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Continuity and invariance of the Sacker-Sell spectrum*

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Abstract The Sacker-Sell (also called dichotomy or dynamical) spectrum Σ is a fundamental concept in the geometric, as well as for a developing bifurcation theory of nonautonomous dynamical systems. In general, it behaves merely upper-semi-continuously and a perturbation theory is therefore delicate. This paper explores an operator-theoretical approach to obtain invariance and continuity conditions for both Σ and its dynamically relevant subsets. Our criteria allow to avoid non-autonomous bifurcations due to collapsing spectral intervals and justify numerical approximation schemes for Σ.

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1 Introduction and motivation

One of the central questions in the qualitative analysis of a dynamical system is its local behavior near fixed reference solutions. This, first of all requires an ambient spectral theory for the associate variational equation in order to determine e.g. stable or (non-) hyperbolic behavior. In case of constant (or periodic) reference solutions to autonomous (resp. periodic) equations the suitable spectral notion is undoubtedly given in terms of conventional eigenvalues (resp. Floquet multipliers). Yet, when dealing with more general time dependencies (quasi-periodic, almost periodic, almost automorphic, asymptotically constant in both time directions, random, etc.) the question for an adequate spectral concept becomes more subtle.

* the first author dedicates this paper to the memory of George R. Sell; in gratitude for many inspirations, discussions and his hospitality

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For this reason several spectra have been suggested (see [28] or [13, pp. 183ff] for a survey), which capture different features of a nonautonomous linear equation. Here, a first choice appears to be the widely known Lyapunov spectrum due to its property that corresponding upper Lyapunov exponents in the negative half line (resp. the unit interval $(0,1)$ for discrete time) yield asymptotic stability of a linear equation. Nevertheless, as classical examples illustrate (cf. [41,35]) this kind of stability does not persist under perturbations, even if they are merely of order $o(|x|)$ as $x \to 0$ or decay (exponentially) to 0 as $t \to \infty$. Hence, Lyapunov exponents are problematic when it comes to the development of an applicable nonlinear theory and related tools such as invariant manifolds, topological linearization, or normal forms (existing in uniform neighborhoods of the reference solution). One actually needs the additional assumption of *regularity* (i.e. the equation and its adjoint satisfy a symmetry property w.r.t. their Lyapunov spectra, cf. [4, pp. 113ff]). It yields a more satisfactory perturbation theory or an appropriate "linear algebra" in form of Oseledets Multiplicative Ergodic Theorem (see for example, [28]). In general, regularity is hard to verify for concrete examples, but on the other hand holds for large problem classes like e.g. linear random dynamical systems (cf. [4]).

Rather than asymptotic stability, the stronger concept of *uniform* asymptotic stability has more convenient robustness properties. The related spectral notion is the *Sacker-Sell* (or *dichotomy* or *dynamical*) *spectrum* $\Sigma \subset \mathbb{R}$, which can be coarser than the Lyapunov spectrum. Following [46,6] the dichotomy spectrum of a linear system in $\mathbb{C}^d$ is the disjoint union of up to d closed intervals. These *spectral intervals* generalize the real parts (or moduli in the discrete time case) of eigenvalues from the autonomous theory. Accordingly, in a comprehensive local theory of nonautonomous dynamical systems the prominent role of the dichotomy spectrum is underlined by the following facts:

– The inclusion $\Sigma \subset (-\infty, 0)$ (or $\Sigma \subset (0,1)$ in discrete time) for the dichotomy spectrum of a variational equation along a reference solution guarantees its uniform asymptotic stability, while the existence of a spectral interval in $\mathbb{R}^+$ (resp. $(1,\infty)$) yields instability.
– Each gap in Σ gives rise to a pseudo-stable and a pseudo-unstable invariant manifold (see [49,43]). Its particular location yields the classical hierarchy of stable, center-stable, center, center-unstable and unstable manifolds.
– Hyperbolicity in form of $0 \notin \Sigma$ (resp. $1 \notin \Sigma$) implies that solutions can be locally continued in parameters and that topological linearization results by means of a Hartman-Grobman theorem hold.
– Finally, information on the fine structure of Σ allows to classify various types of nonautonomous bifurcations on a linear basis already (see [44]).

These striking features of the theory initiated by [46] stimulated further work on evolutionary equations in discrete time [10,5], in infinite dimensions [12], as well as on the numerical approximation of Σ (cf. [17,18,26]) — among other references.

This trend went along with an increasing interest in the asymptotic behavior of nonautonomous equations and particularly a corresponding bifurcation theory. As opposed to the conventional autonomous case, where the behavior of eigenvalues under parameter variation is well-understood (cf., for instance, [24]), any related knowledge on the dichotomy spectrum Σ is rather underdeveloped. Little has been established besides the upper-semicontinuous dependence of Σ on the system and its invariance under kinematic similarity; to say nothing on a smooth dependence for boundary points of spectral intervals on parameters. And yet such behavior hints to qualitative changes or bifurcations when dichotomy intervals touch or cross the stability boundary.

Driven by this motivation, our goal is to obtain further information on Σ and to narrow the above mentioned wide gap when extending the established autonomous to a nonautonomous theory. We begin with necessary preparations on Bohl exponents as a concept to describe uniform exponential growth yielding upper and lower bounds for the dichotomy spectrum. The subsequent Sect. 3 introduces the central interconnection of our overall approach, namely a link between dichotomies and weighted shifts. Moreover, certain classes of coefficient functions are introduced, which become important in later investigations on perturbation properties of Σ. It is shown that additive perturbations leave Σ invariant, provided they fulfill appropriate commutativity assumptions and have vanishing Bohl exponents (for this, see Prop. 4.6) *(invariance)*. We admit that our results are somewhat academic here, since the perturbations have to be from a finite-dimensional space — on the other hand, this is what a direct application of general operator-theoretical tools to our specific situation is able to provide. Finally, sufficient conditions are given that the dichotomy spectrum behaves continuously. This means for instance that a spectral interval cannot collapse to its boundary points or two subintervals under (small) perturbations. Furthermore, certain types of coefficient matrices yielding continuous dependence of the dichotomy spectrum are identified, which for example rule out bifurcations due to a sudden collapse of a spectral interval. Note that these conditions validate numerical approximation techniques *(continuity)*. The attained results are illustrated using the instructive Ex. 5.10.

Our overall analysis partly extends to the dynamically relevant subsets of Σ, namely the Fredholm dichotomy spectra Σ_F, Σ_{F_0}, the surjectivity spectrum Σ_s (see [44] for details), as well as the approximate point spectrum Σ_π. These subsets of Σ are meaningful for various reasons: (1) They yield a classification of nonautonomous bifurcations on a linear basis, (2) the boundary points of Σ are contained in the Fredholm spectrum Σ_F, which therefore indicates qualitative and stability changes, (3) Σ_s allows to describe an intrinsically nonautonomous form of nonhyperbolic behavior, namely an exponential trichotomy, and (4) sufficient continuity conditions for Σ are based on relations between its subspectra (cf. Thm. 5.4).

Rather than using typical dynamical systems techniques, our approach is exclusively based on a close connection between nonautonomous linear dynamics and operator theory due to [10] or [7,42]; we also refer to [11] for the merits of functional analytical methods to tackle dichotomies of nonautonomous evolutionary differential equations. In our setting, the dichotomy spectrum is actually the intersection of the positive real axis with the spectrum of an appropriate matrix-weighted shift operator defined on an ambient sequence space — this intersection property also extends to the subspectra. Hence, when tackling properties of Σ, we benefit from a rich and well-developed related theory for general bounded operators. The required results are summarized in two extensive appendices dealing with bounded operators on Hilbert spaces and weighted shifts.

Since the corresponding techniques apply immediately, our emphasis in this paper are discrete time dynamical systems, i.e. nonautonomous linear difference equations. For the sake of a clear presentation, we furthermore restrict to the finite-dimensional invertible situation. However, in Sect. 6 we indicate how accordant results can be applied to nonautonomous ordinary differential equations. The tool of choice is a characterization of exponential dichotomies from [48] and an ambient linear time-1-mapping.

2 Preliminaries, Bohl exponents and dichotomies

A *discrete interval* is the intersection of a real interval with the integers $\mathbb{Z}$ and we frequently use the abbreviations $\mathbb{Z}_\kappa^+ := \{k \in \mathbb{Z} : \kappa \le k\}$, $\mathbb{Z}_\kappa^- := \{k \in \mathbb{Z} : k \le \kappa\}$.

For a Banach space X with norm $|\cdot|$, the Banach algebra of bounded linear operators on X is denoted by $L(X)$, id_X is the unit element, i.e. the identity mapping, and $GL(X)$ is the group of invertible elements. In our subsequent considerations, X will typically be the unitary space $\mathbb{C}^d$ with the inner product $\langle x, y\rangle := \sum_{j=1}^d x_j \bar{y}_j$ for all $x, y \in \mathbb{C}^d$ and induced norm $|x| := \sqrt{\langle x, x\rangle}$, or the bounded sequences $\ell^\infty(\mathbb{C}^d)$ in $\mathbb{C}^d$ or the Hilbert space of square-summable sequences $\ell^2(\mathbb{C}^d)$ in $\mathbb{C}^d$ equipped with the inner product resp. norm

$$\langle\!\langle \phi, \psi\rangle\!\rangle := \sum_{j\in\mathbb{Z}} \langle \phi_j, \psi_j\rangle, \qquad \|\phi\| := \sqrt{\langle\!\langle \phi, \phi\rangle\!\rangle} \quad \text{for all } \phi = (\phi_j)_{j\in\mathbb{Z}}, \psi = (\psi_j)_{j\in\mathbb{Z}}.$$

We abbreviate $\ell^2 = \ell^2(\mathbb{C}^d)$ throughout and $\mathcal{L}^\infty(\mathbb{C}^d)$ is the linear space $\ell^\infty(L(\mathbb{C}^d))$ of all bounded matrix sequences $A = (A_k)_{k\in\mathbb{Z}}$ in $L(\mathbb{C}^d)$ endowed with the canonical norm $\|A\| := \sup_{k\in\mathbb{Z}} |A_k|$. For the sequence in $\mathcal{L}^\infty(\mathbb{C}^d)$ consisting of identity mappings $\mathrm{id}_{\mathbb{C}^d}$ it is convenient to write $I := (\mathrm{id}_{\mathbb{C}^d})_{k\in\mathbb{Z}}$.

The (topological) closure of a set $\Omega \subseteq X$ is denoted by $\overline{\Omega}$.

2.1 Bohl exponents

Our presentation initially rests upon the following abstract framework: Suppose that $\mathcal{A}$ is a normed unital algebra over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with norm $|\cdot|$. For the family of discrete intervals $\mathbb{J} \subseteq \mathbb{Z}$ with fixed length n one writes $\mathbb{Z}_n$, $n \in \mathbb{N}$. We define the *lower* resp. *upper Bohl* exponent of a sequence $a = (a_k)_{k\in\mathbb{Z}}$ in $\mathcal{A}$ as

$$\underline{\beta}(a) := \liminf_{n\to\infty} \inf_{\mathbb{J}\in\mathbb{Z}_n} \sqrt[n]{\Big|\prod_{j\in\mathbb{J}} a_j\Big|}, \qquad \overline{\beta}(a) := \limsup_{n\to\infty} \sup_{\mathbb{J}\in\mathbb{Z}_n} \sqrt[n]{\Big|\prod_{j\in\mathbb{J}} a_j\Big|} \tag{2.1}$$

and easily obtain the bounds $\underline{\beta}(a) \le \overline{\beta}(a) \le \sup_{k\in\mathbb{Z}} |a_k|$. Their positive homogeneity $\overline{\beta}(\lambda a) = |\lambda|\,\overline{\beta}(a)$ and $\underline{\beta}(\lambda a) = |\lambda|\,\underline{\beta}(a)$ for all $\lambda \in \mathbb{K}$ is obvious. Moreover, the right-hand limit in (2.1) exists and (cf. [45, Prop. 2.2])

$$\begin{aligned} \overline{\beta}(a) &= \lim_{n\to\infty} \sup_{\mathbb{J}\in\mathbb{Z}_n} \sqrt[n]{\Big|\prod_{j\in\mathbb{J}} a_j\Big|} = \inf_{n\in\mathbb{N}} \sup_{\mathbb{J}\in\mathbb{Z}_n} \sqrt[n]{\Big|\prod_{j\in\mathbb{J}} a_j\Big|} \\ &= \inf\left\{\rho > 0 \,\middle|\, \exists K \ge 1 : \forall n \in \mathbb{N} : \sup_{\mathbb{J}\in\mathbb{Z}_n} \Big|\prod_{j\in\mathbb{J}} a_j\Big| \le K\rho^n \right\} \end{aligned} \tag{2.2}$$

holds, where the latter characterization (2.2) requires a to be bounded.

Sequences $(a_k)_{k\in\mathbb{Z}}$ with $\lim_{k\to\pm\infty} a_k = 0$ fulfill $\overline{\beta}(a) = 0$, but as demonstrated in the subsequent Ex. 3.2 the converse fails.

2.2 Exponential dichotomies

We are focussed on linear nonautonomous difference equations

$$\boxed{x_{k+1} = A_k x_k} \tag{Δ_A}$$

with invertible coefficients $A_k \in GL(\mathbb{C}^d)$, $k \in \mathbb{Z}$. It is understood that our results also hold for real eqns. (Δ_A) by applying them to its complexification. Throughout the entire paper, let us impose the global boundedness assumption

$$\sup_{k\in\mathbb{Z}} |A_k| < \infty,$$

which is justifiable since nonautonomous problems (Δ_A) typically occur as variational equations along bounded solutions. The solutions to (Δ_A) can be expressed in terms of the *transition matrix* $\Phi : \mathbb{Z} \times \mathbb{Z} \to GL(\mathbb{C}^d)$,

$$\Phi(k, l) := \begin{cases} A_{k-1} \cdots A_l, & l < k, \\ \mathrm{id}_{\mathbb{C}^d}, & k = l, \\ A_k^{-1} \cdots A_{l-1}^{-1}, & k < l; \end{cases} \tag{2.3}$$

in order to indicate the dependence on A we sometimes write $\Phi_A(k, l)$.

A difference eqn. (Δ_A) is said to possess an *exponential dichotomy* (ED for short, cf. [22, p. 229, Def. 7.6.4] or [10, 5]) on a discrete interval $\mathbb{J}$ being unbounded above, provided there exists a sequence of projections

$$P_k \in L(\mathbb{C}^d), \qquad P_{k+1}A_k = A_k P_k \quad \text{for all } k \in \mathbb{J},$$

as well as reals $\alpha \in (0, 1)$, $K \geq 1$ guaranteeing the hyperbolic splitting

$$|\Phi(k, l)P_l| \leq K\alpha^{k-l}, \qquad |\Phi(l, k)[\mathrm{id}_{\mathbb{C}^d} - P_k]| \leq K\alpha^{k-l} \quad \text{for all } l \leq k$$

and $k, l \in \mathbb{J}$. Unless otherwise noted, we always act on the assumption $\mathbb{J} = \mathbb{Z}$ throughout the paper. On this basis, the *dichotomy spectrum* of (Δ_A) is given as

$$\Sigma(A) := \left\{ \gamma > 0 : \, x_{k+1} = \gamma^{-1} A_k x_k \text{ does not have an ED} \right\}.$$

It captures the exponential growth behavior of solutions to (Δ_A). Referring to [7, Thm. 4] and [6, Thm. 3.4], the dichotomy spectrum consists of $m \leq d$ disjoint *spectral intervals* and is of the form

$$\Sigma(A) = \begin{cases} (0, \beta_m] \\ \text{or} \\ [\alpha_m, \beta_m] \end{cases} \cup \bigcup_{i=1}^{m-1} [\alpha_i, \beta_i] \tag{2.4}$$

with real numbers $0 < \alpha_m \leq \beta_m < \alpha_{m-1} \leq \ldots \leq \beta_1$. The invertibility assumption on A_k ensures that a spectral interval $(0, \beta_m]$ can be avoided precisely for

$$\sup_{k\in\mathbb{Z}} \left| A_k^{-1} \right| < \infty. \tag{2.5}$$

One says a difference eqn. (Δ_A) has *discrete spectrum*, if all spectral intervals $[\alpha_i, \beta_i]$ are singletons, i.e. $\alpha_i = \beta_i$ for $1 \leq i \leq m$. For reasons becoming apparent shortly, we avoid the commonly used term *point spectrum* in this context.

Scalar difference equations indicate the close relation between Bohl exponents and their dichotomy spectrum:

$$\begin{array}{ccccc}
\partial\Sigma(A) & \subseteq & \Sigma_\pi(A) & \subseteq & \Sigma_\pi(A)\cup\Sigma_s(A)\\
\cap\! | & & & & \|\\
\Sigma_F(A) & \subseteq & \Sigma_{F_0}(A) & \subseteq & \Sigma(A) = \Sigma_{F_0}(A)\cup\Sigma_s(A)\\
\cap\! | & & & & \|\\
\Sigma_s(A) & & \subseteq & & \Sigma_\pi(A)\cup\Sigma_{F_0}(A)
\end{array}$$

Fig. 3.1 Fine structure of the dichotomy spectrum as a consequence of [44, Cor. 4.37] and the well-known inclusions $\partial\sigma(T_A) \subseteq \sigma_\pi(T_A) \subseteq \sigma(T_A)$

Example 2.1 (scalar equations) Provided $a \in \mathcal{L}^\infty(\mathbb{C})$ holds, then the dichotomy spectrum of $x_{k+1} = a_k x_k$ reads as (see [7, Thm. 4(ii)] and [45, Prop. 2.4])

$$\Sigma(a) = \begin{cases} (0, \overline{\beta}(a)], & \text{if } \inf_{k\in\mathbb{Z}} |a_k| = 0, \\ [\underline{\beta}(a), \overline{\beta}(a)], & \text{else.} \end{cases} \tag{2.6}$$

3 Weighted shifts and system classes

In order to obtain information on the dichotomy spectrum beyond its basic structure (2.4), we employ a relation between $\Sigma(A)$ and the spectra of matrix-weighted shifts henceforth, which can be traced back to [10,8,7]. De facto, our spectral theory for difference eqns. (Δ_A) is based on the bounded operators

$$\begin{aligned} &S_\lambda \in L(\ell^2), && (S_\lambda\phi)_k := \phi_{k+1} - \lambda^{-1}A_k\phi_k, \\ &T_A \in L(\ell^2), && (T_A\phi)_k := A_{k-1}\phi_{k-1} \end{aligned} \tag{3.1}$$

for all $k \in \mathbb{Z}$, $\phi \in \ell^2$ and $\lambda \in \mathbb{C}\setminus\{0\}$. Thanks to [44, Sect. 4.3] it holds

$$\Sigma_a(A) := \Sigma(A) = \left\{\gamma > 0 : \, S_\gamma \notin GL(\ell^2)\right\}$$

and one furthermore distinguishes the subspectra

$$\begin{aligned} \Sigma_s(A) &:= \{\gamma > 0 : \, S_\gamma \text{ is not onto}\}, \\ \Sigma_F(A) &:= \{\gamma > 0 : \, S_\gamma \text{ is not Fredholm}\}, \\ \Sigma_{F_0}(A) &:= \{\gamma > 0 : \, S_\gamma \text{ is not Weyl}\}, \\ \Sigma_\pi(A) &:= \{\gamma > 0 : \, S_\gamma \text{ is not bounded below}\} \end{aligned}$$

of $\Sigma(A)$. While $\Sigma_a(A) \subseteq (0,\infty)$ is the dichotomy spectrum (see [10,42]), its subsets $\Sigma_s(A)$, $\Sigma_F(A)$, $\Sigma_{F_0}(A)$ and $\Sigma_\pi(A)$ are called *surjectivity*, *Fredholm* (or *essential*), *Weyl* resp. *approximate point dichotomy spectrum* of (Δ_A); the components of these sets are also denoted as *spectral intervals*. Here S_γ is a *Weyl operator*, if it is Fredholm with index 0. Further information and a thorough motivation of the latter spectra can be found in [44]; relations between them are illustrated in Fig. 3.1.

Besides the spectrum $\sigma_a(T_A) := \sigma(T_A) \subseteq \mathbb{C}$ of T_A we need the

$$\begin{aligned} \textit{surjectivity spectrum } \sigma_s(T_A) &:= \{\lambda \in \mathbb{C} : \, T_A - \lambda\,\mathrm{id} \text{ is not onto}\}, \\ \textit{Fredholm spectrum } \sigma_F(T_A) &:= \{\lambda \in \mathbb{C} : \, T_A - \lambda\,\mathrm{id} \text{ is not Fredholm}\}, \\ \textit{Weyl spectrum } \sigma_{F_0}(T_A) &:= \{\lambda \in \mathbb{C} : \, T_A - \lambda\,\mathrm{id} \text{ is not Weyl}\}, \\ \textit{approximate point spectrum } \sigma_\pi(T_A) &:= \{\lambda \in \mathbb{C} : \, T_A - \lambda\,\mathrm{id} \text{ is not bounded below}\} \end{aligned}$$

and obtain the following central relations between σ_α and Σ_α:

Proposition 3.1 *Keep $\alpha \in \{a, s, F_0, F, \pi\}$ fixed and let $A, B \in \mathcal{L}^\infty(\mathbb{C}^d)$. The spectra $\sigma_\alpha(T_A)$ are rotationally invariant w.r.t.* 0 *and fulfill:*

(a) One has the characterization

$$\Sigma_\alpha(A) = \sigma_\alpha(T_A) \cap \mathbb{R}^+. \tag{3.2}$$

(b) $\Sigma_\alpha(A) = \Sigma_\alpha(B) \Leftrightarrow \sigma_\alpha(T_A) = \sigma_\alpha(T_B)$.

Proof Besides (3.2) it suffices to verify that the sets $\sigma_\alpha(T_A) \subset \mathbb{C}$ are rotationally invariant w.r.t. 0. For $\alpha \neq \pi$ this was shown in [44, Sect. 4.3] already and the remaining case results as follows: As in the proof of [7, Thm. 1] we obtain that T_A and $e^{i\nu}T_A$ are similar for each $\nu \in \mathbb{R}$. Thus, [1, p. 253, Exercise 8] yields the rotational invariance of the approximate point spectrum $\sigma_\pi(T_A)$. Since

$$\begin{aligned}
\|(T_A - \lambda\,\mathrm{id}_{\ell^2})\phi\|^2 &= \sum_{k\in\mathbb{Z}} \langle (T_A\phi - \lambda\phi)_k, (T_A\phi - \lambda\phi)_k\rangle \\
&= \sum_{k\in\mathbb{Z}} \langle A_{k-1}\phi_{k-1} - \lambda\phi_k, A_{k-1}\phi_{k-1} - \lambda\phi_k\rangle \\
&= |\lambda|^2 \sum_{k\in\mathbb{Z}} \langle \phi_{k+1} - \lambda^{-1}A_k\phi_k, \phi_{k+1} - \lambda^{-1}A_k\phi_k\rangle = |\lambda|^2\,\|S_\lambda\phi\|^2
\end{aligned}$$

holds for all $\phi \in \ell^2$, $\lambda \in \mathbb{C}\setminus\{0\}$, we deduce the equivalences

$$\begin{aligned}
\lambda \in \sigma_\pi(T_A) &\Leftrightarrow |\lambda| \in \sigma_\pi(T_A) \\
&\Leftrightarrow \forall\varepsilon > 0\ \exists\phi \in \ell^2 \text{ with } \|\phi\| = 1 :\ \|(T_A - |\lambda|\,\mathrm{id}_{\ell^2})\phi\| < \varepsilon \\
&\Leftrightarrow \forall\varepsilon > 0\ \exists\phi \in \ell^2 \text{ with } \|\phi\| = 1 :\ \left\|S_{|\lambda|}\phi\right\| < \varepsilon \\
&\Leftrightarrow |\lambda| \in \Sigma_\pi(A) \quad \text{for all } \lambda \in \mathbb{C}\setminus\{0\}.
\end{aligned}$$

Consequently the claim follows. □

Thanks to Prop. B.4(a), for all $\alpha \in \{a, s, F, F_0, \pi\}$ one can generalize (2.6) to

$$\Sigma_\alpha(A) \subseteq \begin{cases} (0, \overline{\beta}(A)], & \text{if } \sup_{k\in\mathbb{Z}} \left|A_k^{-1}\right| = \infty, \\ [\underline{\beta}(A), \overline{\beta}(A)], & \text{else.} \end{cases} \tag{3.3}$$

Our next goal is to identify coefficient sequences $A = (A_k)_{k\in\mathbb{Z}}$ yielding invariance or continuity properties for the dichotomy spectrum $\Sigma(A)$ and its subsets. To this end, addressing invariance first, let us introduce ambient perturbation classes:

– The set of all matrix sequences commuting with A is defined as

$$\mathcal{C}(A) := \left\{ B \in \mathcal{L}^\infty(\mathbb{C}^d) :\ A_{k+1}B_k = B_{k+1}A_k \quad \text{for all } k \in \mathbb{Z} \right\}.$$

One has $\mathcal{C}(0) = \mathcal{L}^\infty(\mathbb{C}^d)$ and moreover $\mathcal{C}(\lambda I)$, $\lambda \in \mathbb{C}\setminus\{0\}$, consists of constant sequences in $L(\mathbb{C}^d)$. It is not hard to see that $\mathcal{C}(A)$ is a linear space over $\mathbb{C}$ containing A and thus $1 \le \dim\mathcal{C}(A)$. On the other hand, every element of $\mathcal{C}(A)$ is a solution to the linear difference eqn. $X_{k+1} = A_{k+1}X_kA_k^{-1}$ in $L(\mathbb{C}^d)$, or equivalently of

$$X_{k+1} = \tilde{A}_kX_k \tag{3.4}$$

with the linear operator $\tilde{A}_k \in L(L(\mathbb{C}^d))$, $\tilde{A}_kX := A_{k+1}XA_k^{-1}$. This in turn yields the estimate $\dim\mathcal{C}(A) \le \dim L(\mathbb{C}^d) = d^2$ and accordingly $\mathcal{C}(A)$ is finite-dimensional. Since a nontrivial bounded solution exists in form of $(A_k)_{k\in\mathbb{Z}}$, the linear difference eqn. (3.4) cannot have an ED and thus $1 \in \Sigma(\tilde{A})$.

- Given a subset $\mathcal{X} \subseteq \mathcal{L}^\infty(\mathbb{C}^d)$ we furthermore introduce the set

$$\mathcal{E}\mathcal{X} := \left\{X + K \in \mathcal{L}^\infty(\mathbb{C}^d) : X \in \mathcal{X},\, K \in \mathcal{L}^\infty(\mathbb{C}^d) \text{ with } \lim_{k\to\pm\infty} K_k = 0\right\}$$

  of compact perturbations to sequences in $\mathcal{X}$. One has $\mathcal{X} \subseteq \mathcal{E}\mathcal{X}$ and elements of $\mathcal{E}\mathcal{X}$ are said to be *essentially in* $\mathcal{X}$. For instance, $\mathcal{E}\{0\}$ consists of all matrix sequences with two-sided limit 0, or it is $\mathcal{E}\mathcal{L}^\infty(\mathbb{C}^d) = \mathcal{L}^\infty(\mathbb{C}^d)$.
- The matrix sequences from

$$\mathcal{Q}(A) := \left\{Q \in \mathcal{C}(A) : \overline{\beta}(Q) = 0\right\}$$

  commute with A and have vanishing Bohl exponents. As an example, the set $\mathcal{Q}(0)$ contains all sequences in $\mathcal{L}^\infty(\mathbb{C}^d)$ with Bohl exponent 0.

Indeed it is $\mathcal{E}\{0\} \subset \mathcal{Q}(0)$, but the inclusion can be strict:

Example 3.2 In the Banach algebra $\mathcal{A} = \mathbb{R}$ define the sequence $(a_k)_{k\in\mathbb{Z}}$ as

$$a_k := \begin{cases} 1, & k = 0 \text{ or } \log_{10}|k| \in \mathbb{N}_0, \\ \frac{1}{|k|}, & \text{else}. \end{cases}$$

Although it satisfies the limit relation $\limsup_{k\to\pm\infty} a_k = 1$, nonetheless $\overline{\beta}(a) = 0$ holds, because the values of $|a_k|$ become arbitrarily small on increasingly larger discrete intervals. Consequently, we have $a \in \mathcal{Q}(0)$, but $a \notin \mathcal{E}\{0\}$.

In the following, we often make use of the convenient abbreviation $X^{(*)}$, where consistently either the symbol X or X^* is meant. On this basis, for every $p \in \mathbb{N}$ certain increasingly larger subsets of $\mathcal{L}^\infty(\mathbb{C}^d)$ matter, which rely on the definition

$$T \geq S \quad :\Leftrightarrow \quad T - S \text{ is positive semi-definite Hermitian.} \tag{3.5}$$

3.1 The classes $\mathcal{H}_p^{(*)}(\mathbb{C}^d)$

With the transition matrix $\Phi(k,l)$ of (Δ_A) defined in (2.3), we introduce

$$\begin{aligned}\mathcal{H}_p(\mathbb{C}^d) := \Big\{A \in \mathcal{L}^\infty(\mathbb{C}^d) : &\Phi(k+2p,k+p)^*\Phi(k+2p,k+p) \\ &\geq \Phi(k+p,k)\Phi(k+p,k)^* \text{ for all } k \in \mathbb{Z}\Big\}, \\ \mathcal{H}_p^*(\mathbb{C}^d) := \Big\{A \in \mathcal{L}^\infty(\mathbb{C}^d) : &\Phi(k+p,k)\Phi(k+p,k)^* \\ &\geq \Phi(k+2p,k+p)^*\Phi(k+2p,k+p) \text{ for all } k \in \mathbb{Z}\Big\}.\end{aligned}$$

The sets $\mathcal{H}_p^{(*)}(\mathbb{C}^d)$ are topologically closed and cover several examples:

Example 3.3 (scalar equations) For $d = 1$ it is $L(\mathbb{C}^1) = \mathbb{C}$ and commutativity yields

$$\begin{aligned}\mathcal{H}_p(\mathbb{C}) &= \left\{a \in \ell^\infty(\mathbb{C}) : \prod_{j=k+p}^{k+2p-1} |a_j| \geq \prod_{j=k}^{k+p-1} |a_j| \quad \text{for all } k \in \mathbb{Z}\right\}, \\ \mathcal{H}_p^*(\mathbb{C}) &= \left\{a \in \ell^\infty(\mathbb{C}) : \prod_{j=k}^{k+p-1} |a_j| \geq \prod_{j=k+p}^{k+2p-1} |a_j| \quad \text{for all } k \in \mathbb{Z}\right\}.\end{aligned} \tag{3.6}$$

Thus, each $a \in \mathcal{H}_p^{(*)}(\mathbb{C})$ is determined by the following fact: Consecutive geometric means over the reals $|a_k|, \dots, |a_{k+p-1}|$ and $|a_{k+p}|, \dots, |a_{k+2p-1}|$ preserve their order for all $k \in \mathbb{Z}$. The characterization (3.6) yields that $a \in \mathcal{H}_p^{(*)}(\mathbb{C})$ holds if and only if $|a| \in \mathcal{H}_p^{(*)}(\mathbb{C})$. Moreover, one has $\mathcal{H}_1^{(*)}(\mathbb{C}) \subseteq \mathcal{H}_p^{(*)}(\mathbb{C})$. Note that (3.6) extends to the diagonal elements of diagonal matrix sequences in $\mathcal{H}_p^{(*)}(\mathbb{C}^d)$.

Example 3.4 (periodic equations) If (Δ_A) is a p-periodic difference equation, whose transition matrix satisfies the normality assumption

$$\Phi(k+p,k)^*\Phi(k+p,k) = \Phi(k+p,k)\Phi(k+p,k)^* \quad \text{for all } k \in \mathbb{Z},$$

then $A \in \mathcal{H}_p(\mathbb{C}^d) \cap \mathcal{H}_p^*(\mathbb{C}^d)$. In particular, the intersection $\mathcal{H}_1(\mathbb{C}^d) \cap \mathcal{H}_1^*(\mathbb{C}^d)$ contains all coefficient sequences consisting of normal matrices and especially incorporates such autonomous difference equations.

Example 3.5 (unitary equations) Difference eqns. (Δ_A) with unitary coefficient matrices $A_k^* = A_k^{-1}$, $k \in \mathbb{Z}$, are contained in the intersection $\bigcap_{p \in \mathbb{N}} (\mathcal{H}_p(\mathbb{C}^d) \cap \mathcal{H}_p^*(\mathbb{C}^d))$.

To tackle invariance properties for $\mathcal{H}_p^{(*)}(\mathbb{C}^d)$ and further sets, we remind the reader that two difference eqns. (Δ_A) and (Δ_B) are denoted as *kinematically similar*, provided there exists a sequence $(\Lambda_k)_{k \in \mathbb{Z}}$ (called *Lyapunov transformation*) in $\mathbb{C}^{d \times d}$ such that beyond $\Lambda_{k+1} B_k = A_k \Lambda_k$ also

$$\Lambda_k \in GL(\mathbb{C}^d) \quad \text{for all } k \in \mathbb{Z}, \qquad \sup_{k \in \mathbb{Z}} \max \left\{ |\Lambda_k|, \left|\Lambda_k^{-1}\right| \right\} < \infty \tag{3.7}$$

holds. In this case we write $A \simeq_\Lambda B$ and point out that kinematic similarity defines an equivalence relation on the space $\mathcal{L}^\infty(\mathbb{C}^d)$ of all difference eqns. (Δ_A).

Proposition 3.6 *Assume that $A \in \mathcal{H}_p^{(*)}(\mathbb{C}^d)$.*

(a) If $a \in \mathcal{H}_p^{(*)}(\mathbb{C})$, then $aA \in \mathcal{H}_p^{(*)}(\mathbb{C}^d)$.
(b) If $A \simeq_U B$ and U consists of unitary matrices $U_k \in L(\mathbb{C}^d)$, then $B \in \mathcal{H}_p^{(*)}(\mathbb{C}^d)$.

Proof Due to Props. B.8(a) and B.9(a) our assumption means that the associated shifts satisfy $T_A \in H_p^{(*)}(\ell^2)$.

(a) Above all, if $A \in \mathcal{H}_p(\mathbb{C}^d)$, then for every fixed $k \in \mathbb{Z}$ one has

$$\begin{aligned}
\Psi_1 &:= \Phi_{aA}(k+2p,k+p)^*\Phi_{aA}(k+2p,k+p) \\
&= \underbrace{\Phi_{|a|^2}(k+2p,k+p)}_{=:\theta_1 \in \mathbb{R}} \underbrace{\Phi_A(k+2p,k+p)^*\Phi_A(k+2p,k+p)}_{=:\Theta_1}, \\
\Psi_2 &:= \Phi_{aA}(k+p,k)\Phi_{aA}(k+p,k)^* = \underbrace{\Phi_{|a|^2}(k+p,k)}_{=:\theta_2 \in \mathbb{R}} \underbrace{\Phi_A(k+p,k)\Phi_A(k+p,k)^*}_{=:\Theta_2},
\end{aligned}$$

where our assumptions ensure $\theta_1 \geq \theta_2$ and $\Theta_1 \geq \Theta_2$. Due to the evident relations $\theta_2 \geq 0$ and $\Theta_1 \geq 0$ this yields (cf. (A.5))

$$\Psi_1 - \Psi_2 = (\theta_1 - \theta_2)\Theta_1 + \theta_2(\Theta_1 - \Theta_2) \geq \theta_2(\Theta_1 - \Theta_2) \geq 0$$

and therefore $aA \in \mathcal{H}_p(\mathbb{C}^d)$. The proof in case $A \in \mathcal{H}_p^*(\mathbb{C}^d)$ follows accordingly.

(b) The assumption $A \simeq_U B$ means that the shifts $T_A, T_B \in L(\ell^2)$ are similar by virtue of the unitary multiplication operator $M_U \in L(\ell^2)$ (see Appendix B.1), i.e. $T_B = M_U^* T_A M_U$ and thus $T_B^p = M_U^* T_A^p M_U$. In case $A \in \mathcal{H}_p(\mathbb{C}^d)$ this implies

$$T_B^{*p} T_B^p = M_U^* T_A^{*p} T_A^p M_U \overset{(A.6)}{\geq} M_U^* T_A^p T_A^{*p} M_U = T_B^p T_B^{*p}$$

and Prop. B.8(a) ensures $B \in \mathcal{H}_p(\mathbb{C}^d)$. In the dual situation $A \in \mathcal{H}_p^*(\mathbb{C}^d)$ one makes use of Prop. B.9(a) and proceeds analogously to establish $B \in \mathcal{H}_p^*(\mathbb{C}^d)$. □

The elements of $\mathcal{H}_1^{(*)}(\mathbb{C}^d)$ have distinguished representatives w.r.t. kinematic similarity. Indeed they are equivalent to both-sided asymptotically autonomous equations, where the convergence is even monotone w.r.t. the relation (3.5).

Theorem 3.7 *If $A \in \mathcal{H}_1^{(*)}(\mathbb{C}^d)$, then $A \simeq_U B$ with an eqn. (Δ_B) satisfying:*

(a) The coefficient matrices $B_k \in GL(\mathbb{C}^d)$ are positive-semidefinite Hermitian and fulfill $B^- \leq B_k \leq B^+$ (if $A \in \mathcal{H}_1(\mathbb{C}^d)$) resp. $B^+ \leq B_k \leq B^-$ (if $A \in \mathcal{H}_1^*(\mathbb{C}^d)$) for all $k \in \mathbb{Z}$ with the existing limits

$$B^- := \lim_{k \to -\infty} B_k, \qquad B^+ := \lim_{k \to \infty} B_k. \tag{3.8}$$

(b) The corresponding Lyapunov transformation U consists of unitary matrices.

Proof Due to Prop. B.7 the shift operator $T_A \in L(\ell^2)$ is unitarily equivalent to a weighted shift $T_B \in L(\ell^2)$ by means of a multiplication operator $M_U \in L(\ell^2)$ with unitary $U_k \in L(\mathbb{C}^d)$ and positive-semidefinite Hermitian B_k. In particular, it is $B_k = U_{k+1}^* A_k U_k$ for all $k \in \mathbb{Z}$ and (Δ_A) is kinematically similar to (Δ_B). One has

$$0 \leq B_k \leq \sup_{n \in \mathbb{Z}} |B_n| \, \mathrm{id}_{\mathbb{C}^d} \quad \text{for all } k \in \mathbb{Z} \tag{3.9}$$

and we show the convergence assertions for B_k: Our assumption $A \in \mathcal{H}_1(\mathbb{C}^d)$ yields

$$B_{k+1}^* B_{k+1} - B_k B_k^* = U_{k+1}^* \left(A_{k+1}^* A_{k+1} - A_k A_k^* \right) U_{k+1} \geq 0 \quad \text{for all } k \in \mathbb{Z}$$

and thus $B_{k+1}^2 = B_{k+1}^* B_{k+1} \geq B_k B_k^* = B_k^2 \geq 0$ follows from (A.6). Thanks to (3.9) the Löwner-Heinz inequality (see [40]) applies and shows that $(B_k)_{k \in \mathbb{Z}}$ is bounded nondecreasing. Hence, the limits (3.8) exist with $0 \leq B^- \leq B^+$. In case $A \in \mathcal{H}_1^*(\mathbb{C}^d)$ one proceeds accordingly, since $(B_k)_{k \in \mathbb{Z}}$ is bounded and nonincreasing. □

3.2 The classes $\mathcal{A}_p^{(*)}(\mathbb{C}^d)$

Let us furthermore introduce the sets of matrix sequences

$$\begin{aligned}
\mathcal{A}_p(\mathbb{C}^d) := \Big\{ A \in \mathcal{L}^\infty(\mathbb{C}^d) : \; & \Phi(k+2p,k)^* \Phi(k+2p,k) \\
& \geq \left[\Phi(k+p,k)^* \Phi(k+p,k) \right]^2 \text{ for all } k \in \mathbb{Z} \Big\}, \\
\mathcal{A}_p^*(\mathbb{C}^d) := \Big\{ A \in \mathcal{L}^\infty(\mathbb{C}^d) : \; & \Phi(k+2p,k) \Phi(k+2p,k)^* \\
& \geq \left[\Phi(k+2p,k+p) \Phi(k+2p,k+p)^* \right]^2 \text{ for all } k \in \mathbb{Z} \Big\},
\end{aligned}$$

which inherit certain properties from the previous classes $\mathcal{H}_p^{(*)}(\mathbb{C}^d)$:

Proposition 3.8 *Assume that* $A \in \mathcal{A}_p^{(*)}(\mathbb{C}^d)$.

(a) If $a \in \mathcal{H}_p^{(*)}(\mathbb{C})$*, then* $aA \in \mathcal{A}_p^{(*)}(\mathbb{C}^d)$.
(b) If $A \simeq_U B$ *and* U *consists of unitary matrices* $U_k \in L(\mathbb{C}^d)$*, then* $B \in \mathcal{A}_p^{(*)}(\mathbb{C}^d)$.

Proof The assertions can be shown as in the proof of Prop. 3.6, but presently by means of Props. B.8(b) and B.9(b). □

The following result illustrates that a discrete dichotomy spectrum $\Sigma(A)$ is exceptional for coefficients in $\mathcal{A}_p(\mathbb{C}^d)$ or $\mathcal{A}_p^*(\mathbb{C}^d)$:

Proposition 3.9 *If* (Δ_A) *has discrete dichotomy spectrum and* $A \in \mathcal{A}_p^{(*)}(\mathbb{C}^d)$*, then*

$$\Phi(k+2p,k+p)^*\Phi(k+2p,k+p) = \Phi(k+p,k)\Phi(k+p,k)^* \quad \text{for all } k \in \mathbb{Z}. \tag{3.10}$$

Proof Given $A \in \mathcal{A}_p(\mathbb{C}^d)$ our Prop. B.8(b) establishes that $S_A := T_A^p$ is of class A. Thanks to Putnam's inequality $\|S_A^*S_A - S_AS_A^*\| \le \frac{1}{\pi}\lambda_2(\sigma(S_A))$ for class A operators (see [39, Cor. 3.2]) one obtains

$$S_A^*S_A = S_AS_A^*, \tag{3.11}$$

since the Lebesgue measure $\lambda_2(\sigma(S_A)) = \lambda_2(\sigma(T_A)^p)$ (cf. the spectral mapping theorem [37, p. 13, Thm. 34]) vanishes due to

$$\sigma(T_A) \overset{(3.2)}{=} \{e^{it}\lambda \in \mathbb{C} : t \in [0,2\pi), \lambda \in \Sigma(A)\}$$

with the finite dichotomy spectrum $\Sigma(A)$. From (3.11) and Prop. B.4(f) we readily deduce (3.10). Tackling the remaining case $A \in \mathcal{A}_p^*(\mathbb{C}^d)$, one sees that the adjoint S_A^* is of class A and by means of (A.3) the claim follows as above. □

3.3 The classes $\mathcal{P}_p^{(*)}(\mathbb{C}^d)$

We eventually investigate the classes

$$\begin{aligned}
\mathcal{P}_p(\mathbb{C}^d) := \Big\{ & A \in \mathcal{L}^\infty(\mathbb{C}^d) : \Phi(k+2p,k)^*\Phi(k+2p,k) \\
& - 2r\Phi(k+p,k)^*\Phi(k+p,k) + r^2\,\mathrm{id}_{\mathbb{C}^d} \ge 0 \text{ for all } k \in \mathbb{Z},\ r > 0 \Big\}, \\
\mathcal{P}_p^*(\mathbb{C}^d) := \Big\{ & A \in \mathcal{L}^\infty(\mathbb{C}^d) : \Phi(k+2p,k)\Phi(k+2p,k)^* \\
& - 2r\Phi(k+2p,k+p)\Phi(k+2p,k+p)^* + r^2\,\mathrm{id}_{\mathbb{C}^d} \ge 0 \text{ for all } k \in \mathbb{Z},\ r > 0 \Big\}
\end{aligned}$$

and obtain their subsequent invariance properties (cf. Props. 3.6 and 3.8):

Proposition 3.10 *Assume that* $A \in \mathcal{P}_p^{(*)}(\mathbb{C}^d)$.

(a) $\lambda A \in \mathcal{P}_p^{(*)}(\mathbb{C}^d)$ *for all* $\lambda \in \mathbb{C}$.
(b) If $A \simeq_U B$ *and* U *consists of unitary matrices* $U_k \in L(\mathbb{C}^d)$*, then* $B \in \mathcal{P}_p^{(*)}(\mathbb{C}^d)$.

Proof Assertion (a) immediately follows from the definition. In order to show (b) we observe that $\Phi_B(k,l) = U_k^*\Phi_A(k,l)U_l$ for all $k,l \in \mathbb{Z}$ implies the relations

$$\begin{aligned}\Phi_B(k+2p,k)^*\Phi_B(k+2p,k) &= U_k^*\Phi_A(k+2p,k)^*\Phi_A(k+2p,k)U_k,\\ \Phi_B(k+p,k)^*\Phi_B(k+p,k) &= U_k^*\Phi_A(k+p,k)^*\Phi_A(k+p,k)U_k\end{aligned}$$

for all $k \in \mathbb{Z}$ and therefore (A.6) ensures the claim. □

The sets $\mathcal{P}_p^{(*)}(\mathbb{C}^d)$ do not only contain the classes $\mathcal{H}_p^{(*)}(\mathbb{C}^d)$ and $\mathcal{A}_p^{(*)}(\mathbb{C}^d)$, they are additionally nested in the following sense:

Theorem 3.11 *One has the inclusions*

$$\mathcal{H}_p^{(*)}(\mathbb{C}^d) \subseteq \mathcal{A}_p^{(*)}(\mathbb{C}^d) \subseteq \mathcal{P}_p^{(*)}(\mathbb{C}^d) \subseteq \mathcal{P}_{np}^{(*)}(\mathbb{C}^d) \quad \textit{for all } n,p \in \mathbb{N}. \tag{3.12}$$

Proof Given $p \in \mathbb{N}$ our Props. B.8 and B.9 yield the characterizations

$$\begin{aligned}\mathcal{H}_p^{(*)}(\mathbb{C}^d) &= \left\{A \in \mathcal{L}^\infty(\mathbb{C}^d) : T_A^{(*)} \in H_p(\ell^2)\right\},\\ \mathcal{A}_p^{(*)}(\mathbb{C}^d) &= \left\{A \in \mathcal{L}^\infty(\mathbb{C}^d) : T_A^{(*)} \in A_p(\ell^2)\right\},\\ \mathcal{P}_p^{(*)}(\mathbb{C}^d) &= \left\{A \in \mathcal{L}^\infty(\mathbb{C}^d) : T_A^{(*)} \in P_p(\ell^2)\right\}\end{aligned}$$

and so the first two inclusions result from [34, p. 74, (2.57)], since every hyponormal operator is class A, and in turn every class A operator is paranormal. The remaining inclusion $\mathcal{P}_p^{(*)}(\mathbb{C}^d) \subseteq \mathcal{P}_{np}^{(*)}(\mathbb{C}^d)$ for all $n \in \mathbb{N}$ is from [21, Thm. 1]. □

Example 3.12 (scalar equations) In the scalar case $d = 1$ it is

$$\mathcal{H}_p^{(*)}(\mathbb{C}) = \mathcal{A}_p^{(*)}(\mathbb{C}) = \mathcal{P}_p^{(*)}(\mathbb{C}) \quad \text{for all } p \in \mathbb{N} \tag{3.13}$$

and the characterization (3.6) from Ex. 3.3 holds. We show $\mathcal{P}_{p+1}(\mathbb{C}) \setminus \mathcal{P}_p(\mathbb{C}) \neq \emptyset$ for each $p \in \mathbb{N}$: Indeed, consider a complex sequence $(a_k)_{k\in\mathbb{Z}}$ with

$$|a_k| = \begin{cases}\alpha_+, & k \mod (p+1) \neq 0,\ k > 0,\\ \alpha, & k \mod (p+1) = 0,\\ \alpha_-, & k \mod (p+1) \neq 0,\ k < 0\end{cases}$$

and reals $\alpha_+, \alpha, \alpha_- > 0$. The corresponding scalar difference eqn. $x_{k+1} = a_k x_k$ is asymptotically periodic (with period $p+1$) and consequently has the dichotomy spectrum (cf. [45, Ex. 2.6(4)])

$$\Sigma(a) = \left[\sqrt[p+1]{\alpha \min\{\alpha_-, \alpha_+\}^p},\ \sqrt[p+1]{\alpha \max\{\alpha_-, \alpha_+\}^p}\right].$$

Due to the $p+1$-periodicity of a on the discrete intervals $\mathbb{Z}_0^-, \mathbb{Z}_0^+$ one obtains

$$\prod_{j=k+p+1}^{k+2p+1} |a_j| = \prod_{j=k}^{k+p} |a_j| = \alpha \begin{cases}\alpha_-^p, & k \leq -1-2p,\\ \alpha_+^p, & 0 \leq k\end{cases}$$

and it remains to check the criterion (3.6) for $-1-2p < k < 0$. This requires the inequalities

$$\alpha_+\alpha\alpha_-^{p-1} \geq \alpha\alpha_-^p, \qquad \alpha_+^p\alpha \geq \alpha_-^p\alpha, \qquad \alpha_+^p\alpha \geq \alpha_-\alpha\alpha_+^{p-1}$$

to be fulfilled, which are all equivalent to $\alpha_+ \geq \alpha_-$. Hence, we arrive at

$$a \in \mathcal{H}_{p+1}(\mathbb{C}) \quad \Leftrightarrow \quad \alpha_+ \geq \alpha_-$$

and dually, $a \in \mathcal{H}^*_{p+1}(\mathbb{C}) \Leftrightarrow \alpha_+ \leq \alpha_-$. On the other hand, in case $k = 1 - p$ it is

$$\prod_{j=1}^{p} |a_j| = \alpha_+^p, \qquad \prod_{j=1-p}^{0} |a_j| = \alpha_-^{p-1}\alpha$$

and for $\alpha > \alpha_- \left(\frac{\alpha_+}{\alpha_-}\right)^p$ the condition (3.6) yields $a \notin \mathcal{H}_p(\mathbb{C})$. Thus, for sufficiently large values of α we derive from (3.13) that $a \in \mathcal{P}_{p+1}(\mathbb{C}) \setminus \mathcal{P}_p(\mathbb{C})$ for $\alpha_+ \geq \alpha_-$. Analogously, one finds small $\alpha > 0$ such that $a \in \mathcal{P}^*_{p+1}(\mathbb{C}) \setminus \mathcal{P}^*_p(\mathbb{C})$ when $\alpha_+ \leq \alpha_-$.

In order to finally illustrate that the first two inclusions in (3.12) are strict in general, let us employ an example adopted from [30, Problem 9.14]:

Example 3.13 We consider coefficient sequences $A \in \mathcal{L}^\infty(\mathbb{C}^2)$ of the form

$$A_k := \begin{cases} A_+, & k \geq 0, \\ A_-, & k < 0 \end{cases} \quad \text{for all } k \in \mathbb{Z}.$$

(1) For the particular choice $A_- := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $A_+ := \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)^{\frac{1}{2}}$ the matrix expression $A_-A_+^2A_- - A_-^4 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is positive semi-definite, whereas the difference $A_+^2 - A_-^2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ is not since it has the eigenvalues ± 1. Consequently, we arrive at $A \in \mathcal{A}_1(\mathbb{C}^2)$, but $A \notin \mathcal{H}_1(\mathbb{C}^2)$.

(2) For $A_- := \frac{1}{2}\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $A_+ := \left(\begin{smallmatrix} 10 & -3 \\ -3 & 1 \end{smallmatrix}\right)^{\frac{1}{2}}$ it is $A_-A_+^2A_- - A_-^4 = \frac{1}{16}\left(\begin{smallmatrix} 82 & 27 \\ 27 & 7 \end{smallmatrix}\right)$, which is not positive semi-definite due to the eigenvalue $\frac{89-3\sqrt{949}}{32} < 0$; therefore $A \notin \mathcal{A}_1(\mathbb{C}^2)$. In order to establish $A \in \mathcal{P}_1(\mathbb{C}^2)$, we investigate

$$X(r) := A_-A_+^2A_- - 2rA_-^2 + r^2\,\mathrm{id}_{\mathbb{C}^2} = \frac{1}{4}\begin{pmatrix} 4r^2 - 10r + 29 & 12 - 6r \\ 12 - 6r & 4r^2 - 4r + 5 \end{pmatrix}.$$

Thanks to $4r^2 - 10r + 29 > 0$ and $4r^2 - 4r + 5 > 0$, it follows from Sylvester's criterion [25, p. 439, Thm. 7.2.5(a)] that $X(r)$ is positive semi-definite for all $r > 0$, because $\det X(r) = \frac{1}{16}(16r^4 - 56r^3 + 140r^2 - 22r + 1) > 0$ and thus, $A \in \mathcal{P}_1(\mathbb{C}^2)$.

4 Invariance of the dichotomy spectrum

This section investigates two weakenings of the classical kinematic similarity as a property leaving the dichotomy spectra invariant (cf. [44, Cor. 4.33]). For sequences $A, B, \Lambda \in \mathcal{L}^\infty(\mathbb{C}^d)$ they are based on the further $L(\mathbb{C}^d)$-valued sequences

$$\Delta^n_{A,B}\Lambda := \sum_{j=0}^{n} \binom{n}{j}(-1)^j \Phi_A(\cdot, \cdot - n + j)\Lambda_{\cdot - n + j}\Phi_B(\cdot - n + j, \cdot - n) : \mathbb{Z} \to L(\mathbb{C}^d)$$

for all $n \in \mathbb{N}_0$. Under commutativity assumptions this expression simplifies:

– If $B \in \mathcal{C}(A)$ and $\Lambda_{k+1}A_k \equiv A_k\Lambda_k$ on $\mathbb{Z}$ it is

$$(\Delta^n_{A,B}\Lambda)_k = \Lambda_k\Phi_{A-B}(k,k-n) \quad \text{for all } k \in \mathbb{Z},\ n \in \mathbb{N}_0.$$

– If $A \in \mathcal{C}(B)$ and $\Lambda_{k+1}B_k \equiv B_k\Lambda_k$ on $\mathbb{Z}$ it is

$$(\Delta^n_{A,B}\Lambda)_k = \Phi_{A-B}(k,k-n)\Lambda_{k-n} \quad \text{for all } k \in \mathbb{Z},\ n \in \mathbb{N}_0.$$

The linear difference eqns. (Δ_A) and (Δ_B) are called *weakly similar*, in symbols

$$A \sim B \qquad (\text{more specific } A \sim_\Lambda B),$$

provided there exists a nonzero sequence $(\Lambda_k)_{k\in\mathbb{Z}}$ in $L(\mathbb{C}^d)$ fulfilling

$$\sup_{k\in\mathbb{Z}} |\Lambda_k| < \infty, \qquad\qquad \limsup_{n\to\infty} \sqrt[n]{\sup_{k\in\mathbb{Z}} \left|(\Delta^n_{A,B}\Lambda)_k\right|} = 0.$$

For our purpose the above limit relation is to be satisfied only in two special cases:

Remark 4.1 (1) The linear conjugacy relation

$$A_k\Lambda_k \equiv \Lambda_{k+1}B_k \quad \text{on } \mathbb{Z} \tag{4.1}$$

shows $\Phi_A(k,l)\Lambda_l = \Lambda_k\Phi_B(k,l)$, $k,l \in \mathbb{Z}$, and therefore $\Delta^n_{A,B}\Lambda = 0$ for all $n \in \mathbb{N}$ holds due to the trivial identity $\sum_{j=0}^n \binom{n}{j}(-1)^j = (1-1)^n = 0$. Hence, (4.1) implies weak similarity and if in addition also (3.7) holds, then the difference eqns. (Δ_A) and (Δ_B) are even kinematically similar.

(2) For $B \in \mathcal{C}(A)$ and under one of the conditions

$$A_k\Lambda_k \equiv \Lambda_{k+1}A_k \quad \text{or} \quad B_k\Lambda_k \equiv \Lambda_{k+1}B_k \quad \text{on } \mathbb{Z} \tag{4.2}$$

(which particularly hold for the identity transformation $\Lambda = I$) it results from Cor. B.6 that (Δ_A) and (Δ_B) are weakly similar, if $\overline{\beta}(A-B) = 0$ is fulfilled.

Dichotomy spectra of weakly similar equations are not disjoint; one even has

Proposition 4.2 *Difference eqns. (Δ_A) and (Δ_B) with $A \sim_\Lambda B$ satisfy:*

(a) $\Sigma_\pi(A) \cap \Sigma_s(B) \neq \emptyset$ and in case $\Lambda_k \in GL(\mathbb{C}^d)$, $k \in \mathbb{Z}$, every spectral interval of $\Sigma_s(A)$ touches $\Sigma_\pi(B)$.

(b) If (3.7) holds, then $\Sigma_s(A) \subseteq \Sigma_s(B)$.

Proof Since (Δ_A) and (Δ_B) are weakly similar with $A \sim_\Lambda B$, our Lemma B.5 ensures that the weighted shifts T_A and T_B intertwine asymptotically by means of the multiplication operator M_Λ defined in (B.1).

(a) Here [32, Prop. 2.3] yields $\sigma_\pi(T_A) \cap \sigma_s(T_B) \neq \emptyset$ and hence

$$\Sigma_\pi(A) \cap \Sigma_s(B) \overset{(3.2)}{=} \sigma_\pi(T_A) \cap \sigma_s(T_B) \cap \mathbb{R}^+ \neq \emptyset,$$

because both spectra $\sigma_\pi(T_A), \sigma_s(T_B)$ are rotationally invariant (cf. Prop. 3.1). For invertible matrices Λ_k, $k \in \mathbb{Z}$, the multiplication operator M_Λ is one-to-one and the remaining assertion results analogously from [33, p. 265, Cor. 3.5.8].

(b) Due to (3.7) the operator $M_\Lambda \in L(\ell^2)$ is onto and [32, Prop. 3.1] yields the inclusion $\sigma_s(T_A) \subseteq \sigma_s(T_B)$. This gives

$$\Sigma_s(A) \overset{(3.2)}{=} \sigma_s(T_A) \cap \mathbb{R}^+ \subseteq \sigma_s(T_B) \cap \mathbb{R}^+ \overset{(3.2)}{=} \Sigma_s(B)$$

and therefore the claim. □

In order to formulate our next result, let us assume that $(\varDelta_A)$ has the Lyapunov spectrum and filtration (cf. [45, Sect. 2.1] for details)

$$\left\{\lambda_1^+,\ldots,\lambda_n^+\right\}, \qquad 0=:W_0\subset W_1\subset\ldots\subset W_n=\mathbb{C}^d \quad \text{with } n\leq d$$

with forward Lyapunov exponents ordered according to $0<\lambda_1^+<\ldots<\lambda_n^+$.

Proposition 4.3 *Difference eqns. $(\varDelta_A)$ and $(\varDelta_B)$ with $A\sim_\varLambda B$ and (3.7) satisfy the inclusion $\varSigma(A)\subseteq\varSigma(B)$, provided one of the following assumption holds:*

(i) One has the estimate

$$1\leq\lambda_j^+\limsup_{k\to\infty}\sqrt[k]{|\varPhi_A(-k,\kappa)x|} \quad \text{for all } x\in W_j\setminus W_{j-1},\ 1\leq j\leq n. \tag{4.3}$$

(ii) $B\sim_{\bar\varLambda}A$ with a sequence of matrices $\bar\varLambda_k\in GL(\mathbb{C}^d)$ for all $k\in\mathbb{Z}$.

The assumption (i) for instance holds in autonomous or periodic equations.

Proof Referring to our assumption and Lemma B.5, the onto multiplication operator $M_\varLambda\in L(\ell^2)$ asymptotically intertwines T_A and T_B. On the one hand, under assumption (i) it was shown in [45, Proof of Thm. 4.8] that T_A has the SVEP and thus [32, Prop. 3.1] implies $\sigma(T_A)\subseteq\sigma(T_B)$. On the other hand, assumption (ii) ensures a one-to-one $M_{\bar\varLambda}$, which asymptotically intertwines T_B and T_A; from [32, Cor. 3.2] one also gets $\sigma(T_A)\subseteq\sigma(T_B)$. In both cases we conclude

$$\varSigma(A)\overset{(3.2)}{=}\sigma(T_A)\cap\mathbb{R}^+\subseteq\sigma(T_B)\cap\mathbb{R}^+\overset{(3.2)}{=}\varSigma(B)$$

and this was our claim. □

As second approach to weaken the notion of kinematic similarity, we say that $(\varDelta_A)$ and $(\varDelta_B)$ are *asymptotically similar*, in symbols

$$A\approx B \qquad (\text{more detailled } A\approx_\varLambda B),$$

if there exists a sequence $(\varLambda_k)_{k\in\mathbb{Z}}$ satisfying both (3.7) and the limit relations

$$\limsup_{n\to\infty}\sqrt[n]{\sup_{k\in\mathbb{Z}}\left|(\varDelta^n_{A,B}\varLambda)_k\right|}=0, \qquad \limsup_{n\to\infty}\sqrt[n]{\sup_{k\in\mathbb{Z}}\left|(\varDelta^n_{B,A}\varLambda^{-1})_k\right|}=0. \tag{4.4}$$

Remark 4.4 (1) Asymptotic similarity is an equivalence relation on the space of linear difference eqns. $(\varDelta_A)$ in $\mathbb{C}^d$, i.e. on $\mathcal{L}^\infty(\mathbb{C}^d)$.

(2) Kinematic similarity implies asymptotic similarity, which is in turn sufficient for weak similarity. Indeed, if (4.1) and (3.7) hold, then $\varLambda_{k+1}^{-1}A_k=B_k\varLambda_k^{-1}$ yields $\varPhi_B(k,l)\varLambda_l^{-1}=\varLambda_k^{-1}\varPhi_A(k,l)$, $k,l\in\mathbb{Z}$, and consequently $\varDelta^n_{B,A}\varLambda^{-1}=0$ for all $n\in\mathbb{N}$. Hence it follows that both limit relations in (4.4) are satisfied.

(3) For difference eqns. $(\varDelta_B)$ with $B\in\mathcal{C}(A)$ and sequences $\varLambda\in\mathcal{L}^\infty(\mathbb{C}^d)$ such that beyond (3.7) also (4.2) holds, it results as in Rem. 4.1(2) that $\overline{\beta}(A-B)=0$ implies asymptotic similarity. The specific situation $\varLambda=I$ is going to be tackled in the proof of our subsequent Prop. 4.6.

Theorem 4.5 *If $A\approx B$, then $\varSigma_\alpha(A)=\varSigma_\alpha(B)$ for all $\alpha\in\{a,s,\pi\}$.*

Proof Keep $\alpha \in \{a, s, \pi\}$ fixed and suppose that (Δ_A) and (Δ_B) are asymptotically similar by means of $A \approx_\Lambda B$. With the spectral radius r_{T_A,T_B} defined in (A.2) these conditions particularly mean

- the boundedness assumption (3.7) holds and therefore $M_\Lambda \in GL(\ell^2)$
- $r_{T_A,T_B}(M_\Lambda) = 0$, and with Lemma B.5 the multiplication operator M_Λ asymptotically intertwines T_A and T_B
- $r_{T_B,T_A}(M_\Lambda^{-1}) = 0$ and anew Lemma B.5 combined with Prop. B.1(a) establishes M_Λ^{-1} as an intertwiner between T_B and T_A.

This yields asymptotic similarity of the shifts T_A and T_B. Thus, [32, Thm. 3.5] implies $\sigma_\alpha(T_A) = \sigma_\alpha(T_B)$ and (3.2) yields the assertion as above. □

For difference equations on the half line it is common knowledge that their dichotomy spectrum is not affected by linear-homogeneous perturbations decaying to 0 as $k \to \infty$ (cf. [44, Cor. 3.26]). Yet, this situation changes on the entire line and we refer to [42] for a corresponding example. With perturbations having vanishing Bohl exponents and fulfilling commutativity assumptions, one still obtains

Proposition 4.6 *If $\alpha \in \{a, s, \pi\}$, then $\Sigma_\alpha(A + Q) = \Sigma_\alpha(A)$ for all $Q \in \mathcal{Q}(A)$.*

Proof Keep $\alpha \in \{a, s, \pi\}$ fixed and define $B := A + Q$. Due to $Q \in \mathcal{Q}(A)$ one has both $A_{k+1}B_k = B_{k+1}A_k$, as well as $\overline{\beta}(B - A) = 0$. First, Cor. B.6 implies $r_{T_A,T_B}(I) = 0$ and $r_{T_B,T_A}(I) = 0$. Second, Lemma B.5 ensures that $M_I = \mathrm{id}_{\ell^2}$ asymptotically intertwines T_A and T_B, and at the same time T_B and T_A. It follows that (Δ_A) and (Δ_B) are asymptotically similar and Thm. 4.5 yields the claim. □

The applicability of Prop. 4.6 is strongly limited by the commutativity assumption in the definition of $\mathcal{Q}(A)$ — a subset of a finite-dimensional space. Nevertheless, it applies to Σ_F, Σ_{F_0} under the limit relation $\lim_{k\to\pm\infty} Q_k = 0$. Concerning the other spectra we address structured perturbations of upper-triangular equations. Different from the half line situation their dichotomy spectrum is not necessarily the union of the diagonal spectra (see [45, Ex. 2.7(2)]) and we have

Example 4.7 Suppose that $(a_k)_{k\in\mathbb{Z}}$, $(b_k)_{k\in\mathbb{Z}}$ and $(c_k)_{k\in\mathbb{Z}}$ are complex sequences with $b_k \neq 0$ for all $k \in \mathbb{Z}$. We consider matrix sequences

$$A_k := \begin{pmatrix} a_k & c_k \\ 0 & b_k \end{pmatrix}, \qquad Q_k := \begin{pmatrix} 0 & q_k \\ 0 & 0 \end{pmatrix}$$

in $\mathbb{C}^{2\times 2}$ and arrive at the commutativity relation

$$A_{k+1}Q_k - Q_kA_k = \begin{pmatrix} 0 & a_{k+1}q_k - b_kq_{k+1} \\ 0 & 0 \end{pmatrix} \quad \text{for all } k \in \mathbb{Z}.$$

Therefore, in order to obtain $Q \in \mathcal{Q}(A)$ the complex sequence $(q_k)_{k\in\mathbb{Z}}$ has to be an entire bounded solution of the linear scalar difference eqn. $x_{k+1} = \frac{a_{k+1}}{b_k}x_k$ satisfying $\overline{\beta}(q) = 0$. Such a sequence can be constructed under the assumption

$$\gamma^\pm := \lim_{k\to\pm\infty} \left|\tfrac{a_{k+1}}{b_k}\right| \quad \text{with positive reals } \gamma^+ < 1 < \gamma^-.$$

This implies $\lim_{k\to\pm\infty} q_k = 0$ (even exponentially) and so $\overline{\beta}(q) = 0$. By Prop. 4.6 these conditions lead to $\Sigma_\alpha(A) = \Sigma_\alpha(A + \rho Q)$ for all $\alpha \in \{a, s, \pi\}$ and $\rho \in \mathbb{C}$.

5 Continuity of the dichotomy spectrum

Given a difference eqn. (Δ_A) it is obvious from (2.4) that the "closure"

$$\bar{\Sigma}(A) := \Sigma(A) \cup \{0\}$$

of its dichotomy spectrum is a compact nonempty subset of $\mathbb{R}$; this property is shared by the accordingly defined subspectra $\Sigma_\alpha(A)$ for $\alpha \in \{s, F, F_0, \pi\}$ (see [44]). Both for the sake of nonautonomous bifurcation theory, as well as to verify numerical approximations, it is an interesting problem to study continuity properties of the mappings $A \mapsto \bar{\Sigma}_\alpha(A)$. Thereto, let $K(X)$ denote the family of compact subsets of a metric space X. Equipped with the *Hausdorff distance* $h : K(X) \times K(X) \to \mathbb{R}$,

$$h(M_1, M_2) := \max\{d(M_1, M_2), d(M_2, M_1)\}, \quad d(M_1, M_2) := \sup_{x \in M_1} \operatorname{dist}(x, M_2),$$

the pair $(K(X), h)$ is a metric space (cf. [9, p. 37, Thm. 1]).

Example 5.1 (periodic equations) For a p-periodic difference eqn. (Δ_A) one has

$$\Sigma(A) = \left\{ \sqrt[p]{|\lambda|} : \lambda \in \sigma(\Phi(p, 0)) \right\}$$

and since the eigenvalues of the period matrix $\Phi(p, 0)$ depend continuously on the coefficients $A_0, \ldots, A_{p-1} \in GL(\mathbb{C}^d)$, also the dichotomy spectrum is continuous in the class of p-periodic equations in $\mathbb{C}^d$ (see [25, p. 122, Thm. 2.4.9.2]).

For more general time dependencies such a regular behavior cannot be expected and $\bar{\Sigma} : \mathcal{L}^\infty(\mathbb{C}^d) \to K(\mathbb{R})$ is only upper-semicontinuous (cf. [42, Cor. 4]), i.e.

$$\lim_{B \to A} d(\bar{\Sigma}(B), \bar{\Sigma}(A)) = 0 \quad \text{for all } A \in \mathcal{L}^\infty(\mathbb{C}^d)$$

holds; explicit examples with suddenly shrinking spectrum can be found in Ex. 5.10 below or [42, Ex. 5]. The following accentuation ensures that even single spectral intervals behave upper-semicontinuously.

Theorem 5.2 *Keep $\alpha \in \{a, F\}$ fixed. Suppose a sequence $(A^n)_{n \in \mathbb{N}}$ in $\mathcal{L}^\infty(\mathbb{C}^d)$ converges to A. If $\Sigma_\alpha(A)$ is of the form (2.4), then for every $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that*

$$\Sigma_\alpha(A^n) \subseteq \begin{cases} (0, \beta_m + \varepsilon] \\ [\alpha_m - \varepsilon, \beta_m + \varepsilon] \end{cases} \cup \bigcup_{i=1}^{m-1} [\alpha_i - \varepsilon, \beta_i + \varepsilon] \quad \textit{for all } n \geq N.$$

Proof For every spectral interval $J \subseteq \Sigma(A)$ the annulus $C := \{\lambda \in \mathbb{C} : |\lambda| \in J\}$ is a component of $\sigma(T_A)$ due to (3.2) and Prop. 3.1. Then [14, Lemma 1.5(a)] implies that each neighborhood $U \subseteq \mathbb{C}$ of C contains a component of $\sigma(T_{A^n})$, if n is sufficiently large. Again (3.2) yields the claim, since $J \subseteq \Sigma(A)$ was arbitrary.

In case $\alpha = F$ the assertion follows analogously using [14, Lemma 1.5(b)]. □

In order to proceed to continuity properties for Σ, we rest upon (3.2) and follow an approach based on the weighted shifts $T_A \in L(\ell^2)$ defined in (3.1):

Proposition 5.3 *Keep $\alpha \in \{a, s, F_0, F, \pi\}$ fixed. If $\sigma_\alpha : L(\ell^2) \to K(\mathbb{C})$ is continuous at T_A, then $\bar{\Sigma}_\alpha : \mathcal{L}^\infty(\mathbb{C}^d) \to K(\mathbb{R})$ is continuous at A.*

Proof Given $\alpha \in \{a, s, F_0, F, \pi\}$, let $(A^n)_{n\in\mathbb{N}}$ converge to A in $\mathcal{L}^\infty(\mathbb{C}^d)$ and define $\bar{\sigma}_\alpha(T_A) := \sigma_\alpha(T_A) \cup \{0\}$. Thus, Lemma B.3 shows $\lim_{n\to\infty} T_{A^n} = T_A$ in $L(\ell^2)$ and

$$\lim_{n\to\infty} h(\sigma_\alpha(T_{A^n}), \sigma_\alpha(T_A)) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} h(\bar{\sigma}_\alpha(T_{A^n}), \bar{\sigma}_\alpha(T_A)) = 0.$$

Because the sets $\bar{\sigma}_\alpha(T_{A^n}), \bar{\sigma}_\alpha(T_A) \subset \mathbb{C}$ are rotationally symmetric w.r.t. 0 (for this, see Prop. 3.1), elementary geometrical considerations yield

$$\begin{aligned} \operatorname{dist}(x_n, \bar{\Sigma}_\alpha(A)) &= \inf_{y\in\bar{\Sigma}_\alpha(A)} |x_n - y| = \inf_{y\in\bar{\sigma}_\alpha(T_A)} |x_n - y| = \operatorname{dist}(x_n, \bar{\sigma}_\alpha(T_A)) \\ &\le \sup_{y_n\in\bar{\sigma}(T_{A^n})} \operatorname{dist}(y_n, \bar{\sigma}_\alpha(T_A)) \le h(\bar{\sigma}_\alpha(T_{A^n}), \bar{\sigma}_\alpha(T_A)) \end{aligned}$$

for all $x_n \in \bar{\Sigma}_\alpha(A^n)$ and by passing over to the least upper bound

$$d(\bar{\Sigma}_\alpha(A^n), \bar{\Sigma}_\alpha(A)) = \sup_{x_n\in\bar{\Sigma}_\alpha(A^n)} \operatorname{dist}(x_n, \bar{\Sigma}_\alpha(A)) \le h(\bar{\sigma}_\alpha(T_{A^n}), \bar{\sigma}_\alpha(T_A)) \xrightarrow[n\to\infty]{} 0.$$

A completely dual argument leads to

$$d(\bar{\Sigma}_\alpha(A), \bar{\Sigma}_\alpha(A^n)) = \sup_{x\in\bar{\Sigma}_\alpha(A)} \operatorname{dist}(x, \bar{\Sigma}_\alpha(A^n)) \le h(\bar{\sigma}_\alpha(T_A), \bar{\sigma}_\alpha(T_{A^n})) \xrightarrow[n\to\infty]{} 0$$

and consequently the assertion holds. □

To assess system classes guaranteeing continuity, let us briefly highlight the embracing abstract theory of bounded operators $L(X)$ defined on a (separable) Hilbert space X. On the one hand, the set-valued functions $\sigma, \sigma_F : L(X) \to K(\mathbb{C})$ are continuous on a dense G_δ-set, just as they are discontinuous on a dense F_σ-set (cf. [14]); thanks to [23, Thms. 2, 3] this erratic behavior even extends to further subspectra. On the other hand, [37, p. 57, Thm. 14] identifies the discontinuity points for $\sigma, \sigma_F, \sigma_s$ to be of first category, i.e. as meagre. The latter property ensures that eqns. (Δ_A) with discontinuous dichotomy spectra form a meagre set.

One manifestation of the upper-semicontinuity of the dichotomy spectrum from Thm. 5.2 is that a spectral interval shrinks to its boundary points. A condition to avoid this behavior is the following relation between the different spectra:

Theorem 5.4 *The dichotomy spectrum Σ is continuous at A, if beyond* (2.5) *one has*

$$\Sigma(A) = \overline{\Sigma_{F_0}(A) \setminus \Sigma_F(A)}. \tag{5.1}$$

Condition (5.1) rules out cases when the Weyl spectrum is smaller than $\Sigma(A)$ and a large Fredholm spectrum $\Sigma_F(A)$ compared to $\Sigma_{F_0}(A)$ occurs. We furthermore point out that (5.1) is not purely academic. Indeed, using Palmer's characterization of the operator S_γ from (3.1) to be Fredholm (see [44, Props. 4.9 and 4.12] in discrete time), the spectra $\Sigma_{F_0}(A)$ and $\Sigma_F(A)$ can actually be computed on a numerical basis using methods from [26] or [18].

Proof Let us abbreviate $\mathrm{id} = \mathrm{id}_{\ell^2}$. We apply the characterization [15, Thm. 3.3] of continuity points for the spectrum $\sigma : L(\ell^2) \to K(\mathbb{C})$ in form of

$$\sigma(T_A) = \overline{\bigcup_{1\le|n|\le\infty} \{\lambda \in \sigma(T_A) : T_A - \lambda\,\mathrm{id} \text{ is semi-Fredholm with index} = n\}}$$

$$\cup\overline{\{\lambda\in\mathbb{C}:\ \{\lambda\}\ \text{ is a component of }\sigma(T_A)\}}. \tag{5.2}$$

Thereto, we can derive from (3.3) that the spectrum $\sigma(T_A)$ is contained in the closed annulus $\left\{\lambda\in\mathbb{C}:\ \underline{\beta}(A)\le|\lambda|\le\overline{\beta}(A)\right\}$ and its circular symmetry guarantees that $\sigma(T_A)$ has no trivial components, i.e. singletons. This ensures that the second set in the above union (5.2) is empty. The disjoint decomposition

$$\begin{aligned}\sigma_{F_0}(T_A)&=\{\lambda\in\mathbb{C}:\ T_A-\lambda\,\mathrm{id}\ \text{ is not Fredholm or Fredholm with index}\neq 0\}\\&=\sigma_F(T_A)\dot\cup\{\lambda\in\mathbb{C}:\ T_A-\lambda\,\mathrm{id}\ \text{ is Fredholm with index}\neq 0\}\end{aligned}$$

for the Weyl spectrum now implies the representation

$$\{\lambda\in\mathbb{C}:\ T_A-\lambda\,\mathrm{id}\ \text{ is Fredholm with index}\neq 0\}=\sigma_{F_0}(T_A)\setminus\sigma_F(T_A).$$

Thanks to [44, Rem. 4.13(2)] (the operator $T_A-\lambda\,\mathrm{id}$ is semi-Fredholm, if and only if it is Fredholm) we therefore obtain the equivalences

$$\begin{aligned}&\mu\in\bigcup_{1\le|n|\le\infty}\{\lambda\in\sigma(T_A):\ T_A-\lambda\,\mathrm{id}\ \text{ is semi-Fredholm with index}=n\}\\&\Leftrightarrow\mu\in\{\lambda\in\sigma(T_A):\ T_A-\lambda\,\mathrm{id}\ \text{ is semi-Fredholm with index}=n\}\\&\qquad\text{for some } n\in(\mathbb{Z}\setminus\{0\})\cup\{\pm\infty\}\\&\Leftrightarrow\mu\in\sigma(T_A)\text{ and } T_A-\mu\,\mathrm{id}\ \text{ is Fredholm with }\ \mathrm{ind}(T_A-\mu\,\mathrm{id})\neq 0\\&\Leftrightarrow\mu\in\sigma(T_A)\text{ and }\mu\in\sigma_{F_0}(T_A)\setminus\sigma_F(T_A)\\&\Leftrightarrow\mu\in\sigma(T_A)\cap(\sigma_{F_0}(T_A)\setminus\sigma_F(T_A))\end{aligned}$$

and because of $\sigma_{F_0}(T_A)\subseteq\sigma(T_A)$ this means $\mu\in\sigma_{F_0}(T_A)\setminus\sigma_F(T_A)$. Consequently, in our situation of weighted shift operators T_A the relation (5.2) simplifies to $\sigma(T_A)=\overline{\sigma_{F_0}(T_A)\setminus\sigma_F(T_A)}$. Due to Prop. 3.1 this is equivalent to (5.1) and thus T_A is a point of continuity for σ. Now the claim follows from Prop. 5.3. □

Corollary 5.5 *The Weyl dichotomy spectrum Σ_{F_0} is continuous at A, if*

$$\Sigma_{F_0}(A)=\overline{\Sigma_{F_0}(A)\setminus\Sigma_F(A)}. \tag{5.3}$$

Both conditions (5.1) and (5.3) are tailor-made for the situation where the Weyl dichotomy spectrum $\Sigma_{F_0}(A)$ is a union of intervals (with nonempty interior), while $\Sigma_F(A)$ consists of singletons. Given this, (5.1) holds for $\Sigma(A)=\Sigma_{F_0}(A)$.

Proof Presently the operator-theoretical tool guaranteeing that T_A is a point of continuity for $\sigma_{F_0}:L(\ell^2)\to K(\mathbb{C})$ is [15, Thm. 4.4], which involves the condition

$$\begin{aligned}\sigma_{F_0}(T_A)=&\overline{\bigcup_{1\le|n|\le\infty}\{\lambda\in\sigma(T_A):\ T_A-\lambda\,\mathrm{id}\ \text{ is semi-Fredholm with index}=n\}}\\&\cup\overline{\{\lambda\in\mathbb{C}:\ \{\lambda\}\ \text{ is a component of }\sigma_F(T_A)\}}.\end{aligned}$$

Since all components of the Fredholm spectrum $\sigma_F(T_A)\subseteq\mathbb{C}$ are annuli disjoint from $\{0\}$, one obtains as above that $\sigma_{F_0}(T_A)=\overline{\sigma_{F_0}(T_A)\setminus\sigma_F(T_A)}$. Again, Props. 3.1 and 5.3 yield the assertion. □

Corollary 5.6 *Suppose that $\Sigma(A)=\Sigma_{F_0}(A)$ or $A\in\mathcal{P}_p(\mathbb{C}^d)$ holds.*

(a) If (5.1) *is fulfilled, then Σ_{F_0} is continuous at A.*
(b) If (5.3) *is fulfilled, then Σ is continuous at A.*

Proof (I) If we assume $\Sigma(A) = \Sigma_{F_0}(A)$, then Prop. 3.1 implies $\sigma(T_A) = \sigma_{F_0}(T_A)$.
(a) Under (5.1) the proof of Thm. 5.4 shows that σ is continuous and [14, Cor. 3.9] ensures that also σ_{F_0} is continuous at T_A.
(b) Given (5.3) we established in the proof of Cor. 5.5 that σ_{F_0} is continuous and [14, Cor. 3.9] implies the continuity of σ at T_A.

(II) Suppose that $A \in \mathcal{P}_p(\mathbb{C}^d)$. Due to Prop. B.8(c) the operator T_A^p is paranormal and thus $T_A \in P_p(\mathbb{C}^d)$. With Prop. A.3 now Weyl's and therefore Browder's theorem holds for the shift T_A.
(a) The above proof of Thm. 5.4 establishes that σ is continuous at T_A and then Prop. A.2 implies that T_A is also a point of continuity for σ_{F_0}.
(b) One uses the proof of Cor. 5.5 in order to see that T_A is a point of continuity for σ_{F_0}. The characterization Prop. A.2 yields that also σ is continuous at T_A.

In both respective cases (a) and (b), Prop. 5.3 implies the claim. □

While the above results provided sufficient criteria for Σ, Σ_{F_0} to be continuous at a particular $A \in \mathcal{L}^\infty(\mathbb{C}^d)$, in the following we identify whole subsets of $\mathcal{L}^\infty(\mathbb{C}^d)$ on which Σ behaves continuously:

Theorem 5.7 *(a) The dichotomy spectra Σ_α, $\alpha \in \{a, F_0\}$, are continuous on $\mathcal{C}(A)$.*
(b) The Fredholm dichotomy spectrum Σ_F is continuous on $\mathcal{EC}(A)$.

Proof (a) Let $(A^n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}(A)$ converging to A. The corresponding shift operators satisfy $T_{A^n}T_A = T_A T_{A^n}$ for all $n \in \mathbb{N}$. Moreover, Lemma B.3 yields the limit relation $\lim_{n\to\infty} T_{A^n} = T_A$ in $L(\ell^2)$. Thus, [34, p. 36, Thm. 1.12.5] applies in the Banach algebra $L(\ell^2)$ and yields $\lim_{n\to\infty} \sigma(T_{A^n}) = \sigma(T_A)$. By means of [34, p. 37, Thm. 1.12.7] the same limit relation holds for the Weyl spectrum σ_{F_0}.

(b) Suppose that $(A^n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{EC}(A)$ with limit A and thus

$$\lim_{k\to\pm\infty} \left|A_{k+1}^n A_k - A_{k+1} A_k^n\right| = 0 \quad \text{for all } n \in \mathbb{N}.$$

Hence, combining (B.3) with Prop. B.4(c), implies that all $T_{A^n}T_A - T_A T_{A^n} \in L(\ell^2)$, $n \in \mathbb{N}$, are compact. Because Lemma B.3 yields $\lim_{n\to\infty} T_{A^n} = T_A$ we deduce from [34, p. 53, Lemma 2.3.2] that the limit relation $\lim_{n\to\infty} \sigma_F(T_{A^n}) = \sigma_F(T_A)$ holds.

Thanks to (3.2) in both cases (a) and (b) the claim follows from Prop. 5.3. □

Proposition 5.8 *Suppose a sequence $(B^n)_{n\in\mathbb{N}}$ in $\mathcal{L}^\infty(\mathbb{C}^d)$ converges to $B \in \mathcal{C}(A)$. If*

$$\lim_{k\to\pm\infty} B_k^n = 0 \quad \textit{for all } n \in \mathbb{N}, \tag{5.4}$$

then the following holds true:

(a) Under (5.1) *one has $\lim_{n\to\infty} \Sigma(A + B^n) = \Sigma(A)$.*
(b) Under (4.3) *one has $\lim_{n\to\infty} \Sigma_s(A + B^n) = \Sigma_s(A) = \Sigma(A)$.*

Proof Because of $B \in \mathcal{C}(A)$ we have $T_B T_A = T_A T_B$ and (5.4) ensures that every shift operator T_{B^n}, $n \in \mathbb{N}$, is compact by Prop. B.4(c). Since the compact operators form a closed subspace of $L(\ell^2)$ (cf. [1, p. 89]), also the limit T_B is compact. With again Prop. B.4(c) this implies $\lim_{k\to\pm\infty} B_k = 0$ and therefore $\overline{\beta}(B) = 0$. Consequently, T_B is quasi-nilpotent due to Prop. B.4(d).

(a) In the proof of Thm. 5.4 we showed that (5.1) guarantees T_A to be a point of continuity for σ. Since [47, Cor. 3.4] yields $\lim_{n\to\infty}\sigma(T_A+T_{B^n})=\sigma(T_A+T_B)$ w.r.t. the Hausdorff metric and Prop. 4.6 (its proof) shows $\sigma(T_A+T_B)=\sigma(T_A)$.

(b) Similarly, use [47, Thm. 3.5(b)], where T_A has the SVEP due to (4.3) (cf. [45, Proof of Thm. 4.8]). In particular, $\sigma_s(T_A)=\sigma(T_A)$ follows from Lemma A.1.

In both cases the claim follows from Prop. 5.3. □

Theorem 5.9 *The dichotomy spectra Σ_α, $\alpha\in\{a,\pi,s,F_0\}$, are continuous on the sets $\mathcal{P}_p^{(*)}(\mathbb{C}^d)$, while Σ_F is continuous on $\mathcal{E}\mathcal{P}_p^{(*)}(\mathbb{C}^d)$.*

Thanks to Thm. 3.11 the dichotomy spectra Σ_α are also continuous on the subsets $\mathcal{H}_p^{(*)}(\mathbb{C}^d)$ and $\mathcal{A}_p^{(*)}(\mathbb{C}^d)$. Their importance is due to the fact that it might be comparatively easier to verify the inclusion $A\in\mathcal{H}_p^{(*)}(\mathbb{C}^d)$ (or $A\in\mathcal{A}_p^{(*)}(\mathbb{C}^d)$).

Proof Let $p\in\mathbb{N}$ be arbitrary.

(I) If $A\in\mathcal{P}_p(\mathbb{C}^d)$ is limit of a sequence $(A^n)_{n\in\mathbb{N}}$ in $\mathcal{P}_p(\mathbb{C}^d)$, then Prop. B.8(c) shows that T_A^p and $T_{A^n}^p$ are paranormal. Hence, both T_A and T_{A^n} are pth roots of a paranormal operator and [20, Thm. 2.5] guarantees the continuity property

$$\lim_{n\to\infty}\sigma_\alpha(T_{A^n})=\sigma_\alpha(T_A)\quad\text{for all }\alpha\in\{a,\pi,F_0\}\tag{5.5}$$

in the Hausdorff metric. On the other hand, by Prop. A.3 every $T_A,T_{A^n}\in L(\ell^2)$ has the SVEP and thus Lemma A.1 implies

$$\lim_{n\to\infty}\sigma_s(T_{A^n})=\lim_{n\to\infty}\sigma(T_{A^n})\stackrel{(5.5)}{=}\sigma(T_A)=\sigma_s(T_A).$$

(II) For $A,A^n\in\mathcal{P}_p^*(\mathbb{C}^d)$ we derive as above (now see Prop. B.9(c)) that the adjoints T_A^*, $T_{A^n}^*$ are pth roots of paranormal operators and therefore have the SVEP. Moreover, it is not hard to see that $\lim_{n\to\infty}A^n=A$ yields the limit relation $\lim_{n\to\infty}T_{A^n}^*=T_A^*$. Again, [20, Thm. 2.5] implies

$$\lim_{n\to\infty}\sigma_\alpha(T_{A^n})\stackrel{(A.3)}{=}\lim_{n\to\infty}\sigma_\alpha(T_{A^n}^*)=\sigma_\alpha(T_A^*)\stackrel{(A.3)}{=}\sigma_\alpha(T_A)\quad\text{for all }\alpha\in\{a,F_0\},$$

as well as (see Prop. A.3 and Lemma A.1)

$$\lim_{n\to\infty}\sigma_\pi(T_{A^n})\stackrel{(A.4)}{=}\lim_{n\to\infty}\sigma_s(T_{A^n}^*)=\lim_{n\to\infty}\sigma(T_{A^n}^*)=\sigma(T_A^*)=\sigma_s(T_A^*)\stackrel{(A.4)}{=}\sigma_\pi(T_A).$$

Finally, the central [20, Thm. 2.5] also gives us

$$\lim_{n\to\infty}\sigma_s(T_{A^n})\stackrel{(A.4)}{=}\lim_{n\to\infty}\sigma_\pi(T_{A^n}^*)=\sigma_\pi(T_A^*)\stackrel{(A.4)}{=}\sigma_s(T_A).$$

(III) Concerning the Fredholm dichotomy spectrum Σ_F one analogously deduces $\lim_{n\to\infty}\sigma_F(T_{A^n}^{(*)})=\sigma_F(T_A^{(*)})$ using (A.3) and [20, Thm. 2.5].

In all three cases the final assertion is a consequence of Prop. 5.3. □

The subsequent example sums up our results. It illuminates that a spectral interval can collapse into two subintervals and that the dichotomy spectrum of an upper-triangular equation is not necessarily given as union of the diagonal spectra (see [45] for more information on the latter issue).

Example 5.10 Let us consider a difference eqn. (Δ_A) in $\mathbb{R}^2$ with coefficients

$$A_k := \begin{pmatrix} a_k & c_k \\ 0 & b_k \end{pmatrix} \in GL(\mathbb{R}^2)$$

satisfying $A \in \mathcal{L}^\infty(\mathbb{R}^2)$ and involving the real sequences

$$a_k := \begin{cases} \alpha_+, & k \geq 0, \\ \alpha_-, & k < 0, \end{cases} \qquad b_k := \begin{cases} \beta_+, & k \geq 0, \\ \beta_-, & k < 0, \end{cases} \qquad c_k := \begin{cases} \lambda, & k \geq 0, \\ 0, & k < 0 \end{cases}$$

with positive $\alpha_\pm \neq \beta_\pm$ and a parameter $\lambda \in \mathbb{R}$. In particular, for $\lambda = 0$ one has $A \in \mathcal{H}_1(\mathbb{R}^2)$ if and only if $\alpha_- \leq \alpha_+$, $\beta_- \leq \beta_+$, while $A \in \mathcal{H}_1^*(\mathbb{R}^2)$ holds in the dual situation $\alpha_+ \leq \alpha_-$, $\beta_+ \leq \beta_-$. To tackle the related dichotomy spectra we determine the transition matrix as

$$\gamma^{-k}\Phi(k,0) = \begin{cases} \begin{pmatrix} \left(\frac{\alpha_+}{\gamma}\right)^k & \frac{\lambda}{\gamma^k}\frac{\alpha_+^k - \beta_+^k}{\alpha_+ - \beta_+} \\ 0 & \left(\frac{\beta_+}{\gamma}\right)^k \end{pmatrix} & \text{for all } k \geq 0, \\ \begin{pmatrix} \left(\frac{\alpha_-}{\gamma}\right)^k & 0 \\ 0 & \left(\frac{\beta_-}{\gamma}\right)^k \end{pmatrix} & \text{for all } k \leq 0 \end{cases}$$

and every $\gamma > 0$. Therefore, the corresponding stable and unstable bundles

$$V_\gamma^\pm := \left\{ x \in \mathbb{R}^2 : \sup_{k \in \mathbb{Z}_0^\pm} \gamma^{-k} |\Phi(k,0)x| < \infty \right\}$$

of the scaled difference equation

$$x_{k+1} = \gamma^{-1} A_k x_k \tag{5.6}$$

become

$$V_\gamma^+ = \begin{cases} \mathbb{R}^2, & \max\{\alpha_+, \beta_+\} \leq \gamma, \\ \mathbb{R}e_1, & \alpha_+ \leq \gamma < \beta_+, \\ \mathbb{R}\begin{pmatrix} \lambda \\ \beta_+ - \alpha_+ \end{pmatrix}, & \beta_+ \leq \gamma < \alpha_+, \\ \{0\}, & \gamma < \min\{\alpha_+, \beta_+\}, \end{cases} \qquad V_\gamma^- = \begin{cases} \{0\}, & \max\{\alpha_-, \beta_-\} < \gamma, \\ \mathbb{R}^2, & \gamma \leq \min\{\alpha_-, \beta_-\}, \\ \mathbb{R}e_2, & \alpha_- < \gamma \leq \beta_-, \\ \mathbb{R}e_1, & \beta_- < \gamma \leq \alpha_-. \end{cases}$$

According to Palmer's result (cf. [38, Prop. 2.6] in discrete time) the scaled difference eqn. (5.6) admits an ED on the entire line $\mathbb{Z}$, if and only if there are EDs on both semiaxes $\mathbb{Z}_0^-$, $\mathbb{Z}_0^+$ and the decomposition

$$V_\gamma^+ \oplus V_\gamma^- = \mathbb{R}^2$$

holds. Otherwise, for each spectral interval the Fredholm index of S_γ is given by the formula (see [44, Prop. 4.9])

$$\operatorname{ind} S_\gamma = \dim V_\gamma^+ + \dim V_\gamma^- - 2.$$

Due to [44, Prop. 4.29(b)] this yields the discrete Fredholm dichotomy spectrum

$$\Sigma_F(A) = \{\alpha_-, \alpha_+, \beta_-, \beta_+\}.$$

Concerning further spectral properties we distinguish several cases:

(a) $\alpha_+ < \beta_+$:

(a_1) For $\alpha_- < \beta_-$ the dichotomy spectra are given in Fig. 5.1(left) as bold lines. One has $\Sigma(A) = \Sigma_{F_0}(A)$ and therefore the continuity condition for Σ from Thm. 5.4, as well as for Σ_{F_0} from Cor. 5.5 apply, provided no singleton spectral intervals occur.

(a_2) The case $\beta_- < \alpha_-$ is illustrated in Fig. 5.1(right). Here, $\Sigma(A) = \Sigma_{F_0}(A)$ is violated precisely in the situation where $(\alpha_+, \beta_+) \cap (\beta_-, \alpha_-)$ has nonempty interior. Otherwise, Thm. 5.4 yields continuity at A, while due to Cor. 5.5 also the Weyl spectrum is continuous as long as it contains no singletons.

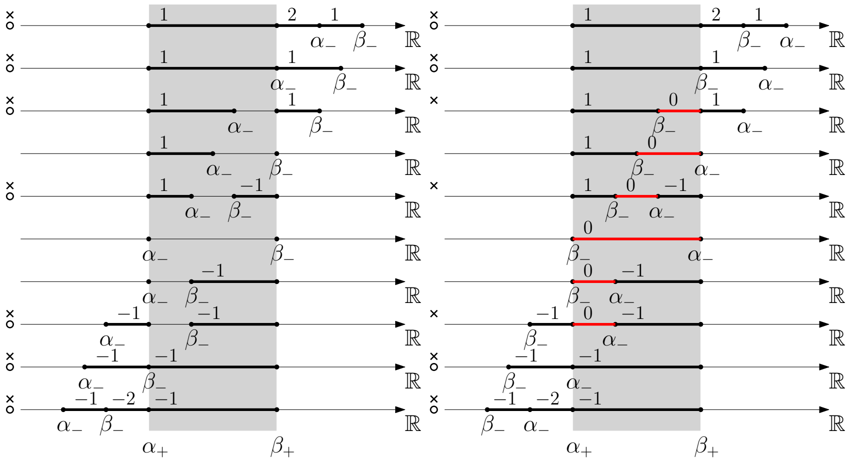

Fig. 5.1 Dichotomy spectra for $\alpha_+ < \beta_+$ and $\alpha_- < \beta_-$ (left) resp. $\beta_- < \alpha_-$ (right). The Weyl spectrum is obtained by excluding the sets indicated in red and the numbers above the spectral intervals are the respective Fredholm index of S_γ.

(b) $\beta_+ < \alpha_+$:

(b_1) $\alpha_- < \beta_-$ (see Fig. 5.2(left)) represents the most interesting constellation. Indeed, one has the dichotomy spectrum

$$\Sigma(A) = [\min\{\alpha_-, \beta_+\}, \max\{\alpha_+, \beta_-\}] \setminus \begin{cases} \emptyset, & \lambda = 0, \\ ((\beta_+, \alpha_+) \cap (\alpha_-, \beta_-)), & \lambda \neq 0, \end{cases}$$

being a compact interval for $\lambda = 0$, which splits into two subintervals for perturbed parameters $\lambda \neq 0$ — this indicates upper-semicontinuity. In any case, the Weyl dichotomy spectrum becomes

$$\Sigma_{F_0}(A) = [\min\{\alpha_-, \beta_+\}, \max\{\alpha_+, \beta_-\}] \setminus ((\beta_+, \alpha_+) \cap (\alpha_-, \beta_-))$$

and thus Thm. 5.4 applies when $\alpha_+ \le \alpha_-$ or $\beta_- \le \beta_+$. For the Weyl spectrum $\Sigma_{F_0}(A)$ continuity is given by Cor. 5.5 when it contains no singletons.

(b_2) Eventually, $\beta_- < \alpha_-$ (see Fig. 5.2(right)) captures a situation as in (a_1).

For the convenience of the reader we eventually use the symbol $\circ$ to indicate in Figs. 5.1 and 5.2 that the continuity condition (5.1) from Thm. 5.4 applies for $\lambda = 0$, while $\times$ points out that Cor. 5.5 can be deployed.

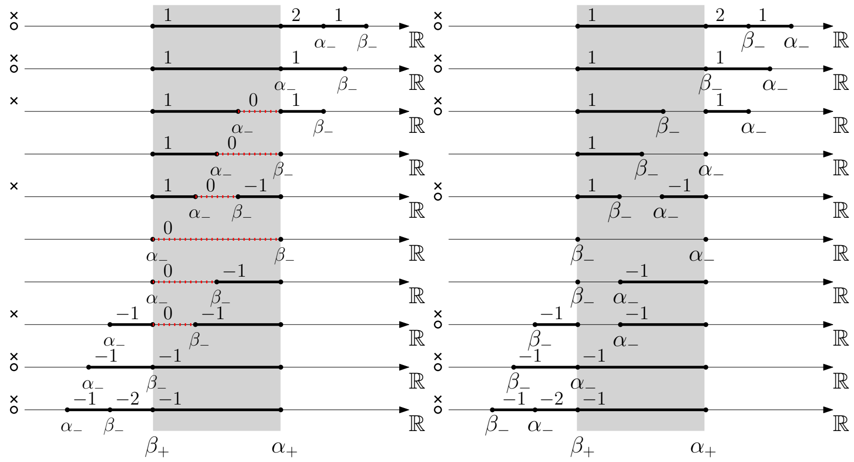

Fig. 5.2 Dichotomy spectra for $\beta_+ < \alpha_+$ and $\alpha_- < \beta_-$ (left) resp. $\alpha_- < \beta_-$ (right). The Weyl spectrum is obtained by excluding the dotted red intervals, being present only for $\lambda = 0$; the dotted intervals also indicate the spectral parts collapsing in case $\lambda \neq 0$. The numbers above the spectral intervals denote the Fredholm indices of S_γ.

6 Perspectives and applications

Despite its operator-theoretical flavor this paper has applications in the field of nonautonomous dynamical systems (not only in discrete time):

– ???justify numerical approximation techniques
– ???Fredholm spectrum is boundary of spectrum and easy to obtain
– ???Avoid solution bifurcations

Besides $\mathcal{P}_p(\mathbb{C}^d)$ and $\mathcal{P}_p^*(\mathbb{C}^d)$ (and its subsets) there are further classes of matrix sequences on which the dichotomy spectrum behaves continuously. Several of them are summarized in [20] and we leave it to the interested reader to characterize appropriate coefficient sequences $(A_k)_{k\in\mathbb{Z}}$ analogously to Props. B.8 and B.9.

Our results can easily be applied to the continuous time situation of ODEs

$$\dot{x} = A(t)x, \qquad A \in L^1_{\mathrm{loc}}(\mathbb{R}, L(\mathbb{C}^d)) \tag{6.1}$$

with transition matrix $U(t,s) \in GL(\mathbb{C}^d)$, $s,t \in \mathbb{R}$, satisfying $\sup_{t\in\mathbb{R}} |U(t,t-1)| < \infty$ as well. The corresponding dichotomy spectrum $\hat{\Sigma}(A) \subseteq \mathbb{R}$ for (6.1) has been studied in [12,13,17,46,50]. If we define

$$A_k := U(k+1,k) \in GL(\mathbb{C}^d) \quad \text{for all } k \in \mathbb{Z}, \tag{6.2}$$

then the characterization [48, Cor. 5.1] enables us to show that the dichotomy spectra of the difference eqn. (Δ_A) and the ODE (6.1) are related by

$$\Sigma(A) = \exp \hat{\Sigma}(A), \qquad \hat{\Sigma}(A) = \ln \Sigma(A).$$

In particular, using these spectral mapping theorems our results transfer to the specific coefficient sequences (6.2) and yield corresponding invariance and continuity information on the continuous time spectrum $\hat{\Sigma}(A)$.

Acknowledgement

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A Operators on Hilbert spaces

For an infinite-dimensional separable and complex Hilbert space X with inner product $\langle\cdot,\cdot\rangle$, let $L(X)$ denote the Banach algebra of bounded linear operators on X with identity id_X.

Given $T \in L(X)$, let $\sigma_a(T) := \sigma(T)$, $\sigma_\pi(T)$, $\sigma_s(T)$, $\sigma_F(T)$ and $\sigma_{F_0}(T)$ be its *spectrum*, *approximate point spectrum*, *surjectivity*, *essential (Fredholm)* and *Weyl spectrum*, respectively (cf. [1,2,34,37]). We write $r(T)$ for the *spectral radius* and define $r_1(T) := \min_{\lambda\in\sigma(T)} |\lambda|$. One speaks of a *quasi-nilpotent operator* T, if $r(T) = 0$ i.e. $\sigma(T) = \{0\}$. In addition, let us define the *derivation* $\delta_{S,T} : L(X) \to L(X)$, $\delta_{S,T}M := SM - MT$ of two operators $S, T \in L(X)$ and obtain that the iterates of this linear operator are of the form

$$\delta^n_{S,T}M = \sum_{j=0}^{n} \binom{n}{j} (-1)^j S^{n-j} M T^j \quad \text{for all } n \in \mathbb{N}_0.$$

When S and T commute one obtains the implications

$$SM = MS \Rightarrow \delta^n_{S,T}M = M(S-T)^n, \qquad TM = MT \Rightarrow \delta^n_{S,T}M = (S-T)^n M \tag{A.1}$$

for all $n \in \mathbb{N}_0$. With the associated *spectral radius*

$$r_{S,T}(M) := \limsup_{n\to\infty} \sqrt[n]{\left\|\delta^n_{S,T}M\right\|} \tag{A.2}$$

let us denote S, T as *asymptotically intertwined* (cf. [32]), if there exists a so-called *intertwiner* $M \in L(X) \setminus \{0\}$ such that $r_{S,T}(M) = 0$.

An operator $T \in L(X)$ has the *single-valued extension property (SVEP for short) at a point* λ_0, provided for every neighborhood $U \subset \mathbb{C}$ of λ_0 the only analytic function $f : U \to X$ satisfying $(\lambda\,\mathrm{id}_X - T)f(\lambda) \equiv 0$ on U is the zero function. If the SVEP holds at every $\lambda_0 \in \mathbb{C}$, the operator T is said to possess the SVEP. The associate set (cf. [3, p. 64ff])

$$\mathfrak{S}(T) := \{\lambda \in \mathbb{C} : T \text{ does not have SVEP at } \lambda\}$$

is open and fulfills $\mathfrak{S}(T) \subseteq \sigma(T)^\circ$ ($^\circ$ denotes the interior of a set). Clearly, T has the SVEP, if and only if $\mathfrak{S}(T) = \emptyset$.

Lemma A.1 (see [2, p. 80, Cor. 2.45]) *If T has the SVEP, then $\sigma(T) = \sigma_s(T)$.*

We say that a bounded operator $T \in L(X)$ fulfills *Weyl's theorem*, if $\sigma(T)\setminus\sigma_{F_0}(T)$ consists of isolated points $\lambda \in \sigma(T)$ being eigenvalues of finite multiplicity, and satisfies *Browder's theorem*, provided $\sigma(T) \setminus \sigma_{F_0}(T)$ is the set of all poles of T with finite rank. It is well-known that Weyl's theorem implies Browder's theorem (cf. [2, p. 166]).

Proposition A.2 (see [19, Thm. 2.2]) *Suppose that $T \in L(X)$ satisfies Browder's theorem. Then σ is continuous at T, if and only if σ_{F_0} has this property.*

If $T^* \in L(X)$ denotes the *adjoint operator* of T, then the spectra of T and T^* are related by (cf. [31, p. 34, Thm. 2.6, p. 145 resp. p. 160], [2, p. 79, Thm. 2.42])

$$\sigma_\alpha(T^*) = \sigma_\alpha(T)^* \qquad \text{for all } \alpha \in \{a, F, F_0\}, \tag{A.3}$$

$$\sigma_s(T) = \sigma_\pi(T^*), \qquad \sigma_s(T^*) = \sigma_\pi(T), \tag{A.4}$$

with set of complex-conjugated values $\Omega^* := \left\{\lambda \in \mathbb{C} : \bar{\lambda} \in \Omega\right\}$ for every $\Omega \subseteq \mathbb{C}$.

A self-adjoint operator $T \in L(X)$ is *positive* (in symbols, $T \geq 0$), if $\langle x, Tx\rangle \geq 0$ holds for all $x \in X$. Furthermore, in case the difference $T - S$ of self-adjoint operators $S, T \in L(X)$ is positive, we write $T \geq S$ and obtain the cone-like conditions

$$\beta T \geq \alpha T \geq \alpha S, \qquad T + R \geq S + R \quad \text{for all } 0 \leq \alpha \leq \beta \tag{A.5}$$

and self-adjoint $R \in L(X)$. With a unitary operator $U \in L(X)$ one moreover has

$$T \geq S \quad \Leftrightarrow \quad U^*TU \geq U^*SU. \tag{A.6}$$

A.1 Hyponormal operators

An operator $T \in L(X)$ is called *hyponormal*, if $T^*T \geq TT^*$ and for every $p \in \mathbb{N}$ we write

$$H_p(X) := \{S \in L(X) : S^p \text{ is hyponormal}\}, \quad H_p^*(X) := \{S \in L(X) : S^{*p} \text{ is hyponormal}\}.$$

The elements of $H_p(X)$ are denoted as *pth roots* of a hyponormal operator and we use a similar terminology for the further operator classes defined below. Both above sets are closed in the norm topology (cf. [29, Prop. 1.5]), while $H_1(X)$ is nowhere dense in $L(X)$ (cf. [36, Thm. 2.4]).

A.2 Class A operators

An operator $T \in L(X)$ is said to be of *class* A (cf. [34, p. 74]), if $T^{*2}T^2 \geq (T^*T)^2$ and with $p \in \mathbb{N}$ we set

$$A_p(X) := \{S \in L(X) : S^p \text{ is of class } A\}, \quad A_p^*(X) := \{S \in L(X) : S^{*p} \text{ is of class } A\}.$$

A.3 Paranormal operators

An operator $T \in L(X)$ satisfying $T^{*2}T^2 - 2rT^*T + r^2 \operatorname{id}_X \geq 0$ for every $r > 0$ is called *paranormal* (cf. [34, p. 50]), and given $p \in \mathbb{N}$ let us write

$$P_p(X) := \{S \in L(X) : S^p \text{ is paranormal}\}, \quad P_p^*(X) := \{S \in L(X) : S^{*p} \text{ is paranormal}\}.$$

The above operator classes are invariant under multiplication with a complex scalar.

Proposition A.3 *Let $p \in \mathbb{N}$. Every $T \in P_p(X)$ satisfies Weyl's theorem and has the SVEP.*

Proof By assumption the operator T is algebraically paranormal. Hence, T fulfills Weyl's theorem due to [16, Thm. 2.4] and has the SVEP by [2, p. 78, Thm. 2.40]. □

B Multiplication and weighted shift operators

We denote by ℓ^2 the linear space of square-summable sequences $\phi = (\phi_k)_{k\in\mathbb{Z}}$ in $\mathbb{C}^d$ equipped with the inner product

$$\langle\!\langle \phi, \psi \rangle\!\rangle := \sum_{k\in\mathbb{Z}} \langle \phi_k, \psi_k \rangle \quad \text{for all } \phi, \psi \in \ell^2$$

and the norm $\|\phi\| = \sqrt{\langle\!\langle \phi, \phi \rangle\!\rangle}$; note that ℓ^2 is the prototype of a separable Hilbert space. The bounded sequences in $L(\mathbb{C}^d)$ are denoted by $\mathcal{L}^\infty(\mathbb{C}^d)$.

B.1 Multiplication operators

Given a bounded *weight sequence* $\Lambda = (\Lambda_k)_{k\in\mathbb{Z}}$ of matrices $\Lambda_k \in L(\mathbb{C}^d)$ we denote

$$M_\Lambda : \ell^2 \to \ell^2, \qquad (M_\Lambda \phi)_k := \Lambda_k \phi_k \quad \text{for all } k \in \mathbb{Z} \tag{B.1}$$

as *multiplication operator*. It is bounded with $\|M_\Lambda\| = \sup_{k\in\mathbb{Z}} |\Lambda_k|$ and the adjoint operator $M_\Lambda^* = M_{\Lambda^*}$. In particular, M_Λ is unitary, if and only if $\Lambda_k^{-1} = \Lambda_k^*$ holds for all $k \in \mathbb{Z}$.

Proposition B.1 (properties of multiplication operators) *Let $\Lambda \in \mathcal{L}^\infty(\mathbb{C}^d)$.*

(a) If every Λ_k, $k \in \mathbb{Z}$, is invertible, then $M_\Lambda \in GL(\ell^2) \Leftrightarrow \sup_{k\in\mathbb{Z}}\left|\Lambda_k^{-1}\right| < \infty$ holds. In particular, it is $(M_\Lambda^{-1}\phi)_k = \Lambda_k^{-1}\phi_k$ for all $k \in \mathbb{Z}$.

(b) An operator $M_\Lambda \in L(\ell^2)$ is compact, if and only if $\lim_{k\to\pm\infty} \Lambda_k = 0$.

Remark B.2 A multiplication operator M_Λ is one-to-one, if and only if each weight Λ_k, $k \in \mathbb{Z}$, is invertible. For an onto M_Λ also every weight fulfills $\Lambda_k \in GL(\mathbb{C}^d)$, while the converse holds under the additional assumption $\sup_{k\in\mathbb{Z}}\left|\Lambda_k^{-1}\right| < \infty$.

The following proof requires the *Kronecker symbol* denoted as $\delta_{k,m}$.

Proof (a) ($\Leftarrow$) From our premise the operator $N_\Lambda \in L(\ell^2)$ given as $(N_\Lambda\phi)_k := \Lambda_k^{-1}\phi_k$ for $k \in \mathbb{Z}$ is well-defined and bounded. Moreover, we readily compute $N_\Lambda M_\Lambda = M_\Lambda N_\Lambda = \mathrm{id}_{\ell^2}$. Thus, M_Λ is invertible with inverse $M_\Lambda^{-1} = N_\Lambda$.

($\Rightarrow$) With invertible M_Λ there exists a $N \in L(\ell^2)$ fulfilling $\phi_k = (M_\Lambda N\phi)_k = \Lambda_k(N\phi)_k$ for all $k \in \mathbb{Z}$, $\phi \in \ell^2$ and this yields $(N\phi)_k = \Lambda_k^{-1}\phi_k$. Since $\bar{B}_1(0) \subseteq \mathbb{C}^d$ is compact, for each $k \in \mathbb{Z}$ there exists a $x_k \in \mathbb{C}^d$ with $|x_k| = 1$ such that $\left|\Lambda_k^{-1}x_k\right| = \sup_{|x|\le 1}\left|\Lambda_k^{-1}\right| = \left|\Lambda_k^{-1}\right|$. Then the ℓ^2-sequence $\tilde\phi^k := (\delta_{j,k}x_k)_{j\in\mathbb{N}}$ fulfills $\left\|\tilde\phi^k\right\| \le 1$ and

$$\left|\Lambda_k^{-1}\right| = \left|\Lambda_k^{-1}x_k\right| = \left|(N\tilde\phi^k)_k\right| \le \left\|N\tilde\phi^k\right\| \le \|N\| \left\|\tilde\phi^k\right\| \le \|N\| \quad \text{for all } k \in \mathbb{Z}$$

concludes the proof of (a).

(b) ($\Rightarrow$) We proceed indirectly and assume $(\Lambda_k)_{k\in\mathbb{Z}}$ does not converge to 0. Then there exists a $\rho > 0$ and a subsequence $(n_k)_{k\in\mathbb{Z}}$ such that $|\Lambda_{n_k}| \ge \rho$ for all $k \in \mathbb{Z}$. As in the above proof one finds $x_k \in \mathbb{C}^d$ satisfying $|x_k| = 1$ and $|\Lambda_k| = |\Lambda_k x_k|$ for all $k \in \mathbb{Z}$. Defining the sequences $\tilde\phi^k := (\delta_{k,n_j}x_{n_j})_{j\in\mathbb{N}}$ it is

$$\left\|M_\Lambda\tilde\phi^k - M_\Lambda\tilde\phi^j\right\|^2 = |\Lambda_{n_k}x_{n_k}|^2 + \left|\Lambda_{n_j}x_{n_j}\right|^2 = |\Lambda_{n_k}|^2 + \left|\Lambda_{n_j}\right|^2, \quad \left\|M_\Lambda\tilde\phi^k - M_\Lambda\tilde\phi^j\right\| \ge \sqrt{2}\rho$$

for all $k \neq j$. Hence, $(M_\Lambda\tilde\phi^k)_{k\in\mathbb{Z}}$ has no convergent subsequence and M_Λ cannot be compact.

($\Leftarrow$) One verifies that M_Λ is the (uniform) limit of a sequence of finite-rank operators. The detailed proof from [44, Lemma 3.7] for operators on ℓ^∞ is literally the same in the present case of square summable sequences ℓ^2. □

B.2 Weighted shifts

For a bounded *weight sequence* $A = (A_k)_{k\in\mathbb{Z}}$ in $L(\mathbb{C}^d)$, we define the weighted left shift

$$T_A : \ell^2 \to \ell^2, \qquad (T_A\phi)_k := A_{k-1}\phi_{k-1} \quad \text{for all } k \in \mathbb{Z}. \tag{B.2}$$

Clearly, T_A is bounded with $\|T_A\| = \sup_{k\in\mathbb{Z}} |A_k|$ and such shift operators form a closed subspace of $L(\ell^2)$. Since the SVEP is invariant under similarity, $\mathfrak{S}(T_A)$ is rotationally symmetric w.r.t. 0. Moreover, for weight sequences $A, B \in \mathcal{L}^\infty(\mathbb{C})$ it is

$$(T_B T_A\phi)_k = B_{k-1}A_{k-2}\phi_{k-2} \quad \text{for all } k \in \mathbb{Z}. \tag{B.3}$$

Lemma B.3 *The mapping $T_\cdot : \mathcal{L}^\infty(\mathbb{C}^d) \to L(\ell^2)$ is linear and continuous.*

Proof The linearity of $T_\cdot$ is clear. For arbitrary $A \in \mathcal{L}^\infty(\mathbb{C}^d)$ one has

$$\|(T_A\phi)\|^2 = \sum_{k\in\mathbb{Z}} |(T_A\phi)_k|^2 \overset{(B.2)}{=} \sum_{k\in\mathbb{Z}} |A_k\phi_k|^2 \le (\sup_{k\in\mathbb{Z}} |A_k|)^2 \|\phi\|^2 \quad \text{for all } \phi \in \ell^2$$

and consequently $\|T_A\| \le \sup_{k\in\mathbb{Z}} |A_k|$. □

The following result summarizes the essential properties of weighted bilateral shifts, and notably guarantees that every compact shift operator is quasi-nilpotent.

Proposition B.4 (properties of shift operators) *Let $p \in \mathbb{N}_0$ and $A \in \mathcal{L}^\infty(\mathbb{C}^d)$.*

(a) $r_1(T_A) = \underline{\beta}(A)$ *and* $r(T_A) = \overline{\beta}(A)$.
(b) If every A_k, $k \in \mathbb{Z}$, is invertible, then $T_A \in GL(\ell^2) \Leftrightarrow \sup_{k\in\mathbb{Z}}\left|A_k^{-1}\right| < \infty$ and under one of these conditions it holds

$$(T_A^{-1}\phi)_k = A_k^{-1}\phi_{k+1} \quad \textit{for all } k \in \mathbb{Z}.$$

(c) $T_A \in L(\ell^2)$ is compact, if and only if $\lim_{k\to\pm\infty} A_k = 0$.
(d) T_A is quasi-nilpotent, if and only if $\overline{\beta}(A) = 0$.
(e) The adjoint of T_A is given by $T_A^* \in L(\ell^2)$, $(T_A^*\phi)_k = A_k^*\phi_{k+1}$ for all $k \in \mathbb{Z}$.
(f) For every $k \in \mathbb{Z}$ one has $(T_A^{*p}\phi)_k = \Phi(k+p,k)^*\phi_{k+p}$ and $(T_A^p\phi)_k = \Phi(k,k-p)\phi_{k-p}$.

Proof Let I denote the constant sequence $(\mathrm{id}_{\mathbb{C}^d})_{k\in\mathbb{Z}}$ in $\mathcal{L}^\infty(\mathbb{C}^d)$.

(a) See [27, p. 127] resp. [8, Thm. 1(i)] combined with the characterization (2.2).

(b) The shift $T_I \in L(\ell^2)$, $(T_I\phi)_k = \phi_{k-1}$ satisfies $T \in GL(\ell^2)$. Due to $T_A = T_I M_A$ the assertion is a result of Prop. B.1(a).

(c) Thanks to the representation $T_A = T_I M_A$ the claim follows from Prop. B.1(b).

(d) is immediate from (a).

(e) For arbitrary $\phi, \psi \in \ell^2$ we obtain

$$\langle\!\langle T_A\phi, \psi\rangle\!\rangle = \sum_{k\in\mathbb{Z}}\langle A_k\phi_k, \psi_{k+1}\rangle = \sum_{k\in\mathbb{Z}}\langle \phi_k, A_k^*\psi_{k+1}\rangle = \langle\!\langle \phi, T_A^*\psi\rangle\!\rangle$$

with $(T_A^*\psi)_k := A_k^*\psi_{k+1}$ for all $k \in \mathbb{Z}$.

(f) We proceed by induction; the claim holds for $p = 0$. As induction step $p \to p+1$ it is

$$(T_A^{p+1}\phi)_k = (T_A(T_A^p\phi))_k = A_{k-1}\Phi(k-1,k-1-p)\phi_{k-1-p} = \Phi(k,k-(p+1))\phi_{k-(p+1)}$$

and

$$\begin{aligned}((T_A^*)^{p+1})\phi)_k &= (T_A^*T_A^{*p}\phi)_k = A_k^*(T_A^{*p}\phi)_{k+1} = A_k^*\Phi(k+1+p,k+1)^*\phi_{k+p+1}\\ &= (\Phi(k+1+p,k+1)A_k)^*\phi_{k+p+1} = \Phi(k+p+1,k)^*\phi_{k+p+1}\end{aligned}$$

for all $k \in \mathbb{Z}$. □

Lemma B.5 *Let $A, B, \Lambda \in \mathcal{L}^\infty(\mathbb{C}^d)$. The multiplication operator $M_\Lambda \in L(\ell^2)$ asymptotically intertwines T_A and T_B, if and only if*

$$\limsup_{n\to\infty}\sup_{k\in\mathbb{Z}}\sqrt[n]{\left|\sum_{j=0}^n\binom{n}{j}(-1)^j\Phi_A(k,k-n+j)\Lambda_{k-n+j}\Phi_B(k-n+j,k-n)\right|} = 0.$$

Proof For each $n \in \mathbb{N}_0$ and $j \in \{0,\dots,n\}$ it easily follows from Prop. B.4(f) that

$$(T_A^{n-j}M_\Lambda T_B^j\phi)_k = \Phi_A(k,k-n+j)\Lambda_{k-n+j}\Phi_B(k-n+j,k-n)\phi_{k-n} \quad \text{for all } k \in \mathbb{Z}.$$

Consequently, given $\phi \in \ell^2$ due to

$$\left(\delta^n_{T_A,T_B}M_\Lambda\phi\right)_k = \left(\sum_{j=0}^n\binom{n}{j}(-1)^j\Phi_A(k,k-n+j)\Lambda_{k-n+j}\Phi_B(k-n+j,k-n)\right)\phi_{k-n}$$

the iterated derivations $\delta^n_{T_A,T_B}M_\Lambda$ are left shifts and have the norm

$$\left\|\delta^n_{T_A,T_B}M_\Lambda\right\| = \sup_{k\in\mathbb{Z}}\left|\sum_{j=0}^n\binom{n}{j}(-1)^j\Phi_A(k,k-n+j)\Lambda_{k-n+j}\Phi_B(k-n+j,k-n)\right|.$$

Then the assertion follows by definition. □

Corollary B.6 *Let $B \in \mathcal{C}(A)$. If (4.2) holds, then $r_{T_A,T_B}(M_\Lambda) \leq \overline{\beta}(B-A)$.*

Proof Thanks to $B \in \mathcal{C}(A)$ it is $T_A T_B = T_B T_A$. The further commutativity relations (4.2) ensure $T_A M_\Lambda = M_\Lambda T_A$ resp. $T_B M_\Lambda = M_\Lambda T_B$ and in both cases (A.1) implies

$$\sqrt[n]{\left\|\delta^n_{T_A,T_B} M_\Lambda\right\|} \leq \sqrt[n]{\left\|T^n_{A-B}\right\|}\sqrt[n]{\|M_\Lambda\|} \quad \text{for all } n \in \mathbb{N}_0.$$

Passing over to the lim sup as $n \to \infty$ in this inequality yields $r_{T_A,T_B}(M_\Lambda) \leq r(T_{A-B})$ and the claim results with Prop. B.4(a). □

Proposition B.7 *Every $T_A \in L(\ell^2)$ is unitarily equivalent to a weighted left shift T_B with*

(a) $B_k \in L(\mathbb{C}^d)$ is positive-semidefinite Hermitian,
(b) $\sup_{k\in\mathbb{Z}} |B_k| < \infty$.

Proof Above all, choose some $\kappa \in \mathbb{Z}$, set $U_\kappa := \mathrm{id}_{\mathbb{C}^d}$ and we claim that there exists a sequence $U_k \in L(\mathbb{C}^d)$ of unitary matrices such that for all $\kappa \leq k$ it is

$$B_k = U^{-1}_{k+1} A_k U_k. \tag{B.4}$$

(I) The matrix A_κ has a polar decomposition, i.e. there exists a unitary $U_{\kappa+1} \in L(\mathbb{C}^d)$ and a positive-semidefinite Hermitian $B_\kappa \in L(\mathbb{C}^d)$ such that $A_\kappa = U_{\kappa+1}B_\kappa$ (cf. [25, p. 449, Thm. 7.3.1(b)]). This yields (B.4) for $k = \kappa$. In the induction step $k-1 \to k$ we invest that $A_k U_k$ possesses a polar decomposition $A_k U_k = U_{k+1} B_k$ with unitary U_{k+1} and an positive-semidefinite B_k, and U_k is known by the induction hypothesis. Thus, (B.4) holds for $k \geq \kappa$.

(II) For $k < \kappa$ we get U_k as follows: If the polar decomposition of $(U^*_k A_{k-1})^*$ reads as $(U^*_k A_{k-1})^* = U_{k-1} B^*_{k-1}$ with unitary U_{k-1} and positive-semidefinite B^*_{k-1}, then (B.4) holds.

(III) Given the associate multiplication operator M_U,

$$\langle M_U\phi, \psi\rangle = \sum_{k\in\mathbb{Z}} \langle U_k\phi_k, \psi_k\rangle = \sum_{k\in\mathbb{Z}} \langle \phi_k, U^*_k\psi_k\rangle \quad \text{for all } \phi, \psi \in \ell^2$$

yields the adjoint $(M^*_U\phi)_k = U^*_k\phi_k$ for all $k \in \mathbb{Z}$. For $k \geq \kappa$ we obtain

$$(M^*_U T_A M_U \phi)_k = U^*_k A_{k-1} U_{k-1}\phi_{k-1} \overset{(B.4)}{=} (T_B\phi)_k \quad \text{and} \quad M^*_U T_A M_U = T_B \quad \text{for all } k < \kappa.$$

Moreover, (B.4) shows that the boundedness of A_k carries over to B_k. □

Proposition B.8 *Let $p \in \mathbb{N}_0$ and $A \in \mathcal{L}^\infty(\mathbb{C}^d)$. The shift operator T^p_A is*

(a) hyponormal, if and only if for all $k \in \mathbb{Z}$ one has

$$\Phi(k+2p, k+p)^*\Phi(k+2p, k+p) \geq \Phi(k+p, k)\Phi(k+p, k)^*, \tag{B.5}$$

(b) of class A, if and only if for all $k \in \mathbb{Z}$ one has

$$\Phi(k+2p, k)^*\Phi(k+2p, k) \geq \left[\Phi(k+p, k)^*\Phi(k+p, k)\right]^2, \tag{B.6}$$

(c) paranormal, if and only if for all $k \in \mathbb{Z}$ and $r > 0$ it holds

$$\Phi(k+2p, k)^*\Phi(k+2p, k) - 2r\Phi(k+p, k)^*\Phi(k+p, k) + r^2\,\mathrm{id}_{\mathbb{C}^d} \geq 0. \tag{B.7}$$

Proof For $p = 0$ the claims are trivial. Given $p \in \mathbb{N}$ we define $S := T^p_A$ and choose $\phi \in \ell^2$. We obtain from Prop. B.4(f) the multiplication operators

$$\begin{aligned}
(S^*S\phi)_k &= \Phi(k+p, k)^*\Phi(k+p, k)\phi_k,\\
(SS^*\phi)_k &= \Phi(k, k-p)\Phi(k, k-p)^*\phi_k,\\
(S^{*2}S^2\phi)_k &= \Phi(k+2p, k)^*\Phi(k+2p, k)\phi_k,\\
((S^*S)^2\phi)_k &= (\Phi(k+p, k)^*\Phi(k+p, k))^2\phi_k \quad \text{for all } k \in \mathbb{Z}
\end{aligned}$$

by means of the weighted shift operators

$$(S^{*2}\phi)_k = \Phi(k+2p,k)^*\phi_{k+2p}, \qquad (S^2\phi)_k = \Phi(k,k-2p)\phi_{k-2p} \quad \text{for all } k \in \mathbb{Z}.$$

(a) Because of

$$\langle\!\langle S^*S\phi - SS^*\phi, \phi\rangle\!\rangle = \sum_{k\in\mathbb{Z}} \langle (\Phi(k+p,k)^*\Phi(k+p,k) - \Phi(k,k-p)\Phi(k,k-p)^*)\phi_k, \phi_k\rangle$$

the relation $S^*S - SS^* \geq 0$ holds, if and only if (B.5) is satisfied.

(b) Thanks to the relation

$$\langle\!\langle S^{*2}S^2\phi - (S^*S)^2\phi, \phi\rangle\!\rangle = \sum_{k\in\mathbb{Z}} \langle \left[\Phi(k+2p,k)^*\Phi(k+2p,k) - (\Phi(k+p,k)^*\Phi(k+p,k))^2\right]\phi_k, \phi_k\rangle$$

the inequality $S^{*2}S^2 \geq (S^*S)^2$ is necessary and sufficient for (B.6)

(c) The relation $S^{*2}S^2 - 2rS^*S\phi + r^2 \operatorname{id}_{\mathbb{C}^d} \geq 0$ for all $r > 0$ characterizes (B.7), due to

$$\begin{aligned}\langle\!\langle (S^{*2}S^2)\phi - 2rS^*S\phi + r^2\phi, \phi\rangle\!\rangle &= \sum_{k\in\mathbb{Z}} \langle \Phi(k+2p,k)^*\Phi(k+2p,k)\phi_k, \phi_k\rangle \\ &\quad - 2r\sum_{k\in\mathbb{Z}} \langle \Phi(k+p,k)^*\Phi(k+p,k)\phi_k, \phi_k\rangle + r^2\sum_{k\in\mathbb{Z}} \langle \phi_k, \phi_k\rangle\end{aligned}$$

and this completes the proof. □

Proposition B.9 *Let $p \in \mathbb{N}_0$ and $A \in \mathcal{L}^\infty(\mathbb{C}^d)$. The adjoint shift operator T_A^{*p} is*
(a) hyponormal, if and only if for all $k \in \mathbb{Z}$ one has

$$\Phi(k+p,k)\Phi(k+p,k)^* \geq \Phi(k+2p,k+p)^*\Phi(k+2p,k+p),$$

(b) of class A, if and only if for all $k \in \mathbb{Z}$ one has

$$\Phi(k+2p,k)\Phi(k+2p,k)^* \geq \left[\Phi(k+2p,k+p)\Phi(k+2p,k+p)^*\right]^2,$$

(c) paranormal, if and only if for all $k \in \mathbb{Z}$ and $r > 0$ it holds

$$\Phi(k+2p,k)\Phi(k+2p,k)^* - 2r\Phi(k+2p,k+p)\Phi(k+2p,k+p) + r^2 \operatorname{id}_{\mathbb{C}^d} \geq 0.$$

Proof W.l.o.g. we suppose $p \in \mathbb{N}$. The abbreviation $R := T_A^{*p}$ and Prop. B.4(f) yields

$$\begin{aligned}(R^*R\phi)_k &= \Phi(k,k-p)\Phi(k,k-p)^*, & (RR^*\phi)_k &= \Phi(k+p,k)^*\Phi(k+p,k), \\ (R^{*2}R^2\phi)_k &= \Phi(k,k-2p)\Phi(k,k-2p)^*, & (R^*R)^2 &= (\Phi(k,k-p)\Phi(k,k-p)^*)^2\end{aligned}$$

for all $k \in \mathbb{Z}$ and arbitrary sequences $\phi \in \ell^2$.

(a) results from the identity

$$\langle\!\langle (R^*R - RR^*)\phi, \phi\rangle\!\rangle = \sum_{k\in\mathbb{Z}} \langle (\Phi(k,k-p)\Phi(k,k-p)^* - \Phi(k+p,k)^*\Phi(k+p,k))\phi_k, \phi_k\rangle.$$

(b) is a consequence of

$$\langle\!\langle (R^{*2}R^2 - (R^*R)^2)\phi, \phi\rangle\!\rangle = \sum_{k\in\mathbb{Z}} \langle (\Phi(k,k-2p)\Phi(k,k-2p)^* - [\Phi(k,k-p)\Phi(k,k-p)^*]^2)\phi_k, \phi_k\rangle.$$

(c) can be seen from

$$\begin{aligned}&\langle\!\langle (R^{*2}R^2 - 2rR^*R + r^2)\phi, \phi\rangle\!\rangle \\ &\quad = \sum_{k\in\mathbb{Z}} \langle (\Phi(k,k-2p)\Phi(k,k-2p)^* - 2r\Phi(k,k-2p)\Phi(k,k-p)^* + r^2)\phi_k, \phi_k\rangle\end{aligned}$$

and this concludes the proof. □

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