigma_i < \inf \sigma_{i+1} \quad \textit{for all } 1 \leq i < n$$

and $\sigma_2, \ldots, \sigma_n$ are compact. Moreover, if each A_k, $k \in \mathbb{Z}$, is invertible, then also σ_1 is compact with $\inf \sigma_1 > 0$*; otherwise $\sigma_1 = (0, a]$ for some $a > 0$.*

Proof. It is shown in [3, Theorem 4] that $\sigma(T_A)$ consists of no more than d concentric rings in $\mathbb{C}$. Then the claim follows in combination with Theorem 1. □

The invariance of $\Sigma(A)$ under Lyapunov transformations follows immediately:

Corollary 3. *If $B_k \in L(X)$, $C_k \in GL(X)$ satisfy $C_{k+1}A_k = B_kC_k$, $k \in \mathbb{Z}$, then*

$$\omega_+(C), \omega_+(C^{-1}) < \infty \quad \Rightarrow \quad \Sigma(A) = \Sigma(B).$$

Proof. Due to $\omega_+(C) < \infty$, the diagonal operator $D \in L(\ell)$, $(D\phi)_k := C_k\phi_k$ is well-defined and thanks to $\omega_+(C^{-1}) < \infty$ also invertible. Moreover, one has $DT_A = T_B D$ and consequently $\sigma(T_A) = \sigma(T_B)$. □

We continue with a series of perturbation results for the dichotomy spectrum, where the distance between subsets $\Sigma_1, \Sigma_2 \subseteq \mathbb{K}$ is measured using the *Hausdorff semidistance*

$$\operatorname{dist}(\Sigma_1, \Sigma_2) := \sup_{\lambda_1\in\Sigma_1} \inf_{\lambda_2\in\Sigma_2} |\lambda_1 - \lambda_2| .$$

For a nonautonomous continuation and bifurcation theory it is of crucial importance to understand the behavior of $\Sigma(A)$ under variation of A. Thus, perturbation results are useful and first we guarantee that exponentially dichotomous equations (L) are open in the set of all linear equations satisfying (1) equipped with the uniform operator topology.

Corollary 4. *(ℓ^∞-roughness) The dichotomy spectrum $\Sigma(A)$ of (L) is upper-semicontinuous. If reals $\gamma_1, \dots, \gamma_n > 0$ satisfy*

$$\{\gamma_1, \dots, \gamma_n\} \cap \Sigma(A) = \emptyset,$$

then there exists a $\varepsilon > 0$ such that

$$B_k \in L(X) \text{ and } \sup_{k\in\mathbb{Z}} \|A_k - B_k\| \le \varepsilon \quad \Rightarrow \quad \{\gamma_1, \dots, \gamma_n\} \cap \Sigma(B) = \emptyset.$$

Proof. The first claim follows from Theorem 1 and the upper-semicontinuity of the spectrum $\sigma(T_A)$ (cf. [9, pp. 208–209, Remark 3.3]). By assumption and Theorem 1, the circles with radii γ_i, $1 \le i \le n$, centered around 0 form a spectral decomposition of $\sigma(T_A)$. Thanks to [9, p. 212, Theorem 3.16] this splitting persists under small perturbations of T_A in $L(\ell)$ and therefore, Theorem 1 yields the assertion. □

The following linear difference system illustrates the upper-semicontinuity of $\Sigma(A)$, i.e., the fact that it can suddenly shrink under perturbation.

Example 1. Suppose $X = \mathbb{R}^2$ and let $A_k(\lambda) \in \mathbb{R}^{2\times 2}$ be given by

$$A_k(\lambda) :\equiv D(\lambda)^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} D(\lambda) \quad \text{for all } k < 0, \quad A_k(\lambda) :\equiv \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

for all $k \ge 0$, with $D(\lambda) := \left(\begin{smallmatrix} -\sin\lambda & \cos\lambda \\ -\cos\lambda & -\sin\lambda \end{smallmatrix}\right)$ and parameters $\lambda \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\alpha \in (0,1)$. It is easy to see that (L) admits an ED on $\mathbb{Z}_{-1}^-$, as well as on $\mathbb{Z}_0^+$, with respective invariant projectors $P_k^- \equiv \left(\begin{smallmatrix} \sin^2\lambda & -\sin\lambda\cos\lambda \\ -\sin\lambda\cos\lambda & \cos^2\lambda \end{smallmatrix}\right)$ and $P_k^+ \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. For $\lambda \ne 0$ we use [4, Theorem 4.8] to compute $\sigma(T_{A(\lambda)})$ and Theorem 1 yields $\Sigma(A(\lambda)) = \{\alpha, \alpha^{-1}\}$. On the other hand, for $\lambda = 0$ we employ the explicit transition matrix $\Phi(k, l)$ to observe that the scaled system $(L)_\gamma$ has a nontrivial complete bounded solution (and no ED on

$\mathbb{Z}$, cf. (P_3)), if and only if $\alpha \leq \gamma \leq \alpha^{-1}$; so $\left[\alpha, \alpha^{-1}\right] \subseteq \Sigma(A(0))$. By (P_4) the property $|\Phi(k,l)| \leq K\alpha^{-|k-l|}$ for all $k,l \in \mathbb{Z}$ additionally yields $\Sigma(A(0)) \subseteq [\alpha, \alpha^{-1}]$ and we conclude

$$\Sigma(A(\lambda)) = \begin{cases} \left[\alpha, \alpha^{-1}\right], & \lambda = 0, \\ \left\{\alpha, \alpha^{-1}\right\}, & \lambda \neq 0. \end{cases}$$

Yet, for particular perturbations it is possible to deduce Lipschitz continuity of $\Sigma(A)$:

Corollary 5. *If $B_k \in L(X)$ satisfies $B_{k+1}A_k = A_{k+1}B_k$, $k \in \mathbb{Z}$, then*

$$\omega_+(B) < \infty \quad \Rightarrow \quad \operatorname{dist}(\Sigma(A), \Sigma(A+B)) \leq \omega_+(B).$$

Proof. The weighted shift operator T_B commutes with T_A, is bounded $r(T_B) \leq \omega_+(B)$ (cf. (P_4)) and we have $T_{A+B} = T_A + T_B$. Consequently, [9, p. 209, Theorem 3.6] yields the continuity estimate

$$\operatorname{dist}(\sigma(T_A), \sigma(T_A + T_B)) \leq \omega_+(B)$$

and thus our claim. □

Lemma 6. *If the transition operator associated with A_k satisfies*

$$N(\Phi(k, k-n)) = \{0\} \quad \text{for all } k \in \mathbb{Z},\ n \in \mathbb{N}, \tag{2}$$

then the spectrum and the essential spectrum of T_A coincide, i.e.,

$$\sigma(T_A) = \sigma_{\text{ess}}(T_A).$$

Proof. We proceed indirectly and assume that there exists a spectral point $\lambda \in \sigma(T_A) \setminus \sigma_{\text{ess}}(T_A)$. Referring to [9, p. 242, Theorem 5.3] the spectral point λ must be an isolated eigenvalue of T_A with finite multiplicity. Thanks to (P_2), this implies $\lambda = 0$ and there exists a nonzero vector $\phi \in \ell$ with $T_A^n \phi = 0$ for some $n \in \mathbb{N}$, which contradicts our assumption. □

Theorem 7. *(ℓ_0-roughness) Suppose $\dim X < \infty$ and $B_k \in L(X)$, $k \in \mathbb{Z}$, is a sequence of compact operators satisfying*

$$\lim_{k \to \pm\infty} \|B_k\| = 0.$$

If the transition operators associated with A_k and $A_k + B_k$ both satisfy (2)*, then $\Sigma(A) = \Sigma(A+B)$.*

Proof. We deduce, e.g., from [7, Lemma 5.3, 5.6] that $T_B \in L(\ell)$ is compact and therefore [9, p. 244, Theorem 5.35] guarantees

$$\sigma_{\text{ess}}(T_A) = \sigma_{\text{ess}}(T_A + T_B) = \sigma_{\text{ess}}(T_{A+B}).$$

Consequently, the above Lemma 6 and Theorem 1 yield the assertion. □

For a sequence $B_k \in L(X)$ as in Theorem 7 the corresponding shift operator $T_B \in L(\ell)$ is compact and (P_2) yields $\sigma(T_B) = \{0\}$. Therefore, Theorem 1 implies $\Sigma(B) = \emptyset$.

References

[1] B. Aulbach and S. Siegmund, *The dichotomy spectrum for noninvertible systems of linear difference equations*, Journal of Difference Equations and Applications **7** (2001), no. 6, 895–913.

[2] B. Aulbach, N. Van Minh, and P. Zabreiko, *The concept of spectral dichotomy for linear difference equations*, Journal of Mathematical Analysis and Applications **185** (1994), 275–287.

[3] B. Aulbach and N. Van Minh, *The concept of spectral dichotomy for linear difference equations II*, Journal of Difference Equations and Applications **2** (1996), 251–262.

[4] A. Ben-Artzi and I. Gohberg, *Dichotomy, discrete Bohl exponents, and spectrum of block weighted shifts*, Integral Equations and Operator Theory **14** (1991), no. 5, 613–677.

[5] ———, *Dichotomies of perturbed time varying systems and the power method*, Indiana University Mathematics Journal **42** (1993), no. 3, 699–720.

[6] C. Chicone and Y. Latushkin, *Evolution Semigroups in Dynamical Systems and Differential Equations*, Mathematical Surveys and Monographs 70, American Mathematical Society, 1999.

[7] K. Ey and C. Pötzsche, *Asymptotic behavior of recursions via fixed point theory*, Journal of Mathematical Analysis and Applications **337** (2008) 1125–1141.

[8] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics 840, Springer, Berlin etc., 1981.

[9] T. Kato, *Perturbation theory for linear operators*, corrected 2nd ed., Grundlehren der mathematischen Wissenschaften 132, Springer, Berlin etc., 1980.

[10] R. Sacker, *The splitting index for linear differential systems*, Journal of Differential Equations **33** (1979), no. 3, 368–405.

[11] R. Sacker and G. Sell, *A spectral theory for linear differential systems*, Journal of Differential Equations **27** (1978), 320–358.

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