Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

DICHOTOMY SPECTRA OF TRIANGULAR EQUATIONS

Christian Pötzsche

Institut für Mathematik, Alpen-Adria Universität Klagenfurt
Universitätsstraße 65–67
9020 Klagenfurt, Austria

(Communicated by the associate editor name)

Abstract. Without question, the dichotomy spectrum is a central tool in the stability, qualitative and geometric theory of nonautonomous dynamical systems. When dealing with such linear equations having triangular coefficient matrices, their dichotomy spectrum associated to the whole time axis is not fully determined by the diagonal entries. On the one hand, this is surprising because such behavior differs from both the half line situation, as well as the classical autonomous and periodic cases. On the other hand, triangular problems surely occur in various applications and particularly numerical techniques.

Based on operator-theoretical tools, this paper provides various sufficient criteria to obtain a corresponding *diagonal significance* for finite-dimensional difference equations in the following sense: Spectral and continuity properties of the diagonal elements extend to the whole triangular system.

1. **Introduction.**

At least in finite dimensions, the local behavior of dynamical systems near constant or periodic solutions is generically determined by the spectrum of its linearization, i.e. by eigenvalues or Floquet multipliers. Provided the (Floquet) spectrum is disjoint from the stability boundary (the unit circle in discrete time resp. the imaginary axis in continuous time) one speaks of *hyperbolicity*. When extending this setting and dealing with general nonautonomous systems or aperiodic solutions, hyperbolicity is not a generic property anymore and cannot be characterized in terms of eigenvalues. Nevertheless for various reasons, an appropriate spectral notion is given in terms of the dichotomy (or Sacker-Sell) spectrum $\Sigma \subseteq \mathbb{R}$ (cf. [36, 6]). This concept is particularly suitable to obtain stability information, and far beyond that to develop a geometric theory for time-dependent equations involving invariant manifolds and topological linearizations [24], as well as normal forms [38, 39]. In addition, it turned out to be beneficial to investigate several dynamically relevant subsets of the dichotomy spectrum for the following reasons: (1) They allow to classify nonautonomous bifurcations on a linear level [32]. (2) While Σ is only upper-semicontinuous under general perturbations, appropriate relations between its dichotomy subspectra yield even continuity for Σ (see [34]).

Typically, the dichotomy spectrum is only accessible on a numerical basis. As a result, both the approximate computation (cf. [12, 20]), and also further properties (see [32]) of Σ received attention over the recent years. Indeed, many of

2010 *Mathematics Subject Classification.* Primary 34D09; Secondary 37C60, 39A30, 47B37.

Key words and phrases. Dichotomy spectrum, Sacker-Sell spectrum, exponential dichotomy, Bohl exponent, block triangular equation, nonautonomous hyperbolicity, diagonal significance.

December 30, 2015.

the computational methods are based on the strategy to transform a linear difference or differential equation to triangular form without affecting its spectrum (or stability properties), and then to extract the spectrum from their resulting diagonal: Since the diagonal elements are scalar functions, their dichotomy spectra are single intervals whose boundary consists of lower and upper Bohl exponents. The spectrum of the whole system then results as the union of the diagonal spectra. This is a valid technique, as long as the equations are merely dichotomic on a half line. Nonetheless, when dealing with exponential dichotomies and related spectra on the whole time axis, also information on elements off the diagonal is needed, or specific assumptions on the diagonal are necessary. Besides numerical techniques, another source for (block-) triangular linear problems are variational equations related to extinction equilibria of nonlinear models in e.g. population dynamics (we refer to [21]). In summary, the full axis dichotomy spectrum has more subtle (and weaker) perturbation properties than the related half line concept.

These observations motivate our deeper analysis of spectral properties for w.l.o.g. (block) upper-triangular linear nonautonomous dynamical systems. For this purpose, the paper restricts to time-dependent finite-dimensional difference equations, since they provide a setting tailor-made to apply convenient operator-theoretical tools as previously exemplified in [10, 5] or [31, 32, 34]. Our presentation begins in the subsequent Sect. 2 with preparations on characteristic, Lyapunov and Bohl exponents, as well as exponential dichotomies in discrete time. These concepts are illustrated by several examples to which we will return throughout the text. We also emphasize the close relationship between the dichotomy spectrum and weighted shift operators on the Hilbert space of square-summable sequences. The following Sects. 3–4 illuminate how results from operator theory provide sufficient conditions on the diagonal sequences, as well as on the off-diagonal elements, such that our desired *diagonal significance* holds: This means

- the dichotomy spectrum and its dynamically relevant subspectra are determined by the union of the corresponding diagonal spectra,
- continuity of the diagonal spectra w.r.t. the Hausdorff distance yields continuity of the full spectrum.

Among others, these conditions are based on ambient compatibility conditions comparing a system's growth in forward and backward time by means of their Lyapunov filtrations (cf. [8]). For instance, Sect. 3 tackles the basic situation of diagonal systems, whereas Sect. 4 studies upper block-triangular equations. Sufficient conditions for diagonal significance depending on the diagonal systems, or the off-diagonal entries are provided. The obtained prototype results extend to triangular equations by means of inductive arguments, which can be found in Sect. 5. For the reader's convenience the paper closes with two appendices covering the required basics of operator theory and matrix-weighted shifts.

Although the present paper sticks to a discrete time situation, it was explained [34, Sect. 6] already, to what extend the results are useful in an ODE context as well. In addition, we recently became aware of Flaviano Battelli's and Ken Palmer's preprint [9] dealing with dichotomies and the related spectrum of block-triangular equations in continuous time. They allow unbounded coefficients and obtain also necessary conditions for diagonal significance in the dichotomy spectrum. Moreover, a procedure to determine the full-axis spectrum from the half line spectra is given. The methods in [9] are different from ours though.

We start with the necessary terminology: Given a real interval $I \subseteq \mathbb{R}$, we denote an intersection $I_{\mathbb{Z}} := I \cap \mathbb{Z}$ with the integers $\mathbb{Z}$ as *discrete interval*; for such a discrete interval $\mathbb{I}$, set $\mathbb{I}' := \{k \in \mathbb{Z} : k+1 \in \mathbb{I}\}$. Here, $\mathbb{I}$ will typically be unbounded, and e.g. of the form $\mathbb{Z}_\kappa^+ := [\kappa, \infty)_{\mathbb{Z}}$, $\mathbb{Z}_\kappa^- := (-\infty, \kappa]_{\mathbb{Z}}$ $\kappa \in \mathbb{Z}$, or $\mathbb{Z}$. Let us write $\mathbb{K}$ for one of the fields $\mathbb{R}$ or $\mathbb{C}$. On $\mathbb{K}^d$ we denote the Euclidean resp. unitary norm by $|\cdot|$, write $L(\mathbb{K}^d)$ for the $d \times d$-matrices and $GL(\mathbb{K}^d)$ for the invertible matrices. The space of square-summable sequences in $\mathbb{K}^d$ is abbreviated as $\ell^2 = \ell^2(\mathbb{K}^d)$ throughout.

Let $K(\mathbb{K})$ denote the family of nonempty compact subsets of $\mathbb{K}$ and

$$h : K(\mathbb{K}) \times K(\mathbb{K}) \to \mathbb{R}, \quad h(M_1, M_2) := \max\left\{\sup_{x \in M_1} \operatorname{dist}(x, M_2), \sup_{x \in M_2} \operatorname{dist}(x, M_1)\right\}$$

be the *Hausdorff distance*. Then the pair $(K(\mathbb{K}), h)$ becomes a metric space. Finally, the *closure* of a subset $M \subseteq \mathbb{K}^d$ is denoted by $\overline{M}$, and M° is its *interior*.

2. **Preliminaries.**

Consider a linear nonautonomous difference equation

$$\boxed{x_{k+1} = A_k x_k} \tag{Δ_A}$$

with coefficient matrices $A_k \in GL(\mathbb{K}^d)$, $k \in \mathbb{I}'$, fulfilling the assumption

$$\sup_{k \in \mathbb{I}'} |A_k| < \infty.$$

We often identify (Δ_A) with the matrix sequence $A = (A_k)_{k \in \mathbb{I}'}$ in the Banach space

$$\mathcal{L}^\infty(\mathbb{K}^d) := \ell^\infty(\mathbb{K}^{d \times d}), \qquad \|A\| := \sup_{k \in \mathbb{I}'} |A_k|.$$

The solutions to (Δ_A) can be expressed in terms of the *transition matrix*

$$\Phi : \mathbb{I} \times \mathbb{I} \to GL(\mathbb{K}^d), \qquad \Phi(k, l) := \begin{cases} A_{k-1} \cdots A_l, & l < k, \\ \mathrm{id}_{\mathbb{K}^d}, & k = l, \\ A_k^{-1} \cdots A_{l-1}^{-1}, & k < l. \end{cases}$$

Along with (Δ_A) let us introduce the *adjoint difference equation*

$$\boxed{x_k = A_{k+1}^* x_{k+1},} \tag{Δ_A^*}$$

whose *(adjoint) transition matrix* is given by $\Phi^*(k, \kappa) = \Phi(\kappa + 1, k + 1)^*$.

2.1. **Characteristic exponents and Lyapunov filtration.**

Assume that the discrete interval $\mathbb{I}$ is unbounded above. In order to capture the long-term behavior of (Δ_A) consider the *(upper) characteristic exponent*

$$\chi_A(x) := \limsup_{k \to \infty} \sqrt[k]{|\Phi(k, \kappa)x|}$$

of its solution starting in $x \in \mathbb{K}^d$; this exponent is independent of the initial time $\kappa \in \mathbb{I}$ and clearly fulfills $\chi_A(0) = 0$. A difference eqn. (Δ_A) possesses up to d characteristic exponents which form its *(upper) Lyapunov spectrum*

$$\left\{\chi_A(x) > 0 : x \in \mathbb{K}^d \setminus \{0\}\right\} = \{\lambda_1, \ldots, \lambda_n\}$$

with $n \leq d$. We suppose that the positive reals λ_j are ordered according to

$$0 < \lambda_1 < \ldots < \lambda_n.$$

The sublevel sets $W_j := \left\{x \in \mathbb{K}^d : \chi_A(x) \leq \lambda_j\right\}$ are linear subspaces of $\mathbb{K}^d$ yielding the *Lyapunov filtration* of strict inclusions

$$0 =: W_0 \subset W_1 \subset \ldots \subset W_n = \mathbb{K}^d.$$

Concerning this, and more details on Lyapunov spectra we refer to [8, pp. 56ff].

For the adjoint difference eqn. (Δ_A^*) the characteristic exponent is defined by

$$\chi_A^*(x) := \limsup_{k\to\infty} \sqrt[k]{|\Phi^*(k,\kappa)x|} = \limsup_{k\to\infty} \sqrt[k]{|\Phi(\kappa+1,k+1)^*x|} \quad \text{for all } x \in \mathbb{K}^d.$$

As above one obtains a finite Lyapunov spectrum

$$\left\{\chi_A^*(x) > 0 : \, x \in \mathbb{K}^d \setminus \{0\}\right\} = \{\mu_1, \dots, \mu_{n^*}\}$$

and a Lyapunov filtration $0 =: V_0 \subset V_1 \subset \ldots \subset V_{n^*} = \mathbb{K}^d$ with $n^* \le d$ for (Δ_A^*).

2.2. **Exponential Dichotomies.** Besides characteristic exponents and Lyapunov filtrations, a further and arguably more appropriate tool to capture the asymptotics of nonautonomous equations are exponential dichotomies.

Given an unbounded discrete interval $\mathbb{I}$, a linear difference eqn. (Δ_A) has an *exponential dichotomy* on $\mathbb{I}$ (ED for short, cf. [17, 6]), if there exists a sequence of projections $P_k \in L(\mathbb{K}^d)$, $k \in \mathbb{I}$, with $P_{k+1}A_k = A_k P_k$ for all $k \in \mathbb{I}'$, growth rates $\alpha \in (0,1)$ and a constant $K \ge 1$ such that the estimates

$$|\Phi(k,l)P_l| \le K\alpha^{k-l}, \qquad |\Phi(l,k)[\mathrm{id}_{\mathbb{K}^d} - P_k]| \le K\alpha^{k-l} \quad \text{for all } l \le k$$

and $k, l \in \mathbb{I}$ hold. Then *dichotomy spectrum* of (Δ_A) is defined as

$$\Sigma(A) = \left\{\gamma > 0 : \, x_{k+1} = \gamma^{-1} A_k x_k \text{ does not have an ED on } \mathbb{I}\right\};$$

it is empty or consists of up to d disjoint *spectral intervals* (cf. [6, Thm. 3.4])

$$\Sigma(A) = \bigcup_{i=1}^{m-1} [\alpha_i, \beta_i] \cup \begin{cases} (0, \beta_m] \\ [\alpha_m, \beta_m] \end{cases}$$

with real numbers $0 < \alpha_m \le \beta_m < \alpha_{m-1} \le \ldots \le \beta_1$, $m \le d$. The invertibility assumption on A_k ensures that an empty spectrum or a spectral interval $(0, \beta_m]$ can be avoided precisely in case

$$\sup_{k\in\mathbb{I}'} \left|A_k^{-1}\right| < \infty. \tag{2.1}$$

Due to its role for stability properties, $\max \Sigma(A)$ is called *stability radius* of (Δ_A). We speak of a discrete spectrum, if $\Sigma(A)$ is finite. Discrete spectra on the half line $\mathbb{I} = \mathbb{Z}_\kappa^+$ typically occur for asymptotically periodic equations, whereas on the whole line $\mathbb{I} = \mathbb{Z}$, periodic (or autonomous) equations possess a discrete spectrum.

If we denote the dichotomy spectra associated with the discrete intervals $\mathbb{Z}_\kappa^+$, $\mathbb{Z}_\kappa^-$ or $\mathbb{Z}$ by $\Sigma^+(A)$, $\Sigma^-(A)$ resp. $\Sigma(A)$, then the inclusions

$$\{\lambda_1, \dots, \lambda_n\} \subseteq \Sigma^+(A) \subseteq \Sigma(A), \qquad \Sigma^-(A) \subseteq \Sigma(A)$$

hold (see [24, p. 88, Thm. 5.13]). Thus, the Lyapunov spectrum is finer than the dichotomy spectra, and we refer to our concluding Ex. 2.6 for concrete examples illustrating these inclusions.

2.3. **Lyapunov and Bohl exponents.** While the often studied Lyapunov exponents measure exponential growth in a straight-forward manner, the related Bohl exponents (cf. [18, pp. 253ff, Sect. 3.3]) rather determine uniform growth of linear equations or individual solutions.

For the family of discrete intervals $\mathbb{J} \subseteq \mathbb{I}$ with fixed length $n \in \mathbb{N}$ one writes

$$\mathbb{I}_n := \{\mathbb{J} \subseteq \mathbb{I} : \, \mathbb{J} \text{ is a discrete interval with } \#\mathbb{J} = n\} \quad \text{for all } n \in \mathbb{N}.$$

It has advantages to introduce Lyapunov and Bohl exponents abstractly: Suppose thereto that $\mathcal{A}$ is a normed unital algebra over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with norm $|\cdot|$. Let us define the *lower* resp. *upper Bohl* exponent of a sequence $a = (a_k)_{k\in\mathbb{I}}$ in $\mathcal{A}$ as

$$\overline{\beta}_{\mathbb{I}}(a) := \limsup_{n\to\infty} \sup_{\mathbb{J}\in\mathbb{I}_n} \sqrt[n]{\Big|\prod_{j\in\mathbb{J}} a_j\Big|}, \qquad \underline{\beta}_{\mathbb{I}}(a) := \liminf_{n\to\infty} \inf_{\mathbb{J}\in\mathbb{I}_n} \sqrt[n]{\Big|\prod_{j\in\mathbb{J}} a_j\Big|}. \tag{2.2}$$

For fixed $\mathbb{I}$ one writes $\underline{\beta}(a)$ resp. $\overline{\beta}(a)$. In comparison, *Lyapunov exponents* are

$$\underline{\lambda}_+(a) := \liminf_{n\to\infty} \sqrt[n]{\Big|\prod_{j=\kappa}^{n-1} a_j\Big|}, \qquad \overline{\lambda}_+(a) := \limsup_{n\to\infty} \sqrt[n]{\Big|\prod_{j=\kappa}^{n-1} a_j\Big|},$$

$$\underline{\lambda}_-(a) := \liminf_{n\to\infty} \sqrt[n]{\Big|\prod_{j=-n}^{\kappa-1} a_j\Big|}, \qquad \overline{\lambda}_-(a) := \limsup_{n\to\infty} \sqrt[n]{\Big|\prod_{j=-n}^{\kappa-1} a_j\Big|}$$

and independent of $\kappa \in \mathbb{I}$, as long as all $a_j \in \mathcal{A}$, $j \in \mathbb{I}$, are invertible. It goes without saying that $\mathbb{I}$ has to be unbounded above in order to introduce $\underline{\lambda}_+(a), \overline{\lambda}_+(a)$, while the definition of $\underline{\lambda}_-(a), \overline{\lambda}_-(a)$ only makes sense for $\mathbb{I}$ being unbounded below. When dealing with Bohl exponents $\underline{\beta}_{\mathbb{I}}(a), \overline{\beta}_{\mathbb{I}}(a)$ it suffices that $\mathbb{I}$ is solely unbounded.

Remark 2.1. In the Banach algebra $\mathcal{A} = \mathbb{K}$ the Lyapunov exponents of a sequence a and the characteristic exponents of the corresponding scalar difference equation

$$\boxed{x_{k+1} = a_k x_k} \tag{Δ_a}$$

are related by

$$\overline{\lambda}_+(a) = \chi_a(x), \qquad \underline{\lambda}_+(a) = \chi_a^*(x)^{-1} \quad \text{for all } x \neq 0. \tag{2.3}$$

Properties of Lyapunov and particularly Bohl exponents, as well as the fact that the limits in (2.2) exist under natural assumptions, are given in

Proposition 2.2. *On unbounded discrete subintervals $\mathbb{J} \subseteq \mathbb{I}$ one has*

$$\underline{\beta}_{\mathbb{I}}(a) \le \underline{\beta}_{\mathbb{J}}(a) \le \overline{\beta}_{\mathbb{J}}(a) \le \overline{\beta}_{\mathbb{I}}(a) \le \sup_{k\in\mathbb{I}} |a_k|,$$

$$\underline{\beta}_{\mathbb{Z}}(a) \le \underline{\beta}_{\mathbb{Z}_\kappa^\pm}(a) \le \underline{\lambda}_\pm(a) \le \overline{\lambda}_\pm(a) \le \overline{\beta}_{\mathbb{Z}_\kappa^\pm}(a) \le \overline{\beta}_{\mathbb{Z}}(a) \quad \textit{for all } \kappa \in \mathbb{Z} \tag{2.4}$$

and the positive homogeneity

$$\begin{aligned} \overline{\beta}(\mu a) &= |\mu|\,\overline{\beta}(a), & \underline{\beta}(\mu a) &= |\mu|\,\underline{\beta}(a), \\ \underline{\lambda}_\pm(\mu a) &= |\mu|\,\underline{\lambda}_\pm(a), & \overline{\lambda}_\pm(\mu a) &= |\mu|\,\overline{\lambda}_\pm(a) \quad \textit{for all } \mu \in \mathbb{K}. \end{aligned} \tag{2.5}$$

Moreover, the left-hand limit in (2.2) exists and the characterizations

$$\overline{\beta}_{\mathbb{I}}(a) = \lim_{n\to\infty} \sup_{\mathbb{J}\in\mathbb{I}_n} \sqrt[n]{\Big|\prod_{j\in\mathbb{J}} a_j\Big|} = \inf_{n\in\mathbb{N}} \sup_{\mathbb{J}\in\mathbb{I}_n} \sqrt[n]{\Big|\prod_{j\in\mathbb{J}} a_j\Big|} \tag{2.6}$$

$$= \inf\left\{\rho > 0 \,\middle|\, \exists K \ge 1 : \forall n \in \mathbb{N} : \sup_{\mathbb{J}\in\mathbb{I}_n} \Big|\prod_{j\in\mathbb{J}} a_j\Big| \le K\rho^n\right\} \tag{2.7}$$

hold, where (2.7) necessitates the sequence a to be bounded.

Proof. The inequalities relating Bohl exponents on different discrete intervals are evident from (2.2), as well as their homogeneity relations (2.5). Furthermore, define $\alpha := \sup_{j\in\mathbb{I}} |a_j|$ and for the sake of a convenient notation abbreviate

$$\phi_n := \sup_{\mathbb{J}\in\mathbb{I}_n} \sqrt[n]{\Big|\prod_{j\in\mathbb{J}} a_j\Big|} = \sqrt[n]{\sup_{\mathbb{J}\in\mathbb{I}_n}\Big|\prod_{j\in\mathbb{J}} a_j\Big|} \quad \text{for all } n\in\mathbb{N}.$$

Therefore, because the norm $|\cdot|$ is submultiplicative, $\big|\prod_{j\in\mathbb{J}} a_j\big| \le \prod_{j\in\mathbb{J}} |a_j| \le \alpha^n$ for all $\mathbb{J}\in\mathbb{I}_n$ implies $\sqrt[n]{\phi_n} \le \alpha$ for every $n\in\mathbb{N}$ and consequently $\overline{\beta}(a)\le\alpha$.

(a) Let $m,n\in\mathbb{N}$ and suppose $\mathbb{J}\in\mathbb{I}_{m+n}$ denotes an arbitrary discrete interval, e.g. of the form $\mathbb{J} = [\kappa,\kappa+m)_{\mathbb{Z}} \cup [\kappa+m,\kappa+m+n)_{\mathbb{Z}}$ with some $\kappa\in\mathbb{I}$. Again the submultiplicativity of the norm allows us to obtain

$$\Big|\prod_{j\in\mathbb{J}} a_j\Big| \le \Big|\prod_{j=\kappa}^{\kappa+m-1} a_j\Big|\Big|\prod_{j=\kappa+m}^{\kappa+m+n-1} a_j\Big| \le \phi_m\phi_n \quad \text{for all } m,n\in\mathbb{N}$$

and since $\mathbb{J}\in\mathbb{I}_{m+n}$ was arbitrary, we can pass to the least upper bound over all such discrete intervals $\mathbb{J}$ yielding $0\le\phi_{m+n}\le\phi_m\phi_n$ for all $m,n\in\mathbb{N}$. Now it is well-known (see, e.g., [1, p. 246]) that the real sequence $(\sqrt[n]{\phi_n})_{n\in\mathbb{N}}$ converges to the value $\inf_{n\in\mathbb{N}}\sqrt[n]{\phi_n}$, which establishes (2.6).

In order to deduce the characterization (2.7), we abbreviate the right-hand side of the inequality required in (2.7) by R. Thus, for every $\varepsilon>0$ there exists a $K\ge 0$ such that $\phi_n \le K(R+\varepsilon)^n$ for all $n\in\mathbb{N}$ and $\overline{\beta}(a)\le R$ follows from

$$\overline{\beta}(a) = \limsup_{n\to\infty}\sqrt[n]{\phi_n} \le \limsup_{n\to\infty}\sqrt[n]{K}(R+\varepsilon) = R+\varepsilon \quad \text{for all } \varepsilon>0.$$

Conversely, it remains to show $R\le\overline{\beta}(a)$. From $\overline{\beta}(a) = \lim_{\nu\to\infty}\sup_{n\ge\nu}\sqrt[n]{\phi_n}$ we see that for every sufficiently small $\varepsilon>0$ there exists a $N\in\mathbb{N}$ such that the inequality $\sup_{n\ge\nu}\sqrt[n]{\phi_n}\le\overline{\beta}(a)+\varepsilon$ holds for all $\nu\ge N$. Hence, it is

$$\phi_n \le (\overline{\beta}(a)+\varepsilon)^n \quad \text{for all } n\ge N \tag{2.8}$$

and if we define $K := \sup_{1\le n<N}\left(\frac{\sup_{j\in\mathbb{I}}|a_j|}{\overline{\beta}(a)+\varepsilon}\right)^n \ge 1$, then

$$\phi_n \le \big(\sup_{j\in\mathbb{I}}|a_j|\big)^n \le K\left(\overline{\beta}(a)+\varepsilon\right)^n \quad \text{for all } 1\le n<N.$$

Combining this with (2.8), and since $\varepsilon>0$ was arbitrary, we get $R\le\overline{\beta}(a)$. □

Corollary 2.3. *In algebras $\mathcal{A}$ with multiplicative norm (i.e. $|ab| = |a|\,|b|$, $a,b\in\mathcal{A}$) one has $\overline{\beta}(|a|) = \overline{\beta}(a)$, $\underline{\beta}(|a|) = \underline{\beta}(a)$ and if every $a_k\in\mathcal{A}$ is invertible, then*

$$\inf_{k\in\mathbb{I}}|a_k| \le \underline{\beta}(a) = \lim_{n\to\infty}\inf_{\mathbb{J}\in\mathbb{I}_n}\sqrt[n]{\Big|\prod_{j\in\mathbb{J}} a_j\Big|} = \sup_{n\in\mathbb{N}}\inf_{\mathbb{J}\in\mathbb{I}_n}\sqrt[n]{\Big|\prod_{j\in\mathbb{J}} a_j\Big|}. \tag{2.9}$$

Proof. It is clear that both sequences a and $|a|$ have the same Bohl exponents. When each $a_k\in\mathcal{A}$, $k\in\mathbb{I}$, is invertible, then (2.9) follows from the proof of Prop. 2.2 applied to $\tilde{a}_k := a_k^{-1}$. □

As temporary conclusion, we present the close connection between the dichotomy spectrum and Bohl exponents of scalar difference eqns. (Δ_a):

Proposition 2.4 (see [21, Prop. B.4]). *If $a = (a_k)_{k\in\mathbb{I}'}$ is a sequence in $\mathbb{K}$ with*

$$0 < \inf_{k\in\mathbb{I}'} |a_k| \le \sup_{k\in\mathbb{I}'} |a_k| < \infty, \tag{2.10}$$

then (Δ_a) has the dichotomy spectrum $\Sigma(a) = [\underline{\beta}(a), \overline{\beta}(a)]$.

2.4. **Weighted shift operators.** For one-sided time $\mathbb{I} = \mathbb{Z}^+_\kappa$, $\kappa \in \mathbb{Z}$, the dichotomy spectrum $\Sigma^+(A)$ of difference eqns. (Δ_A) and the essential (Fredholm) spectrum σ_F of the unilateral matrix-weighted shift

$$T_A\phi := (0, A_\kappa\phi_\kappa, A_{\kappa+1}\phi_{\kappa+1}, \dots) \quad \text{for all } \phi \in \ell^2$$

are related by (cf. [10] or [32, Thm. 3.22])

$$\Sigma^+(A) = \sigma_F(T_A) \cap \mathbb{R}^+. \tag{2.11}$$

The set $\sigma_F(T_A)$ is rotationally invariant, i.e. consists of concentric rings and annuli in the complex plane. This observation has the striking advantage that information on the dichotomy spectrum can be obtained from results on shifts, like for instance

Example 2.5 (asymptotically periodic scalar equations). *Let $p \in \mathbb{N}$ and suppose $(a_k)_{\kappa\le k}$ is a sequence in $\mathbb{K}$. If $(|a_k|)_{\kappa\le k}$ is* asymptotically p-periodic, *i.e. there is some p-periodic positive real sequence $(p_k)_{k\in\mathbb{I}}$ satisfying $\lim_{k\to\infty}(|a_k| - p_k) = 0$, then Ex. B.3 yields the dichotomy spectrum*

$$\Sigma^+(a) = \{c\}$$

with the asymptotic mean $c := \sqrt[p]{p_{\kappa+p-1}\cdots p_\kappa}$.

Difference eqns. (Δ_A) defined on the whole axis $\mathbb{I} = \mathbb{Z}$ exhibit a richer spectral theory; it is based on bilateral matrix-weighted shifts

$$(T_A\phi)_k := A_{k-1}\phi_{k-1} \quad \text{for all } k \in \mathbb{Z},\ \phi \in \ell^2.$$

As motivated in Sect. 1, it is advisable to distinguish different dichotomy spectra

$$\Sigma_\alpha(A) = \sigma_\alpha(T_A) \cap \mathbb{R}^+ \quad \text{for all } \alpha \in \{a, s, F, F_0, \pi\}, \tag{2.12}$$

where $\Sigma_a(A) := \Sigma(A)$ denotes the dichotomy spectrum of (Δ_A), while its subspectra $\Sigma_s(A), \Sigma_F(A), \Sigma_{F_0}(A)$ and $\Sigma_\pi(A)$ are called *surjectivity*, *Fredholm*, *Weyl* resp. *approximate point spectrum* (see [32, 34]). They consist of all reals $\gamma > 0$ such that $L_\gamma \in L(\ell^2)$, $(L_\gamma\phi)_k := \phi_{k+1} - \gamma^{-1}A_k\phi_k$, $k \in \mathbb{Z}$, fails to be onto, Fredholm, Weyl resp. bounded below. The corresponding spectra σ_α are introduced in Sect. A.

2.5. **Examples for $\mathbb{I} = \mathbb{Z}$.** The upcoming examples allow to obtain Lyapunov and Bohl exponents explicitly. This equips us with a number of difference equations sufficiently flexible to illustrate our results later on.

Example 2.6 (scalar equations). *Choose $\kappa \in \mathbb{Z}$ fixed and let $(a_k)_{k\in\mathbb{Z}}$ be a sequence in $\mathbb{K}$ satisfying (2.10). The inclusions $\partial\Sigma(a) \subseteq \Sigma_F(a) \subseteq \Sigma_{F_0}(a) \subseteq \Sigma(a)$, as well as $\partial\Sigma(a) \subseteq \Sigma_s(a) \subseteq \Sigma(a)$ are fulfilled due to* [32, Cors. 4.26(d) and 4.31]. *We can apply Prop. 2.4 in order to determine the dichotomy spectra of the eqns. (Δ_a). Moreover, based on the relations (2.3) one automatically has information concerning their characteristic exponents χ_a, χ_a^*. Finally, combining (2.12) with the abstract results provided in Sect. B.2 implies the following concrete examples:*

(1) If $|a_k| \equiv \bar\alpha$ with $\bar\alpha > 0$, then all Bohl and Lyapunov exponents coincide, i.e.,

$$\underline{\beta}_{\mathbb{Z}^\pm_\kappa}(a) = \overline{\beta}_{\mathbb{Z}^\pm_\kappa}(a) = \underline{\beta}_{\mathbb{Z}}(a) = \overline{\beta}_{\mathbb{Z}}(a) = \underline{\lambda}_+(a) = \overline{\lambda}_+(a) = \underline{\lambda}_-(a) = \overline{\lambda}_-(a) = \bar\alpha.$$

One has discrete dichotomy spectra $\Sigma_\alpha(a) = \{\bar\alpha\}$ for all $\alpha \in \{a, F_0, F, s, \pi\}$.

(2) If $|a|$ is p-periodic, $p \in \mathbb{N}$, then the Bohl and Lyapunov exponents become

$$\begin{aligned}\underline{\beta}_{\mathbb{Z}_\kappa^\pm}(a) = \overline{\beta}_{\mathbb{Z}_\kappa^\pm}(a) = \underline{\beta}_{\mathbb{Z}}(a) = \overline{\beta}_{\mathbb{Z}}(a) \\ = \underline{\lambda}_+(a) = \overline{\lambda}_+(a) = \underline{\lambda}_-(a) = \overline{\lambda}_-(a) = \sqrt[p]{|a_{k+p-1}\cdots a_k|}\end{aligned}$$

and we also arrive at the discrete spectra

$$\Sigma_\alpha(a) = \left\{\sqrt[p]{|a_{p-1+k}\cdots a_k|}\right\} \quad \text{for all } \alpha \in \{a, F_0, F, s, \pi\} \text{ and } k \in \mathbb{Z}.$$

(3) The both-sided asymptotically constant situation $\lim_{k\to\pm\infty}|a_k| = \alpha_\pm$ with reals $\alpha_\pm > 0$ now illustrates a distinction between Bohl and Lyapunov exponents

$$\begin{aligned}&\underline{\beta}_{\mathbb{Z}_\kappa^+}(a) = \overline{\beta}_{\mathbb{Z}_\kappa^+}(a) = \underline{\lambda}_+(a) = \overline{\lambda}_+(a) = \alpha_+, && \underline{\lambda}_-(a) = \overline{\lambda}_-(a) = \alpha_-, \\ &\underline{\beta}_{\mathbb{Z}}(a) = \min\{\alpha_-, \alpha_+\}, && \overline{\beta}_{\mathbb{Z}}(a) = \max\{\alpha_-, \alpha_+\},\end{aligned}$$

as well as their dependence on the discrete interval. Using [32, Ex. 5.3] *one deduces*

$$\Sigma(a) = [\min\{\alpha_-, \alpha_+\}, \max\{\alpha_-, \alpha_+\}], \quad \Sigma_\pi(a) = \begin{cases}\{\alpha_-, \alpha_+\}, & \alpha_- \le \alpha_+, \\ [\alpha_+, \alpha_-], & \alpha_+ \le \alpha_-,\end{cases}$$

$$\Sigma_{F_0}(a) = [\min\{\alpha_-, \alpha_+\}, \max\{\alpha_-, \alpha_+\}], \quad \Sigma_F(a) = \{\alpha_-, \alpha_+\}.$$

(4) Let $p^+, p^- \in \mathbb{N}$. If $|a|$ is asymptotically p^+- resp. p^--periodic to real positive sequences $(p_k^+)_{k\ge\kappa}$, $(p_k^-)_{k\le\kappa}$ on $\mathbb{Z}_\kappa^+$ or $\mathbb{Z}_\kappa^-$ by means of Ex. 2.5, then it follows

$$\Sigma(a) = [\min\{c_+, c_-\}, \max\{c_-, c_+\}], \qquad \Sigma_\pi(a) = \begin{cases}\{c_+, c_-\}, & c_- \le c_+, \\ [c_+, c_-], & c_+ \le c_-\end{cases}$$

from Ex. B.4 with the asymptotic means $c_\pm := \sqrt[p^\pm]{p^\pm_{\kappa+p^\pm-1}\cdots p^\pm_\kappa}$.

(5) To clarify that the inequalities (2.4) can be strict, for given reals $\alpha, \beta > 0$ consider a sequence $(a_k)_{k\ge 0}$ satisfying

$$|a_k| = \begin{cases}\alpha, & k \in \left[\frac{n(n+1)}{2}, \frac{(n+1)(n+2)}{2}\right)_{\mathbb{Z}}, n \text{ even}, \\ \beta, & k \in \left[\frac{n(n+1)}{2}, \frac{(n+1)(n+2)}{2}\right)_{\mathbb{Z}}, n \text{ odd}\end{cases} \quad \text{for all } k \in \mathbb{Z}_0^+.$$

Hence, the modulus of a_k is alternately equal to the constant values α resp. β on arithmetically increasing intervals (see Fig. 1). This yields the Bohl exponents

$$\underline{\beta}(a) = \min\{\alpha, \beta\}, \qquad \overline{\beta}(a) = \max\{\alpha, \beta\}.$$

In order to obtain the Lyapunov exponents of a, we observe that for every $k \ge 1$ there exist unique $n \in \mathbb{N}$ and $l \in [0, n)_{\mathbb{Z}}$ with $k = \frac{n(n+1)}{2} + l$. For even n this implies

$$\left|\prod_{j=0}^{k-1} a_j\right| = \prod_{j=0}^{\frac{n(n+1)}{2}+l} |a_j| = \beta^l \alpha\beta^2\alpha^3\beta^4\cdots\alpha^{n-1}\beta^n = \beta^l \alpha^{\frac{n^2}{4}} \beta^{\frac{n^2}{4}-\frac{n}{2}}$$

and for odd n it is

$$\left|\prod_{j=0}^{k-1} a_j\right| = \prod_{j=0}^{\frac{n(n+1)}{2}+l} |a_j| = \beta^l \alpha\beta^2\alpha^3\beta^4\cdots\beta^{n-1}\alpha^n = \beta^l \alpha^{\frac{(n+1)^2}{4}} \beta^{\frac{n^2-1}{4}-\frac{n}{2}}.$$

By means of these representations it is not difficult to deduce that the Lyapunov exponents are given as geometric mean $\overline{\lambda}_+(a) = \underline{\lambda}_+(a) = \sqrt{\alpha\beta}$ and fulfill the inequality $\underline{\beta}(a) \le \underline{\lambda}_+(a) = \overline{\lambda}_+(a) \le \overline{\beta}(a)$; this corresponds with (2.4).

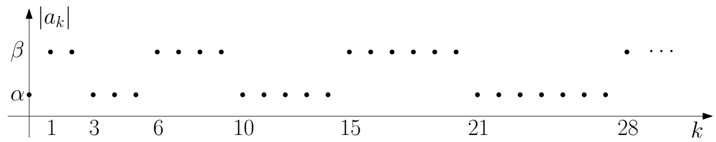

FIGURE 1. The sequence $(a_k)_{k\in\mathbb{Z}_0^+}$ from Ex. 2.6(5)

We continue with a 2-dimensional problem from [34, Ex. 5.10] being useful for several reasons: It illustrates both that the dichotomy spectrum is upper-semicontinuous, and that it might be smaller than the union of its diagonal spectra:

Example 2.7. *Consider the planar real difference eqn.* (Δ_A) *with coefficients*

$$A_k := \begin{pmatrix} a_k^1 & c_k \\ 0 & a_k^2 \end{pmatrix} \in GL(\mathbb{R}^2)$$

satisfying $A \in \mathcal{L}^\infty(\mathbb{R}^2)$ *and involving the real sequences*

$$a_k^1 := \begin{cases} \alpha_+, & k \ge 0, \\ \alpha_-, & k < 0, \end{cases} \qquad a_k^2 := \begin{cases} \beta_+, & k \ge 0, \\ \beta_-, & k < 0, \end{cases} \qquad c_k := \begin{cases} \lambda, & k \ge 0, \\ 0, & k < 0 \end{cases} \tag{2.13}$$

with reals $\alpha_\pm, \beta_\pm > 0$ *and a parameter* $\lambda \in \mathbb{R}$.

(1) Evidently, the background to obtain the Lyapunov spectra and filtrations are the transition matrices $\Phi(k,\kappa)$ *of* (Δ_A) *and* $\Phi^*(k,\kappa)$ *of* (Δ_A^*)*:*

- *For* $\alpha_+ \ne \beta_+$ *they are given by*

$$\Phi(k,\kappa) = \begin{cases} \begin{pmatrix} \alpha_+^{k-\kappa} & \lambda\frac{\alpha_+^{k-\kappa}-\beta_+^{k-\kappa}}{\alpha_+-\beta_+} \\ 0 & \beta_+^{k-\kappa} \end{pmatrix} & \text{for all } k,\kappa \ge 0, \\ \begin{pmatrix} \alpha_-^{k-\kappa} & 0 \\ 0 & \beta_-^{k-\kappa} \end{pmatrix} & \text{for all } k,\kappa \le 0, \end{cases}$$

$$\Phi^*(k,\kappa) = \begin{cases} \begin{pmatrix} \alpha_+^{\kappa-k} & 0 \\ \lambda\frac{\alpha_+^{\kappa-k}-\beta_+^{\kappa-k}}{\alpha_+-\beta_+} & \beta_+^{\kappa-k} \end{pmatrix} & \text{for all } k,\kappa \ge 0, \\ \begin{pmatrix} \alpha_-^{\kappa-k} & 0 \\ 0 & \beta_-^{\kappa-k} \end{pmatrix} & \text{for all } k,\kappa \le 0 \end{cases}$$

and consequently yield

$$n = 2, \qquad \{\lambda_1, \lambda_2\}, \qquad \{0\} = W_0 \subset W_1 \subset W_2 = \mathbb{R}^2,$$
$$n^* = 2, \qquad \{\mu_1, \mu_2\}, \qquad \{0\} = V_0 \subset V_1 \subset V_2 = \mathbb{R}^2.$$

The explicit values for these quantities can be found in Tab. 1.

	λ_1	λ_2	μ_1	μ_2	W_1	V_1
$\alpha_+ < \beta_+$	α_+	β_+	$\frac{1}{\beta_+}$	$\frac{1}{\alpha_+}$	$\mathbb{R}e_1$	$\mathbb{R}e_2$
$\alpha_+ > \beta_+$	β_+	α_+	$\frac{1}{\alpha_+}$	$\frac{1}{\beta_+}$	$\mathbb{R}\binom{\lambda}{\beta_+-\alpha_+}$	$\mathbb{R}\binom{\beta_+-\alpha_+}{\lambda}$

TABLE 1. Lyapunov spectra and filtrations for Ex. 2.7 with $\alpha_+ \ne \beta_+$

- *For $\alpha_+ = \beta_+$ the transition matrices become*

$$\Phi(k,\kappa) = \begin{cases} \begin{pmatrix} \alpha_+^{k-\kappa} & \lambda(k-\kappa)\alpha_+^{k-\kappa-1} \\ 0 & \alpha_+^{k-\kappa} \end{pmatrix} & \text{for all } k,\kappa \geq 0, \\ \begin{pmatrix} \alpha_-^{k-\kappa} & 0 \\ 0 & \beta_-^{k-\kappa} \end{pmatrix} & \text{for all } k,\kappa \leq 0, \end{cases}$$

$$\Phi^*(k,\kappa) = \begin{cases} \begin{pmatrix} \alpha_+^{\kappa-k} & 0 \\ \lambda(\kappa-k)\alpha_+^{k-\kappa-1} & \alpha_+^{\kappa-k} \end{pmatrix} & \text{for all } k,\kappa \geq 0, \\ \begin{pmatrix} \alpha_-^{\kappa-k} & 0 \\ 0 & \beta_-^{\kappa-k} \end{pmatrix} & \text{for all } k,\kappa \leq 0 \end{cases}$$

and consequently yield

$$n = 1, \qquad \{\alpha_+\}, \qquad \{0\} = W_0 \subset W_1 = \mathbb{R}^2,$$
$$n^* = 1, \qquad \left\{\tfrac{1}{\alpha_+}\right\}, \qquad \{0\} = V_0 \subset V_1 = \mathbb{R}^2.$$

The related dichotomy spectra $\Sigma(A)$ for the various constellations of $\alpha_\pm \neq \beta_\pm$ were computed in [34, Ex. 5.10] *already.*

(2) We particularly focus on the situation

$$\alpha_+ := \delta, \qquad \alpha_- := \delta^{-1}, \qquad \beta_- := \alpha_+, \qquad \beta_+ := \alpha_- \tag{2.14}$$

for some real $\delta > 1$, where the diagonal sequences satisfy $\Sigma(a^1) = \Sigma(a^2) = [\alpha_-, \alpha_+]$ (cf. Prop. 2.4). One therefore obtains from [32, Ex. 5.5] *that*

$$\Sigma(A) = \begin{cases} \{\alpha_-, \alpha_+\}, & \lambda \neq 0, \\ [\alpha_-, \alpha_+], & \lambda = 0. \end{cases}$$

Hence, $\Sigma(A)$ suddenly shrinks and fulfills $\Sigma(A) \subset \Sigma(a^1) \cup \Sigma(a^2)$ for $\lambda \neq 0$.

3. **Spectra of diagonal equations.** Before discussing the general situation of triangular equations, let us initially tackle a simpler case: Results on scalar difference eqns. (Δ_a) extend to (Δ_A) with diagonal coefficient matrices A_k, i.e.

$$x_{k+1} = A_k x_k, \qquad A_k = \begin{pmatrix} a_k^1 & & \\ & \ddots & \\ & & a_k^d \end{pmatrix} \tag{D}$$

and bounded diagonal sequences $(a_k^1)_{k\in\mathbb{I}'}, \ldots, (a_k^d)_{k\in\mathbb{I}'}$. Since the half line situation $\mathbb{I} = \mathbb{Z}_\kappa^+$ was tackled in [32, Cor. 3.25] already, we restrict to the whole axis $\mathbb{I} = \mathbb{Z}$.

Theorem 3.1. *Keep $\alpha \in \{a, s, F\}$ fixed. Diagonal difference eqns. (D) fulfill $\Sigma_\alpha(A) = \bigcup_{i=1}^d \Sigma_\alpha(a^i)$.*

Proof. For $1 \leq i \leq d$ we define the linear operators

$$L_\gamma \in L(\ell^\infty(\mathbb{K}^d)), \qquad (L_\gamma\phi)_k := \phi_{k+1} - \gamma^{-1}A_k\phi_k,$$
$$L_\gamma^i \in L(\ell^\infty(\mathbb{K})), \qquad (L_\gamma^i\phi)_k := \phi_{k+1} - \gamma^{-1}a_k^i\phi_k.$$

If $\alpha = s$, then due to [30, Prop. 1] we obtain the equivalences

$$\gamma \notin \Sigma_s(A) \Leftrightarrow L_\gamma \text{ is onto}$$
$$\Leftrightarrow \forall\psi \in \ell^\infty(\mathbb{K}^d) : \exists\phi \in \ell^\infty(\mathbb{K}^d) :\ L_\gamma\phi = \psi$$

$\Leftrightarrow \forall\psi \in \ell^\infty(\mathbb{K}^d) :\ x_{k+1} = \gamma^{-1}A_k x_k + \psi_k$ has a bounded solution ϕ

$\Leftrightarrow \forall 1 \le i \le d : \forall \psi^i \in \ell^\infty(\mathbb{K}) :\ x_{k+1} = \gamma^{-1}a_k^i x_k + \psi_k^i$ has a bounded solution ϕ^i

$\Leftrightarrow \forall 1 \le i \le d : \forall \psi^i \in \ell^\infty(\mathbb{K}) : \exists \phi^i \in \ell^\infty(\mathbb{K}) :\ L_\gamma^i \phi^i = \psi^i$

$\Leftrightarrow L_\gamma^i$ is onto for all $1 \le i \le d$

$\Leftrightarrow x_{k+1} = \gamma^{-1}a_k^i x_k$ has an ED for all $1 \le i \le d \Leftrightarrow \gamma \notin \bigcup_{i=1}^d \Sigma_s(a^i)$

and the claim results in the logical contraposition. The situation $\alpha = a$ follows similarly by means of the characterization [17, p. 230, Thm. 7.6.5] applied to the operators L_γ and L_γ^i, while $\alpha = F$ was tackled in [32, Cor. 4.30]. □

4. **Block-triangular equations.** Our following analysis focusses on block-triangular systems (Δ_A), capturing essential phenomena present in triangular equations. Thereto, let $d_1, d_2 \in \mathbb{N}$ be integers with $d_1 + d_2 = d$ and suppose w.l.o.g. that (Δ_A) is in upper block-triangular form

$$x_{k+1} = A_k x_k, \qquad A_k := \begin{pmatrix} A_k^1 & C_k \\ 0 & A_k^2 \end{pmatrix} \tag{B}$$

with blocks $A_k^1 \in \mathbb{K}^{d_1\times d_1}$, $A_k^2 \in \mathbb{K}^{d_2\times d_2}$, $C_k \in \mathbb{K}^{d_1\times d_2}$ and $k \in \mathbb{I}$ unbounded above. Due to $\det A_k = \det A_k^1 \det A_k^2$ one has $A_k \in GL(\mathbb{K}^d)$ if and only if both diagonal blocks A_k^1, A_k^2 are invertible.

4.1. **Equations on the half line $\mathbb{I} = \mathbb{Z}_\kappa^+$.** Here, diagonal significance holds:

Theorem 4.1. *Block-triangular eqns. (B) satisfy $\Sigma^+(A) = \Sigma^+(A^1) \cup \Sigma^+(A^2)$.*

Proof. We represent points $x \in \mathbb{K}^d$ as pairs (x^1, x^2) with the components $x^i \in \mathbb{K}^{d_i}$, $i = 1, 2$, and introduce the unilateral shift operator $T_A \in L(\ell^2)$ as

$$(T_A\phi)_k = \begin{cases} 0, & k = \kappa, \\ A_{k-1}\phi_{k-1}, & k > \kappa \end{cases} = \begin{cases} 0, & k = \kappa, \\ \begin{pmatrix} A_{k-1}^1\phi_{k-1}^1 + C_{k-1}\phi_{k-1}^2 \\ A_{k-1}^2\phi_{k-1}^2 \end{pmatrix}, & k > \kappa. \end{cases}$$

With the bounded projection $P \in L(\ell^2)$, $(P\phi)_k := \begin{pmatrix} \phi_k^1 \\ 0 \end{pmatrix}$ and the closed subspaces $X := R(P)$, $Y := N(P)$ of ℓ^2 one obtains the direct sum $\ell^2 = X \oplus Y$ and

$$(T_A P\phi)_k = \begin{cases} 0, & k = \kappa, \\ \begin{pmatrix} A_{k-1}^1\phi_{k-1}^1 \\ 0 \end{pmatrix}, & k > \kappa, \end{cases} \quad (T_A(\mathrm{id}_{\ell^2} - P)\phi)_k = \begin{cases} 0, & k = \kappa, \\ \begin{pmatrix} C_{k-1}\phi_{k-1}^2 \\ A_{k-1}^2\phi_{k-1}^2 \end{pmatrix}, & k > \kappa. \end{cases}$$

Furthermore, $T_A \in L(\ell^2)$ can be represented as upper-triangular matrix operator $T_A = \begin{pmatrix} T_{A^1} & T_C \\ 0 & T_{A^2} \end{pmatrix} \in L(X \oplus Y)$ with the unilateral shifts $T_{A^1} \in L(X)$, $T_{A^2} \in L(Y)$,

$$(T_{A^1}\phi)_k := \begin{cases} 0, & k = \kappa, \\ A_{k-1}^1\phi_{k-1}, & k > \kappa \end{cases} \qquad (T_{A^2}\phi)_k := \begin{cases} 0, & k = \kappa, \\ A_{k-1}^2\phi_{k-1}, & k > \kappa \end{cases}$$

and $T_C \in L(Y, X)$, $(T_C\phi)_k := \begin{cases} 0, & k = \kappa, \\ C_{k-1}\phi_{k-1}, & k > \kappa \end{cases}$ as blocks. Due to Prop. B.2(c) the operator T_{A^2} has SVEP and [13, Thm. 2.3] implies $\sigma_F(T_A) = \sigma_F(T_{A^1}) \cup \sigma_F(T_{A^2})$. With (2.11) in mind this yields the claim. □

4.2. **Equations on the whole line $\mathbb{I} = \mathbb{Z}$.** On the whole integer axis $\mathbb{I} = \mathbb{Z}$ the statement of Thm. 4.1 is in general false and additional assumptions are required to obtain *diagonal significance*

$$\Sigma_\alpha(A) = \Sigma_\alpha(A^1) \cup \Sigma_\alpha(A^2) \quad \text{for all } \alpha \in \{a, F, F_0, s, \pi\}\,. \tag{DS_α}$$

Indeed, the Ex. 2.7(2) shows that the dichotomy spectrum $\Sigma(A)$ of a block-triangular eqn. (B) can be strictly smaller than the union $\Sigma(A^1) \cup \Sigma(A^2)$. However, there are two approaches to determine subsets of $\Sigma(A)$. The first one is based on well-known relations between the half line spectra and the spectra on $\mathbb{Z}$:

Proposition 4.2. *Keep $\alpha \in \{a, F_0, s\}$ fixed. Block-triangular eqns. (B) satisfy*

$$\Sigma_F(A) = \Sigma^+(A^1) \cup \Sigma^+(A^2) \cup \Sigma^-(A) \subseteq \Sigma_\alpha(A) \subseteq \Sigma(A)$$

and under (2.1) one can replace $\Sigma^-(A)$ by $\Sigma^-(A^1) \cup \Sigma^-(A^2)$.

Proof. Thanks to [32, Cor. 4.30] one has $\Sigma_F(A) = \Sigma^+(A) \cup \Sigma^-(A) \subseteq \Sigma_\alpha(A) \subseteq \Sigma(A)$, while the above Thm. 4.1 implies $\Sigma^+(A) = \Sigma^+(A^1) \cup \Sigma^+(A^2)$. Under the assumption (2.1), with the aid of [33, Prop. 2.1] one can also show $\Sigma^-(A) = \Sigma^-(A^1) \cup \Sigma^-(A^2)$ and the claim follows. □

Concerning a second method to determine one set inclusion in (DS_α) and a subset of $\Sigma_\alpha(A)$, we remind the reader that the *symmetric difference* of sets M_1, M_2 is

$$M_1 \triangle M_2 := (M_1 \cup M_2) \setminus (M_1 \cap M_2)$$

and contains all elements which are either in M_1 or in M_2. The intersection of sets distributes over the symmetric difference, i.e. for arbitrary sets M one has

$$(M_1 \triangle M_2) \cap M = (M_1 \cap M) \triangle (M_2 \cap M). \tag{4.1}$$

Theorem 4.3. *Keep $\alpha \in \{a, F, F_0\}$ fixed. Block-triangular eqns. (B) satisfy:*

(a) $\Sigma_\alpha(A^1) \triangle \Sigma_\alpha(A^2) \subseteq \Sigma_\alpha(A) \subseteq \Sigma_\alpha(A^1) \cup \Sigma_\alpha(A^2)$.
(b) If $\Sigma_\alpha(A^1) \cap \Sigma_\alpha(A^2)$ has no interior points, then (DS_α) holds.

The following construction has prototype character for our investigations and closely resembles the proof of Thm. 4.1.

Proof. Let us represent $x \in \mathbb{K}^d$ as pairs (x^1, x^2) with the components $x^i \in \mathbb{K}^{d_i}$ for $i = 1, 2$. First, this allows us to introduce a bilateral shift $T_A \in L(\ell^2)$,

$$(T_A\phi)_k = A_{k-1}\phi_{k-1} = \begin{pmatrix} A^1_{k-1}\phi^1_{k-1} + C_{k-1}\phi^2_{k-1} \\ A^2_{k-1}\phi^2_{k-1} \end{pmatrix} \quad \text{for all } k \in \mathbb{Z},$$

and second, $P \in L(\ell^2)$, $(P\phi)_k := \binom{\phi^1_k}{0}$ defines a projection. Therefore, $X := R(P)$, $Y := N(P)$ are closed subspaces of $\ell^2 = X \oplus Y$ and it holds

$$(T_A P\phi)_k = \begin{pmatrix} A^1_{k-1}\phi^1_{k-1} \\ 0 \end{pmatrix}, \quad (T_A(\mathrm{id}_{\ell^2} - P)\phi)_k = \begin{pmatrix} C_{k-1}\phi^2_{k-1} \\ A^2_{k-1}\phi^2_{k-1} \end{pmatrix} \quad \text{for all } k \in \mathbb{Z}.$$

Moreover, T_A can be written as $T_A = \begin{pmatrix} T_{A^1} & T_C \\ 0 & T_{A^2} \end{pmatrix} \in L(X \oplus Y)$ with

$$\begin{aligned} &T_{A^1} \in L(X), && T_{A^2} \in L(Y), && T_C \in L(Y, X), \\ &(T_{A^1}\phi)_k := A^1_{k-1}\phi_{k-1}, && (T_{A^2}\phi)_k := A^2_{k-1}\phi_{k-1}, && (T_C\phi)_k := C_{k-1}\phi_{k-1}. \end{aligned}$$

(a) It is $\sigma_\alpha(T_{A^1}) \triangle \sigma_\alpha(T_{A^2}) \subseteq \sigma_\alpha(T_A) \subseteq \sigma(T_{A^1}) \cup \sigma(T_{A^2})$ for $\alpha \in \{a, F, F_0\}$ due to [16, Cor. 4 for $\alpha = a$], [41, proof of Thm. 3.1 for $\alpha = F$] and [26, (6.1) for $\alpha = F_0$]. The first claimed inclusion results from (4.1), if we set $M_i := \sigma(T_{A^i})$, $M := \mathbb{R}^+$ and use $\Sigma_\alpha(A^i) = \sigma_\alpha(T_{A^i}) \cap \mathbb{R}^+$, $i = 1, 2$, as well as $\Sigma_\alpha(A) = \sigma_\alpha(T_A) \cap \mathbb{R}^+$ (cf. (2.12)). The second claimed inclusion anew follows using (2.12).

(b) Thanks to $\sigma(T_{A^1}) \cap \sigma(T_{A^2}) = \{\lambda \in \mathbb{C} : |\lambda| \in \Sigma(A^1) \cap \Sigma(A^2)\}$ and our assumption on interior points the intersection $\sigma(T_{A^1}) \cap \sigma(T_{A^2}) \subseteq \mathbb{C}$ is a finite union of circles centered around 0 (or $\emptyset$) and has thus no interior points. Then [16, Cor. 8] shows $\sigma(T_A) = \sigma(T_{A^1}) \cup \sigma(T_{A^2})$ and (2.12) yields the claim for $a = \alpha$. In case $\alpha \in \{F, F_0\}$ one proceeds accordingly with [41, Cor. 3.2] resp. [26, Cor. 7]. □

Rather than its whole line dichotomy spectrum $\Sigma(A)$, the stability radius of (B) turns out to be fully determined by the diagonal blocks:

Corollary 4.4. $\max \Sigma(A) = \max \{\max \Sigma(A^1), \max \Sigma(A^2)\}$

Proof. The angular symmetry of $\sigma(T_A)$ and [19, Prop. 4] guarantee

$$\max \Sigma(A) \overset{(2.12)}{=} r(T_A) = \max_{i=1}^{2} r(T_{A^i}) = \max_{i=1}^{2} \max \Sigma(A^i)$$

and this implies the claim. □

Let us continue with general spectral inclusions:

Proposition 4.5. *Block-triangular difference eqns.* (B) *satisfy:*

(a) $\Sigma_\alpha(A) \subseteq \Sigma_\alpha(A^1) \cup \Sigma_\alpha(A^2)$ *for all* $\alpha \in \{\pi, s\}$
(b) $\Sigma(A^2) \subseteq \Sigma_s(A^1) \cup \Sigma(A)$
(c) $\Sigma_\pi(A^1) \cup \Sigma_s(A^2) \subseteq \Sigma(A)$

Combined with Thm. 4.3(a) the inclusion (c) implies (DS_a), provided the identity $\Sigma_\pi(A^1) \cup \Sigma_s(A^2) = \Sigma(A^1) \cup \Sigma(A^2)$ holds.

Proof. (a) follows as in the proof of Thm. 4.3 using [15, Prop. 1.1]. Concerning (b) let us apply [15, Cor. 2.2] and [19, Proof of Prop. 4] yields assertion (c). □

For the following it is advisable to introduce the *defect set*

$$\mathfrak{D}_\alpha(A) := \big(\Sigma_\alpha(A^1) \cup \Sigma_\alpha(A^2)\big) \setminus \Sigma_\alpha(A) \quad \text{for all } \alpha \in \{a, F, F_0, \pi, s\}$$

of a block-triangular eqn. (B). By Thm. 4.3(a) and Prop. 4.5(a) one observes that diagonal significance (DS_α) precisely holds for $\mathfrak{D}_\alpha(A) = \emptyset$.

Theorem 4.6 (diagonal significance for Σ)**.** *One has*

$$\Sigma(A) \cup \big(\Sigma(A^1) \setminus \Sigma_\pi(A^1) \cap \Sigma(A^2) \setminus \Sigma_s(A^2)\big) = \Sigma(A^1) \cup \Sigma(A^2).$$

We immediately locate the defect set as $\mathfrak{D}_a(A) \subseteq \Sigma(A^1) \setminus \Sigma_\pi(A^1) \cap \Sigma(A^2) \setminus \Sigma_s(A^2)$.

Proof. With the shifts T_A, T_{A^1} and T_{A^2} defined in the proof of Thm. 4.3, we obtain

$$\sigma(T_A) \cup (\sigma(T_{A^1}) \setminus \sigma_\pi(T_{A^1}) \cap \sigma(T_{A^2}) \setminus \sigma_s(T_{A^2})) = \sigma(T_{A^1}) \cup \sigma(T_{A^2})$$

from [40, (7)]. Because the intersection of both sides in this relation with $\mathbb{R}^+$ distributes over the set operations involved, the claim results with (2.12). □

Corollary 4.7. *If* $(\Sigma_s(A^1) \cap \Sigma_\pi(A^2)) \setminus (\Sigma_\pi(A^1) \cap \Sigma_s(A^2))$ *possesses no interior point, then* (DS_a) *holds.*

Proof. By means of (A.1) this follows as above using [40, Cor. 3.2] and (2.12). □

A problem with the above criteria for diagonal significance is that certain dichotomy spectra, as well as their subspectra, have to be known in advance. In the following, we will thus obtain sufficient conditions on the basis of Lyapunov filtrations alone. As a further advantage, these criteria also provide diagonal significance of subspectra. To be precise, let us suppose that the diagonal systems (Δ_{A^i}) and their adjoint eqns. $(\Delta^*_{A^i})$ have Lyapunov spectra and filtrations

$$\begin{aligned} &\left\{\lambda_1^i,\ldots,\lambda_{n_i}^i\right\}, && W_0^i\subset\ldots\subset W_{n_i}^i=\mathbb{K}^{d_i}\quad\text{with } n_i\leq d_i,\\ &\left\{\mu_1^i,\ldots,\mu_{n_i^*}^i\right\}, && V_0^i\subset\ldots\subset V_{n_i^*}^i=\mathbb{K}^{d_i}\quad\text{with } n_i^*\leq d_i \end{aligned}$$

for $i\in\{1,2\}$. Given this, one is able to formulate the following conditions on the Lyapunov spectra of the diagonal systems

$$1\leq\mu_j^1\limsup_{k\to\infty}\sqrt[k]{\left|\Phi^*_{A^1}(-k,\kappa)x\right|}\quad\text{for all } x\in V_j^1\setminus V_{j-1}^1,\ 1\leq j\leq n_1^*,\qquad (S^*_{A^1})$$

$$1\leq\lambda_j^2\limsup_{k\to\infty}\sqrt[k]{\left|\Phi_{A^2}(-k,\kappa)x\right|}\quad\text{for all } x\in W_j^2\setminus W_{j-1}^2,\ 1\leq j\leq n_2.\qquad (S_{A^2})$$

Note that we will illustrate these conditions in the Exs. 4.10 and 4.14 below.

Theorem 4.8 (diagonal significance for Σ_π and Σ_s)**.** *If a block-triangular difference eqn. (B) fulfills*

(a) $(S^*_{A^1})$, *then* $\Sigma_\pi(A)=\Sigma_\alpha(A^1)\cup\Sigma_\pi(A^2)$ *for* $\alpha\in\{a,\pi\}$.
(b) (S_{A^2}), *then* $\Sigma_s(A)=\Sigma_s(A^1)\cup\Sigma_\alpha(A^2)$ *for* $\alpha\in\{a,s\}$.

Proof. We borrow our notation from the proof of Thm. 4.3. Using (B.3) we know that the adjoint shift $T^*_{A^1}$ has the SVEP if and only if

$$1\leq\chi^*_{A^1}(x)\limsup_{k\to\infty}\sqrt[k]{\left|\Phi^*_{A^1}(-k,\kappa)x\right|}\quad\text{for all } x\in\mathbb{K}^d\setminus\{0\}$$

holds. Let us first establish that this inequality is equivalent to $(S^*_{A^1})$:
$(\Rightarrow)$ The definition of the Lyapunov filtration for $(\Delta^*_{A^1})$ guarantees $\chi^*_{A^1}(x)=\mu_j^1$ for $x\in V_j^1\setminus V_{j-1}^1$, $1\leq j\leq n_1^*$ and consequently $(S^*_{A^1})$ holds.
$(\Leftarrow)$ Conversely, assume $(S^*_{A^1})$ is satisfied and choose $x\in\mathbb{K}^d\setminus\{0\}$ arbitrarily. There exists a maximal $1\leq j\leq n_1^*$ such that $x\in V_j^1\setminus V_{j-1}^1$ and thus $\chi^*_{A^1}(x)=\mu_j^1$. Hence,

$$1\leq\mu_j^1\limsup_{k\to\infty}\sqrt[k]{\left|\Phi^*_{A^1}(-k,\kappa)x\right|}=\chi^*_{A^1}(x)\limsup_{k\to\infty}\sqrt[k]{\left|\Phi^*_{A^1}(-k,\kappa)x\right|}$$

and since $x\neq 0$ was arbitrary, $T^*_{A^1}$ has the SVEP.

Analogously one uses (B.2) to show that (S_{A^2}) is equivalent to the SVEP of T_{A^2}.

(a) Since $T^*_{A^1}$ has the SVEP, by means of [40, Cor. 3.13] this implies (DS_s) and the claim results from (2.12) and Lemma A.2(b).

(b) Here, T_{A^2} has the SVEP. Then [40, Prop. 3.2] implies (DS_π) and the assertion follows with (2.12) and Lemma A.2(a). □

Theorem 4.9 (diagonal significance for Σ and Σ_F)**.** *Keep* $\alpha\in\{a,F\}$ *fixed. Then* (DS_α) *holds, if a block-triangular difference eqn. (B) fulfills one of the assumptions*

(i) $\Sigma_\alpha(A^1)\subseteq\Sigma_\alpha(A)$,
(ii) $\Sigma_\alpha(A^2)\subseteq\Sigma_\alpha(A)$,
(iii) $(S^*_{A^1})$,
(iv) (S_{A^2}).

Proof. Thanks to (2.12) it again suffices to establish $\sigma_\alpha(T_A) = \sigma_\alpha(T_{A^1}) \cup \sigma_\alpha(T_{A^2})$ with the shifts T_{A^1}, T_{A^2} defined in the proof of Thm. 4.3. Under the assumption (i) or (ii) this follows from [13, Lemma 2.2]. As in the above proof of Thm. 4.8 one shows that (iii) and (iv) are equivalent to the SVEP of $T_{A^1}^*$ resp. of T_{A^2}. Therefore, [13, Thm. 2.3] applies and yields the claim. □

In order to obtain results on the diagonal significance of the Weyl dichotomy spectrum $\Sigma_{F_0}(A)$, one has to impose assumptions dual to ($S_{A^1}^*$) and (S_{A^2}), namely

$$1 \le \lambda_j^1 \limsup_{k\to\infty} \sqrt[k]{|\Phi_{A^1}(-k,\kappa)x|} \quad \text{for all } x \in W_j^1 \setminus W_{j-1}^1,\ 1 \le j \le n_1, \qquad (S_{A^1})$$

$$1 \le \mu_j^2 \limsup_{k\to\infty} \sqrt[k]{\left|\Phi_{A^2}^*(-k,\kappa)x\right|} \quad \text{for all } x \in V_j^2 \setminus V_{j-1}^2,\ 1 \le j \le n_2^*. \qquad (S_{A^2}^*)$$

Example 4.10. *For the real sequences a^1, a^2 from Ex. 2.7 the equivalences*

$$(S_{a^1}) \quad \Leftrightarrow \quad \alpha_- \le \alpha_+, \qquad\qquad (S_{a^2}^*) \quad \Leftrightarrow \quad \beta_+ \le \beta_-$$

hold. Asymptotically periodic sequences a as in Ex. 2.6(4) fulfill (S_a) or (S_a^*) if and only if their asymptotic means $c_-, c_+ > 0$ satisfy $c_- \le c_+$ resp. $c_+ \le c_-$.

Theorem 4.11 (diagonal significance for Σ_{F_0})**.** *It holds (DS_{F_0}), if a block-triangular difference eqn. (B) fulfills both of the assumptions*

(i) ($S_{A^1}^*$) or (S_{A^2}),
(ii) (S_{A^1}) or ($S_{A^2}^*$).

Proof. In the proof of Thm. 4.8 we have shown that ($S_{A^1}^*$) is equivalent to the SVEP of $T_{A^1}^*$ and that (S_{A^2}) holds if and only if T_{A^2} has the SVEP. Along the same lines one verifies the equivalence of (S_{A^1}) to the SVEP of T_{A^1} resp. that ($S_{A^2}^*$) is equivalent to a SVEP of $T_{A^2}^*$. Given this, in a formal logical language our assumptions can be formulated as $((S_{A^1}^*) \vee (S_{A^2})) \wedge ((S_{A^1}) \vee (S_{A^2}^*))$, which is synonymous to the expression

$$((S_{A^1}) \wedge (S_{A^2})) \vee ((S_{A^1}^*) \wedge (S_{A^2}^*)) \vee ((S_{A^1}^*) \wedge (S_{A^1})) \vee ((S_{A^2}) \wedge (S_{A^2}^*)).$$

Hence, [40, Cors. 3.10 and 3.11] apply and yield $\sigma_{F_0}(T_A) = \sigma_{F_0}(T_{A^1}) \cup \sigma_{F_0}(T_{A^2})$. We intersect both sides of this equation with $\mathbb{R}^+$ and from (2.12) one gets

$$\Sigma_{F_0}(A) = (\sigma_{F_0}(T_{A^1}) \cap \mathbb{R}^+) \cup (\sigma_{F_0}(\sigma_{A^2}) \cap \mathbb{R}^+) = \Sigma_{F_0}(A^1) \cup \Sigma_{F_0}(A^2)$$

due to distributivity of the set relations, and thus the claim. □

We close with several statements concerning the conditions (S_{A^i}) and $(S_{A^i}^*)$, which compare forward and backward growth of a difference equation resp. its adjoint, $i \in \{1, 2\}$. In the classical periodic situation they are fulfilled:

Proposition 4.12. *Let $i \in \{1, 2\}$. If $\Sigma(A^i)$ is discrete, then (S_{A^i}) and $(S_{A^i}^*)$ hold.*

Proof. Since $\Sigma(A^i)$, $i \in \{1, 2\}$, is discrete, due to (2.12) the spectrum $\sigma(T_{A^i})$ consists of finitely many concentric circles. Then the inclusion $\partial\sigma(T_{A^i}) \subseteq \sigma_\alpha(T_{A^i}) \subseteq \sigma(T_{A^i})$ implies $\sigma_\alpha(T_{A^i}) = \partial\sigma(T_{A^i})$ for $\alpha \in \{\pi, s\}$ and Lemma A.3 guarantees that both weighted shifts T_{A^i} and $T_{A^i}^*$ have the SVEP. Thanks to the characterization [11, Thm. 2.1 resp. Cor. 2.2] one shows as in the proof of Thm. 4.8 that this is equivalent to the estimates (S_{A^i}) resp. $(S_{A^i}^*)$. □

Remark 4.13 (the classes $\mathcal{P}_p(\mathbb{K}^d)$ and $\mathcal{P}_p^*(\mathbb{K}^d)$)**.** In [34] we consider linear difference eqns. (Δ_A) with coefficient sequences in the classes

$$\mathcal{P}_p(\mathbb{K}^d) := \Big\{A \in \mathcal{L}^\infty(\mathbb{K}^d) : \Phi(k+2p,k)^*\Phi(k+2p,k) - 2r\Phi(k+p,k)^* \cdot \\ \cdot \Phi(k+p,k) + r^2 \operatorname{id}_{\mathbb{K}^d} \text{ is positively-semidefinite for all } k \in \mathbb{Z},\ r > 0\Big\},$$

$$\mathcal{P}_p^*(\mathbb{K}^d) := \Big\{A \in \mathcal{L}^\infty(\mathbb{K}^d) : \Phi(k+2p,k)\Phi(k+2p,k)^* - 2r\Phi(k+2p,k+p) \cdot \\ \cdot \Phi(k+2p,k+p)^* + r^2 \operatorname{id}_{\mathbb{K}^d} \text{ is positively-semidefinite for all } k \in \mathbb{Z},\ r > 0\Big\},$$

which are related to the above assumptions. Indeed, by means of [34, Prop. A.3] one establishes the implications

$$A^i \in \mathcal{P}_p(\mathbb{K}^{d_i}) \Rightarrow (S_{A^i}), \qquad A^i \in \mathcal{P}_p^*(\mathbb{K}^{d_i}) \Rightarrow (S_{A^i}^*) \quad \text{for all } i \in \{1,2\}.$$

Example 4.14. *We revisit the planar upper-triangular difference eqn.* (Δ_A) *from Ex. 2.7. The dichotomy spectra for its diagonal sequences are given in Ex. 2.6(3) and moreover* [32, Ex. 5.3] *yields the surjectivity dichotomy spectra*

$$\Sigma_s(a^1) = \begin{cases} \{\alpha_-, \alpha_+\}, & \alpha_+ \le \alpha_-, \\ [\alpha_-, \alpha_+], & \alpha_- \le \alpha_+, \end{cases} \qquad \Sigma_s(a^2) = \begin{cases} \{\beta_-, \beta_+\}, & \beta_+ \le \beta_-, \\ [\beta_-, \beta_+], & \beta_- \le \beta_+. \end{cases}$$

(1) Since the Fredholm spectra $\Sigma_F(a^i)$ are discrete, one obtains from Thm. 4.3(b) that (DS_F) holds with $\Sigma_F(A) = \{\alpha_-, \alpha_+, \beta_-, \beta_+\}$ and that the estimates

$$\max\{\alpha_-, \alpha_+\} \le \min\{\beta_-, \beta_+\} \quad \text{or} \quad \max\{\beta_-, \beta_+\} \le \min\{\alpha_-, \alpha_+\}$$

are sufficient for (DS_α), $\alpha \in \{a, F_0\}$, to hold. The inequalities $\alpha_+ \le \alpha_-$ and $\beta_- \le \beta_+$ imply $\Sigma_\pi(a^1) \cup \Sigma_s(a^2) = [\alpha_+, \alpha_-] \cup [\beta_-, \beta_+] = \Sigma(a^1) \cup \Sigma(a^2)$ and therefore Prop. 4.5 guarantees (DS_a). Note that this even holds under the weaker condition

$$\alpha_+ \le \alpha_- \quad \text{or} \quad \beta_- \le \beta_+, \tag{4.2}$$

because Cor. 4.7 applies for discrete subspectra $\Sigma_s(a^1)$, $\Sigma_\pi(a^2)$, i.e. (4.2).

(2) By means of Lyapunov exponent-like conditions we obtain the following criteria for diagonal significance. Analogously to the above Ex. 4.10 the condition

- *$(S_{a^1}^*)$ is equivalent to $\alpha_+ \le \alpha_-$ and so Thm. 4.8(a) leads to (DS_π)*
- *(S_{a^2}) is equivalent to $\beta_- \le \beta_+$ and Thm. 4.8(b) guarantees (DS_s)*

Hence, under one of the conditions (4.2) *our Thm. 4.9 implies (DS_α), $\alpha \in \{a, F\}$. Our above Thm. 4.11 yields (DS_{F_0}), provided both* (4.2) *and the dual condition $\alpha_- \le \alpha_+$ or $\beta_+ \le \beta_-$ (cf. again Ex. 4.10) hold.*

4.3. **Conditions on C.** In order to provide sufficient conditions for diagonal significance on basis of the sequence $C = (C_k)_{k \in \mathbb{Z}}$ alone, we define the linear spaces

$$\mathcal{N}(A) := \left\{X \in \ell^\infty(\mathbb{K}^{d_1 \times d_2}) : X_{k+1}A_k^2 \equiv A_{k+1}^1 X_k \text{ on } \mathbb{Z}\right\},$$

$$\mathcal{R}(A) := \left\{Y \in \ell^\infty(\mathbb{K}^{d_1 \times d_2}) \mid \exists X \in \ell^\infty(\mathbb{K}^{d_1 \times d_2}) : Y_k \equiv A_k^1 X_k - X_{k+1}A_k^2 \text{ on } \mathbb{Z}\right\}$$

and immediately obtain

Theorem 4.15. *If $C \in \mathcal{N}(A) + \mathcal{R}(A)$ is satisfied, then (DS_a) holds.*

Proof. We abbreviate $\ell_i^2 := \ell^2(\mathbb{K}^{d_i})$ and for shifts $T_{A^i} \in L(\ell_i^2)$, $i = 1, 2$, the generalized derivation $\Delta : L(\ell_2^2, \ell_1^2) \to L(\ell_2^2, \ell_1^2)$, $\Delta\Xi := T_{A^1}\Xi - \Xi T_{A^2}$ is bounded. If C denotes a bounded sequence $(C_k)_{k \in \mathbb{Z}}$ of matrices $C_k \in \mathbb{K}^{d_1 \times d_2}$, then the operators

$$(T_C\phi)_k := C_{k-1}\phi_{k-1}, \qquad (M_C\phi)_k := C_k\phi_k \quad \text{for all } k \in \mathbb{Z}$$

fulfill $T_C, M_C \in L(\ell_2^2, \ell_1^2)$. First, in case $C \in \mathcal{N}(A)$ we obtain

$$(\Delta T_C \phi)_k = \big(A^1_{k-1} C_{k-2} - C_{k-1} A^2_{k-2}\big)\phi_{k-2} = 0 \quad \text{for all } k \in \mathbb{Z},\ \phi \in \ell_2^2$$

and thus $\Delta T_C = 0$, i.e. T_C is in the kernel of Δ. Second, in case $C \in \mathcal{R}(A)$ with $C_k = A^1_k X_k - X_{k+1} A^2_k$ for all $k \in \mathbb{Z}$ and some $X \in \ell^\infty(\mathbb{K}^{d_2 \times d_1})$ it is

$$(\Delta M_X \phi)_k = \big(A^1_{k-1} X_{k-1} - X_k A^2_{k-1}\big)\phi_{k-1} = C_{k-1}\phi_{k-1} \quad \text{for all } k \in \mathbb{Z}.$$

This yields $\Delta M_X = T_C$ and hence T_C is in the range of Δ. By linearity we conclude that for elements $C \in \mathcal{N}(A) + \mathcal{R}(A)$ the corresponding shifts T_C are contained in the sum $N(\Delta) + R(\Delta)$. Consequently, [7, Thm. 1] implies $\sigma(T_A) = \sigma(T_{A^1}) \cup \sigma(T_{A^2})$ and the claim follows from (2.12). □

Corollary 4.16. *If $\Sigma(A^1) \cap \Sigma(A^2) = \emptyset$, then $\mathcal{N}(A) = \{0\}$.*

Proof. We use the terminology from the above proof. By [25, p. 256, Thm. 3.4.1] it is $\sigma(\Delta) = \sigma(T_{A^1}) - \sigma(T_{A^2})$. Thus, thanks to (2.12) we obtain that $\Delta \in L(L(\ell_2^2, \ell_1^2))$ is invertible and particularly $N(\Delta) = \{0\}$. The claim follows, because $C \in \mathcal{N}(A)$ implies $T_C \in N(\Delta) = \{0\}$ and therefore $C_k = 0$, $k \in \mathbb{Z}$. □

We finally illuminate the close relation between the assumption of Thm. 4.15 and exponential dichotomies resp. trichotomies as discussed in [30]:

Remark 4.17. Let $d_1 = d_2$ and suppose that all $A^2_k \in \mathbb{K}^{d_2 \times d_2}$ are invertible with

$$\sup_{k \in \mathbb{Z}} \big|(A^2_k)^{-1}\big| < \infty.$$

(1) The linear space $\mathcal{R}(A)$ consists of all matrix sequences $Y \in \mathcal{L}^\infty(\mathbb{K}^{d_1})$ such that the matrix difference eqn. $X_{k+1} = (A^1_k X_k - Y_k)(A^2_k)^{-1}$ has a bounded solution. This, in turn, holds provided the linearly-homogenous equation

$$X_{k+1} = A^1_k X_k (A^2_k)^{-1} \tag{4.3}$$

has an exponential trichotomy on $\mathbb{Z}$ (cf. [30, Prop. 1]).

(2) The stronger assumption that (4.3) is even exponentially dichotomic on $\mathbb{Z}$ corresponds precisely to the case $\mathcal{N}(A) = \{0\}$.

4.4. **Spectral continuity.** While the dichotomy spectra interpreted as mappings $\bar{\Sigma}_\alpha : \mathcal{L}^\infty(\mathbb{K}^d) \to K(\mathbb{R})$, $\bar{\Sigma}_\alpha(A) := \Sigma_\alpha(A) \cup \{0\}$ are only upper-semicontinuous in general (cf. [31, Cor. 4] for $\mathbb{I} = \mathbb{Z}$ and [32, Cor. 3.24] on the half line $\mathbb{I} = \mathbb{Z}^+_\kappa$), let us next provide continuity criteria for triangular coefficient sequences. We particularly present conditions implying that continuity of the block subsystems extends to (B).

Our analysis is fundamentally based on the geometrically evident

Proposition 4.18 (see [34, Prop. 5.3])**.** *Keep $\alpha \in \{a, F_0\}$ fixed. If $\sigma_\alpha : L(\ell^2) \to K(\mathbb{C})$ is continuous at T_A, then $\bar{\Sigma}_\alpha : \mathcal{L}^\infty(\mathbb{K}^d) \to K(\mathbb{R})$ is continuous at A.*

The remaining section is based on the assumption that A, A^1 and A^2 fulfill (2.1). Then, in our preparatory paper [34] we have shown that

$$\Sigma(A) = \overline{\Sigma_{F_0}(A) \setminus \Sigma_F(A)} \tag{C^a_A}$$

is a sufficient condition for Σ to be continuous at (Δ_A), while

$$\Sigma_{F_0}(A) = \overline{\Sigma_{F_0}(A) \setminus \Sigma_F(A)} \tag{$C^{F_0}_A$}$$

guarantees the corresponding continuity of the Weyl spectrum Σ_{F_0}.

Theorem 4.19. *Keep $\alpha \in \{a, F_0\}$ fixed. If a block-triangular eqn. (B) satisfies* *(i)* *both the continuity conditions* ($C^\alpha_{A^1}$) *and* ($C^\alpha_{A^2}$),

(ii) $\Sigma_\alpha(A^1) \cap \Sigma_\alpha(A^2) = \emptyset$,

then Σ_α *is continuous at* A.

Proof. Let $\alpha \in \{a, F_0\}$ be fixed. On the one hand, condition (i) ensures that the shift T_{A^i} is a point of continuity for σ (cf. [34, Proof of Thm. 5.4]) resp. σ_{F_0} (see [34, Proof of Cor. 5.5]) with $i = 1, 2$. On the other hand, assumption (ii) guarantees $\sigma_\alpha(T_{A^1}) \cap \sigma(T_{A^2}) = \emptyset$. From [37, Thm. 7] we get that σ_α is continuous at T_A. Then Prop. 4.18 implies that also the dichotomy spectrum Σ_α is continuous at (Δ_A). ☐

Theorem 4.20. *If a block-triangular difference eqn.* (B) *satisfies*

(i) *both the continuity conditions* ($C^a_{A^1}$) *and* ($C^a_{A^2}$),

(ii) *the estimates* ($S^*_{A^1}$),

then Σ *is continuous at* A.

Proof. In the previous proof of Thm. 4.19 we justified that σ is continuous at T_{A^1} and T_{A^2}. Moreover, in the proof of Thm. 4.8 it was shown that the adjoint $T^*_{A^1}$ has the SVEP. If we combine Prop. 4.18 with [37, Cor. 9], then the claim follows. ☐

Let us eventually discuss the upper-triangular difference equation from Ex. 2.7(2) having a discontinuous dichotomy spectrum in the light of Thm. 4.19 and 4.20:

Example 4.21. *Given some real* $\delta > 1$ *we consider* (Δ_A) *as defined in Ex. 2.7 with parameters satisfying the conditions* (2.14). *Then Ex. 2.6(3) yields the spectra*

$$\Sigma(a^i) = \Sigma_{F_0}(a^i) = [\alpha_-, \alpha_+], \qquad \Sigma_F(a^i) = \{\alpha_-, \alpha_+\}$$

and therefore the continuity conditions ($C^\alpha_{a^i}$) *hold for* $\alpha \in \{a, F\}$ *and* $i = 1, 2$. *However, the assumption (ii) in both Thms. 4.19 and 4.20 are violated.*

5. **Triangular equations.** This closing section is concerned with linear difference eqns. (Δ_A) whose coefficient matrices A_k are triangular, where w.l.o.g. we restrict to the upper-triangular situation. Hence, they are of the form

$$x_{k+1} = A_k x_k, \qquad A_k = \begin{pmatrix} a^1_k & a^{1,2}_k & a^{1,3}_k & \dots & a^{1,d}_k \\ & a^2_k & a^{2,3}_k & \dots & a^{2,d}_k \\ & & \ddots & & \vdots \\ & & & & a^d_k \end{pmatrix} \tag{T}$$

with bounded diagonal sequences $(a^i_k)_{k\in\mathbb{I}'}$ and bounded super-diagonal sequences $(a^{i,j}_k)_{k\in\mathbb{I}'}$ for indices $1 \le i < j \le d$ in $\mathbb{K}$. On the half line $\mathbb{I} = \mathbb{Z}^+_\kappa$ it is known that $\Sigma^+(A)$ is simply the union of spectra for the corresponding diagonal eqns. (Δ_{a^i}) (see [32, Cor. 3.25]). As again demonstrated in Ex. 2.7(2), the situation for $\mathbb{I} = \mathbb{Z}$ is more complicated. Let us consider the whole line case from now on.

Theorem 5.1. *The dichotomy spectrum* $\Sigma(A)$ *of* (T) *satisfies*

$$\Sigma(a^1) \bigtriangleup \Sigma_1 \subseteq \Sigma(A) \subseteq \Sigma(a^1) \cup \Sigma_1,$$

with a set Σ_1 *given by*

$$\Sigma_1 := \begin{cases} \emptyset, & d = 1, \\ \Sigma(a^2), & d = 2 \end{cases}$$

and for $d > 2$ *allowing the recursive construction*

$$\Sigma(a^{d-1}) \bigtriangleup \Sigma(a^d) \subseteq \Sigma_{d-2} \subseteq \Sigma(a^{d-1}) \cup \Sigma(a^d),$$

$$\Sigma(a^j) \triangle \Sigma_j \subseteq \Sigma_{j-1} \subseteq \Sigma(a^j) \cup \Sigma_j \quad \textit{for all } 2 \le j < d-1.$$

Remark 5.2. (1) The above procedure shows $\Sigma(A) \subseteq \bigcup_{i=1}^d \Sigma(a^i)$.

(2) Under the assumption $\inf_{k\in\mathbb{I}}\left|a_k^j\right| > 0$ for all $1 \le j \le d$ one can substitute the sets $\Sigma(a^j)$ by the closed intervals $\left[\underline{\beta}(a^j), \overline{\beta}(a^j)\right]$ (cf. Prop. 2.4). Moreover, the symmetric differences in Thm. 5.1 can be replaced by their closures.

Proof. For $d = 1, 2$ the claim follows directly from Thm. 4.3(a). For $d > 2$ write

$$A_k := \begin{pmatrix} a_k^1 & & & & b_k^1 \\ & a_k^2 & & & b_k^2 \\ & & a_k^3 & & \vdots \\ & & & \ddots & b_k^{d-1} \\ & & & & a_k^d \end{pmatrix} \quad \text{for all } k \in \mathbb{Z}$$

with rows $b_k^j \in \mathbb{K}^{1\times(d-j)}$ and diagonal sequences a^j yielding

$$A_k^{j-1} = \begin{pmatrix} a_k^j & b_k^j \\ & A_k^j \end{pmatrix} \quad \text{for all } 1 \le j < d. \tag{5.1}$$

Here, the square matrix $A_k^j \in \mathbb{K}^{(d-j)\times(d-j)}$ is defined by simultaneously discarding the first $j \in [0,d)_\mathbb{Z}$ columns and rows of A_k; it is $A_k = A_k^0$. Setting $\Sigma_j := \Sigma(A^j)$ and applying Thm. 4.3(a) to (5.1) we deduce

$$\Sigma(a^j) \triangle \Sigma_j \subseteq \Sigma_{j-1} \subseteq \Sigma(a^j) \cup \Sigma_j \quad \text{for all } 1 \le j < d,$$

particularly $\Sigma(a^1) \triangle \Sigma_1 \subseteq \Sigma(A) \subseteq \Sigma(a^1) \cup \Sigma_1$ and $A_k^{d-1} = a_k^d$ yields our claim. □

The Thm. 5.1 allows to circumscribe the dichotomy spectrum using exclusively the diagonal spectral intervals. As a concrete example we consider

Example 5.3. *For a difference eqn. (T) in $\mathbb{R}^4$ with diagonal sequences, the spectra*

$$\Sigma(a^1) = [1,2], \qquad \Sigma(a^2) = [1,3],$$
$$\Sigma(a^3) = [3,5], \qquad \Sigma(a^4) = [4,5]$$

and bounded super-diagonal entries, the following holds: In the terminology of Thm. 5.1 with $d = 4$ we obtain the inclusions

$$[3,4] = \overline{\Sigma(a^3) \triangle \Sigma(a^4)} \subseteq \Sigma_2 \subseteq \Sigma(a^3) \cup \Sigma(a^4) = [3,5],$$
$$\overline{\Sigma(a^2) \triangle \Sigma_2} \subseteq \Sigma_1 \subseteq \Sigma(a^2) \cup \Sigma_2 = [1,3] \cup \Sigma_2 \subseteq [1,5],$$

which guarantee $\Sigma(a^2) \cap \Sigma_2 = \{3\}$, consequently $[1,4] \subseteq \overline{\Sigma(a^2) \triangle \Sigma_2}$ and thus the inclusions $[1,4] \subseteq \Sigma_1 \subseteq [1,5]$. Now $\Sigma(a^1) = [1,2] \subseteq \Sigma_1$ implies

$$[2,4] \subseteq \overline{\Sigma_1 \setminus \Sigma(a^1)} = \overline{\Sigma(a^1) \triangle \Sigma_1} \subseteq \Sigma(A) \subseteq [1,5].$$

Yet, one obtains the stability radius of (T) from the diagonal sequences:

Corollary 5.4. $\max \Sigma(A) = \max_{j=1}^d \overline{\beta}(a^j)$.

Proof. With Cor. 4.4 the formula for $\max \Sigma(A)$ follows by induction over j. □

In the following, we provide several sufficient criteria for *diagonal significance* of triangular systems (T), i.e. the fact that $\Sigma(A)$ can be obtained from the diagonal sequences. The corresponding results yield from our preparations for block-triangular equations (B) in Subsect. 4.2 by means of mathematical induction. We exemplify this in case of Thm. 4.3 and leave it to the interested reader to deduce counterparts to e.g. Thms. 4.8, 4.9 or 4.11:

Corollary 5.5 (exponential separation)**.** *If all the intersections $\Sigma(a^i) \cap \Sigma(a^j)$ for $i \neq j$ have empty interior, then $\Sigma(A) = \bigcup_{i=1}^d \Sigma(a^i)$.*

According to Ex. 2.7(2) one knows that Cor. 5.5 is wrong without the additional assumption on interior points.

Proof. Retaining the notation from the proof of Thm. 5.1, set $\sigma_j := \Sigma(a^j)$. By assumption, the intersection $\bigcap_{i=d-1}^d \sigma_i$ has no interior points and Thm. 4.3(b) implies $\Sigma_{d-2} = \bigcup_{i=d-1}^d \sigma_i$. Our induction is based on the hypothesis $\Sigma_j = \bigcup_{i=j+1}^d \sigma_i$ and for the induction step $j \to j-1$ we proceed as follows: Thanks to (5.1) it is

$$\sigma_j \triangle \Sigma_j \subseteq \Sigma_{j-1} \subseteq \sigma_j \cup \Sigma_j,$$

the induction hypothesis guarantees $\sigma_j \cap \Sigma_j = \sigma_j \cap \bigcup_{i=j+1}^d \sigma_i = \bigcup_{i=j+1}^d (\sigma_i \cap \sigma_j)$, and if we invest our assumption, $\sigma_j \cap \Sigma_j$ has no interior points. Hence, Thm. 4.3(b) yields $\Sigma_{j-1} = \sigma_j \cup \Sigma_j = \bigcup_{i=j}^d \sigma_i$. □

The following criteria for diagonal significance involve only Lyapunov exponents of the first resp. last $d-1$ diagonal sequences. For instance, using Ex. 2.6(5) one easily constructs diagonally significant equations with overlapping diagonal spectral intervals.

Theorem 5.6. *If a triangular difference eqn. (T) fulfills one of the assumptions*

(i) $\underline{\lambda}_-(a^i) \leq \overline{\lambda}_+(a^i)$ *for* $1 < i \leq d$,

(ii) $\underline{\lambda}_+(a^i) \leq \overline{\lambda}_-(a^i)$ *for* $1 \leq i < d$,

then

$$\Sigma_\alpha(A) = \bigcup_{i=1}^d \Sigma_\alpha(a^i) \quad \text{for all } \alpha \in \{a, F\}\,. \tag{5.2}$$

Proof. First of all, using Lemma B.5 we obtain from assumption (i) that every T_{a^i} has the SVEP for $1 < i \leq d$, while assumption (ii) guarantees the SVEP of $T_{a^i}^*$ for all $1 \leq i < d$. Because mathematical induction on basis of [13, Thm. 2.3] implies the relation $\sigma_\alpha(T_A) = \bigcup_{i=1}^d \sigma_\alpha(T_{a^i})$, the claim results from (2.12). □

Corollary 5.7. *(a) If* $\underline{\lambda}_-(a^i) \leq \overline{\lambda}_+(a^i)$ *holds for* $1 \leq i \leq d$, *then*

$$\Sigma(a^i) = \Sigma_s(a^i) \quad \text{for all } 1 \leq i \leq d, \qquad \Sigma(A) = \Sigma_s(A) = \bigcup_{i=1}^d \Sigma_s(a^i). \tag{5.3}$$

(b) If $\underline{\lambda}_+(a^i) \leq \overline{\lambda}_-(a^i)$ *holds for* $1 \leq i \leq d$, *then*

$$\Sigma(a^i) = \Sigma_\pi(a^i) \quad \text{for all } 1 \leq i \leq d, \qquad \Sigma(A) = \Sigma_\pi(A) = \bigcup_{i=1}^d \Sigma_\pi(a^i).$$

In both cases one has $\Sigma_\alpha(A) = \bigcup_{i=1}^d \Sigma_\alpha(a^i)$ *for* $\alpha \in \{F, F_0\}$.

Proof. (a) As in the proof of Thm. 5.6(a) one argues that every T_{a^i} has the SVEP and thus Lemma A.2(a) implies $\Sigma_s(a^i) = \sigma_s(T_{a^i}) \cap \mathbb{R}^+ = \sigma(T_{a^i}) \cap \mathbb{R}^+ = \Sigma(a^i)$ for all $1 \leq i \leq d$ (cf. (2.12)). Moreover, an inductive argument based on [2, p. 62, Thm. 2.9] establishes the SVEP for T_A and once more Lemma A.2(a) with (2.12) implies $\Sigma(A) = \Sigma_s(A)$. Then (5.3) follows from (5.2).

(b) The assertions are "dual" to (a) and based on Lemma A.2(b).

Finally, given $\alpha \in \{F, F_0\}$ the representation $\Sigma_\alpha(A) = \bigcup_{i=1}^d \Sigma_\alpha(a^i)$ follows from [14, Prop. 3.5(ii) and (iii)] for (a) and from [14, Prop. 3.6(ii) and (iii)] for (b). □

Appendix A. Operators on Hilbert spaces. Let X be an infinite-dimensional separable and complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. The set of linear bounded operators between X and a normed space Y is abbreviated as $L(X, Y)$; we write $N(S) \subseteq X$ for the kernel and $R(S) \subseteq Y$ for the range of some $S \in L(X, Y)$. Moreover, $L(X) := L(X, X)$ is the Banach algebra of bounded linear operators on X with identity id_X.

Given an operator $T \in L(X)$, let $\sigma_a(T) := \sigma(T)$, $\sigma_\pi(T)$, $\sigma_s(T)$, $\sigma_F(T)$ and $\sigma_{F_0}(T)$ be its *spectrum*, *approximate point spectrum*, *surjectivity*, *essential* and *Weyl spectrum*, respectively (see [1, 2, 3, 25]). Since X is a Hilbert space, the *left spectrum* $\sigma_l(T)$ resp. the *right spectrum* $\sigma_r(T)$ satisfy

$$\sigma_l(T) = \sigma_\pi(T), \qquad\qquad \sigma_r(T) = \sigma_s(T). \tag{A.1}$$

Let us write $r(T)$ for the *spectral radius* and $r_F(T) := \sup_{\lambda\in\sigma_F(T)} |\lambda|$ for the *essential spectral radius* of T. When $T^* \in L(X)$ denotes the (Hilbert space) *adjoint operator* of T, then the spectra of T and T^* are related by (cf. [1, p. 244, Thm. 6.14 and p. 300, Lemma 7.41], [2, p. 79, Thm. 2.42])

$$\begin{aligned} \sigma_\alpha(T^*) &= \sigma_\alpha(T)^* &\qquad& \text{for all } \alpha \in \{a, F, F_0\}, \\ \sigma_s(T) &= \sigma_\pi(T^*), && \sigma_s(T^*) = \sigma_\pi(T), \end{aligned} \tag{A.2}$$

where $\Omega^* := \{\lambda \in \mathbb{C} : \bar{\lambda} \in \Omega\}$ for every $\Omega \subseteq \mathbb{C}$.

Lemma A.1 ([2, p. 79, Thm. 2.42])**.** *$\partial\sigma(T) \subseteq \sigma_\pi(T) \cap \sigma_s(T)$.*

An operator $T \in L(X)$ possesses the *single-valued extension property* (SVEP for short) *at a point* λ_0, provided for every neighborhood $U \subseteq \mathbb{C}$ of λ_0 the only analytic function $f : U \to X$ satisfying $(\lambda\,\mathrm{id}_X - T)f(\lambda) \equiv 0$ on U is identically vanishing. If the SVEP holds for every $\lambda_0 \in \mathbb{C}$, then the operator T is said to have the SVEP. The associate set (cf. [3, p. 65ff])

$$\mathfrak{S}(T) := \{\lambda \in \mathbb{C} : T \text{ does not have the SVEP at } \lambda\}$$

is open and fulfills $\mathfrak{S}(T) \subseteq \sigma(T)^\circ$. Clearly, T has the SVEP, if and only if $\mathfrak{S}(T) = \emptyset$.

Lemma A.2 ([2, p. 80, Cor. 2.45])**.** *(a) If T has the SVEP, then $\sigma(T) = \sigma_s(T)$.*
(b) If T^* has the SVEP, then $\sigma(T) = \sigma_\pi(T)$.

Lemma A.3 ([2, p. 85, Thm. 2.52])**.** *(a) If $\partial\sigma(T) = \sigma_s(T)$, then T has the SVEP.*
(b) If $\partial\sigma(T) = \sigma_\pi(T)$, then T^* has the SVEP.

Appendix B. Weighted shift operators. Let $\mathbb{I}$ be a discrete interval unbounded above. We denote by ℓ^2 the linear space of square-summable sequences $\phi = (\phi_k)_{k\in\mathbb{I}}$ in $\mathbb{K}^d$ equipped with the inner product

$$\langle\phi, \psi\rangle := \sum_{k\in\mathbb{I}} \langle\phi_k, \psi_k\rangle \quad \text{for all } \phi, \psi \in \ell^2$$

and norm $\|\phi\| = \sqrt{\langle\phi, \phi\rangle}$; ℓ^2 is the prototype of a separable Hilbert space.

For a bounded *weight sequence* $A = (A_k)_{k\in\mathbb{I}}$ in $L(\mathbb{K}^d)$, we define the left shift

$$\begin{aligned} T_A : \ell^2 \to \ell^2, \quad (T_A\phi)_k &:= A_{k-1}\phi_{k-1} \quad \text{if } \mathbb{I} = \mathbb{Z}, \\ T_A\phi &:= (0, A_\kappa\phi_\kappa, A_{\kappa+1}\phi_{\kappa+1}, \ldots) \quad \text{if } \mathbb{I} = \mathbb{Z}_\kappa^+; \end{aligned} \tag{B.1}$$

T_A is bounded with $\|T_A\| = \sup_{k\in\mathbb{I}} |A_k|$ and $\overline{\beta}(A) = r(T_A)$ (cf. [5, Thm. 1(i)]). Since the SVEP is invariant under similarity, $\mathfrak{S}(T_A)$ is rotationally symmetric w.r.t. 0.

Lemma B.1. *The adjoint of T_A is given by $T_A^* \in L(\ell^2)$, $(T_A^*\phi)_k = A_k^*\phi_{k+1}$, $k \in \mathbb{I}$.*

Proof. For arbitrary $\phi, \psi \in \ell^2$ we obtain

$$\langle\!\langle T_A\phi, \psi\rangle\!\rangle = \sum_{k\in\mathbb{I}} \langle A_k\phi_k, \psi_{k+1}\rangle = \sum_{k\in\mathbb{I}} \langle \phi_k, A_k^*\psi_{k+1}\rangle = \langle\!\langle \phi, T_A^*\psi\rangle\!\rangle$$

with $(T_A^*\psi)_k := A_k^*\psi_{k+1}$ for all $k \in \mathbb{I}$. □

B.1. **Unilateral shifts.** On a discrete interval $\mathbb{I} = \mathbb{Z}_\kappa^+$, $\kappa \in \mathbb{Z}$, one denotes (B.1) as *unilateral shift*. The essential properties of unilateral shifts are summarized in

Proposition B.2. *If each A_k, $k \in \mathbb{Z}_\kappa^+$, is invertible, then the following holds:*

(a) $\sigma_F(T_A) = \sigma_\pi(T_A) \subseteq \left\{\lambda \in \mathbb{C} : \underline{\beta}(A) \le |\lambda| \le \overline{\beta}(A)\right\}$

(b) $\underline{\beta}(A) > 0$ *if and only if* $\sup_{\kappa\le k}\left|A_k^{-1}\right| < \infty$

(c) $\mathfrak{S}(T_A) = \emptyset$ *and* $\mathfrak{S}(T_A^*) = \left\{\lambda \in \mathbb{C} : |\lambda| < \underline{\lambda}_+(A)\right\}$

Proof. (a) is from [28], (b) by [23, p. 134, Thm. 4.6.10] and (c) by [27, Thm. 2.1]. □

Assume now that $a = (a_k)_{\kappa\le k}$ is a bounded sequence in $\mathbb{K}$. Then the results [35, Thm. 1] combined with Prop. B.2(a) yield the relations

$$\sigma_\alpha(T_a) = \begin{cases} \left\{\lambda \in \mathbb{C} : \underline{\beta}(a) \le |\lambda| \le \overline{\beta}(a)\right\}, & a_k \neq 0 \text{ for all } k \in \mathbb{Z}_\kappa^+, \\ \{0\} \cup \left\{\lambda \in \mathbb{C} : \underline{\beta}(\tilde{a}) \le |\lambda| \le \overline{\beta}(\tilde{a})\right\}, & a_k = 0 \text{ for finitely many } k \in \mathbb{Z}_\kappa^+, \\ \left\{\lambda \in \mathbb{C} : |\lambda| \le \overline{\beta}(a)\right\}, & a_k = 0 \text{ for infinitely many } k \in \mathbb{Z}_\kappa^+ \end{cases}$$

with $\alpha \in \{F, \pi\}$, where the $\mathbb{K}$-valued sequence $\tilde{a}$ is defined as $\tilde{a}_k := a_{k-\kappa+K}$ for every $\kappa \le k$ with K being the minimal integer such that $a_k \neq 0$ for all $k \ge K$.

Example B.3 (asymptotically periodic case)**.** *Let $p \in \mathbb{N}$. A sequence $(a_k)_{\kappa\le k}$ in $\mathbb{K}$ is* asymptotically p-periodic, *if there exists a p-periodic sequence $(p_k)_{\kappa\le k}$ with*

$$\lim_{k\to\infty} |a_k - p_k| = 0$$

and $c := \sqrt[p]{|p_{\kappa+p-1}\cdots p_\kappa|}$ is denoted as its asymptotic mean. *Provided $(|a_k|)_{\kappa\le k}$ is asymptotically p-periodic, then* [35, Thm. 2] *and Prop. B.2(a) yield*

$$\sigma_F(T_a) = \sigma_\pi(T_a) = \begin{cases} \{\lambda \in \mathbb{C} : |\lambda| = c\}, & a_k \neq 0 \textit{ for all } k \in \mathbb{Z}_\kappa^+, \\ \{0\} \cup \{\lambda \in \mathbb{C} : |\lambda| = c\}, & a_k = 0 \textit{ for some } k \in \mathbb{Z}_\kappa^+. \end{cases}$$

B.2. **Bilateral shifts.** For $\mathbb{I} = \mathbb{Z}$ one speaks of *bilateral shifts* $T_A \in L(\ell^2)$. In case of invertible weights $A_k \in \mathbb{K}^{d\times d}$ the condition $\sup_{k\in\mathbb{Z}}\left|A_k^{-1}\right| < \infty$ implies $0 \notin \sigma(T_A)$.

As opposed to unilateral shifts, a characterization of the SVEP is more involved:

- T_A has the SVEP if and only if (cf. [11, Thm. 2.1])

$$\liminf_{k\to\infty} \sqrt[k]{|\Phi(-k,\kappa)x|}^{-1} \le \limsup_{k\to\infty} \sqrt[k]{|\Phi(k,\kappa)x|} \quad \text{for all } x \in \mathbb{K}^d \setminus \{0\}. \tag{B.2}$$

- T_A^* has the SVEP if and only if (cf. [11, Cor. 2.2])

$$\liminf_{k\to\infty} \sqrt[k]{|\Phi(\kappa,k)^*x|}^{-1} \le \limsup_{k\to\infty} \sqrt[k]{|\Phi(\kappa,-k)^*x|} \quad \text{for all } x \in \mathbb{K}^d \setminus \{0\} \tag{B.3}$$

for some $\kappa \in \mathbb{Z}$. Different from the scalar situation (cf. Lemma B.5 below), both T_A and T_A^* might fail to possess the SVEP (see [11, Ex. 2.3]).

Particularly for scalar shifts with weights $a \in \ell^\infty(\mathbb{K})$, thanks to [35, Thm. 3] it is

$$\sigma(T_a) \overset{(A.2)}{=} \sigma(T_a^*) = \left\{\lambda \in \mathbb{C} : \underline{\beta}(a) \le |\lambda| \le \overline{\beta}(a)\right\},$$

$$\sigma_\pi(T_a) = \begin{cases} \left\{\lambda \in \mathbb{C} : \underline{\beta}_{\mathbb{Z}_\kappa^-}(a) \le |\lambda| \le \overline{\beta}_{\mathbb{Z}_\kappa^-}(a)\right\} & \\ \quad \cup \left\{\lambda \in \mathbb{C} : \underline{\beta}_{\mathbb{Z}_\kappa^+}(a) \le |\lambda| \le \overline{\beta}_{\mathbb{Z}_\kappa^+}(a)\right\}, & \overline{\beta}_{\mathbb{Z}^-}(a) < \underline{\beta}_{\mathbb{Z}^+}(a), \\ \sigma(T_a), & \text{otherwise.} \end{cases}$$

Example B.4 (asymptotically periodic case)**.** *Let $\kappa \in \mathbb{Z}$, $p^+, p^- \in \mathbb{N}$, and $(a_k)_{k\in\mathbb{Z}}$ be a sequence in $\mathbb{K}$. If $(|a_k|)_{k\ge\kappa}$ is asymptotically p_+-periodic and $(|a_k|)_{k\le\kappa}$ is asymptotically p_--periodic with asymptotic means c_+, c_-, then* [35, Thm. 5] *showed*

$$\sigma_\pi(T_a) = \begin{cases} \{\lambda \in \mathbb{C} : c_+ \le |\lambda| \le c_-\}, & c_+ \le c_-, \\ \{\lambda \in \mathbb{C} : |\lambda| \in \{c_+, c_-\}\}, & c_- < c_+, \end{cases}$$

$$\sigma(T_a) = \{\lambda \in \mathbb{C} : \min\{c_+, c_-\} \le |\lambda| \le \max\{c_-, c_+\}\}.$$

Lemma B.5 ([29, Prop. 2.5] and [4, Thm. 18 and Cor. 19])**.** *Either T_a or T_a^* has the SVEP and*

$$\mathfrak{S}(T_a) = \left\{\lambda \in \mathbb{C} : \overline{\lambda}_+(a) < |\lambda| < \underline{\lambda}_-(a)\right\},$$
$$\mathfrak{S}(T_a^*) = \left\{\lambda \in \mathbb{C} : \overline{\lambda}_-(a) < |\lambda| < \underline{\lambda}_+(a)\right\}.$$

REFERENCES

[1] Abramovich, Y., Aliprantis, C.: An invitation to operator theory, *Graduate Studies in Mathematics* 50. AMS, Providence, RI, 2002

[2] Aiena, P.: Fredholm and local spectral theory, with applications to multipliers. Kluwer, Dordrecht, 2004

[3] Aiena, P.: Semi-Fredholm operators, perturbation theory and localized SVEP. XX Escuela Venezolana de Matemáticas, Caracas, Venezuela, 2007

[4] Aiena, P., Miller, T., Neumann, M.: On a localized single-valued extension property. Math. Proc. R. Ir. Acad. **104A**(1), 17–34 (2004)

[5] Aulbach, B., Van Minh, N., Zabreiko, P.: The concept of spectral dichotomy for linear difference equations. J. Math. Anal. Appl. **185**, 275–287 (1994)

[6] Aulbach, B., Siegmund, S.: A spectral theory for nonautonomous difference equations. In: López-Fenner J., e.a. (ed.) Proceedings of the 5th Intern. Conference of Difference Eqns. and Application (Temuco, Chile, 2000), pp. 45–55. Taylor & Francis, London, 2002

[7] Barraa, M., Boumazgour, M.: A note on the spectrum of an upper triangular operator matrix. Proc. Am. Math. Soc. **131** (10), 2131–2137 (2006)

[8] Barreira, L., Pesin, Y.: Introduction to Smooth Ergodic Theory, *Graduate Studies in Mathematics* 148, AMS, Providence, RI, 2013

[9] Battelli, F., Palmer, K.: Dichotomy and triangular systems. Preprint (2014)

[10] Ben-Artzi, A., Gohberg, I.: Dichotomies of perturbed time varying systems and the power method. Indiana Univ. Math. J. **42**(3), 699–720 (1993)

[11] Bourhim, A., Chidume, C.: The single-valued extension property for bilateral operator weighted shifts. Proc. Am. Math. Soc. **133** (2), 3083–3088 (2003)

[12] Dieci, L., van Vleck, E.: Lyapunov and other spectra: A survey. In: Collected Lectures on the Preservation of Stability under Discretization, pp. 197–218. SIAM, 2002

[13] Djordjević, S., Han, Y.: A note on Weyl's theorem for operator matrices. Proc. Am. Math. Soc. **131**(8), 2543–2547 (2002)

[14] Duggal, B.: Upper triangular operators with SVEP: Spectral properties. Filomat **21**(1), 25–37 (2007)

[15] Elbjaoui, H., Zerouali, E.: Local spectral theory for 2 × 2 operator matrices. Int. J. Math. Math. Sci. **42**, 2667–2672 (2003)

[16] Han, J., Lee, H., Lee, W.: Invertible completitions of 2×2 upper triangular operator matrices. Proc. Am. Math. Soc. **128**(1), 119–123 (1999)

[17] Henry, D.: Geometric Theory of Semilinear Parabolic Equations, *Lect. Notes Math.* 840. Springer, Berlin etc., 1981

[18] Hinrichsen, D., Pritchard, A.: Mathematical Systems Theory I – Modelling, State Space Analysis, Stability and Robustness, *Texts in Applied Mathematics* 48. Springer, Heidelberg etc., 2005

[19] Hong-Ke, D., Jin, P.: Perturbations of spectrums of 2×2 operator matrices. Proc. Am. Math. Soc. **121**(3), 761–766 (1994)

[20] Hüls, T.: Computing Sacker-Sell spectra in discrete time dynamical systems. SIAM Journal on Numerical Analysis **48**(6), 2043–2064 (2010)

[21] Hüls, T., Pötzsche, C.: Qualitative Analysis of a Nonautonomous Beverton-Holt Ricker Model, 44p., to appear in SIAM Journal of Applied Dynamical Systems (2014)

[22] Huy, N., Van Minh, N.: Exponential dichotomy of difference equations and applications to evolution equations on the half-line. Comput. Math. Appl. **42**, 301–311 (2001)

[23] Jiang, C., Wang, Z.: Structure of Hilbert space operators. World Scientific, New Jersey, 2006

[24] Kloeden, K., Rasmussen, M.: Nonautonomous Dynamical Systems. *Mathematical Surveys and Monographs* 176. AMS, Providence, RI, 2011

[25] Laursen, K., Neumann, M.: An Introduction to Local Spectral Theory. Oxford Science Publications, Oxford, 2000

[26] Lee, W.: Weyl spectra of operator matrices. Proc. Am. Math. Soc. **129**(1), 131–138 (2000)

[27] Li, J.: The single valued extension property for operator weighted shifts. Northeast. Math. J. **10**(1), 99–103 (1994)

[28] Li, J., Ji, Y., Sun, S.: The essential spectrum and Banach reducibility of operator weighted shifts. Acta Mathematica Sinica **17**(3), 413–424 (2001)

[29] Miller, T., Miller, V., Neumann, M.: Local spectral properties of weighted shifts. J. Operator Theory **51**, 71–88 (2004)

[30] Papaschinopoulos, G.: On exponential trichotomy of linear difference equations. Appl. Anal. **40**, 89–109 (1991)

[31] Pötzsche, C.: A note on the dichotomy spectrum. J. Difference Equ. Appl. **15**(10), 1021–1025 (see also the corrigendum in J. Difference Equ. Appl. **18**(7), 1257–1261 (2012)) (2009)

[32] Pötzsche, C.: Fine structure of the dichotomy spectrum. Integral Equations Oper. Theory **73**(1), 107–151 (2012)

[33] Pötzsche, C.: Continuity of the dichotomy spectrum on the half line. Submitted (2014)

[34] Pötzsche, C., Russ, E.: Continuity and invariance of the dichotomy spectrum. Submitted (2014).

[35] Ridge, W.: Approximate point spectrum of a weighted shift. Trans. Am. Math. Soc. **147**, 349–356 (1970)

[36] Sacker, R., Sell, G.: A spectral theory for linear differential systems. J. Differ. Equations **27**, 320–358 (1978)

[37] Sánchez-Perales, S., Djordjević, S.: Continuity of spectrum and approximate point spectrum on operator matrices. J. Math. Anal. Appl. **378**, 289–294 (2011)

[38] Siegmund, S.: Normal forms for nonautonomous differential equations. J. Differ. Equations **178(2)**, 541–573 (2002)

[39] Siegmund, S.: Normal forms for nonautonomous difference equations. Comput. Math. Appl. **45(6–9)**, 1059–1073 (2003)

[40] Zerouali, E., Zguitti, H.: Perturbation of spectra of operator matrices and local spectral theory. J. Math. Anal. Appl. **324**, 992–1005 (2006)

[41] Zhang, Y., Zhong, H., Lin, L.: Browder spectra and essential spectra of operator matrices. Acta Mathematica Sinica **24**(6), 947–955 (2008)

E-mail address: christian.poetzsche@aau.at