# Dichotomy spectra of nonautonomous linear integrodifference equations

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**Abstract** We give examples of dichotomy spectra for nonautonomous linear difference equations in infinite-dimensional spaces. Particular focus is on the spectrum of integrodifference equations having compact coefficients. Concrete systems with explicitly known spectra are discussed for several purposes: (1) They yield reference examples for numerical approximation schemes. (2) The asymptotic behavior of spectral intervals is tackled illustrating their merging.

**Key words:** Integrodifference equations; Dichotomy spectrum; Sacker-Sell spectrum

## 1 Motivation and introduction

Over the last decades, integrodifference equations (IDEs, for short) became popular models in theoretical ecology, since they provide a flexible tool to describe the growth and dispersal of populations with discrete nonoverlapping generations. In the simplest case, where growth precedes dispersal, they are of Hammerstein type

$$u_{t+1}(x) = \int_\Omega k_t(x,y) f_t(y,u_t(y))\,\mathrm{d}y \quad \text{for all } t \in \mathbb{Z},\, x \in \Omega \tag{1}$$

(see [17]). Here, the real-valued function $u_t$ represents the density of a population at discrete time $t$ over some spatial habitat $\Omega \subseteq \mathbb{R}^\kappa$, the kernels $k_t$ are probability density functions describing the dispersal and $f_t$ is a growth function of e.g. Beverton-Holt or Ricker type. Both functions $k_t$ and $f_t$ are allowed to depend on time in order to include temporally changing environments into our analysis; we re-

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fer to [16] for a concrete application. Typical state spaces for (1) are the continuous or the $p$-integrable functions over $\Omega$.

Apparently, linear IDEs are of fundamental nature. First, they describe Malthusian growth $f_t(y,u) = c_t(y)u$ with ambient growth functions $c_t$. Second, and more importantly, when linearizing (1) along a reference solution $(\phi_t^*)_{t\in\mathbb{Z}}$, one arrives at a linear variational equation

$$v_{t+1}(x) = \int_\Omega k_t(x,y) D_2 f_t(y, \phi_t^*(y)) v_t(y)\,\mathrm{d}y \quad \text{for all } t \in \mathbb{Z}, x \in \Omega. \tag{2}$$

This is a nonautonomous linear difference equation in the infinite-dimensional state space of (1) and alone a local analysis near $\phi^*$ requires a thorough insight into the dynamical behavior of (2). Theoretically the *dichotomy spectrum* $\Sigma \subseteq (0,\infty)$ (also denoted as *dynamical* or *Sacker-Sell spectrum*) of (2) provides such an insight and hence an adequate "linear algebra" well-suited to establish a geometric theory of nonautonomous difference equations (cf. [21]) and particularly (1). In terms of *spectral intervals* it indeed gives nonautonomous counterparts to eigenvalue moduli, while the *spectral bundles* extend (generalized) eigenspaces to a time-variant setting. Specific applications of the dichotomy spectrum are as follows:

- The solution $\phi^*$ is uniformly asymptotically stable, if and only if $\Sigma \subseteq (0,1)$ holds, while a spectral interval in $(1,\infty)$ implies instability.
- If $1 \notin \Sigma$, then the solution $\phi^*$ is robust and persists locally as unique bounded entire solution to (1) under variation of the system.
- For each gap in $\Sigma$ one can construct a pair of invariant fiber bundles, which generalize the classical hierarchy of invariant manifolds to a nonautonomous setting. In case $1 \in \Sigma$ stability is determined by the behavior on such a center fiber bundle. Hence, the gaps determine the number of invariant fiber bundles corresponding to an entire solution $\phi^*$ to (1).

While the dichotomy spectrum dates back to [25, 4], a detailed analysis of its structure for difference equations in infinite-dimensional spaces is of more recent origin [24]. Nevertheless the motivation for this text is two-fold: First, already in finite dimensions only numerical methods allow an approximation of the spectrum (see [15]). It is thus handy to have a class of reference examples with explicitly known spectra available in order to verify computational methods. Second, we illustrate the structure of several spectra arising for nonautonomous IDEs and investigate the asymptotics of their spectral intervals.

The organization of this paper is as follows: We begin reviewing the dichotomy spectrum and some of its central properties for difference equations in infinite-dimensional state spaces. Particular focus is on the situation of compact operators, which was established in [24]. We then concentrate on operators having a discrete spectrum and provide the spectra for associate systems with multiplicative time-varying perturbations. As concrete application we consider IDEs. Sufficient criteria for their well-definedness in $L^p$- and $C$-spaces are quoted, we address the asymptotic behavior of the spectral intervals accumulating at 0, and finally present operators with explicitly known spectra or at least explicitly known asymptotics. The latter case applies to various equations relevant in applications.

As reference for difference equations in Banach spaces we mention [11, 21]. Corresponding results for nonautonomous parabolic evolutionary equations were obtained in [22].

Notation

Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. The Kronecker symbol is denoted by $\delta_{kl}$. A *discrete interval* $\mathbb{I}$ is the intersection of a real interval with $\mathbb{Z}$, i.e. a set of consecutive integers. We write $\mathbb{I}' := \{t \in \mathbb{I} : t+1 \in \mathbb{I}\}$ and suppose throughout that $\mathbb{I}$ is unbounded. For nonempty subsets $A, B \subseteq \mathbb{R}$ and $\lambda \in \mathbb{R}$ let us abbreviate

$$AB := \{ab \in \mathbb{R} : a \in A, b \in B\}, \qquad \lambda A := \{\lambda a \in \mathbb{R} : a \in A\}.$$

Unless further noted, $X, Y$ are Banach spaces, resp. their complexification, if spectral theoretical matters are addressed. Let $X'$ be the dual space of $X$ with duality pairing $\langle \cdot, \cdot \rangle$. The bounded linear maps from $X$ to $Y$ are denoted by $L(X,Y)$, $L(X) := L(X,X)$ and $I_X$ is the identity mapping on $X$. We write $N(T) := T^{-1}(\{0\})$ for the *kernel* and $R(T) := TX$ for the *range* of $T \in L(X,Y)$. The *spectrum* of $S \in L(X)$ is $\sigma(S) \subset \mathbb{C}$.

A subset $\mathscr{A} \subseteq \mathbb{I} \times X$ is called a *nonautonomous set*, if all *t-fibers*

$$\mathscr{A}(t) := \{x \in X : (t,x) \in \mathscr{A}\}, t \in \mathbb{I}$$

are nonempty. One speaks of a *vector bundle* $\mathscr{V} \subseteq \mathbb{I} \times X$, if every fiber $\mathscr{V}(t) \subseteq X$ is a linear subspace and in case all $\mathscr{V}(t)$ have the same dimension, it determines the *dimension* $\dim \mathscr{V}$ of $\mathscr{V}$. *Constant vector bundles* are of the form $\mathscr{V} = \mathbb{I} \times X_0$ with a subspace $X_0 \subseteq X$ and particular examples are

$$\mathscr{O} := \mathbb{I} \times \{0\}, \qquad \mathscr{X} := \mathbb{I} \times X.$$

## 2 Dichotomy spectrum

Given a sequence $(\mathcal{K}_t)_{t \in \mathbb{I}'}$ of bounded linear operators in $L(X)$ as coefficients, we consider linear nonautonomous equations

$$\boxed{u_{t+1} = \mathcal{K}_t u_t} \tag{L}$$

in an infinite-dimensional Banach space $X$. A vector bundle $\mathscr{V}$ is called *forward invariant* resp. *invariant*, provided $\mathcal{K}_t \mathscr{V}(t) \subseteq \mathscr{V}(t+1)$ or $\mathcal{K}_t \mathscr{V}(t) = \mathscr{V}(t+1)$ hold for all $t \in \mathbb{I}'$. Their *evolution operator* is the mapping

$$\Phi_{\mathcal{K}} : \{(t,s) \in \mathbb{I} \times \mathbb{I} : s \le t\} \to L(X), \qquad \Phi_{\mathcal{K}}(t,s) := \begin{cases} \mathcal{K}_{t-1} \cdots \mathcal{K}_s, & s < t, \\ I_X, & s = t. \end{cases}$$

For simplicity we suppose from now on that ($L$) has *bounded (forward) growth*, i.e.

$$\alpha_0 := \sup_{t \in \mathbb{I}'} \|\mathcal{K}_t\| < \infty. \tag{3}$$

One says a linear difference eqn. ($L$) has an *exponential dichotomy* (ED for short, cf. [14, p. 229, Def. 7.6.4]) on $\mathbb{I}$, if there exists a projector $P : \mathbb{I} \to L(X)$ and reals $K \ge 1$, $\alpha \in (0,1)$ such that

- $\mathcal{K}_t P(t) = P(t+1)\mathcal{K}_t$ for all $t \in \mathbb{I}'$ ($P$ is an *invariant projector*)
- $\bar{\Phi}_{\mathcal{K}}(t,s) := \Phi_{\mathcal{K}}(t,s)|_{N(P(s))} : N(P(s)) \to N(P(t))$ is a topological isomorphism for $s < t$[1]
- $\|\Phi_{\mathcal{K}}(t,s)P(s)\| \le K\alpha^{t-s}$ and $\left\|\bar{\Phi}_{\mathcal{K}}(s,t)\,[I_X - P(t)]\right\| \le K\alpha^{t-s}$ for $s \le t$.

The *dichotomy spectrum* of ($L$) is defined as

$$\Sigma_{\mathbb{I}}(\mathcal{K}) := \left\{\gamma > 0 : u_{t+1} = \gamma^{-1}\mathcal{K}_t u_t \text{ admits no ED on } \mathbb{I}\right\}$$

and $\rho_{\mathbb{I}}(\mathcal{K}) := (0,\infty) \setminus \Sigma_{\mathbb{I}}(\mathcal{K})$ denotes the *dichotomy resolvent*. If the discrete interval $\mathbb{I}$ is fixed, then we simply write $\Sigma(\mathcal{K})$ resp. $\rho(\mathcal{K})$.

Due to the bounded growth (3) one has $\Sigma(\mathcal{K}) \subseteq (0,\alpha_0]$. The components of $\Sigma(\mathcal{K})$ are called *spectral intervals* and the *dominant spectral interval* contains the largest elements. If $\Sigma(\mathcal{K})$ consists of isolated points, one speaks of a *discrete spectrum*.

Essential properties of the dichotomy spectrum can be summarized as follows:

- $\Sigma(\mathcal{K}) \cup \{0\}$ is compact, $\Sigma_{\mathbb{I}}(\mathcal{K}) \subseteq \Sigma_{\mathbb{Z}}(\mathcal{K})$ for unbounded subintervals $\mathbb{I} \subseteq \mathbb{Z}$ and
$$\Sigma(\lambda\mathcal{K}) = |\lambda|\,\Sigma(\mathcal{K}) \quad \text{for all } \lambda \in \mathbb{C} \setminus \{0\}$$
- It is upper-semicontinuous, i.e. for every $\varepsilon > 0$ there exists a $\delta > 0$ such that every sequence $(\bar{\mathcal{K}}_t)_{t \in \mathbb{I}'}$ in $L(X)$ fulfills
$$\sup_{t \in \mathbb{I}'} \left\|\bar{\mathcal{K}}_t - \mathcal{K}_t\right\| < \delta \quad \Rightarrow \quad \Sigma(\bar{\mathcal{K}}) \subseteq B_\varepsilon(\Sigma(\mathcal{K}))$$
- $\Sigma(\mathcal{K})$ is invariant under *kinematic similarity*, i.e. if there exists a sequence $(\mathcal{S}_t)_{t \in \mathbb{I}}$ of invertible operators $\mathcal{S}_t \in L(X,Y)$ with $\sup_{t \in \mathbb{I}} \max\left\{\|\mathcal{S}_t\|, \left\|\mathcal{S}_t^{-1}\right\|\right\} < \infty$, then ($L$) and $v_{t+1} = \mathcal{S}_{t+1}^{-1}\mathcal{K}_t\mathcal{S}_t v_t$ have the same dichotomy spectrum. The sequence $(\mathcal{S}_t)_{t \in \mathbb{I}}$ is called *Lyapunov transformation*.

Finally, for every $\gamma > 0$ we define the vector bundles

$$\mathcal{V}_\gamma^+ := \left\{(\tau,\xi) \in \mathcal{X} : \sup_{\tau \le t} \|\Phi_{\mathcal{K}}(t,\tau)\xi\|\,\gamma^{\tau-t} < \infty\right\},$$

[1] for this it suffices to assume that $\mathcal{K}_t|_{N(P(t))} : N(P(t)) \to N(P(t+1))$, $t \in \mathbb{I}'$, are isomorphisms

$$\mathscr{V}_\gamma^- := \left\{ (\tau,\xi) \in \mathscr{X} : \begin{array}{c} \text{there exists a solution } (\phi_t)_{t\in\mathbb{I}} \text{ of } (L) \\ \text{with } \phi_\tau = \xi \text{ and } \sup_{\tau\le t} \|\phi_t\|\, \gamma^{\tau-t} < \infty \end{array} \right\};$$

in case $\gamma$ is chosen from the dichotomy resolvent $\rho(\mathcal{K})$, one denotes $\mathscr{V}_\gamma^+$ as a *pseudo-stable* and $\mathscr{V}_\gamma^-$ as a *pseudo-unstable* bundle of $(L)$.

The subsequent classes of linear difference equations allow more detailed statements and insights into the structure of their dichotomy spectrum:

## 2.1 Periodic difference equations

Let $(L)$ be $p$-periodic, i.e. there exists a $p \in \mathbb{N}$ such that $\mathcal{K}_t = \mathcal{K}_{t+p}$ for all $t \in \mathbb{Z}$.

Then the dichotomy spectrum reads as

$$\Sigma_{\mathbb{Z}}(\mathcal{K}) = |\{\lambda \in \mathbb{C} : \lambda \in \sigma(\Phi_{\mathcal{K}}(p,0))\} \setminus \{0\}|^{1/p} \tag{4}$$

and in particular for autonomous equations $(p = 1)$ it consists of the positive moduli of the spectral points for $\mathcal{K}$. The pseudo-stable and -unstable bundles of $(L)$ can be be characterized in terms of Riesz projections (see [8, p. 30, Thm. 1.5.4]) associated to the components of $\sigma(\Phi_{\mathcal{K}}(p,0))$, but need not to be finite-dimensional.

Rather explicit information can be obtained in

*Example 1 (multiplication operator).* Suppose $(\Omega,\Sigma,\mu)$ is a $\sigma$-finite measure space and $1 \le p < \infty$. For $\mathbb{K}$-valued functions $a_t \in L^\infty(\Omega,\mu)$ we define the *essential range*

$$\rho_{\mathrm{ess}}(a_t) := \left\{ \lambda \in \mathbb{C} : \mu\left( \left\{ x \in \Omega : \left| a_t(x) - \lambda \right| < \varepsilon \right\} \right) \neq 0 \text{ for all } \varepsilon > 0 \right\}$$

for all $t \in \mathbb{I}'$. On $X = L^p(\Omega,\mu)$ the multiplication operators

$$\mathcal{K}_t \in L(L^p(\Omega,\mu)), \qquad [\mathcal{K}_t v](x) := a_t(x)v(x) \quad \text{for all } t \in \mathbb{I}', x \in \Omega$$

are well-defined and yield an evolution operator of $(L)$ given by

$$[\Phi_{\mathcal{K}}(t,\tau)v](x) = \left( \prod_{s=\tau}^{t-1} a_s(x) \right) v(x) \quad \text{for all } \tau \le t,\, v \in L^p(\Omega,\mu),$$

which is a multiplication operator again. In the periodic situation $a_t = a_{t+p}$, $t \in \mathbb{Z}$, the spectrum of $\Phi_{\mathcal{K}}(p,0)$ is the essential range of the product $\prod_{s=0}^{p-1} a_s : \Omega \to \mathbb{K}$ (see [10, pp. 30ff]) and due to (4) we arrive at

$$\Sigma(\mathcal{K}) = \left| \rho_{\mathrm{ess}}\left( \prod_{s=0}^{p-1} a_s \right) \setminus \{0\} \right|^{1/p}.$$

*Example 2 (shift operator).* Suppose that $(B_t)_{t\in\mathbb{Z}}$ is a bounded sequence in $L(Y)$ such that the difference eqn. $y_{t+1} = B_t y_t$ in $Y$ has a nonempty dichotomy spectrum

$\Sigma_{\mathbb{Z}}(B)$. Furthermore, let $X := \ell^p(Y)$ be the space of $p$-summable sequences $(y_t)_{t\in\mathbb{Z}}$ in $Y$ for $p \in [1,\infty]$ and define the shift

$$\mathcal{K} \in L(\ell^p(Y)), \qquad [\mathcal{K}v]_s := B_{s-1}v_{s-1} \quad \text{for all } s \in \mathbb{Z},\, v \in \ell^p(Y).$$

In [20, Thm. 1] it is shown that $\sigma(\mathcal{K}) = \overline{\{\lambda \in \mathbb{C} : |\lambda| \in \Sigma_{\mathbb{Z}}(B)\}}$ and we hence obtain from (4) for $p = 1$ that $\Sigma_{\mathbb{I}}(\mathcal{K}) = \Sigma_{\mathbb{Z}}(B)$.

### 2.2 Compact difference equations

Let (*L*) be compact, i.e. the coefficients $\mathcal{K}_t \in L(X)$, $t \in \mathbb{I}'$, are compact operators.

Due to our global bounded growth assumption (3) the spectrum $\Sigma(\mathcal{K})$ is bounded above by $\alpha_0$ and there exists a $\gamma_0 > 0$ such that $(\gamma_0,\infty) \subseteq \rho(\mathcal{K})$; we set

$$\mathscr{V}^+_{\gamma_0} := \mathscr{X}, \qquad \mathscr{V}^-_{\gamma_0} := \mathscr{O}.$$

Furthermore, in [24, Cor. 4.13] it is shown that $\Sigma(\mathcal{K})$ is a union of at most countably many intervals which can only accumulate at a number $\bar{\mu} \geq 0$ and that the pseudo-unstable bundles $\mathscr{V}^-_\gamma$ are finite-dimensional. In detail, one of the cases holds:

($\mathfrak{S}_0$) $\Sigma(\mathcal{K}) = \emptyset$

($\mathfrak{S}_1$) $\Sigma(\mathcal{K})$ consists of finitely many closed spectral intervals:

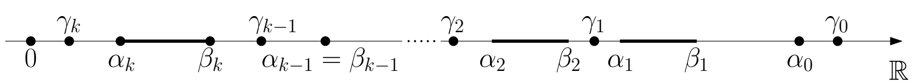


**Fig. 1** Case ($\mathfrak{S}_1^1$) with $k$ compact spectral intervals

($\mathfrak{S}_1^1$) There exists a $k \in \mathbb{N}$ and reals $0 < \alpha_k \leq \beta_k < \ldots < \alpha_1 \leq \beta_1 \leq \alpha_0$ with

$$\Sigma(\mathcal{K}) = \bigcup_{j=1}^{k} [\alpha_j, \beta_j]$$

and we choose reals $\gamma_j \in (\beta_{j+1}, \alpha_j)$, $1 \leq j < k$, and $\gamma_k \in (0, \alpha_k)$ (see Fig. 1)

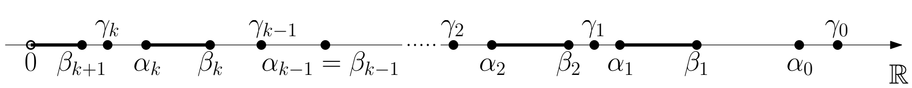


**Fig. 2** Case ($\mathfrak{S}_1^2$) with $k+1$ spectral intervals

($\mathfrak{S}_1^2$) There exists a $k \in \mathbb{N}_0$ and reals $0 < \beta_{k+1} < \alpha_k \leq \beta_k < \ldots < \alpha_1 \leq \beta_1 \leq \alpha_0$ with

$$\Sigma(\mathcal{K}) = (0, \beta_{k+1}] \cup \bigcup_{j=1}^{k} [\alpha_j, \beta_j]$$

and we choose reals $\gamma_j \in (\beta_{j+1}, \alpha_j)$, $1 \leq j \leq k$ (see Fig. 2).

In both cases the *spectral bundles*

$$\mathscr{X}_0 := \mathscr{V}_{\gamma_0}^-, \qquad \mathscr{X}_j := \mathscr{V}_{\gamma_{j-1}}^+ \cap \mathscr{V}_{\gamma_j}^- \neq \mathscr{O} \quad \text{for all } 1 \leq j \leq k$$

are finite-dimensional invariant vector bundles of ($L$) with the finite *Whitney sum*

$$\mathscr{X} = \bigoplus_{j=0}^{k} \mathscr{X}_j \oplus \mathscr{V}_{\gamma_k}^+$$

and the bundle $\mathscr{V}_{\gamma_k}^- = \bigoplus_{j=0}^{k} \mathscr{X}_j$ satisfying $k \leq \dim \mathscr{V}_{\gamma_k}^- = \sum_{j=0}^{k} \dim \mathscr{X}_j$

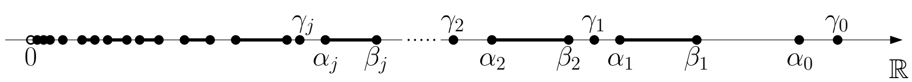


**Fig. 3** Case ($\mathfrak{S}_2$) with infinitely many spectral intervals $[\alpha_j, \beta_j]$ accumulating at $\bar{\mu} = 0$ i.e. $\sigma_\infty = \emptyset$

($\mathfrak{S}_2$) $\Sigma(\mathcal{K})$ consists of infinitely many spectral intervals: There exist strictly decreasing sequences $(\alpha_j)_{j \in \mathbb{N}}$, $(\beta_j)_{j \in \mathbb{N}}$ such that

$$\Sigma(\mathcal{K}) = \sigma_\infty \cup \bigcup_{j=1}^{\infty} [\alpha_j, \beta_j],$$

where $\bar{\mu} < \alpha_j \leq \beta_j$, $\lim_{j \to \infty} \alpha_j = \bar{\mu}$, $\sigma_\infty = \emptyset$ for $\bar{\mu} = 0$ and $\sigma_\infty = (0, \bar{\mu}]$ otherwise (see Fig. 3). If we choose reals $\gamma_j \in (\beta_{j+1}, \alpha_j)$, $j \in \mathbb{N}$, then the *spectral bundles*

$$\mathscr{X}_0 := \mathscr{V}_{\gamma_0}^-, \qquad \mathscr{X}_j := \mathscr{V}_{\gamma_{j-1}}^+ \cap \mathscr{V}_{\gamma_j}^- \neq \mathscr{O} \quad \text{for all } j \in \mathbb{N}$$

are finite-dimensional invariant vector bundles of ($L$) and for every $k \in \mathbb{N}$ one has the finite *Whitney sum*

$$\mathscr{X} = \bigoplus_{j=0}^{k} \mathscr{X}_j \oplus \mathscr{V}_{\gamma_k}^+$$

and the bundle $\mathscr{V}_{\gamma_k}^- = \bigoplus_{j=0}^{k} \mathscr{X}_j$ satisfying $k \leq \dim \mathscr{V}_{\gamma_k}^- = \sum_{j=0}^{k} \dim \mathscr{X}_j$.

By construction, the dominant interval is $[\alpha_1, \beta_1]$. The *order* of a spectral interval with maximum $\beta_j$ is the dimension of the associate spectral bundle $\mathscr{X}_j$; a *simple* spectral interval has order 1.

### 2.3 Finite-rank difference equations

Let $(L)$ be of finite rank, i.e. there exists a finite-dimensional subspace $X_0 \subset X$ such that $R(\mathcal{K}_t) = X_0$ for all $t \in \mathbb{I}'$. In particular, every $\mathcal{K}_t$ is compact and $(L)$ essentially behave like finite-dimensional equations.

If $d := \dim X_0$, then $\Sigma(\mathcal{K})$ is a union of at most $d$ intervals (cf. [24, Thm. 4.14]), i.e. either $(\mathfrak{S}_0)$ holds or $\Sigma(\mathcal{K})$ consists of $k \in \{1, \ldots, d\}$ spectral intervals: There exist reals $0 < \alpha_k \le \beta_k < \ldots < \alpha_1 \le \beta_1 \le \alpha_0$ with closed spectral intervals:

$$\Sigma(\mathcal{K}) = \begin{cases} [\alpha_k, \beta_k] \\ (0, \beta_k] \end{cases} \cup \bigcup_{j=1}^{k-1} [\alpha_j, \beta_j]. \tag{5}$$

If possible, we choose $\gamma_k \in \rho(\mathcal{K})$ such that $(0, \gamma_k) \subseteq \rho(\mathcal{K})$ and otherwise, we define $\mathcal{V}_{\gamma_k}^+ = \mathcal{O}$ and $\mathcal{V}_{\gamma_k}^- = \mathcal{X}$. Then $\mathcal{X}_{k+1} = \mathcal{V}_{\gamma_k}^+$ and $\mathcal{X}_0 = \mathcal{V}_{\gamma_0}^-$ are invariant vector bundles of $(L)$. For $k > 1$ we choose reals $\gamma_j \in (\beta_{j+1}, a_j)$, $1 \le j < k$. Then the sets

$$\mathcal{X}_j := \mathcal{V}_{\gamma_{j-1}}^+ \cap \mathcal{V}_{\gamma_j}^- \neq \mathcal{O} \quad \text{for all } 1 \le j \le k$$

are finite-dimensional invariant vector bundles of $(L)$ with the *Whitney sum*

$$\mathcal{X} = \bigoplus_{j=0}^{k+1} \mathcal{X}_j.$$

*Remark 1.* Note that the above situation differs from the dichotomy spectrum introduced in [4] for finite-dimensional equations. Indeed, [4] work with the dichotomy concept from [3], which is not $\ell^\infty$-robust and yields a finer spectrum than ours.

### 2.4 Finite-dimensional and difference equations

Suppose that $(B_t)_{t \in \mathbb{I}'}$ is a bounded sequence in $\mathbb{K}^{n \times n}$ and consider a linear equation

$$\boxed{y_{t+1} = B_t y_t} \tag{6}$$

with evolution operator $\Phi_B(t,s) \in \mathbb{K}^{n \times n}$, $s \le t$. Its dichotomy spectrum $\Sigma(B)$ fits in the above framework of Sect. 2.3. Each spectral interval in (5) corresponds to an invariant vector bundle

$$\mathcal{Y}_j := \left\{ (t,x) \in \mathbb{I} \times \mathbb{K}^n : x \in R(p_j(t)) \right\} \quad \text{for all } 1 \le j \le k,$$

where $p_j : \mathbb{I} \to L(\mathbb{K}^n)$ is an invariant projector for (6), and $\mathbb{I} \times \mathbb{K}^n = \bigoplus_{j=1}^k \mathcal{Y}_j$.

For scalar difference equations the following notion of Bohl exponents is central. Assume $(a_t)_{t \in \mathbb{I}'}$ is a *tempered* sequence in $\mathbb{K}$, i.e. it satisfies $a_t \neq 0$ for all $t \in \mathbb{I}'$ and

$$\sup_{t\in\mathbb{I}'}\max\left\{|a_t|,\left|a_t^{-1}\right|\right\}<\infty.$$

Let $I_T(\mathbb{I}) := \{\mathbb{J}\subseteq\mathbb{I} : \mathbb{J}$ is a discrete interval with $\#\mathbb{J} = T\}$ denote the family of all discrete subintervals of $\mathbb{I}$ with $T\in\mathbb{N}$ elements. The *upper* resp. *lower Bohl exponent* of $a$ are given by

$$\overline{\beta}(a) := \lim_{T\to\infty}\sup_{\mathbb{J}\in I_T(\mathbb{I})}\sqrt[T]{\left|\prod_{s\in\mathbb{J}}a_s\right|}, \qquad \underline{\beta}(a) := \lim_{T\to\infty}\inf_{\mathbb{J}\in I_T(\mathbb{I})}\sqrt[T]{\left|\prod_{s\in\mathbb{J}}a_s\right|}$$

and one clearly has the homogeneity relations

$$\underline{\beta}(\lambda a) = |\lambda|\,\underline{\beta}(a), \qquad \overline{\beta}(\lambda a) = |\lambda|\,\overline{\beta}(a) \quad \text{for all } \lambda\in\mathbb{C}\setminus\{0\}.$$

Especially for $\mathcal{K}_t := a_t I_X$, $t\in\mathbb{I}$, one has the spectrum

$$\Sigma(\mathcal{K}) = [\underline{\beta}(a),\overline{\beta}(a)]$$

and we refer to [23] for further properties of Bohl exponents.

## 3 Operators with discrete spectrum

Assume now that $\mathcal{K}\in L(X)$ is a single linear operator. Given an eigenvalue $\lambda\in\mathbb{C}$ of $\mathcal{K}$, we denote its *order* as

$$o_\lambda = \min\left\{o\in\mathbb{N} : N(\mathcal{K}-\lambda I_X)^o = N(\mathcal{K}-\lambda I_X)^{o+1}\right\}$$

and our future analysis is based on the following properties:

$(H_1)$ There exist nonempty discrete intervals $\mathbb{J}(\mathcal{K})\subseteq I(\mathcal{K})\subseteq\mathbb{N}$ such that

- $\sigma(\mathcal{K})\setminus\{0\} = \{\lambda_i : i\in I(\mathcal{K})\}$ consists of eigenvalues $\lambda_i$ such that $(|\lambda_i|)_{i\in I(\mathcal{K})}$ is a decreasing sequence
- $|\sigma(\mathcal{K})\setminus\{0\}| = \left\{\rho_j : j\in\mathbb{J}(\mathcal{K})\right\}$ with a strictly deceasing sequence $(\rho_j)_{j\in\mathbb{J}(\mathcal{K})}$ of positive reals and $s_j := \#\left\{\lambda\in\sigma(\mathcal{K}) : |\lambda| = \rho_j\right\} < \infty$ for $j\in\mathbb{J}(\mathcal{K})$

$(H_2)$ Given bases of generalized (and norm 1) eigenvectors such that

$$N(\mathcal{K}-\lambda I_X)^{o_\lambda} = \operatorname{span}\left\{e_\lambda^1,\dots,e_\lambda^{o_\lambda}\right\} \quad \text{for all } \lambda\in\sigma(\mathcal{K})\setminus\{0\},$$

the sequence $(e_n)_{n\in N} := (e_{\lambda_1}^1,\dots,e_{\lambda_1}^{o_{\lambda_1}},e_{\lambda_2}^1,\dots,e_{\lambda_2}^{o_{\lambda_2}},\dots)$ is a basis of $X$.

According to [8, p. 80, Lemma 3.3.1] one can complement the basis $(e_n)_{n\in N}$ of $X$ to a biorthonormal system $(e_n,f_n)_{n\in N}$, where $N\subseteq\mathbb{N}$ is a discrete interval. This means there exists a sequence $(f_n)_{n\in N} := (f_{\lambda_1}^1,\dots,f_{\lambda_1}^{o_{\lambda_1}},f_{\lambda_2}^1,\dots,f_{\lambda_2}^{o_{\lambda_2}},\dots)$ of functionals $f_n\in X'$ satisfying $\langle e_n,f_m\rangle = \delta_{nm}$ for all $m,n\in N$. Then

$$\Pi(\lambda) := \sum_{n=1}^{o_\lambda} \left\langle \cdot, f_\lambda^n \right\rangle e_\lambda^n \quad \text{for all } \lambda \in \sigma(\mathcal{K}) \setminus \{0\}$$

is a bounded projector onto $N(\mathcal{K} - \lambda I_X)^{o_\lambda}$ with

$$\Pi(\lambda_i)\Pi(\lambda_j) = \delta_{ij}\Pi(\lambda_i), \qquad \Pi(\lambda_i)\mathcal{K} = \mathcal{K}\Pi(\lambda_i) \quad \text{for all } i, j \in I(\mathcal{K}), \tag{7}$$

since $(e_n, f_n)_{n\in N}$ is a biorthonormal system. We next define the *spectral spaces*

$$X_j := \bigoplus_{|\lambda|=\rho_j} N(\mathcal{K} - \lambda I_X)^{o_\lambda} \quad \text{for all } j \in \mathbb{J}(\mathcal{K}),$$

which are invariant and of dimension $\sum_{|\lambda|=\rho_j} o_\lambda$, as well as finite rank mappings

$$\Pi_j : X \to X_j, \qquad \Pi_j := \sum_{|\lambda|=\rho_j} \Pi(\lambda) \quad \text{for all } j \in \mathbb{J}(\mathcal{K}).$$

From (7) we readily obtain the commutativity relations

$$\Pi_j\Pi_i = \delta_{ij}\Pi_j, \qquad \mathcal{K}\Pi_j = \Pi_j\mathcal{K} \quad \text{for all } i, j \in \mathbb{J}(\mathcal{K}).$$

Thus, $\Pi_j$, $j \in \mathbb{J}(\mathcal{K})$, are a family of complementary projections onto the spectral spaces $X_j$.

*Example 3 (normal compact operators).* If $\mathcal{K} \in L(X)$ is a compact operator with $I(\mathcal{K}) = \mathbb{J}(\mathcal{K}) = \mathbb{N}$, then $\lim_{i\to\infty} \lambda_i = \lim_{j\to\infty} \rho_j = 0$ holds. In case $X$ is an infinite-dimensional Hilbert space and $\mathcal{K}$ is normal, we identify $X'$ with $X$ by means of the Riesz representation theorem. One chooses $f_n := e_n$, $n \in N$, and the projections $\Pi_j$, as well as the eigenspaces $X_j$ are pairwise orthonormal (see [18, p. 484ff, Sect. 6.7]).

*Example 4 (finite rank operators).* Suppose that $X_0 := R(\mathcal{K})$ is finite-dimensional with a basis $(x_1, \dots, x_d)$ and let $S : X_0 \to \mathbb{C}^d$ be an isomorphism. Following [1, p. 274, Thm. 7.4] and using the representation

$$\mathcal{K}v = \sum_{j=1}^{d} \left\langle v, x_j' \right\rangle x_j \quad \text{for all } v \in X$$

we define the matrix $K := (x_i'(x_j))_{i,j=1}^d \in \mathbb{C}^{d\times d}$ and obtain $\sigma(\mathcal{K}) = \sigma(K) \cup \{0\}$. By means of e.g. the Jordan form there exists an invertible matrix $T \in \mathbb{C}^{d\times d}$ such that

$$T^{-1}KT = \begin{pmatrix} S_k & & \\ & \ddots & \\ & & S_1 \end{pmatrix} \quad \text{and} \quad k \leq d.$$

The eigenvalues of each block matrix $S_j \in \mathbb{C}^{d_j\times d_j}$ have the same moduli and satisfy $\left|\sigma(S_{j+1})\right| < \left|\sigma(S_j)\right|$ for $1 \leq j < k$. One obtains the spectral spaces

$$X_j := ST(\{0\} \times \mathbb{C}^{d_j} \times \{0\}) \subset X \quad \text{for all } 1 \le j \le k$$

and $\Pi_j := ST\operatorname{diag}(0, I_{\mathbb{C}^{d_j}}, 0)(ST)^{-1}$ as corresponding projections.

In conclusion, we arrive at a weighted sum

$$\mathcal{K}v = \sum_{j \in \mathbb{J}(\mathcal{K})} \sum_{|\lambda| = \rho_j} \lambda \Pi(\lambda) v \quad \text{for all } v \in X$$

and the discrete semigroup $(\mathcal{K}^t)_{t \ge 0}$ generated by $\mathcal{K}$ has the Fourier representation

$$\mathcal{K}^t v = \sum_{j \in \mathbb{J}(\mathcal{K})} \sum_{|\lambda| = \rho_j} \lambda^t \Pi(\lambda) v \quad \text{for all } t \ge 0,\, v \in X. \tag{8}$$

For autonomous difference equations

$$\boxed{u_{t+1} = \mathcal{K} u_t}$$

in $X$ with coefficients $\mathcal{K} \in L(X)$ satisfying $(H_1)$-$(H_2)$ the above notions translate into the language of Sect. 2.2 as follows: We obtain a discrete dichotomy spectrum

$$\Sigma(\mathcal{K}) = \bigcup_{j \in \mathbb{J}(\mathcal{K})} \{\rho_j\}$$

and constant spectral bundles $\mathscr{X}_j = \mathbb{I} \times X_j$, $j \in \mathbb{J}(\mathcal{K})$, from (4). An immediate nonautonomous generalization is treated in

**Theorem 1 (multiplicative perturbation 1).** *If a sequence $(a_t)_{t \in \mathbb{I}}$ is tempered, then the difference equation*

$$\boxed{u_{t+1} = a_t \mathcal{K} u_t} \tag{9}$$

*has the dichotomy spectrum $\Sigma(a\mathcal{K}) = \left[\underline{\beta}(a), \overline{\beta}(a)\right] \bigcup_{j \in \mathbb{J}(\mathcal{K})} \{\rho_j\}$ and constant spectral bundles.*

*Proof.* Using the Fourier representation (8) we obtain that the evolution operator of (9) reads as

$$\Phi_{a\mathcal{K}}(t,s) = \sum_{j \in \mathbb{J}(\mathcal{K})} \left( \prod_{r=s}^{t-1} a_r \right) \sum_{|\lambda| = \rho_j} \lambda^{t-s} \Pi(\lambda) \quad \text{for all } s \le t.$$

If $\left\{\lambda_j^1, \ldots, \lambda_j^{s_j}\right\} \subseteq \sigma(\mathcal{K})$ is the set of eigenvalues with absolute value $\rho_j$, we obtain

$$\Pi_j \Phi_{a\mathcal{K}}(t,s) = \sum_{j \in \mathbb{J}(\mathcal{K})} \left( \prod_{r=s}^{t-1} a_r \right) \sum_{i=1}^{s_j} (\lambda_j^i)^{t-s} P_j \Pi(\lambda_j^i) = \Phi_{a\mathcal{K}}(t,s) \Pi_j \quad \text{for all } s \le t.$$

Hence, the finite-dimensional vector bundles $\mathscr{P}_j := \left\{(t,v) \in \mathscr{X} : v \in R(\Pi_j)\right\}$ are invariant w.r.t. (9) for all $j \in \mathbb{J}(\mathcal{K})$. Inside of each $\mathscr{P}_j$ the dynamics is given by

$$u_{t+1} = a_t \sum_{i=1}^{s_j} \lambda_j^i \Pi(\lambda_j^i) u_t,$$

having an evolution operator $\Phi^j(t,s) := \Phi_{a\mathcal{K}}(t,s)\Pi_j$ and the spectrum $\rho_j\left[\underline{\beta}(a), \overline{\beta}(a)\right]$. Thanks to $\Phi_{a\mathcal{K}}(t,s) = \sum_{j\in\mathbb{J}(\mathcal{K})} \Phi^j(t,s)$ for all $s \le t$ we thus obtain the assertion. □

**Corollary 1.** *If a sequence $(a_t)_{t\in\mathbb{Z}}$ in $\mathbb{K}$ is p-periodic with nonzero values, then*

$$\Sigma(a\mathcal{K}) = \sqrt[p]{\prod_{s=0}^{p-1} |a_s|} \bigcup_{j\in\mathbb{J}(\mathcal{K})} \{\rho_j\}.$$

*Proof.* The upper and lower Bohl exponents of $a$ are given by $\sqrt[p]{\prod_{s=0}^{p-1} |a_s|}$.

In the following, we are interested in systems of difference equation

$$\boxed{U_{t+1} = \tilde{\mathcal{K}}_t U_t} \tag{10}$$

on the state space $X^n$ for coefficient sequences $(\tilde{\mathcal{K}}_t)_{t\in\mathbb{I}'}$ in $L(X^n)$. We conveniently abbreviate $U = (u_1, \dots, u_n) \in X^n$ throughout. Suppose that $(B_t)_{t\in\mathbb{I}'}$ is a sequence of invertible matrices in $\mathbb{K}^{n\times n}$ satisfying

$$\sup_{t\in\mathbb{I}'} \|B_t\| < \infty, \qquad \sup_{t\in\mathbb{I}'} \left\|B_t^{-1}\right\| < \infty \tag{11}$$

and having the entries $b_{ij}(t)$, $1 \le i, j \le n$. In [4, Thm. 2.1] and Sect. 2.4 it is shown that $\Sigma(B)$ consists of compact intervals in $(0,\infty)$.

**Theorem 2 (multiplicative perturbation 2).** *Suppose that* (11) *holds. If* (6) *possesses* full spectrum*, i.e.*

$$\Sigma(B) = \bigcup_{i=1}^{n} \sigma_i \tag{12}$$

*with compact, decreasing and disjoint spectral intervals $\sigma_i \subset (0,\infty)$, then the difference eqn.* (10) *with*

$$\tilde{\mathcal{K}}_t U := \begin{pmatrix} b_{11}(t)\mathcal{K}u_1 + \ldots + b_{1n}(t)\mathcal{K}u_n \\ \vdots \\ b_{n1}(t)\mathcal{K}u_1 + \ldots + b_{nn}(t)\mathcal{K}u_n \end{pmatrix} \quad \text{for all } t \in \mathbb{I}', U \in X^n$$

*has the dichotomy spectrum $\Sigma(\tilde{\mathcal{K}}) = \bigcup_{j\in\mathbb{J}(\mathcal{K})} \rho_j \bigcup_{i=1}^n \sigma_i = \bigcup_{j\in\mathbb{J}(\mathcal{K})} \rho_j \Sigma(B)$.*

*Remark 2 (computation of* (12)*).* For general coefficient sequences in (6) the computation of the dichotomy spectrum $\Sigma(B)$ is only possible using numerical schemes, as developed in [15, 9].

For the remaining section it is convenient to define the operator

$$\hat{\mathcal{K}} := \begin{pmatrix} \mathcal{K} & & \\ & \ddots & \\ & & \mathcal{K} \end{pmatrix} \in L(X^n).$$

*Proof.* First of all, we obtain from [26, Reduction Theorem] that (6) is kinematically similar to a diagonal system in $\mathbb{K}^n$. More precisely, there exists a Lyapunov transformation $(S_t)_{t\in\mathbb{I}}$ in $\mathbb{K}^{n\times n}$ such that $S_{t+1}^{-1}B_tS_t = \operatorname{diag}(b_t^1,\dots,b_t^n)$ with tempered sequences $(b_t^i)_{t\in\mathbb{I}'}$ such that $\sigma_i = [\underline{\beta}(b^i), \overline{\beta}(b^i)]$, $1 \le i \le n$. One has

$$\hat{\mathcal{K}}S_tU = S_t\hat{\mathcal{K}}U \quad \text{for all } t \in \mathbb{I}, U \in X^n$$

and consequently we arrive at

$$S_{t+1}^{-1}\tilde{\mathcal{K}}_tS_t = S_{t+1}^{-1}B_t\hat{\mathcal{K}}S_t = S_{t+1}^{-1}B_tS_t\hat{\mathcal{K}} = \begin{pmatrix} b_t^1\mathcal{K} & & \\ & \ddots & \\ & & b_t^n\mathcal{K} \end{pmatrix} \quad \text{for all } t \in \mathbb{I}'.$$

Hence, (10) is kinematically similar to a diagonal difference system in $X^n$ and therefore $\Sigma(\tilde{\mathcal{K}}) = \bigcup_{i=1}^n \Sigma(b^i\mathcal{K})$. Then the assertion follows from Thm. 1 yielding the spectra $\Sigma(b^i\mathcal{K})$. □

We next investigate scalar multiplicative and time-dependent perturbations. The situation is related to Thm. 2, but allows a different proof.

**Theorem 3 (multiplicative perturbation 3).** *Suppose $D \in \mathbb{K}^{n\times n}$ is diagonalizable and $\sigma(D) = \{d_1,\dots,d_n\}$. If $(a_t)_{t\in\mathbb{I}'}$ is tempered, then the difference eqn.* (10) *with*

$$\tilde{\mathcal{K}}_tU := a_t \begin{pmatrix} d_{11}\mathcal{K}u^1 + \ldots + d_{1n}\mathcal{K}u^n \\ \vdots \\ d_{n1}\mathcal{K}u^1 + \ldots + d_{nn}\mathcal{K}u^n \end{pmatrix} \quad \text{for all } t \in \mathbb{I}', U \in X^n$$

*has the dichotomy spectrum $\Sigma(a\tilde{\mathcal{K}}) = \left[\underline{\beta}(a), \overline{\beta}(a)\right] \bigcup_{j\in\mathbb{J}(\mathcal{K})} \rho_j \bigcup_{i=1}^n |d_i|$ and constant spectral bundles.*

*Proof.* First of all, one has the representation $\tilde{\mathcal{K}}_t = a_tD\hat{\mathcal{K}}$ and therefore

$$\Phi_{\tilde{\mathcal{K}}}(t,s) = \left(\prod_{r=s}^{t-1} a_r\right)(D\hat{\mathcal{K}})^{t-s} \quad \text{for all } s \le t.$$

Since $D$ and $\hat{\mathcal{K}}$ commute, we arrive at

$$\Phi_{\tilde{\mathcal{K}}}(t,s) = \left(\prod_{r=s}^{t-1} a_r\right)D^{t-s}\hat{\mathcal{K}}^{t-s} \quad \text{for all } s \le t.$$

By assumption $D$ is diagonalizable and hence there is an invertible $T \in \mathbb{K}^{n\times n}$ with $D = T\operatorname{diag}(d_1,\dots,d_n)T^{-1}$. From $\hat{\mathcal{K}}T^{-1} = T^{-1}\hat{\mathcal{K}}$ we get

$$\begin{aligned} T\Phi_{\tilde{\mathcal{K}}}(t,s)T^{-1} &= \left(\prod_{r=s}^{t-1} a_r\right) TD^{t-s}\hat{\mathcal{K}}^{t-s}T^{-1} = \left(\prod_{r=s}^{t-1} a_r\right) TD^{t-s}T^{-1}\hat{\mathcal{K}}^{t-s} \\ &= \left(\prod_{r=s}^{t-1} a_r\right) (TDT^{-1})^{t-s}\hat{\mathcal{K}}^{t-s} \\ &= \left(\prod_{r=s}^{t-1} a_r\right) \operatorname{diag}((d_1\mathcal{K})^{t-s}, \dots, (d_n\mathcal{K})^{t-s}) \quad \text{for all } s \le t. \end{aligned}$$

Thus, (10) is kinematically similar to the $n$ systems $u_{t+1} = d_i a_t \mathcal{K} u_t$ for all $1 \le i \le n$ and therefore has the dichotomy spectrum $\Sigma(\tilde{\mathcal{K}}) = \bigcup_{i=1}^n \Sigma(d_i a \mathcal{K})$. Using Thm. 1 again, this implies the assertion. □

On the basis of Cor. 1 it is easy to conclude the special case of a periodic eqn. (10) in Thm. 3.

## 4 Linear integrodifference equations

Throughout this section, we suppose that $(\Omega, \Sigma, \mu)$ is a measure space. From now on the coefficients in our difference eqn. ($L$) are assumed to be integral operators

$$\mathcal{K}_t v := \int_\Omega k_t(\cdot,y)v(y)\,\mathrm{d}\mu(y) : \Omega \to \mathbb{K} \quad \text{for all } t \in \mathbb{I}'$$

of Fredholm type with appropriate *kernels* $k_t : \Omega^2 \to \mathbb{K}$. Such equations for instance occur as right-hand sides of variational eqns. (2). Consequently, ($L$) is an IDE and well-definedness of the coefficients $\mathcal{K}_t$ on various function spaces will be tacked in Sect. 4.1. On a purely formal level, the evolution operator of ($L$) is again an integral operator

$$\Phi_{\mathcal{K}}(t,\tau) = \int_\Omega k_\tau^{t-1}(\cdot,y)v(y)\,\mathrm{d}\mu(y) : \Omega \to \mathbb{K} \quad \text{for all } \tau < t$$

with the *iterated kernels* for all $x, y \in \Omega$ and $\tau, \tau + n \in \mathbb{I}'$ given by

$$k_\tau^{\tau+n}(x,y) := \begin{cases} k_\tau(x,y), & n = 1, \\ \underbrace{\int_\Omega \cdots \int_\Omega}_{n-1 \text{ times}} k_{\tau+n-1}(x,y_{n-1}) \dots k_{\tau+1}(y_2,y_1) k_\tau(y_1,y) \cdot \\ \qquad \cdot \,\mathrm{d}\mu(y_{n-1}) \dots \mathrm{d}\mu(y_2)\,\mathrm{d}\mu(y_1), & n > 1. \end{cases}$$

### *4.1 Integral operators*

We now summarize basic properties of the integral operators $\mathcal{K}_t$. For this purpose it suffices to focus on the time-invariant situation

$$\mathcal{K}v := \int_\Omega k(\cdot,y)v(y)\,\mathrm{d}\mu(y). \tag{13}$$

**Theorem 4 ([1, p. 275, Thm. 7.7]).** *Let $\Omega$ be a compact metric space, $\mu$ be the Borel measure and $p \in [1,\infty]$. If $k \in C(\Omega^2)$, then $\mathcal{K} \in L(L^p(\Omega,\mu))$ is well-defined and compact.*

The Hilbert space $L^2(\Omega) = L^2(\Omega,\mu)$ with the Lebesgue measure $\mu$ is tackled in

**Theorem 5 ([12, p. 47, Thm. 3.2.7]).** *Let $\Omega \subseteq \mathbb{R}^\kappa$ be measurable. If $k \in L^2(\Omega^2)$, then $\mathcal{K} \in L(L^2(\Omega))$ is well-defined and compact with*

$$\|\mathcal{K}\| \le \sqrt{\int_\Omega \int_\Omega |k(x,y)|^2\,\mathrm{d}y\,\mathrm{d}x}.$$

In the setting of Thm. 4 and 5 the adjoint operator $\mathcal{K}^* \in L(L^2(\Omega))$ of $\mathcal{K}$ becomes

$$\mathcal{K}^* v = \int_\Omega \overline{k(y,\cdot)}v(y)\,\mathrm{d}y$$

and consequently $\mathcal{K}$ is

- *self-adjoint*, if and only if $k(x,y) = \overline{k(y,x)}$ for $\mu$-almost all $(x,y) \in \Omega^2$. In this case one denotes the kernel $k$ as *symmetric* and it follows that $\sigma(\mathcal{K}) \subset \mathbb{R}$
- *normal*, if and only if $k(x,y)\overline{k(z,y)} = \overline{k(y,x)}k(y,z)$ for $\mu$-almost all $x,y,z \in \Omega$.

On the continuous functions we eventually obtain

**Theorem 6 ([12, p. 45, Thm. 3.2.6]).** *Let $\Omega \subset \mathbb{R}^\kappa$ be compact. If $k : \Omega^2 \to \mathbb{K}$ satisfies*

*(i)* $\int_\Omega |k(x,y)|\,\mathrm{d}y < \infty$
*(ii)* $\lim_{\xi \to x} \int_\Omega |k(\xi,y) - k(x,y)|\,\mathrm{d}y = 0$ *for all $x \in \Omega$,*

*then $\mathcal{K} \in L(C(\Omega))$ is well-defined and compact.*

The following consequence of Thms. 4 and 6 ensures that the spectrum of an integral operator $\mathcal{K}$ is independent of the state space:

**Corollary 2.** *For $k \in C(\Omega^2)$ one has $\|\mathcal{K}\|_{L(C(\Omega))} = \max_{x\in\Omega} \int_\Omega |k(x,y)|\,\mathrm{d}y$ and the spectrum of $\mathcal{K}$ is independent whether $\mathcal{K}$ is considered in $L(L^2(\Omega))$ or $L(C(\Omega))$.*

*Proof.* See [12, p. 45, Lemma 3.2.2] for the assertion on the norm and [8, p. 113, Thm. 4.2.20]) concerning the spectrum. □

## 4.2 Asymptotics of spectral intervals

It is not difficult to construct difference eqns. $(L)$ having an empty dichotomy spectrum (e.g. $\mathcal{K}_t \equiv 0$). However, whether $\Sigma(\mathcal{K})$ consists of a finite (case $(\mathfrak{S}_1)$, see Fig. 1 and 2) or an infinite number of spectral intervals (case $(\mathfrak{S}_2)$, see Fig. 3) depends on various factors. The relevance of this question is due to the fact that the gaps in the dichotomy spectrum $\Sigma(\mathcal{K})$ of a variational equation determines the number of invariant fiber bundles associated to the entire solution along which e.g. (1) is linearized.

In the prototypical situation of a multiplicative perturbation

$$\boxed{u_{t+1} = a_t \mathcal{K} u_t}$$

with a tempered sequence $(a_t)_{t\in\mathbb{I}'}$ in $\mathbb{K}$ it results from Thm. 3 that

$$\Sigma(a\mathcal{K}) = \bigcup_{j\in\mathbb{J}(\mathcal{K})} \sigma_j, \qquad \sigma_j := \left[\left|\lambda_j\right| \underline{\beta}(a), \left|\lambda_j\right| \overline{\beta}(a)\right].$$

Even for $\mathbb{J}(\mathcal{K}) = \mathbb{N}$ it is possible that consecutive intervals $\sigma_j$ eventually overlap and yield a finite number of components and hence spectral intervals in $\Sigma(a\mathcal{K})$. Since the eigenvalues $\lambda_j$ are ordered as in $(H_1)$ we obtain: The intervals $\sigma_j, \sigma_{j+1}$

- merge in case $\max \sigma_{j+1} \geq \min \sigma_j$, which is equivalent to

$$\left|\lambda_j\right| \leq \tfrac{\overline{\beta}(a)}{\underline{\beta}(a)} \left|\lambda_{j+1}\right| \tag{14}$$

- stay apart for $\max \sigma_{j+1} < \min \sigma_j$, which holds if and only if

$$\left|\lambda_{j+1}\right| < \tfrac{\underline{\beta}(a)}{\overline{\beta}(a)} \left|\lambda_j\right|. \tag{15}$$

Hence, in order to have an infinite number of spectral intervals, one needs exponentially decaying eigenvalues of $\mathcal{K}$ with a suitable decay rate. This property depends on the smoothness of the kernel, as the following results illustrate:

- Let the compact set $\Omega \subset \mathbb{R}^\kappa$ be equipped with the Borel measure. If a continuous kernel $k : \Omega^2 \to \mathbb{K}$ satisfies a Hölder condition in the second variable with

$$\int_\Omega \|k(x,\cdot)\|_{C^\gamma} \,\mathrm{d}x < \infty$$

  for some exponent $\gamma \in (0,1]$, then the eigenvalues of $\mathcal{K} \in L(L^2(\Omega,\mu))$ behave asymptotically like $\lambda_i = O(i^{-1/2-\gamma/\kappa})$ as $i \to \infty$ (see [13, Thm. 3]). For such positively definite kernels this can be improved to $\lambda_i = O(i^{-1-\gamma/\kappa})$ (see [7, Thm. 4]), which still cannot guarantee (15)
- Let $\Omega = [-1,1]$ and $k : \Omega^2 \to \mathbb{R}$ be of class $C^1$. If $k$ is symmetric, $k(\cdot,y)$ has an analytic extension from $[-1,1]$ to the ellipse (foci $\pm 1$, axis sum $R > 1$)

$$E_R := \left\{ z \in \mathbb{C} : \tfrac{(\Re z)^2}{a^2} + \tfrac{(\Im z)^2}{b^2} < 1 \right\}, \qquad a := \tfrac{1}{2}(R + \tfrac{1}{R}), \qquad b := \tfrac{1}{2}(R - \tfrac{1}{R})$$

and $k$ is bounded on $E_R \times [-1,1]$, then $\lambda_i = O(R^{-i})$ (see [5, p. 68, Thm. 4.22]). An analytic extension to every such set thus yields super-exponential decay.

Further information on the asymptotic behavior of eigenvalues to integral operators can be found in the monograph [6].

## 4.3 Examples

In this section, we first collect miscellaneous examples of time-invariant integral operators (13) resp. corresponding kernel functions, for which both eigenvalues and -functions are explicitly known. Then several convolution kernels relevant for applications are discussed, which also allow to obtain information on the asymptotics of their spectrum. These operators fulfill the properties $(H_1)$-$(H_2)$ from Sect. 3 and consequently the dichotomy spectra of the nonautonomous eqns. (9) and (10) tackled in Thm. 1, 2 resp. 3 — which are now linear IDEs — can be determined.

By means of the following remark these results extend to wider classes of IDEs:

*Remark 3 (kinematic similarity).* Let $1 \le p < \infty$ and $\mathcal{K}_t \in L(L^p(\Omega,\mu))$. Suppose that $m_t \in L^\infty(\Omega,\mu)$ are $\mathbb{K}$-valued functions with $0 \notin \rho_{\text{ess}}(m_t)$ for all $t \in \mathbb{I}'$ and

$$\sup_{t \in \mathbb{I}'} \rho_{\text{ess}}(m_t) < \infty, \qquad \sup_{t \in \mathbb{I}'} \rho_{\text{ess}}(m_t^{-1}) < \infty.$$

According to [10, pp. 30ff] the multiplication operators

$$\mathcal{M}_t \in L(L^p(\Omega,\mu)), \qquad [\mathcal{M}_t v](x) := m_t(x)v(x) \quad \text{for all } t \in \mathbb{I}', x \in \Omega$$

are well-defined and invertible. Consequently, due to

$$[\mathcal{M}_{t+1}^{-1}\mathcal{K}_t\mathcal{M}_t v](x) = \int_\Omega k_t(x,y)\frac{m_t(y)}{m_{t+1}(x)}v(y)\,\mathrm{d}\mu(y) \quad \text{for all } t \in \mathbb{I}', x \in \Omega$$

the linear IDE $(L)$ and

$$u_{t+1} = \int_\Omega \frac{k_t(\cdot,y)}{m_{t+1}(\cdot)} m_t(y)u_t(y)\,\mathrm{d}\mu(y)$$

are kinematically similar and thus have the same dichotomy spectrum.

### 4.3.1 Explicitly known spectra

Assume that $(a_t)_{t \in \mathbb{I}'}$ is a tempered sequence in $\mathbb{K}$ with $\underline{\beta}(a) < \overline{\beta}(a)$.

*Example 5.* The Sturm-Liouville problem $-u'' = \lambda u$, $u(\alpha) = u(\beta) = 0$ leads to a continuous, symmetric Green's function (see Fig. 4 (left))

$$k(x,y) := \begin{cases} (y-\alpha)(\beta-x), & \alpha \le y \le x \le \beta, \\ (x-\alpha)(\beta-y), & \alpha \le x < y \le \beta. \end{cases}$$

Thanks to Thm. 5, on the interval $\Omega := (\alpha,\beta)$ the operator $\mathcal{K} \in L(L^2(\alpha,\beta))$ is compact with real eigenvalues $\lambda_j := \frac{(\beta-\alpha)^3}{\pi^2 j^2}$ of order $o_j = 1$ and normed eigenfunctions $e_j(x) := \sqrt{\frac{2}{\beta-\alpha}} \sin(\frac{\pi j}{\beta-\alpha}(x-\alpha))$, $j \in \mathbb{N}$. From (4) we obtain a discrete spectrum

$$\Sigma(\mathcal{K}) = \left\{ \tfrac{(\beta-\alpha)^3}{\pi^2 j^2} : j \in \mathbb{N} \right\}, \qquad \mathscr{X}_j := \mathbb{I} \times \operatorname{span}\left\{e_j\right\}$$

with simple spectral intervals. Moreover, (14) shows that $\Sigma(a\mathcal{K})$ is of the form $(\mathfrak{S}_1^2)$.

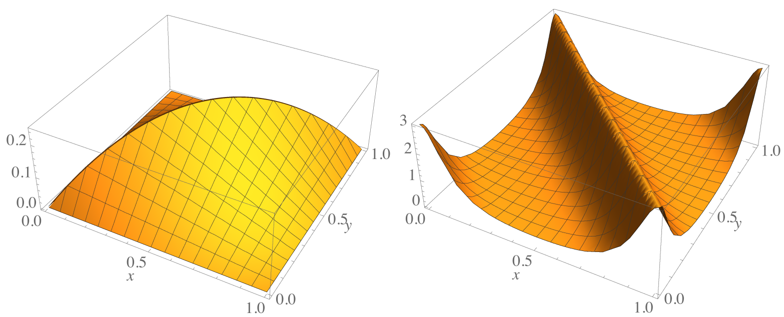


**Fig. 4** The symmetric kernels $k : (0,1)^2 \to \mathbb{R}$ from Exam. 5 (left) and Exam. 6 (right, for $\gamma = \frac{1}{2}$)

*Example 6.* On $\Omega := (\alpha,\beta)$ the analytical function (see Fig. 4 (right))

$$k(x,y) := \frac{1-\gamma^2}{1+\gamma^2 - 2\gamma\cos(\frac{2\pi}{\beta-\alpha}(x+y-2\alpha))} \quad \text{for all } \gamma \in (0,1)$$

defines a symmetric kernel. By Thm. 5 the operator $\mathcal{K} \in L(L^2(\alpha,\beta))$ is compact, has real eigenvalues (of order $o_j = 1$) and eigenfunctions (cf. [2, pp. 254–255])

$$\lambda_j := (\beta-\alpha) \begin{cases} \gamma^j, & j \ge 0, \\ -\gamma^{-j}, & j < 0. \end{cases}, \quad e_j(x) := \begin{cases} \sqrt{\frac{2}{\beta-\alpha}} \cos(\frac{2\pi j}{\beta-\alpha}(x-\alpha)), & j > 0, \\ \sqrt{\frac{1}{\beta-\alpha}}, & j = 0, \\ \sqrt{\frac{2}{\beta-\alpha}} \sin(\frac{2\pi j}{\beta-\alpha}(\alpha-x)), & j < 0. \end{cases}$$

Note that the reals $\lambda_j$ are exponentially decaying and symmetrically distributed around 0. It follows from $|\lambda_j| = |\lambda_{-j}|$ and (4) that

$$\Sigma(\mathcal{K}) = \left\{(\beta-\alpha)\gamma^j : j \in \mathbb{N}_0\right\}, \qquad \mathscr{X}_j = \mathbb{I} \times \begin{cases} \operatorname{span}\{e_0\}, & j=0,\\ \operatorname{span}\{e_j, e_{-j}\}, & j>0; \end{cases}$$

the dominant interval $\{\beta-\alpha\}$ is simple, while the other intervals have order 2. Furthermore, the concrete structure of $\Sigma(a\mathcal{K})$ depends on the ratio of the Bohl exponents. In case $\frac{\underline{\beta}(a)}{\overline{\beta}(a)} \le \gamma$ it follows from (14) that $\Sigma(a\mathcal{K})$ is of the form $(\mathfrak{S}_1^2)$. For $\gamma < \frac{\underline{\beta}(a)}{\overline{\beta}(a)}$ however, (15) implies a countably infinite number of spectral intervals, where the dominant one $(\beta-\alpha)\left[\underline{\beta}(a), \overline{\beta}(a)\right]$ is simple, while the remaining ones are of order 2.

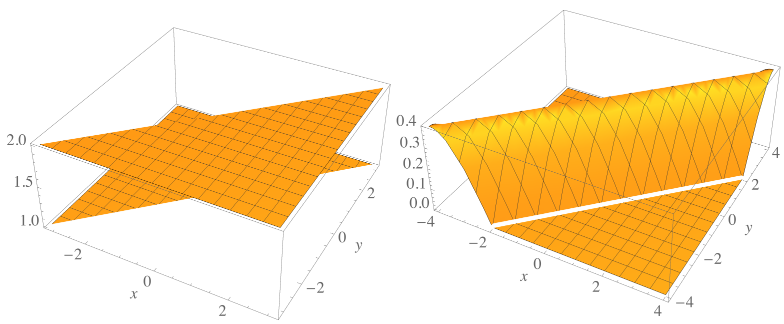


**Fig. 5** The asymmetric kernel $k : (-\pi,\pi)^2 \to \mathbb{R}$ from Exam. 7 with $\alpha = 1$, $\beta = 2$ (left) and symmetric finite radius dispersal kernel from Exam. 8 (right) for $\alpha = 2$

*Example 7.* On $\Omega := (-\pi,\pi)$ consider the discontinuous kernel (see Fig. 5 (left))

$$k(x,y) := \begin{cases} 2, & -\pi \le y \le x \le \pi,\\ 1, & -\pi \le x < y \le \pi, \end{cases}$$

which fails to be symmetric. It has the complex eigenvalues and -functions

$$\lambda_j = \frac{2\pi}{\ln 2 + 2\pi\imath j}, \qquad e_j(x) = \exp\left(\left(\tfrac{\ln 2}{2\pi} + \imath j\right)x\right) \quad \text{for all } j \in \mathbb{Z}.$$

Due to [8, p. 89, Thm. 3.3.15] the set $\left\{e_j\right\}_{j\in\mathbb{Z}}$ is a minimal complete set in $L^2(\Omega)$. Moreover, $\left|\lambda_j\right| = \left|\lambda_{-j}\right|$ and (4) imply

$$\Sigma(\mathcal{K}) = \left\{\tfrac{2\pi}{\sqrt{(\ln 2)^2 + (2\pi j)^2}} : j \in \mathbb{N}_0\right\}, \qquad \mathscr{X}_j = \mathbb{I} \times \begin{cases} \operatorname{span}\{e_0\}, & j=0,\\ \operatorname{span}\{e_j, e_{-j}\}, & j>0; \end{cases}$$

consequently, the dominant spectral interval $\left\{\frac{2\pi}{\ln 2}\right\}$ is simple, while the other spectral intervals have order 2. Moreover, since the eigenvalues decay merely linearly, it results that $\Sigma(a\mathcal{K})$ is of the form $(\mathfrak{S}_1^2)$.

We next discuss a class of kernels, where also a spectrum of the form $(\mathfrak{S}_1^1)$ (see Fig. 1) can be realized. Thereto, a kernel $k : \Omega^2 \to \mathbb{K}$ is denoted as *degenerate*, if it can be written as

$$k(x,y) := \sum_{j=1}^{d} a_j(y)x_j(x) \quad \text{for all } x,y \in \Omega$$

with linearly independent functions $x_1,\dots,x_d : \Omega \to \mathbb{K}$. This brings us into the framework of finite rank operators discussed in Sect. 2.3 and Exam. 4 with

$$\mathcal{K}v = \int_\Omega \sum_{j=1}^{d} a_j(y)v(y)\,\mathrm{d}\mu(y)x_j = \sum_{j=1}^{d} \left\langle v, x_j' \right\rangle x_j : \Omega \to \mathbb{K}$$

and functionals $\left\langle v, x_j' \right\rangle := \int_\Omega a_j(y)v(v)\,\mathrm{d}\mu(y)$. The entries of the matrix $K \in \mathbb{K}^{d\times d}$ from Exam. 4 are $k_{ij} := \int_\Omega a_i(y)x_j(y)\,\mathrm{d}\mu(y)$, $1 \le i,j \le d$, yield the discrete spectrum

$$\Sigma(\mathcal{K}) = \left|\{\lambda \in \mathbb{C} : \det(\lambda I_{\mathbb{C}^d} - K) = 0\} \setminus \{0\}\right|.$$

*Example 8 (finite radius dispersal kernel).* Let $\Omega = (-1,1)$. The kernel

$$k(x,y) := \begin{cases} \frac{\pi}{4\alpha}\cos\left(\frac{\pi(x-y)}{2\alpha}\right), & |x-y| \le \alpha, \\ 0, & |x-y| > \alpha \end{cases}$$

(cf. [17], see Fig. 5 (right)) is continuous and symmetric. Moreover, due to

$$k(x,y) = \begin{cases} \frac{\pi}{4\alpha}\left(\cos\frac{\pi x}{2\alpha}\cos\frac{\pi y}{2\alpha} + \sin\frac{\pi x}{2\alpha}\sin\frac{\pi y}{2\alpha}\right), & |x-y| \le \alpha, \\ 0, & |x-y| > \alpha \end{cases}$$

it is degenerate. Hence, for $\alpha \ge 2$ the integral operator $\mathcal{K}$ allows the representation

$$\mathcal{K}v = \sum_{j=1}^{2}\left(\int_{-1}^{1} a_j(y)v(y)\,\mathrm{d}y\right)x_j : \Omega \to \mathbb{K}$$

with $a_1(x) := \cos\frac{\pi x}{2\alpha}$, $a_2(x) = \sin\frac{\pi x}{2\alpha}$ and the linearly independent functions

$$x_1(x) := \frac{\pi}{4\alpha}\cos\frac{\pi x}{2\alpha}, \qquad x_2(x) := \frac{\pi}{4\alpha}\sin\frac{\pi x}{2\alpha}.$$

Therefore, $\mathcal{K}$ is a rank 2 operator and its eigenvalues $\lambda$ are the roots of the equation

$$\det\begin{pmatrix} \lambda - \int_{-1}^{1} a_1(y)x_1(y)\,\mathrm{d}y & -\int_{-1}^{1} a_1(y)x_2(y)\,\mathrm{d}y \\ -\int_{-1}^{1} a_2(y)x_1(y)\,\mathrm{d}y & \lambda - \int_{-1}^{1} a_2(y)x_2(y)\,\mathrm{d}y \end{pmatrix} = 0.$$

In the following example the eigenvalues are not explicitly known, but can be obtained as solutions of a transcendental equation in $\mathbb{R}$ yielding also their asymptotics.

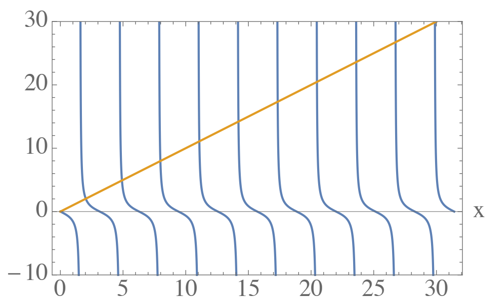


**Fig. 6** The points of intersection $\nu_j > 0$ of the graphs to $x \mapsto x$ and $x \mapsto -\tan x$ yield the eigenvalues in Exam. 9

*Example 9.* On $\Omega := (0,1)$ the continuous kernel

$$k(x,y) := \tfrac{1}{2}\min\{x,y\}\,(2-\max\{x,y\})$$

is symmetric. Suppose that $(\nu_j)_{j\in\mathbb{N}}$ denotes the strictly increasing sequence of positive real solutions to the transcendental equation $\nu + \tan\nu = 0$ (see Fig. 6). The associate integral operator $\mathcal{K}$ has the eigenvalues $\lambda_j := \frac{1}{\nu_j^2}$ of order $o_j = 1$ with normed eigenfunctions $e_j(x) = 2\sqrt{\frac{\nu_j}{(2\nu_j - \sin(2\nu_j))}}\sin(\nu_j x)$, $j \in \mathbb{N}$ (see [19, p. 438]). This yields a discrete dichotomy spectrum with simple spectral intervals

$$\Sigma(\mathcal{K}) = \left\{\nu_j^{-2} : j \in \mathbb{N}\right\}, \qquad \mathscr{X}_j := \mathbb{I} \times \operatorname{span}\left\{e_j\right\}.$$

In addition, (14) implies that $\Sigma(a\mathcal{K})$ is of the form $(\mathfrak{S}_1^2)$.

### 4.3.2 Spectra of convolutive operators

In the remaining, we suppose $\Omega = (-1,1)$ and consider kernels of convolution type

$$[\mathcal{K}v](x) := \int_{-1}^{1} k_0(x-y)v(y)\,\mathrm{d}y \quad \text{for all } x \in (-1,1)$$

with a real, even and integrable function $k_0 : \mathbb{R} \to \mathbb{R}$. These kernels frequently arise in applications [17] from theoretical ecology and have a real spectrum. In addition, we approximate their (largest) eigenvalues numerically using a Nyström method with the rectangular rule as quadrature and 1000 nodes.

Following [27], the (scaled) *Fourier transformation* of $k_0$ becomes

$$\tilde{k}_0(\xi) := \int_{\mathbb{R}} e^{\imath\xi x} k_0(x)\,\mathrm{d}x$$

and provided it is positive, we define $\Gamma(\xi) := -\ln\tilde{k}_0(\xi)$.

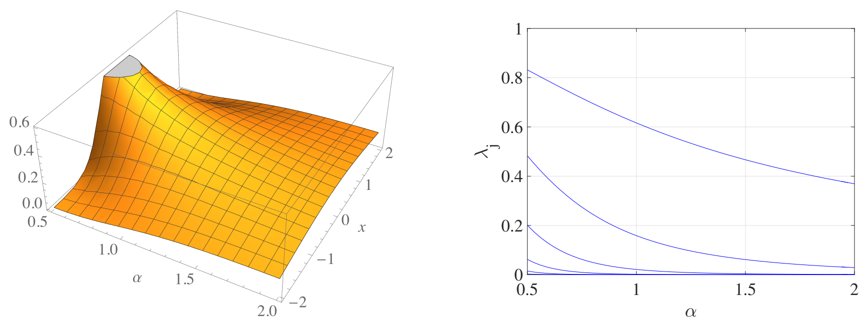


**Fig. 7** The Gaussian convolution kernel $k_0 : \mathbb{R} \to \mathbb{R}$ from Exam. 10 (left) and the super-exponentially decaying largest eigenvalues $\lambda_j$ depending on $\alpha \in [\frac{1}{2}, 2]$

*Example 10 (Gauß kernel).* As archetypical mesokurtic distribution consider

$$k_0(x) := \tfrac{1}{\sqrt{2\pi\alpha^2}} \exp\left(-\tfrac{x^2}{2\alpha^2}\right) \quad \text{for all } \alpha > 0 \tag{16}$$

(see Fig. 7) with standard deviation $\alpha > 0$. It is real analytical with $\operatorname{lip} k_0 \le \frac{1}{\sqrt{2e\pi\alpha^2}}$, the Fourier transformation $\tilde{k}_0(\xi) = e^{-\frac{\alpha^2}{2}\xi^2}$ is bounded, even and positive, whence it is $\Gamma(\xi) = \frac{\alpha^2}{2}\xi^2$. Since $\Gamma$ is convex and satisfies $\lim_{\xi\to\infty} \frac{\Gamma(\xi)}{\xi} = \infty$, it follows from [27, Cor. 1] that $\ln\lambda_j \sim -j\ln j$ as $j \to \infty$. Consequently, $\Sigma(\mathcal{K})$ and $\Sigma(a\mathcal{K})$ consists of an infinite number of spectral intervals accumulating at 0, i.e. both dichotomy spectra are of the form $(\mathfrak{S}_2)$ with $\bar{\mu} = 0$.

*Example 11 (Cauchy kernel).* Another smooth kernel is the *Cauchy kernel*

$$k_0(x) := \frac{\alpha}{\pi(\alpha^2 + x^2)} \quad \text{for all } \alpha > 0$$

(see Fig. 8) resembling the Gauß kernel (16). The Fourier transform $\tilde{k}_0(\xi) = e^{-\alpha|\xi|}$ is bounded, even and positive with $\Gamma(\xi) = \alpha|\xi|$. From [27, Thm. 2] we hence obtain $\ln\lambda_j \sim -j\psi(\alpha)$ as $j \to \infty$ with the function $\psi(\alpha) := \pi\frac{E(\operatorname{sech}(\pi/\alpha))}{E(\tanh(\pi/\alpha))} > 0$, where $E$ stands for the complete elliptic integral of first kind. It results from (14) that $\Sigma(a\mathcal{K})$ is of the form $(\mathfrak{S}_1^2)$ for $e^{\psi(\alpha)} \le \frac{\overline{\beta}(a)}{\underline{\beta}(a)}$, while (15) and $\frac{\overline{\beta}(a)}{\underline{\beta}(a)} < e^{\psi(\alpha)}$ guarantee $(\mathfrak{S}_2)$, i.e. an infinite number of spectral intervals.

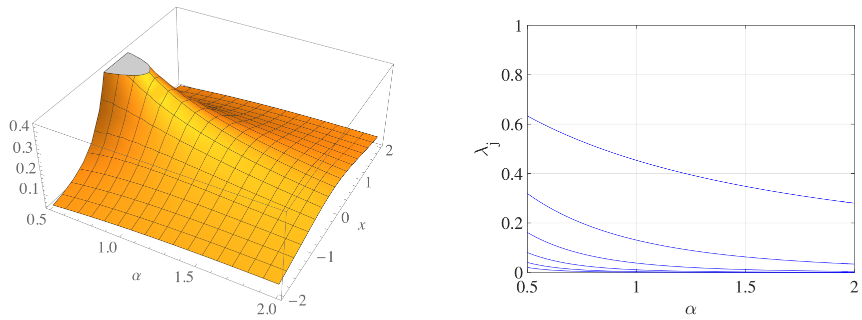


**Fig. 8** The Cauchy convolution kernel $k_0 : \mathbb{R} \to \mathbb{R}$ from Exam. 11 (left) and the exponentially decaying largest eigenvalues $\lambda_j$ depending on $\alpha \in [\frac{1}{2}, 2]$

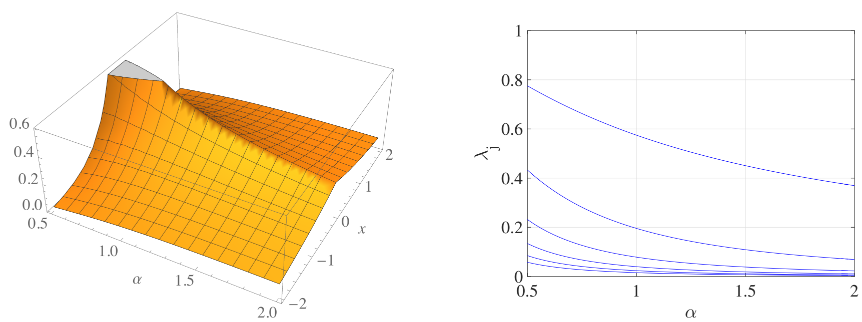


**Fig. 9** The Laplacian convolution kernel $k_0 : \mathbb{R} \to \mathbb{R}$ from Exam. 12 (left) and the quadratically decaying largest eigenvalues $\lambda_j$ depending on $\alpha \in [\frac{1}{2}, 2]$

*Example 12 (Laplace kernel).* The *Laplace kernel* is given by the function

$$k_0(x) := \tfrac{1}{2\alpha} \exp\left(-\tfrac{|x|}{\alpha}\right) \quad \text{for all } \alpha > 0$$

(see Fig. 9), which is continuous with $\operatorname{lip} k_0 \leq \frac{1}{2\alpha^2}$. If $(\nu_j)_{j\in\mathbb{N}}$ denotes the strictly increasing sequence of positive solutions to the transcendental equation $\tan \frac{\nu}{\alpha} = \pm\nu$, then $\mathcal{K}$ possesses the eigenvalues $\lambda_j := \frac{1}{1+\nu_j^2}$, $j \in \mathbb{N}$ (see [17]). On the one hand, this shows that $\lambda_j$ decays quadratically to 0. On the other hand, the Fourier transform of $k_0$ is $\tilde{k}_0(\xi) = \frac{1}{1+\alpha^2\xi^2}$ and hence $\Gamma(\xi) = \ln(1+\alpha^2\xi^2)$. Referring to [27, Thm. I] it results that $\lambda_j \sim \tilde{k}_0(\frac{\pi}{2} j + o(j))$ as $j \to \infty$, which confirms the quadratic decay. Due to (14) this results in a dichotomy spectrum $\Sigma(a\mathcal{K})$ of the form $(\mathfrak{S}_1^2)$.

*Example 13 (exponential square root kernel).* For the kernel

$$k_0(x) := \tfrac{1}{4\alpha} \exp\left(-\sqrt{\tfrac{|x|}{\alpha}}\right) \quad \text{for all } \alpha > 0$$

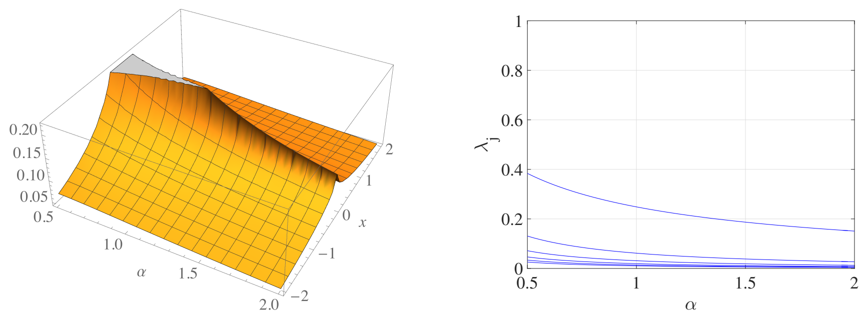


**Fig. 10** The exponential square root convolution kernel $k_0 : \mathbb{R} \to \mathbb{R}$ from Exam. 13 (left) and the six largest eigenvalues $\lambda_j$ depending on $\alpha \in [\frac{1}{2}, 2]$

(see Fig. 10) the tails are not exponentially bounded. It is continuous with a Hölder condition $\operatorname{hol}_{1/2} k_0 \le \frac{1}{4\alpha^{3/2}}$, but not differentiable in 0. The Fourier transformation

$$\tilde{k}_0(\xi) = \sqrt{2\pi}\frac{\sin\left(\frac{1}{4\alpha|\xi|}\right)\left(1-2S\left(\frac{1}{\sqrt{2\pi\alpha|\xi|}}\right)\right)+\cos\left(\frac{1}{4\alpha|\xi|}\right)\left(1-2C\left(\frac{1}{\sqrt{2\pi\alpha|\xi|}}\right)\right)}{|\alpha\xi|^{3/2}}$$

is bounded, even and positive, where $S, C$ denote the Fresnel integrals. In this setting, [27, Thm. I] leads to $\lambda_j \sim \tilde{k}_0(\frac{\pi}{2}j + o(j))$ as $j \to \infty$.

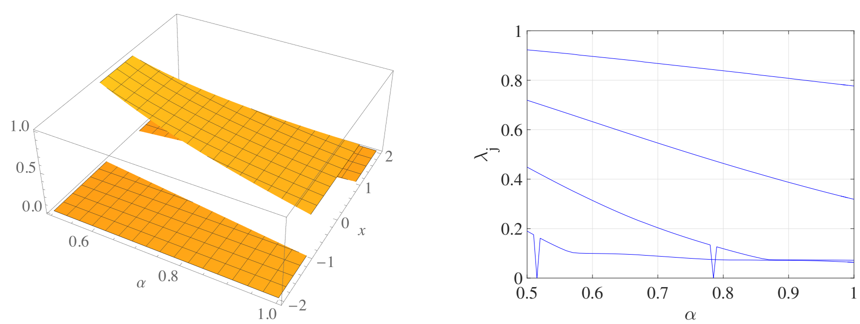


**Fig. 11** The top hat convolution kernel $k_0 : \mathbb{R} \to \mathbb{R}$ from Exam. 14 (left) and the six largest eigenvalues $\lambda_j$ depending on $\alpha \in [\frac{1}{2}, 1]$. The spikes appear to be due to numerical inaccuracies

*Example 14 (top hat kernel).* Let $\alpha \in (0, 1]$. The *top hat kernel* is defined as

$$k_0(x) := \frac{1}{2\alpha}(\theta(x+\alpha) - \theta(x-\alpha)) = \frac{1}{2\alpha}\chi_{[-\alpha,\alpha]}(x) \quad \text{for all } \alpha > 0$$

(see Fig. 11) and has the Fourier transform $\tilde{k}_0(\xi) = \frac{\sin(\alpha\xi)}{\alpha\xi}$, which is bounded, even, but fails to be positive. Hence, the results from [27] do not apply.

## References


1. Y. Abramovich & C. Aliprantis. *An Invitation to Operator Theory*, Graduate Studies in Mathematics 50. AMS, Providence, RI, 2002
2. K. Atkinson. *The Numerical Solution of Integral Equations of the Second Kind*, Monographs on Applied and Computational Mathematics 4. University Press, Cambridge, 1997
3. B. Aulbach & J. Kalkbrenner. Exponential forward splitting for noninvertible difference equations. *Comput. Math. Appl.* 42, 743–754, 2001
4. B. Aulbach & S. Siegmund. The dichotomy spectrum for noninvertible systems of linear difference equations. *J. Difference Equ. Appl.* 7(6), 895–913, 2001
5. J. Baumeister. *Stable Solution of Inverse Problems*, Advanced Lectures in Mathematics, Vieweg & Sohn, Braunschweig, 1987
6. M.S. Birman & M.Z. Solomjak. *Quantitative Analysis in Sobolev Imbedding Theorems and Applications to Spectral Theory*, AMS Translations 114. AMS, Providence, RI, 1980
7. F. Cobos & T. Kühn. *Eigenvalues of integral operators with positive definite kernels satisfying integrated Hölder conditions over metric compacta*. J. Approx. Theory 63, 39–55, 1990
8. E. Davies. *Linear Operators and their Spectra*. University Press, Cambridge, 2007
9. L. Dieci, C. Elia & E. Van Vleck. *Detecting exponential dichotomy on the real line: SVD and QR algorithms*, BIT Numerical Mathematics 51(3), 555–579, 2011
10. K. Engel & R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics 194. Springer, Berlin etc., 2000
11. M. Gil'. *Difference Equations in Normed Spaces — Stability and Oscillation*, *Mathematics Studies* 206. North-Holland, Amsterdam etc., 2007
12. W. Hackbusch. *Integral Equations – Theory and Numerical Treatment*. Birkhäuser, Basel etc., 1995
13. S. Heinrich & T. Kühn. *Embedding maps between Hölder spaces over metric compacta and eigenvalues of integral operators*. Indagationes Mathematicae 88(1), 47–62, 1982
14. D. Henry. *Geometric Theory of Semilinear Parabolic Equations*, Lect. Notes Math. 840. Springer, Berlin etc., 1981
15. T. Hüls. Computing Sacker-Sell spectra in discrete time dynamical systems. *SIAM Journal on Numerical Analysis* 48(6), 2043–2064, 2010
16. J. Jacobsen, Y. Jin & M. Lewis. Integrodifference models for persistence in temporally varying river environments. *J. Math. Biol.* 70, 549–590, 2015
17. M. Kot & W. Schaffer. Discrete-time growth-dispersal models. *Math. Biosci.* 80, 109–136, 1986
18. C. Kubrusly. *The Elements of Operator Theory* (2nd ed.). Birkhäuser, New York etc., 2011
19. P. Kythe & M. Schäferkotter. *Handbook of Computational Methods for Integration*. Chapman & Hall/CRC, Boca Raton etc., 2006
20. C. Pötzsche. A note on the dichotomy spectrum, *J. Difference Equ. Appl.* 15(10), 1021–1025, 2009 (see also the corrigendum in *J. Difference Equ. Appl.* 18(7), 1257–1261, 2012)
21. C. Pötzsche. *Geometric theory of discrete nonautonomous dynamical systems*, Lect. Notes Math. 2002. Springer, Berlin etc., 2010
22. C. Pötzsche & E. Ruß. Notes on spectrum and exponential decay in nonautonomous evolutionary equations. In T. Krisztin, editor, *Proc. 10th Coll. Qualitative Theory of Diff. Equ. (Szeged, Hungary, 2015)*. 15p., 2016
23. C. Pötzsche. Dichotomy spectra of triangular equations. *Discrete Contin. Dyn. Syst. (Series A)* 36(1), 423–450, 2016
24. E. Ruß. Dichotomy spectrum for difference equations in Banach spaces. *J. Difference Equ. Appl.*, 44p., 2016
25. R. Sacker & G. Sell. A spectral theory for linear differential systems. *J. Differ. Equations* 27, 320–358, 1978
26. S. Siegmund. Block diagonalization of linear difference equations, *J. Difference Equ. Appl.* 8(2), 177–189, 2002
27. H. Widom. Asymptotic behavior of eigenvalues of certain integral equations. II. *Arch. Ration. Mech. Anal.* 17(3), 215–229, 1964