Bifurcations in a periodic discrete-time environment

Christian Pötzsche

Institut für Mathematik, Alpen-Adria Universität Klagenfurt, 9020 Klagenfurt, Austria

Abstract

This paper investigates local and global bifurcation, as well as continuation properties for discrete-time periodic dynamical models in arbitrary (finite) dimension. Our focus is to provide explicitly verifiable conditions which guarantee or prevent bifurcations of, say ω_1-periodic solutions for ω_0-periodic difference equations. In doing so, we give concrete branching relations ensuring bifurcations of e.g. fold, transcritical, pitchfork or flip type, including information on the global branches. Beyond that we obtain formulas indicating the local behavior of mean population sizes under parameter variation or bifurcation, and furthermore tackle stability issues. Our results are applied to various real-world population models.

Thus, the paper will be useful for a thorough analysis and understanding of general periodic time-discrete models in population dynamics, life sciences and beyond.

Keywords: Bifurcation, Global Continuation, Periodic difference equation, Floquet spectrum, Population dynamics
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1. Motivation

For a multitude of problems in population dynamics and other fields of mathematical biology, a realistic and therefore successful description is achieved using discrete-time models, i.e. difference equations. This notably concerns scenarios with no overlap between successive generations and classical examples include fisheries (for instance, the Beverton-Holt model [6], or the Ricker model [43]), but we also refer to more recent approaches like the celebrated flour beetle equation from [11], as well as the studies [12, 13, 15, 19, 20, 21]. In many of these models, a realistic influence of the environment on the population is hardly constant over time. Indeed, a periodic time dependence is well-motivated due to extrinsic seasonal influences like the day-night-cycle or effects on longer time scales (seasons, hibernation, harvesting). For this reason the seemingly classical topic of periodic difference equations and their solutions became a contemporary field of research over the recent years.

Particular interest has been devoted to questions of *resonance* and *attenuation* (cf. [21, 22, 18] or [41] in continuous time). These two concepts mean an increase (resp.

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decrease) for the average size of a population in response to an increase in the amplitude of an environment oscillation in comparison to a constant environment. Indeed, a corresponding qualitative theory is unquestionable of high ecological interest, since it guarantees that a fluctuating habitat is deleterious or stimulating to a population in the sense that the average population size is less resp. greater in a periodically oscillating environment than in a constant habitat. As special case, such periodic environments occur when (periodic) harvesting strategies are applied to a discrete-time model (see, e.g., [5] for the Beverton-Hold equation). Beyond that periodic forcing might also be helpful in the fields of global stabilization (cf. [9, 28] and the references therein) or chaos control (see [10, 17] for autonomous examples).

For a rigorous mathematical description of the above issues, a slight generalization of the classical theory of discrete dynamical systems is necessary. Above all, the adequate and natural invariant objects to investigate in this setting are periodic solutions rather than equilibria as previously in the usual time-invariant case. An appropriate spectral theory yielding stability, instability or hyperbolicity is given in terms of Floquet multipliers. The required Floquet theory for linear difference equations can be found in various textbooks like [16, pp. 153ff], [27, pp. 90ff], [35, pp. 108ff] or [14] — in particular, see the thesis [23] for the noninvertible situation. Finally, surveys of results on periodic difference equations are [14, 31].

Biologically motivated questions of continuation and bifurcation were previously tackled in [4, 5, 12, 15, 20, 21, 22] or [19, 30]. More detailed, [22] shows that hyperbolic fixed points persist as periodic solutions under a periodic stimulation and gives criteria that the reference solutions resonate or attenuate locally. A local transcritical bifurcation result is due to [20], where periodic solutions bifurcate from the trivial branch, and a related global version can be found in [12]. Attenuation and resonance under 2-periodic forcing are systematically studied in [18, 21].

Besides these interesting case studies, often restricted to scalar equations, we found it hard to locate explicit and flexible results specifically designed for periodic discrete dynamical systems in $\mathbb{R}^d$ and covering various bifurcation types. Thus, the novelty in this research paper is to provide a comprehensive approach to persistence, as well as fold, transcritical, pitchfork and flip bifurcations of periodic solutions to higher dimensional periodic difference equations, including information on the global solution branches. We present easily applicable corresponding criteria. Their verification, nevertheless, requires to solve two typically highly nonlinear problems first: (i) The computation of a reference invariant object (i.e. the periodic orbit) to persist or bifurcate, and (ii) the eigenvalue problem to determine its Floquet spectrum — both ask for numerical tools (cf. [2, 44]) in real-world problems. In addition, a possibly large period involves many parameters and therefore a high codimension in the sense of bifurcation theory. Nevertheless, provided the periodic orbit and its Floquet multipliers are known, it remains to evaluate (sums of) partial derivatives, which is an easy endeavor in the modern age of computer algebra.

Throughout, we are interested in ω_0-periodic difference equations

$$\boxed{x_{k+1} = f_k(x_k, \lambda),} \tag{Δ_λ}$$

where the C^m-mappings $f_k : \Omega \times \Lambda \to \mathbb{R}^d$, $k \in \mathbb{Z}$, $m \in \mathbb{N}$, are defined on nonempty open convex subsets $\Omega \subseteq \mathbb{R}^d$, $\Lambda \subseteq \mathbb{R}^p$, depend on a parameter $\lambda \in \Lambda$ and fulfill

$$f_k = f_{k+\omega_0} \quad \text{for all } k \in \mathbb{Z}$$

with some *basic period* $\omega_0 \in \mathbb{N}$. Clearly, an autonomous difference eqn. (Δ_λ) has period $\omega_0 = 1$. Moreover, the fact that (Δ_λ) is defined for all times $k \in \mathbb{Z}$ is no restriction, since by virtue of an ω_0-periodic extension our results remain applicable for equations defined only on semiaxes. Rather than being open, it suffices that the set $\Omega \times \Lambda$ has an open neighborhood on which the f_k are sufficiently smooth. This setting includes the typical situation in population biology, where x_k is a vector of population densities or sizes, and hence Ω is the cube $[0,1]^d$ resp. the nonnegative orthant $[0,\infty)^d$.

In common with [20, 12, 22], our presentation has a more functional analytical flavor than the dynamical systems approaches of e.g. [14, 19]. It is largely based on abstract branching theory (cf., for example, [25, 46, 47]), rather than geometric reduction concepts as attractive invariant manifolds. Although we merely apply branching and continuation results for general parameter-dependent equations in Banach spaces from [25, 45, 46, 47], our obtained special cases or concretizations appear to be of interest and feature certain advantages:

- First, they yield an alternative approach to the method used in e.g. [31, 32] and particularly we do not have to compute derivatives of composite mappings, which can be tedious in higher-dimensions. Here our techniques work, as long as we aim to select ω_1-periodic solutions bifurcating into ω-periodic solutions, where the period ω is a multiple of both ω_0 and ω_1.

- To consider difference equations of arbitrary finite dimension right from the beginning hardly causes extra effort. For instance, Thm. 4.3 follows from a quite recent result on global bifurcations in [45].

- It is worth to point out that the bifurcations studied here are not restricted to stability changes from asymptotically stable to unstable, or vice versa. Indeed, also unstable solutions can bifurcate into unstable solutions, where the branching process goes hand in hand with a change in the respective Morse indices, i.e., the dimension of the unstable manifolds associated to the periodic solutions.

An exception to our framework is the Sacker-Neimark bifurcation (cf. [26]), where a whole closed invariant curve rather than a single solution bifurcates; hence, it does not fit into our technical set-up.

We organize this paper in a tutorial way split into three parts. This means the reader primarily interested in applications and applicability, does not have to dive too deeply into the mathematical formulation and machinery. In this spirit, the subsequent Sect. 2 tackles minimal periods of solution branches, provides criteria for stability of periodic solutions to general periodic difference eqns. (Δ_λ) and summarizes some basic terminology. Conditions that solutions, their period and stability persist under parameter variation can be found in Sect. 3, were we also present a result on global continuation resp. the structure of global solution branches. Beyond that a condition for local attenuation or resonance is given. We refer to Sect. 4 for explicit sufficient conditions that periodic solutions bifurcate; they include local bifurcations of fold (saddle-node), transcritical and pitchfork type, as well as remarks on the flip bifurcation. We additionally provide criteria to determine the stability of bifurcating solutions and give information on the global structure of the branches. For a transcritical bifurcation we can check attenuation or resonance locally. As our second part, these results are illustrated in Sect. 5 by means of analytical studies and simulations on scalar and higher-dimensional models — some of them are periodic extensions of problems studied previously in [8, 13, 19, 30]. Finally, as third part the mathematical proofs are summarized in Sect. 7.

As a conclusion we put our approach into the context of a recent nonautonomous bifurcation theory (cf. [36, 37, 40]) dealing with arbitrary rather than merely periodic time-dependencies. In the first instance, a different spectral theory is required, which is based on exponential dichotomies rather than the Floquet spectrum. Second, the abstract branching tools [25, 46, 47] used here are also applicable to guarantee branches of bounded solutions, but require a different Fredholm theory; for instance in the bifurcation criteria from [36] only unstable solutions can bifurcate. On the other hand, the present periodic special case allows a finer insight and a much more detailed description of the bifurcation scenarios, and in particular stability and global assertions.

2. Periodic difference equations

We work with mappings $f_k(\cdot,\lambda)$, $\lambda\in\Lambda$, rather than homeomorphisms as the right-hand side of (Δ_λ). Hence, in general only forward solutions of (Δ_λ) exist. Such a unique forward solution to (Δ_λ) satisfying the *initial condition* $x_\kappa=\xi$ with initial time $\kappa\in\mathbb{Z}$, initial state $\xi\in\Omega$ and parameter $\lambda\in\Lambda$, is called *general solution*, will be denoted by $\varphi_\lambda(\cdot;\kappa,\xi)$ and reads as

$$\varphi_\lambda(k;\kappa,\xi)=\begin{cases}f_{k-1}(\cdot,\lambda)\circ\ldots\circ f_\kappa(\cdot,\lambda)(\xi), & k>\kappa,\\ \xi, & k=\kappa\end{cases}$$

for integers $k\in\mathbb{Z}_\kappa^+:=\{n\in\mathbb{Z}:\ \kappa\le n\}$, as long as the above compositions remain in the *state space* $\Omega\subseteq\mathbb{R}^d$. Furthermore, the map $(\xi,\lambda)\mapsto\varphi_\lambda(k;\kappa,\xi)$ inherits its smoothness from the right-hand side f_k. Thanks to the intrinsic ω_0-periodicity of (Δ_λ) we have the translation invariance property

$$\varphi_\lambda(k+n\omega_0;\kappa+n\omega_0,\xi)=\varphi_\lambda(k;\kappa,\xi)\quad\text{for all } n\in\mathbb{Z},\ \kappa\le k,\ \xi\in\Omega \tag{2.1}$$

and $\lambda\in\Lambda$ (cf. [35, p. 68, Prop. 2.5.3 and p. 22, Prop. 1.4.4]).

2.1. Branches of periodic solutions

Typically parameter-dependent eqns. (Δ_λ) have whole *branches* $\phi(\lambda)$ of, for instance, ω_1-periodic solutions. For ω_1 being a multiple of the basic period ω_0 in (Δ_λ), such a branch can be determined as solutions $x_\lambda^*\in\Omega$ to the fixed point equations

$$x_\lambda^*=\varphi_\lambda(\kappa+\omega_1,\kappa,x_\lambda^*)\quad\text{for all } \lambda\in\Lambda$$

in Ω via $\phi(\lambda)_k := \varphi_\lambda(k;\kappa,x_\lambda^*)$ for all $k \geq \kappa$ and with an ω_1-periodic continuation to the whole axis $\mathbb{Z}$. Nonetheless, in general an explicit computation of x_λ^* is possible only on a numerical basis, e.g. using appropriate continuation methods (cf. [2]).

In order to describe periodic solutions, given $\omega \in \mathbb{N}$ we introduce the set

$$\ell_\omega(\Omega) := \{\phi = (\phi_k)_{k\in\mathbb{Z}} : \phi_k \in \Omega \text{ and } \phi_k = \phi_{k+\omega} \quad \text{for all } k \in \mathbb{Z}\}$$

and abbreviate $\ell_\omega := \ell_\omega(\mathbb{R}^d)$ for the linear space of ω-periodic sequences. One has the embedding $\ell_\omega(\Omega) \subseteq \ell_{m\omega}(\Omega)$ for all $m \in \mathbb{N}$, i.e. a constant or ω-periodic sequence is also $m\omega$-periodic. Indeed, ℓ_ω is isomorphic to $\mathbb{R}^{d\omega}$ by means of the isomorphisms

$$J_\kappa : \ell_\omega \to \mathbb{R}^{d\omega}, \qquad J_\kappa\phi := (\phi_\kappa, \dots, \phi_{\kappa+\omega-1}) \quad \text{for all } \kappa \in \mathbb{Z} \tag{2.2}$$

with the inverses $J_\kappa^{-1}(x_0, \dots, x_{\omega-1}) := x_{\kappa+\cdot \mod \omega}$.

We close this subsection with some consideration on difference equations

$$\boxed{x_{k+1} = g(x_k, \eta_k),} \tag{g_η}$$

where $g : \Omega \times \Lambda \to \mathbb{R}^d$ is of class C^1 and $\eta \in \ell_{\omega_0}(\Lambda)$ denotes a parameter sequence. The following result is inspired by [21, Thm. 3] (see also [31, Cor. 7]):

Proposition 2.1. *Let $U \subseteq \ell_{\omega_0}(\Lambda)$, $\eta^* \in U$ and $\phi^* \in \ell_{\omega_1}(\Omega)$ be a solution to (g_{η^*}), where ω_0, ω_1 are minimal periods. If $\phi : U \to \ell_\omega(\Omega)$ is a continuous solution branch to (g_η) with minimal period ω and $\phi(\eta^*) = \phi^*$, then:*

(a) ω is a multiple of ω_1,

(b) provided g fulfills the injectivity assumption

$$g(\phi_k^*, \lambda) = g(\phi_k^*, \bar\lambda) \quad \Rightarrow \quad \lambda = \bar\lambda \quad \textit{for all } \kappa \leq k < \kappa + \omega_1 \tag{2.3}$$

and $\lambda, \bar\lambda \in \Lambda$, then there exists a $\rho > 0$ such that every solution $\phi(\eta)$ to (g_η) with $\eta \in U$ satisfying $\max_{k\in\mathbb{Z}} \|\eta_k - \eta_k^*\| \leq \rho$ has minimal period $\operatorname{lcm}\{\omega_0, \omega_1\}$, i.e. $\omega = \operatorname{lcm}\{\omega_0, \omega_1\}$.

2.2. Periodic variational equations

We continue to introduce prerequisites on periodic difference eqns. (Δ_λ). Given a solution branch $\phi(\lambda) \in \ell_{\omega_1}(\Omega)$, $\lambda \in \Lambda$, one defines the associate *variational equation*

$$\boxed{x_{k+1} = D_1 f_k(\phi(\lambda)_k, \lambda) x_k} \tag{V_λ}$$

and the *transition operator* $\Phi_\lambda : \{(k,l) \in \mathbb{Z}^2 : l \leq k\} \to \mathbb{R}^{d\times d}$ as product

$$\Phi_\lambda(k,l) := \begin{cases} D_1 f_{k-1}(\phi(\lambda)_{k-1}, \lambda) \cdots D_1 f_l(\phi(\lambda)_l, \lambda), & l < k, \\ I_d, & k = l; \end{cases}$$

the general forward solution of (V_λ) is $\Phi_\lambda(\cdot, \kappa)\xi : \mathbb{Z}_\kappa^+ \to \mathbb{R}^d$ for $\kappa \in \mathbb{Z}$, $\xi \in \mathbb{R}^d$.

For fixed parameters $\lambda \in \Lambda$, we point out that the variational eqn. (V_λ) is ω-periodic with $\omega = \operatorname{lcm}(\omega_0, \omega_1)$ and thus stability properties of (V_λ) as well as of $\phi(\lambda)$ are determined by the *period matrix*

$$\begin{aligned} \Xi_\omega(\lambda) &:= \Phi_\lambda(\kappa+\omega,\kappa) \\ &= D_1 f_{\kappa+\omega-1}(\phi(\lambda)_{\kappa+\omega-1},\lambda)\cdots D_1 f_\kappa(\phi(\lambda)_\kappa,\lambda) \in \mathbb{R}^{d\times d}. \end{aligned} \tag{2.4}$$

Its eigenvalues are called *Floquet multipliers* of a solution $\phi(\lambda) \in \ell_\omega(\Omega)$. The *Floquet spectrum* of $\phi(\lambda)$ is the set of all eigenvalues for $\Xi_\omega(\lambda)$, i.e.

$$\sigma_\omega(\lambda) := \sigma(\Xi_\omega(\lambda)) = \sigma(\Phi_\lambda(\kappa+\omega,\kappa)) \quad \text{for all } \lambda \in \Lambda.$$

The *multiplicity* of a Floquet multiplier ν is the dimension of the corresponding eigenspace $N(\nu I_d - \Xi_\omega(\lambda)) \subseteq \mathbb{R}^d$ and a *simple* Floquet multiplier has multiplicity 1. Hence, the problem to obtain the *critical* parameter values λ^* with $1 \in \sigma_\omega(\lambda^*)$ requires to find the roots to a polynomial of order ωd.

Proposition 2.2. *The Floquet spectrum $\sigma_\omega(\lambda)$ is independent of the initial time $\kappa \in \mathbb{Z}$. Moreover, one has $\sigma_{n\omega}(\lambda) = \sigma_\omega(\lambda)^n$ for all $n \in \mathbb{N}$.*

Under the assumption $\sigma_\omega(\lambda) \cap \mathbb{S}^1 = \emptyset$ a solution $\phi(\lambda) \in \ell_{\omega_1}$ to (Δ_λ) is called *hyperbolic* and for each multiplier $\nu \in \sigma_\omega(\lambda)$ we write $X_\nu(\lambda) \subseteq \mathbb{R}^d$ for the generalized eigenspace associated to ν. The *Morse index* of $\phi(\lambda)$ is the dimension of the direct sum of all linear spaces $X_\nu(\lambda)$ corresponding to Floquet multipliers ν with modulus $|\nu| > 1$. On this basis the following facts are well-known:

Theorem 2.3. *Let $\lambda \in \Lambda$ be fixed. A solution $\phi(\lambda) \in \ell_{\omega_1}$ of (Δ_λ) is*

(a) (uniformly) asymptotically stable, if $\sigma_\omega(\lambda) \subseteq B_1(0)$,

(b) unstable, if there exists a Floquet multiplier $\nu \in \sigma_\omega(\lambda)$ with $|\nu| > 1$.

Remark 2.1. (1) We work with periodic solutions to difference eqns. (Δ_λ) (i.e. sequences in Ω) rather than orbits (meaning subsets of Ω). Yet, for fixed parameters $\lambda \in \Lambda$, an ω-periodic solution $\phi = (\phi_k)_{k\in\mathbb{Z}}$ to (Δ_λ) is asymptotically stable, if and only if the orbit $O(\phi_\kappa) = \{\phi_k \in \Omega : k \in \mathbb{Z}\} = \{\phi_k \in \Omega : \kappa \le k < \kappa+\omega\}$ is an asymptotically stable set, i.e. it is both

- *stable* in the sense that for any neighborhood U of the orbit $O(\phi_\kappa) \subseteq \Omega$ there are neighborhoods U_l of each ϕ_l, $\kappa \le l < \kappa+\omega$, such that for all $x \in U_l$ one has

$$\varphi_\lambda(k; l, x) \in U \quad \text{for all } \kappa \le l < \kappa+\omega,\ l \le k$$

- *attractive* in the sense that there exist neighborhoods V_l of ϕ_l such that

$$\lim_{k\to\infty} \operatorname{dist}(\varphi_\lambda(k; l, x), O(\phi_\kappa)) = 0 \quad \text{for all } \kappa \le l < \kappa+\omega,\ x \in V_l.$$

This is shown in [7, Lemma 3], as well as a statement on merely stability.

(2) A stability analysis of periodic solutions to (Δ_λ) in the critical case of Floquet multipliers ν^* on the complex unit circle $\mathbb{S}^1$ requires a center manifold reduction. It allows to simplify the d-dimensional periodic problem (Δ_λ) to n difference eqns., where $n \le d$ is the dimension of the generalized eigenspaces $X_{\nu^*}(\lambda)$; we refer to e.g. [39] for details.

3. Continuation in periodic equations

It is a folklore and generically valid result that the asymptotic behavior of an autonomous system does not change essentially, if parameters are perturbed by small periodic (in fact even bounded) sequences; the precise assumption for this is a weakened form of hyperbolicity. We present corresponding conditions yielding that an ω_1-periodic solution ϕ^* to an ω_0-periodic eqn. (Δ_{λ^*}) persists under variation of the parameter λ near a fixed value λ^*. We begin with an amalgamation of both [34, Thm. 3.11] and [38, Thm. 2.11]:

Theorem 3.1. *Let $\lambda^* \in \Lambda$, $\omega_1 \in \mathbb{N}$ and $\omega := \operatorname{lcm}(\omega_0, \omega_1)$. If ϕ^* is an ω_1-periodic solution of (Δ_{λ^*}) satisfying the* weak hyperbolicity condition

$$1 \notin \sigma_\omega(\lambda^*), \tag{3.1}$$

then there exist $\rho, \varepsilon > 0$ and a C^m-function $\phi : B_\rho(\lambda^*) \to B_\varepsilon(\phi^*) \subseteq \ell_\omega(\Omega)$ such that the following holds for all $\lambda \in B_\rho(\lambda^*)$:

(a) $\phi(\lambda^*) = \phi^*$ and

$$\phi'(\lambda^*) = \Phi_{\lambda^*}(\cdot, \kappa)\xi_\kappa + \sum_{l=\kappa}^{\cdot-1} \Phi_{\lambda^*}(\cdot, l+1) D_2 f_l(\phi_l^*, \lambda^*) \tag{3.2}$$

with $\xi_\kappa := [I_d - \Xi_\omega(\lambda^*)]^{-1} \sum_{j=\kappa}^{\kappa+\omega-1} \Phi_{\lambda^*}(\kappa+\omega, j+1) D_2 f_j(\phi_j^*, \lambda^*)$,

(b) $\phi(\lambda)$ is the unique ω-periodic solution of (Δ_λ) in $B_\varepsilon(\phi^*)$,

(c) in case the solution ϕ^* is even hyperbolic, then also $\phi(\lambda)$ is hyperbolic with the same Morse index as ϕ^*.

Remark 3.1. (1) In our assertion (c) the local constancy of the Morse index particularly means that asymptotically stable or unstable solutions ϕ^* retain their stability properties under small perturbations of λ^*.

(2) To estimate perturbation bounds, in applications it might be relevant to obtain information on the size of $\rho, \varepsilon > 0$. This can be done on basis of a quantitative implicit function theorem as in [38, Cor. 2.19].

If an eqn. (Δ_λ) is a model from population dynamics, then a central question is to understand the effect of parameter fluctuations on the total or individual mean population sizes. More precisely, for a sequence $\phi \in \ell_\omega$ with values in $\mathbb{R}^d$ and component sequences ϕ^i, $1 \leq i \leq n$, in $\mathbb{R}$, we introduce its *individual* resp. its *total mean value*

$$M(\phi) := \frac{1}{\omega} \sum_{k=0}^{\omega-1} \phi_k, \qquad \hat{M}(\phi) := \frac{1}{\omega} \sum_{k=0}^{\omega-1} \sum_{i=1}^{d} \phi_k^i = \sum_{i=1}^{d} M(\phi)^i. \tag{3.3}$$

When such a periodic sequence ϕ describes the evolution of interacting species, i.e. its components stand for population sizes of the individual species, then the components of the vector $M(\phi) \in \mathbb{R}^d$ contain the means size of each individual population, while the quantity $\hat{M}(\phi) \in \mathbb{R}$ indicates the mean total size of all interacting populations over a period interval. Local information near the reference parameter λ^* can be obtained from the formulas given in

Remark 3.2 (attenuation and resonance). Assume λ is a real parameter, i.e. $\Lambda \subseteq \mathbb{R}$, and consider the solution branch $\phi(\lambda)$ guaranteed by Thm. 3.1. Our goal is to understand how the individual resp. total mean values

$$m(\lambda) := M(\phi(\lambda)), \qquad \hat{m}(\lambda) := \hat{M}(\phi(\lambda)),$$

behave locally under variation of λ. Thereto, we determine the derivatives $m'(\lambda^*) \in \mathbb{R}^d$ and $\hat{m}'(\lambda^*) \in \mathbb{R}$, since their signs indicate monotonicity properties for the individual resp. total population means. Both the mappings $M : \ell_\omega \to \mathbb{R}^d$ and $\hat{M} : \ell_\omega \to \mathbb{R}$ defined in (3.3) are linear, thus $m'(\lambda) = M(\phi'(\lambda))$, $\hat{m}'(\lambda) = \hat{M}(\phi'(\lambda))$ holds and

$$\begin{aligned} m'(\lambda^*) &= \frac{1}{\omega} \sum_{k=\kappa}^{\kappa+\omega-1} \left(\Phi_{\lambda^*}(k,\kappa)\xi_\kappa + \sum_{l=\kappa}^{k-1} \Phi_{\lambda^*}(k,l+1) D_2 f_l(\phi_l^*, \lambda^*) \right), \\ \hat{m}'(\lambda^*) &= \sum_{i=1}^{d} m'(\lambda^*)^i, \end{aligned}$$

using the explicit formula (3.2).

While condition (3.1) is weaker than hyperbolicity guaranteed by

$$\sigma_\omega(\lambda^*) \cap \mathbb{S}^1 = \emptyset, \tag{3.4}$$

the latter allows to exclude $n\omega$-periodic solutions near $\phi(\lambda)$ locally.

Corollary 3.2. *If additionally $1 \notin \sigma_\omega(\lambda^*)^n$ holds for some $n \in \mathbb{N}$, then there exists no $n\omega$-periodic solution to (Δ_λ) in $B_\varepsilon(\phi^*)$ besides $\phi(\lambda)$.*

Example 3.1 (autonomous Ricker equation). We consider the intrinsic growth rate $\lambda > 0$ as parameter in the *Ricker equation*

$$x_{k+1} = f(x_k, \lambda) := x_k e^{\lambda\left(1 - \frac{x_k}{K}\right)} \tag{3.5}$$

with fixed carrying capacity $K > 0$. It has two branches of 1-periodic, i.e. constant solutions $\phi_1(\lambda)_k \equiv 0$, $\phi_2(\lambda)_k \equiv K$ with respective linearizations $D_1 f(\phi_1(\lambda)_k, \lambda) \equiv e^\lambda > 1$ and $D_1 f(\phi_2(\lambda)_k, \lambda) \equiv 1 - \lambda$ on $\mathbb{Z}$. Note that (3.5) can be seen as an ω_0-periodic difference equation for any $\omega_0 \in \mathbb{N}$. We restrict to the branch $\phi_2(\lambda)$, which can be interpreted as family of ω_1-periodic solutions with any $\omega_1 \in \mathbb{N}$, and the period matrix becomes $\Xi_\omega(\lambda) = (1-\lambda)^\omega$ with $\omega = \mathrm{lcm}(\omega_0, \omega_1)$. Thus, Thm. 2.3 shows that $\phi_2(\lambda)$ is asymptotically stable for $\lambda \in (0, 2)$ (with Morse index 0) and unstable for $\lambda > 2$ (with Morse index 1). When ω is odd, condition (3.1) holds for all $\lambda > 0$ and there are no solutions of odd period near $\phi_2(\lambda)$. On the other hand, for $\lambda = 2$ and an even ω, the weak hyperbolicity condition (3.1) is violated; indeed there is a well-known flip bifurcation at $\lambda = 2$. We illustrate this in Fig. 1(left), where there are no further 1-periodic solutions near $\phi_2(\lambda)$ for $\lambda = 2$. Yet, there exist 2-periodic solutions and as the further diagrams of Fig. 1 underline, for values $\lambda > 2$ there are solutions of higher periods. This reflects the chaotic behavior of the Ricker map discussed in various papers (cf., e.g., [33]).

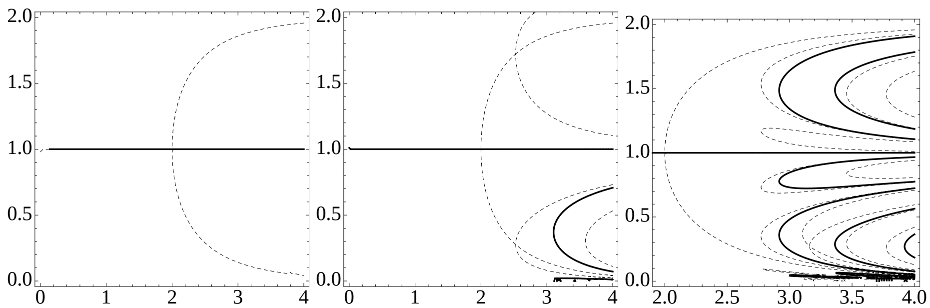

Figure 1: λ-x-plane to illustrate the solution set to the fixed-point eqn. $\varphi_\lambda(\omega, 0, x) = x$ for (3.5) in Ex. 3.1 with $K = 1$, which yield initial values for ω-periodic solutions:
Left: Fixed points for $\omega = 1$ (solid) and $\omega = 2$ (dashed)
Middle: Fixed points for $\omega = 3$ (solid) and $\omega = 4$ (dashed)
Right: Fixed points for $\omega = 5$ (solid) and $\omega = 6$ (dashed)

The subsequent result gives information on the global structure of periodic solutions to (Δ_λ).

Theorem 3.3. *Let $\Omega = \mathbb{R}^d$, $\lambda^* \in \Lambda = \mathbb{R}$, $\omega_1 \in \mathbb{N}$, $\omega := \operatorname{lcm}(\omega_0, \omega_1)$ and define the set of all ω-periodic solutions*

$$S_\omega := \{(\phi, \lambda) \in \ell_\omega \times \mathbb{R} : \phi \text{ solves } (\Delta_\lambda)\} \tag{3.6}$$

to a difference eqn. (Δ_λ). If ϕ^* is an ω_1-periodic solution of (Δ_{λ^*}) satisfying (3.1), then the connected component $C \subseteq S_\omega \subseteq \ell_\omega \times \mathbb{R}$ containing the local branch

$$\{(\phi(\lambda), \lambda) : \lambda \in (\lambda^* - \rho, \lambda^* + \rho)\}$$

from Thm. 3.1 fulfills at least one of the following assertions (cf. Fig. 2):

(a) There exist unbounded disjoint subsets $C^+, C^- \subseteq \ell_\omega \times \mathbb{R}$ satisfying

$$C = \{(\phi^*, \lambda^*)\} \cup C^- \cup C^+,$$

(b) $C \setminus \{(\phi^*, \lambda^*)\}$ is connected.

4. Bifurcations in periodic difference equations

As above let us assume that ω is a multiple of both the periods ω_0 and ω_1 to a difference eqn. (Δ_λ) resp. our reference solution ϕ^*.

In the previous Sect. 3 we learned that qualitative changes in the structure of ω-periodic solutions to (Δ_λ) can only occur when the weak hyperbolicity condition (3.1) is violated. From an applied perspective it is now crucial to locate parameter values λ^* giving rise to such changes and to understand them at least locally. Indeed, dynamically

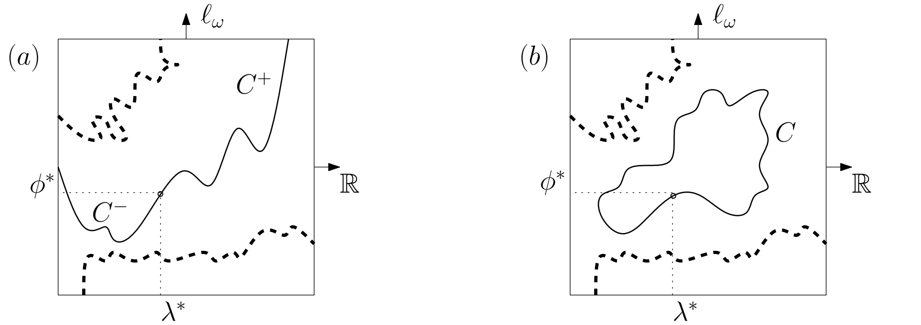

Figure 2: Structure of the global branch C (solid lines) of ω-periodic solutions to (Δ_λ) containing (ϕ^*, λ^*) according to Thm. 3.3: Case (a) with two unbounded components C^-, C^+ (left) and case (b) of a connected component C (right). Other components of S_ω are represented by dashed lines

more complex scenarios can occur in the neighborhood of periodic solutions if the weak hyperbolicity condition (3.1) is violated, i.e. for the critical case

$$1 \in \sigma_\omega(\lambda^*). \tag{4.1}$$

Actually, the existence of a Floquet multiplier 1 in (V_λ) is a necessary condition for a bifurcation of periodic solutions (cf. Thm. 3.1). More detailed, we say an ω-periodic solution ϕ^* to (Δ_{λ^*}) *bifurcates* at the parameter value $\lambda^* \in \Lambda$, if there exists a parameter sequence $(\lambda_n)_{n\in\mathbb{N}}$ with limit λ^* and distinct sequences $(\phi_n^1)_{n\in\mathbb{N}}$, $(\phi_n^1)_{n\in\mathbb{N}}$ of ω-periodic solutions to (Δ_{λ_n}) satisfying $\lim_{n\to\infty} \phi_n^1 = \lim_{n\to\infty} \phi_n^2$.

We again stress that this concept of a bifurcation is purely "algebraic" and independent of stability changes, which will be addressed separately. Making such a general concept more specific, we will describe bifurcations where the pair (ϕ^*, λ^*) is contained in a smooth branch $\Gamma \subseteq \ell_\omega(\Omega) \times \Lambda$ of ω-periodic solutions. This precisely means that there exists a $\rho > 0$, open convex neighborhoods $U \subseteq \ell_\omega(\Omega)$ of ϕ^*, $\Lambda_0 \subseteq \Lambda$ of λ^* and functions $\psi : (-\rho, \rho) \to U$, $\lambda : (-\rho, \rho) \to \Lambda_0$ such that

- $\psi(0) = \phi^*$, $\lambda(0) = \lambda^*$,
- each $\psi(s)$ is an ω-periodic solution of $(\Delta_{\lambda(s)})$ for all $s \in (-\rho, \rho)$.

For later use we can now abbreviate the solution branches

$$\Gamma := (\psi, \lambda)(-\rho, \rho), \qquad \Gamma^+ := (\psi, \lambda)(0, \rho), \qquad \Gamma^- := (\psi, \lambda)(-\rho, 0) \tag{4.2}$$

and assign them a stability property (asymptotically stable or unstable), if all solutions on them possess the respective stability characteristic.

In order to deduce corresponding sufficient conditions for bifurcation, given a parameter value $\lambda^* \in \Lambda$, let us proceed as follows:

(O) Suppose that ϕ^* is an ω_1-periodic solution to (Δ_{λ^*}).

(I) Choose orthonormal vectors $\xi_1, \dots, \xi_n \in \mathbb{R}^d$ such that

$$N(I_d - \Xi_\omega(\lambda^*)) = \operatorname{span}\{\xi_1, \dots, \xi_n\}, \tag{4.3}$$

i.e. the Floquet multiplier 1 has multiplicity n.

(II) Choose orthonormal vectors $\xi'_1, \dots, \xi'_r \in \mathbb{R}^d$ with (cf. [47, p. 294, Prop. 6(ii)])

$$N(I_d - \Xi_\omega(\lambda^*)^T) = R(I_d - \Xi_\omega(\lambda^*))^\perp = \operatorname{span}\{\xi'_1, \dots, \xi'_r\}. \tag{4.4}$$

In the context of this paper, orthonormality always refers to the Euclidean inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^d$, i.e. the *dot product* given by $\langle x, y\rangle := \sum_{j=1}^d x_j y_j$. Under the above assumptions (4.3) and (4.4) we denote λ^* as *critical value*.

Remark 4.1 (period doubling). Suppose that a solution $\phi^* \in \ell_\omega(\Omega)$ to (Δ_{λ^*}) has a Floquet multiplier ν with $\nu^l = 1$ for some $l \in \mathbb{N}$. Then Prop. 2.2 ensures $1 \in \sigma_{l\omega}(\lambda^*)$, which in turn means that the nonhyperbolicity condition (4.1) holds with the period ω replaced by the multiple $l\omega$. Hence, provided their further assumptions are satisfied, the following results yield that $l\omega$-periodic solutions to (Δ_λ) bifurcate from $\phi^* \in \ell_\omega$. In particular, for $l = 2$ (i.e. a Floquet multiplier -1) one speaks of a *flip* or *period-doubling bifurcation*.

Before proceeding to actual bifurcation results, we point out that our theory applies without any invertibility assumptions on the derivatives $D_1 f_k$ or the variational eqn. (V_λ). This requires to introduce the ambient notation

$$\hat{\Phi}_{\lambda^*}(\kappa, j)^T\xi := \left(J_\kappa^{-1}(\Phi_{\lambda^*}(\kappa, k)^T\xi)_{k=\kappa-\omega}^{\kappa-1}\right)_j \quad \text{for all } j \in \mathbb{Z} \tag{4.5}$$

and $\xi \in N(I_d - \Xi_\omega(\lambda^*)^T)$; in words, this means that $(\hat{\Phi}_{\lambda^*}(\kappa, j)^T\xi)_{j\in\mathbb{Z}}$ is the finite sequence $(\Phi_{\lambda^*}(\kappa, k)^T\xi)_{k=\kappa-\omega}^{\kappa-1}$ continued ω-periodically to the whole integer axis $\mathbb{Z}$. The interested reader might consult the appendix (see Rem. 7.2) to see that in case $D_1 f_k(\phi_k^*, \lambda^*) \in GL(\mathbb{R}^d)$, $\kappa \le k < \kappa + \omega$, this becomes

$$\hat{\Phi}_{\lambda^*}(\kappa, j)^T\xi = \Phi_{\lambda^*}(\kappa, j)^T\xi \quad \text{for all } j \in \mathbb{Z}.$$

4.1. Fold bifurcation

The first bifurcation scenario does not require that a whole solution branch to the difference eqn. (Δ_λ) is known in advance. Even though it is also known as *saddle-node bifurcation*, we prefer the terminology *fold bifurcation* since it does not suggest a stability change. Here, we restrict to a real parameter space Λ.

Theorem 4.1 (local fold bifurcation)**.** *Let $\Lambda \subseteq \mathbb{R}$, $m \ge 2$ and suppose 1 is a simple eigenvalue of $\Xi_\omega(\lambda^*)$ with $r = 1$. If*

$$\begin{aligned} g_{01} :=& \langle \xi'_1, D_2 f_{\kappa+\omega-1}(\phi^*_{\kappa+\omega-1}, \lambda^*)\rangle \\ &+ \sum_{j=\kappa+1}^{\kappa+\omega-1} \langle \hat{\Phi}_{\lambda^*}(\kappa, j)^T\xi'_1, D_2 f_{j-1}(\phi^*_{j-1}, \lambda^*)\rangle \neq 0, \end{aligned}$$

then there exists a bifurcating branch Γ *as in* (4.2) *with* C^{m-1}*-functions* ψ, λ *satisfying* $\psi'(0) = \Phi_{\lambda^*}(\cdot,\kappa)\xi_1$ *and* $\lambda'(0) = 0$*. Under the additional assumption*

$$\begin{aligned} g_{20} :=&\langle \xi_1', D_1^2 f_{\kappa+\omega-1}(\phi^*_{\kappa+\omega-1}, \lambda^*)[\Phi_{\lambda^*}(\kappa+\omega-1,\kappa)\xi_1]^2\rangle \\ &+ \sum_{j=\kappa+1}^{\kappa+\omega-1} \langle \hat{\Phi}_{\lambda^*}(\kappa,j)^T \xi_1', D_1^2 f_{j-1}(\phi^*_{j-1},\lambda^*)[\Phi_{\lambda^*}(j-1,\kappa)\xi_1]^2\rangle \neq 0, \end{aligned}$$

the solution ϕ^* *of* (Δ_{λ^*}) *bifurcates at* λ^**, it is* $\lambda''(0) = -\frac{g_{20}}{g_{01}}$ *and one has locally in* $U \times \Lambda_0$ *(cf. Fig. 3):* ϕ^* *is the unique solution of* (Δ_{λ^*}) *in* $\ell_\omega(\Omega)$ *and*

- *(c)* Subcritical case*: If* $g_{20}/g_{01} > 0$*, then* (Δ_λ) *has no* ω*-periodic solution for* $\lambda > \lambda^*$ *and exactly two distinct* ω*-periodic solutions for* $\lambda < \lambda^*$*.*
- *(d)* Supercritical case*: If* $g_{20}/g_{01} < 0$*, then* (Δ_λ) *has no* ω*-periodic solution for* $\lambda < \lambda^*$ *and exactly two distinct* ω*-periodic solutions for* $\lambda > \lambda^*$*.*

If the Floquet multiplier 1 is the unique element of $\sigma_\omega(\lambda^*)$ on the complex unit circle, while the remaining Floquet spectrum is contained $B_1(0)$, i.e.

$$\sigma_\omega(\lambda^*) \setminus \{1\} \subseteq B_1(0), \tag{4.6}$$

then a bifurcation goes hand in hand with a stability change for ϕ^*. Indeed, according to the stability exchange principle, one of the bifurcating branches of ω-periodic solutions in (4.2) is asymptotically stable, while the other one is unstable. Stability properties of the solution ϕ^* to (Δ_{λ^*}) itself, can be obtained on the basis of Rem. 2.1(2) in Sect. 2.2.

Corollary 4.2. *Under* (4.6) *one additionally has (cf. Fig. 3):*

- *(a) If* $g_{20} > 0$*, then* Γ^- *is asymptotically stable and* Γ^+ *is unstable.*
- *(b) If* $g_{20} < 0$*, then* Γ^+ *is asymptotically stable and* Γ^- *is unstable.*

4.2. Bifurcation along solution branches

The above assumptions allow us to formulate further bifurcation results. All of them require a strengthening of condition (O) to the existence of a constant solution branch to (Δ_λ), i.e. in Sect. 4.2 we require

- (O') Assume that ϕ^* is an ω_1-periodic solution to (Δ_λ) for all $\lambda \in \Lambda$, i.e. we have a constant solution branch $\Gamma^* := \{(\phi^*, \lambda) \in \ell_\omega(\Omega) \times \Lambda\}$.

In applications one is often confronted with the situation that a non-constant branch $\phi(\lambda)$ of ω_1-periodic solutions to (Δ_λ) is given. This situation, however, can be reduced to the assumption (O') as follows: Rather than (Δ_λ) one considers the associated *equation of perturbed motion* given by

$$x_{k+1} = f_k(x_k + \phi(\lambda)_k, \lambda) - f_k(\phi(\lambda)_k, \lambda) =: \hat{f}_k(x_k,\lambda), \tag{$\hat{\Delta}_\lambda$}$$

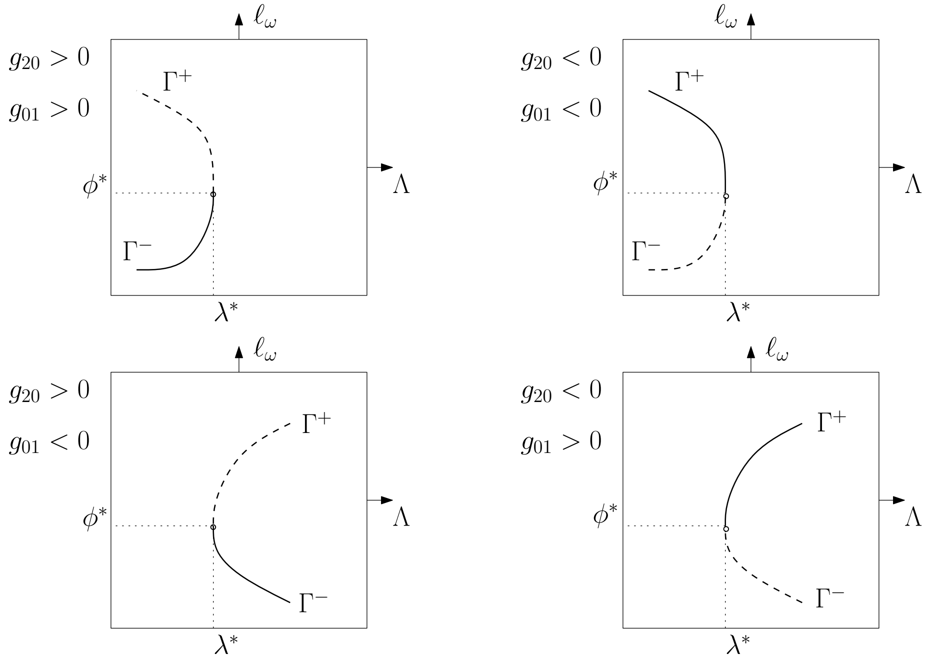

Figure 3: Local subcritical (top) and supercritical (bottom) fold bifurcation of ω-periodic solutions to (Δ_λ) described in Thm. 4.1 and exchange of stability between the branches Γ^+ and Γ^- from unstable (dashed line) to asymptotically stable (solid) covered in Cor. 4.2

which is $\hat{\omega}_0$-periodic with $\hat{\omega}_0 = \operatorname{lcm}(\omega_0, \omega_1)$. Then the following results are applicable to $(\hat{\Delta}_\lambda)$ with ω_0 and f_k replaced by $\hat{\omega}_0$ and $\hat{f}_k$, resp., and the trivial solution as constant solution branch ϕ^*. Here, $\phi : \Lambda \to \ell_{\omega_1}$ has to be smooth.

We retreat to a simple Floquet-multiplier 1 and real parameter spaces. The first result tackles the global structure of the solution set $S_\omega \subseteq \ell_\omega(\Omega) \times \Lambda$ (defined by (3.6) in Thm. 3.1) to (Δ_λ) near a bifurcation point:

Theorem 4.3 (bifurcation with simple Floquet multiplier)**.** *Let $\Lambda \subseteq \mathbb{R}$, $m \geq 2$ and suppose 1 is a simple eigenvalue of $\Xi_\omega(\lambda^*)$ with $r = 1$. If the* transversality condition

$$\begin{aligned} g_{11} :=& \langle \xi_1', D_1 D_2 f_{\kappa+\omega-1}(\phi^*_{\kappa+\omega-1}, \lambda^*) \Phi_{\lambda^*}(\kappa+\omega-1, \kappa)\xi_1 \rangle \\ &+ \sum_{j=\kappa+1}^{\kappa+\omega-1} \langle \hat{\Phi}_{\lambda^*}(\kappa, j)^T \xi_1', D_1 D_2 f_{j-1}(\phi^*_{j-1}, \lambda^*) \Phi_{\lambda^*}(j-1, \kappa)\xi_1 \rangle \neq 0 \end{aligned} \tag{4.7}$$

is satisfied, then the solution ϕ^* of (Δ_{λ^*}) bifurcates at λ^* and there exists a bifurcating branch Γ as in (4.2) *with C^{m-1}-functions ψ, λ satisfying:*

(a) $\psi'(0) = \Phi_{\lambda^*}(\cdot, \kappa)\xi_1$,

(b) each $\psi(s)$, $s \neq 0$, is an ω-periodic solution of $(\Delta_{\lambda(s)})$, distinct from ϕ^*.

Moreover, Γ is contained in a connected component C of $\overline{\{(\phi,\lambda)\in S_\omega:\ \phi\neq\phi^*\}}$ with precisely one of the properties (cf. Fig. 4):

(c) C intersects the boundary $\partial(\ell_\omega(\Omega)\times\Lambda)$ or C is unbounded,

(d) C contains an ω-periodic solution ϕ^* to (Δ_{λ_*}) with $\lambda_*\neq\lambda^*$, i.e. C returns to the constant branch $\Gamma^*=\{(\phi^*,\lambda):\ \lambda\in\Lambda\}$.

Provided C^+ (resp. C^-) is the connected component of $C\setminus\Gamma^-$ containing Γ^+ (resp. the connected component of $C\setminus\Gamma^+$ containing Γ^-), then each of the global solution branches C^+ and C^- has one of the properties:

(e_1) *It intersects the boundary $\partial(\ell_\omega(\Omega)\times\Lambda)$*

(e_2) *it is unbounded*

(e_3) *it contains an ω-periodic solution ϕ^* to (Δ_{λ_*}) with $\lambda_*\neq\lambda^*$, i.e. the branch returns to the constant branch $\{(\phi^*,\lambda):\ \lambda\in\Lambda\}$*

(e_4) *it contains an ω-periodic solution $\phi^\bullet$ to (Δ_λ) different from ϕ^* with*

$$\sum_{j=\kappa}^{\kappa+\omega-1}\langle\phi_j^\bullet-\phi_j^*,\Phi_{\lambda^*}(j,\kappa)\xi_1\rangle=0. \tag{4.8}$$

To give an interpretation in the state space $\Omega\subseteq\mathbb{R}^d$ instead of $\ell_\omega(\Omega)$, the two alternatives (c) and (e_1) mean that the global branch C resp. $C^\pm$ contains ω-periodic solutions $\psi=(\psi_k)_{k\in\mathbb{Z}}$ with values $\psi_k\in\partial\Omega$ for some (hence, infinitely many) $k\in\mathbb{Z}$.

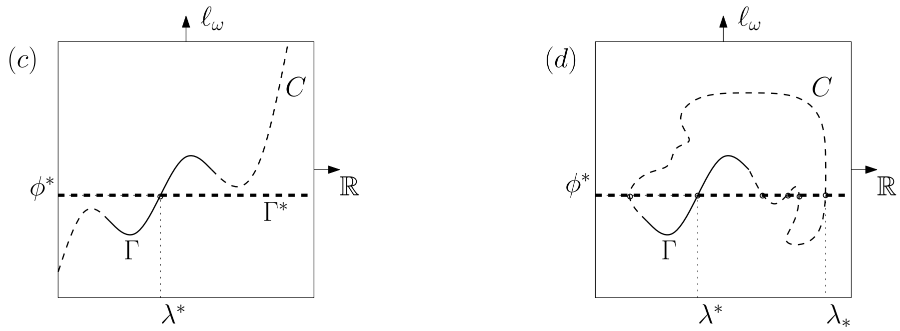

Figure 4: Structure of the global branch C (dashed line) of ω-periodic solutions to (Δ_λ) containing Γ (solid line) according to Thm. 4.3 with $\Omega=\mathbb{R}^d$ and $\Lambda=\mathbb{R}$: The set C is either unbounded (case (c), left) or an eqn. (Δ_{λ_*}) for at least one parameter $\lambda_*\neq\lambda^*$ possesses the ω-periodic solution ϕ^* on C (case (d), right)

Further information on the derivatives of f_k yields a more detailed description of the local branch Γ and yields two well-known bifurcation patterns:

Corollary 4.4 (transcritical bifurcation)**.** *Under the additional assumption*

$$g_{20}:=\langle\xi_1',D_1^2f_{\kappa+\omega-1}(\phi^*_{\kappa+\omega-1},\lambda^*)[\Phi_{\lambda^*}(\kappa+\omega-1,\kappa)\xi_1]^2\rangle$$

$$+\sum_{j=\kappa+1}^{\kappa+\omega-1}\langle\hat{\Phi}_{\lambda^*}(\kappa,j)^T\xi_1',D_1^2f_{j-1}(\phi_{j-1}^*,\lambda^*)[\Phi_{\lambda^*}(j-1,\kappa)\xi_1]^2\rangle\neq 0$$

it is $\lambda'(0)=-\frac{g_{20}}{2g_{11}}$ and locally in $U\times\Lambda_0$, the difference eqn. (Δ_λ) has a unique ω-periodic solution $\chi(\lambda)$ distinct from ϕ^* for $\lambda\neq\lambda^*$ and ϕ^* is the unique ω-periodic solution of (Δ_{λ^*}). Under (4.6) *one additionally has (cf. Fig. 5):*

(a) If $g_{11}>0$, then for $\lambda<\lambda^*$ the solution ϕ^* is unstable, while $\chi(\lambda)$ is asymptotically stable, for $\lambda>\lambda^*$ the solution ϕ^* is asymptotically stable and $\chi(\lambda)$ is unstable.

(b) If $g_{11}<0$, then for $\lambda<\lambda^*$ the solution ϕ^* is asymptotically stable, while $\chi(\lambda)$ is unstable, for $\lambda>\lambda^*$ the solution ϕ^* is unstable and $\chi(\lambda)$ is asymptotically stable.

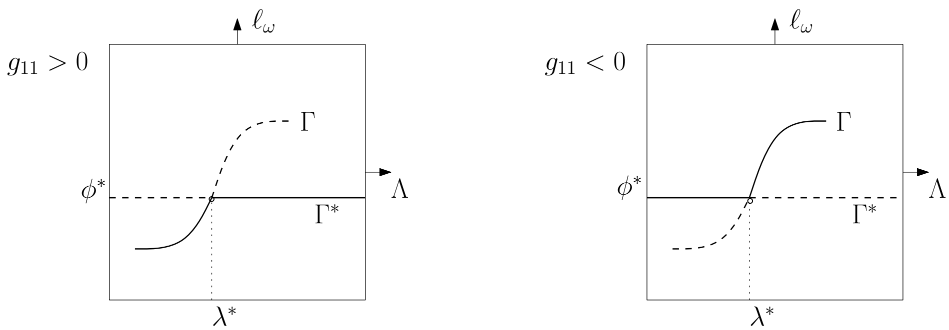

Figure 5: Local transcritical bifurcation of ω-periodic solutions to (Δ_λ) from the trivial branch Γ^* into Γ described in Cor. 4.4 and exchange of stability from unstable (dashed line) to asymptotically stable (solid)

We can investigate the effect of a bifurcation on the individual and total mean, when stability gets transferred from Γ^* to the nontrivial branch $\chi(\lambda)$:

Remark 4.2 (attenuation and resonance). As in Rem. 3.2 we describe the local behavior of the individual resp. total means $m(\lambda):=M(\chi(\lambda))$, $\hat{m}(\lambda):=\hat{M}(\chi(\lambda))$ w.r.t. a changing parameter λ near λ^*. According to the representations given in Thm. 4.3 and Cor. 4.4, one has $\psi(s)=\phi^*+\psi'(0)s+o(s)$ and $\lambda(s)=\lambda^*-\frac{g_{20}}{2g_{11}}s+o(s)$, which in turn yields $s(\lambda)=\frac{g_{20}}{g_{11}}(\lambda^*-\lambda)+o(\lambda)$ and finally (cf. Thm. 4.3(a)) $\chi(\lambda)=\psi(s(\lambda))=\phi^*+\frac{g_{20}}{g_{11}}(\lambda^*-\lambda)\Phi_{\lambda^*}(\cdot,\kappa)\xi_1+o(\lambda)$. From $\chi'(\lambda^*)=-\frac{g_{20}}{g_{11}}\Phi_{\lambda^*}(\cdot,\kappa)\xi_1$ we consequently arrive at the derivatives

$$m'(\lambda^*)=-\frac{g_{20}}{\omega g_{11}}\sum_{k=\kappa}^{\kappa+\omega-1}\Phi_{\lambda^*}(k,\kappa)\xi_1,\qquad \hat{m}'(\lambda^*):=\sum_{k=1}^{d}m'(\lambda^*)^i,\tag{4.9}$$

which indicate the local behavior of the individual and total means under variation of the parameter λ near λ^*.

The degenerate situation $g_{20}=0$ leads to

Corollary 4.5 (pitchfork bifurcation)**.** *For $m \geq 3$ and under the additional assumptions*

$$\begin{aligned} g_{20} &:= 0, \\ g_{30} &:= \langle \xi_1', D_1^3 f_{\kappa+\omega-1}(\phi^*_{\kappa+\omega-1}, \lambda^*)[\Phi_{\lambda^*}(\kappa+\omega-1,\kappa)\xi_1]^3 \rangle \\ &\quad + \sum_{j=\kappa+1}^{\kappa+\omega-1} \langle \hat{\Phi}_{\lambda^*}(\kappa,j)^T \xi_1', D_1^3 f_{j-1}(\phi^*_{j-1}, \lambda^*)[\Phi_{\lambda^*}(j-1,\kappa)\xi_1]^3 \rangle \neq 0 \end{aligned}$$

it is $\lambda'(0) = 0$, $\lambda''(0) = -\frac{g_{30}}{3g_{11}}$ and the following holds locally in $U \times \Lambda_0$ (cf. Fig. 6):

(a) Subcritical case*: If $g_{30}/g_{11} > 0$, then ϕ^* is the unique ω-periodic solution for $\lambda \geq \lambda^*$ and (Δ_λ) has exactly two ω-periodic solutions distinct from ϕ^* for $\lambda < \lambda^*$.*

(b) Supercritical case*: If $g_{30}/g_{11} < 0$, then ϕ^* is the unique ω-periodic solution for $\lambda \leq \lambda^*$ and (Δ_λ) has exactly two ω-periodic solutions distinct from $\phi^*$for $\lambda > \lambda^*$.*

Stability properties of the bifurcating solution branches Γ in Cors. 4.4 and 4.5 can also be tackled using the following criterion.

Proposition 4.6 (stability formula)**.** *Let $\lambda \in \Lambda \subseteq \mathbb{R}$ and suppose 1 is a simple eigenvalue of $\Xi_\omega(\lambda^*)$ satisfying* (4.6)*. If the characteristic polynomial of $\Xi_\omega(\lambda)$ is written as*

$$\det(tI_d - \Xi_\omega(\lambda)) = t^d + \sum_{j=0}^{d-1} p_j(\lambda) t^j \tag{4.10}$$

with coefficients $p_0, \ldots, p_{d-1} : \Lambda \to \mathbb{R}$ and $\theta := \frac{\sum_{j=0}^{d-1} p_j'(\lambda^*)}{d + \sum_{j=1}^{d-1} j p_j(\lambda^*)} \neq 0$, then:

(a) In case $\theta < 0$ the solution ϕ^* to (Δ_λ) is asymptotically stable for $\lambda < \lambda^*$ and unstable for $\lambda > \lambda^*$.

(b) In case $\theta > 0$ the solution ϕ^* to (Δ_λ) is unstable for $\lambda < \lambda^*$ and asymptotically stable for $\lambda > \lambda^*$.

The ω-periodic solutions to (Δ_λ) on the branch Γ different from the constant one Γ^* have opposite stability properties (cf. Fig. 6).

Remark 4.3. From e.g. [42, p. 188] we have $p_0(\lambda) = (-1)^d \det \Xi_\omega(\lambda)$ and $p_{d-1}(\lambda) = -\operatorname{tr} \Xi_\omega(\lambda)$. Thus, the expression for θ simplifies for

- scalar difference eqns. (Δ_λ) (i.e., $d = 1$) to $\theta = -\Xi_\omega'(\lambda^*)$,
- planar difference eqns. (Δ_λ) (i.e., $d = 2$) to $\theta = \frac{p_0'(\lambda^*) - \operatorname{tr} \Xi_\omega'(\lambda^*)}{\operatorname{tr} \Xi_\omega(\lambda^*) - 2}$.

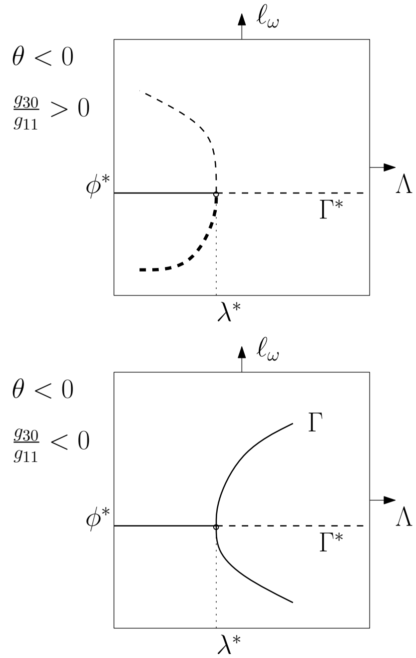

Figure 6: Local subcritical (top) and supercritical (bottom) pitchfork bifurcation of ω-periodic solutions to (Δ_λ) described in Cor. 4.5 and exchange of stability between the trivial branch Γ^* and Γ from unstable (dashed line) to asymptotically stable (solid) covered in Prop. 4.6

4.3. Global bifurcations

This subsection tackles one-parameter bifurcations and we assume beyond $\Lambda = \mathbb{R}$ that the state space Ω for (Δ_λ) is the whole $\mathbb{R}^d$. We suppose that there exists a global solution branch $C^* \subseteq \ell_\omega \times \mathbb{R}$ for the ω_0-periodic difference eqn. (Δ_λ) given as graph of a C^1-function $\phi : \mathbb{R} \to \ell_\omega$.

We consider the continuous function $\delta_\omega : \mathbb{R} \to \mathbb{R}$,

$$\delta_\omega(\lambda) := \det\left(I_d - \Xi_\omega(\lambda)\right) \tag{4.11}$$

and note that sign changes of δ_ω indicate bifurcations of ω-periodic solutions:

Theorem 4.7 (global bifurcation)**.** *If there are parameters $\lambda_1 < \lambda_2$ satisfying a sign change $\delta_\omega(\lambda_1)\delta_\omega(\lambda_2) < 0$, then there exists a parameter value $\lambda^* \in (\lambda_1, \lambda_2)$ such that $\phi(\lambda^*) \in \ell_\omega$ bifurcates at λ^*. More precisely, a connected set $C \subseteq \ell_\omega \times \mathbb{R}$ of ω-periodic solutions to (Δ_λ) branches off from C^* at $(\phi(\lambda^*), \lambda^*)$ with precisely one of the properties (cf. Fig. 7):*

(a) C is unbounded in $\ell_\omega \times \mathbb{R}$

(b) C is bounded and intersects the solution branch C^* at another point.

We point out that merely a zero of the function δ_ω is not sufficient for a bifurcation; see [3, p. 82, Example 1.6] for an example.

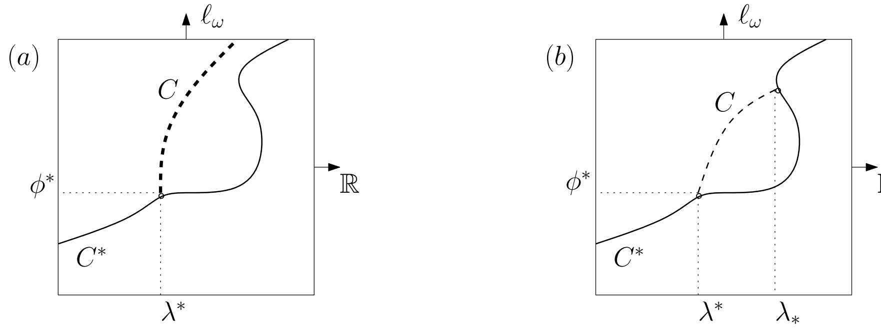

Figure 7: Structure of the global branch C (dashed) of ω-periodic solutions to (Δ_λ) bifurcating from C^* (solid) according to Thm. 4.7: The set C is either unbounded (case (a), left) or an eqn. (Δ_{λ_*}) for at least one $\lambda_* \neq \lambda^*$ possesses an ω-periodic solution $\phi(\lambda_*)$ on C (case (b), right)

5. Applications

5.1. Scalar models

Bifurcations in scalar, i.e. 1-dimensional, periodic difference equations were previously studied in e.g. [20, 4] (logistic equation) or [7, 5] (Beverton-Holt). We consequently focus on other models:

5.1.1. An equation of Castillo-Chavez and Brauer

We consider an ω_0-periodic version

$$x_{k+1} = f_k(x_k, \lambda) := \frac{r_k(\lambda) x_k^2}{a_k(\lambda) + x_k^2} \tag{5.1}$$

of a difference equation suggested in [8, p. 53] to describe populations which die out completely in each generation and have birth rates saturating for large population sizes. Suppose parameters $r_k(\lambda) := \rho + \lambda\rho_k$ and $a_k(\lambda) := \alpha + \lambda\alpha_k$ with reals $\alpha, \rho > 0$ and ω_0-periodic real sequences $(\alpha_k)_{k\in\mathbb{Z}}$, $(\rho_k)_{k\in\mathbb{Z}}$. We choose $\lambda^* := 0$ as critical parameter value, (5.1) becomes autonomous, and we obtain two nontrivial fixed points $x_-^* \leq x_+^*$,

$$x_\pm^* := \frac{\rho \pm w}{2}, \qquad w := \sqrt{\rho^2 - 4\alpha},$$

provided $\rho^2 \geq 4\alpha$. In particular, for $\rho^2 = 4\alpha$ both fixed points coincide with $x^* = \frac{\rho}{2}$. The corresponding linearizations read as

$$D_1 f_k(x_\pm^*, 0) = \frac{4\alpha}{\rho^2 \pm \rho w} =: \zeta_\pm, \qquad \zeta_+ \in (0,1), \qquad \zeta_- \in (1, \infty),$$

we therefore have $D_1 f_k(x_\pm^*, 0) = 1$ if and only if $\alpha = \frac{\rho^2}{4}$, and otherwise x_+^* is asymptotically stable, while x_-^* is unstable. Referring to Thm. 3.1, the hyperbolicity condition $\rho^2 > 4\alpha$ ensures that these fixed points (including their stability properties) persist as ω_0-periodic solutions for small values of the parameter $\lambda \neq 0$ (cf. Fig. 8).

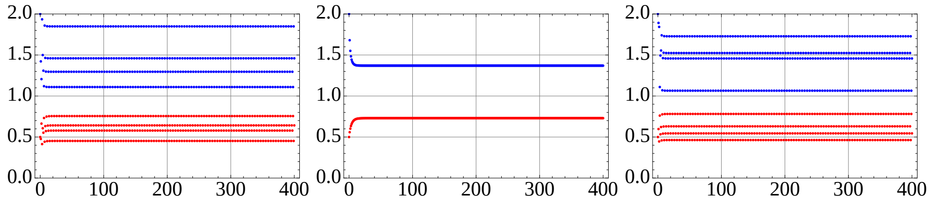

Figure 8: Solution sequences (dotted) of the 4-periodic difference eqn. (5.1) with parameters $\alpha = 1$, $\rho = 2.1$ and sequences $\alpha_k = (-1)^k$, $\rho_k = \cos(\frac{\pi}{2}k)$:
$\lambda = -0.5$ (left): Fixed-points persist as 4-periodic solutions
$\lambda = 0$ (middle): Stable fixed point x_+^* (blue) and unstable fixed-point x_-^* (red) in the autonomous case
$\lambda = 0.5$ (right): Fixed-points persist as 4-periodic solutions

Since the nonzero asymptotically stable solutions to (5.1) are a perturbation of x_+^*, we are interested in attenuation or resonance near this hyperbolic fixed point. Using the formulas from Rem. 3.2, which become

$$\xi_\kappa = \frac{\zeta_+^{\kappa+\omega_0-1}}{1-\zeta_+^{\omega_0}} \sum_{j=\kappa}^{\kappa+\omega_0-1} \frac{(\rho+w)\rho_j - 2\alpha_j}{2\rho\zeta_+^j},$$

$$m'(0) = \frac{1}{\omega_0} \sum_{k=\kappa}^{\kappa+\omega_0-1} \sum_{l=\kappa}^{\kappa+\omega_0-1} \left(\frac{\zeta_+^{k+\omega_0-1}}{1-\zeta_+^{\omega_0}} + \zeta_+^{k-1} \right) \frac{(\rho+w)\rho_l - 2\alpha_l}{2\rho\zeta_+^l}$$

$$= \frac{1}{2\rho\omega_0} \left(\frac{1-2\zeta_+^{\omega_0}}{\zeta_+ - \zeta_+^2} \right) \sum_{l=\kappa}^{\kappa+\omega_0-1} \frac{(\rho+w)\rho_l - 2\alpha_l}{\zeta_+^{l-\kappa}},$$

one obtains that resonance occurs locally for $m'(0) > 0$, while the dual inequality $m'(0) < 0$ implies attenuation (cf. Fig. 9).

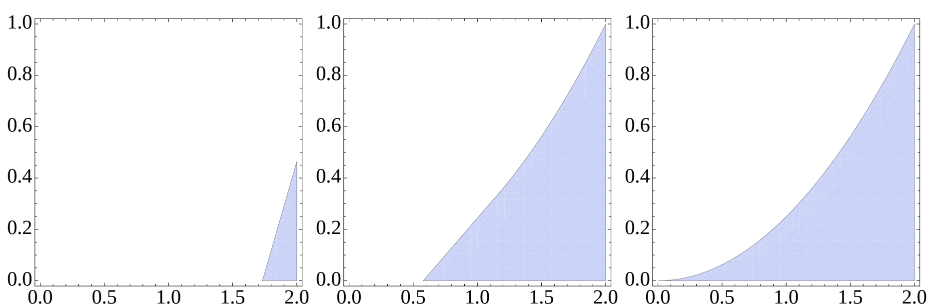

Figure 9: ρ-α-plane to illustrate resonance (shaded region) at the asymptotically stable fixed point x_+^* of (5.1) for different periods: $\omega_0 = 2$ (left), $\omega_0 = 3$ (middle), $\omega_0 = 4$ (right) and $\alpha_k = \cos\left(\frac{2\pi}{\omega_0}k\right)$, $\rho_k = \sin\left(\frac{2\pi}{\omega_0}k\right)$

We now tackle the nonhyperbolic case $\rho^2 = 4\alpha$ and the behavior of the unique nontrivial fixed point $x^* = \frac{\rho}{2}$ for parameters $\lambda \neq 0$. First, as a side note, if we interpret ρ as bifurcation parameter, this indicates a supercritical fold bifurcation of the nontrivial fixed point $x^* = \frac{\rho}{2}$ in the autonomous problem $x_{k+1} = \frac{\rho x_k^2}{\alpha + x_k^2}$.

Yet, in order to illustrate the flexibility of Thm. 4.1, choose $\alpha, \rho > 0$ according to the critical case $\rho^2 = 4\alpha$ and consider the general situation of an ω_0-periodic difference equation with arbitrary period $\omega_0 \in \mathbb{N}$ and λ as bifurcation parameter. We obtain

$$\Xi_\omega(\lambda^*) = \prod_{j=\kappa}^{\kappa+\omega_0-1} D_1 f_k(x^*, \lambda^*) = 1 \quad \text{for all } \kappa \in \mathbb{Z},$$

and $D_2 f_k(x^*, \lambda^*) = \frac{\rho_k}{2} - \frac{\alpha_k}{\rho}$, $D_1^2 f_k(x^*, \lambda^*) = -\frac{2}{\rho}$ for $k \in \mathbb{Z}$. By choosing $\xi_1 = \xi_1' = 1$ we arrive at the bifurcation indicators

$$g_{01} = \sum_{j=0}^{\omega_0-1} \left(\frac{\rho_j}{2} - \frac{\alpha_j}{\rho} \right), \qquad\qquad g_{20} = -\frac{2\omega_0}{\rho} < 0.$$

This allows us to conclude from Thm. 4.1 that the nonhyperbolic fixed point $x^* = \frac{\rho}{2}$ is the unique ω_0-periodic solution of (5.1) for $\lambda = 0$. Furthermore, locally near $\lambda = 0$ one has an ω_0-periodic fold bifurcation (cf. Fig. 10):

- *Subcritical bifurcation*: If $\sum_{j=0}^{\omega_0-1} \frac{\alpha_j}{\rho} > \sum_{j=0}^{\omega_0-1} \frac{\rho_j}{2}$, then (5.1) possesses no ω_0-periodic solution for $\lambda > 0$ and exactly two distinct ω_0-periodic solutions for $\lambda < 0$.
- *Supercritical bifurcation*: If $\sum_{j=0}^{\omega_0-1} \frac{\alpha_j}{\rho} < \sum_{j=0}^{\omega_0-1} \frac{\rho_j}{2}$, then (5.1) possesses no ω_0-periodic solution for $\lambda < 0$ and exactly two distinct ω_0-periodic solutions for $\lambda > 0$.

Due to Cor. 4.2, one of these ω_0-periodic solutions to (5.1) is asymptotically stable, while the other one is unstable.

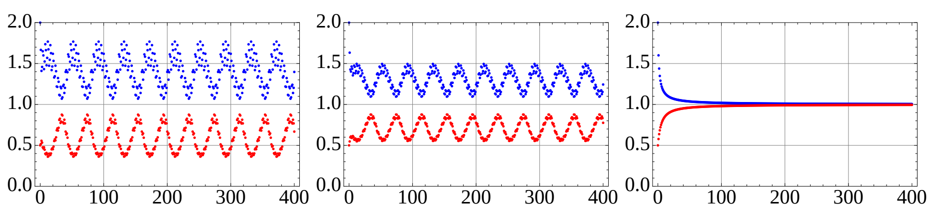

Figure 10: Solution sequences (dotted) of the 40-periodic difference eqn. (5.1) with parameters $\alpha = 1$, $\rho = 2$ and sequences $\alpha_k = 1 + \sin(\frac{\pi}{20}k)$ and $\rho_k = \sin(\frac{\pi}{2}k)$ yielding $g_{10} = 20$ and thus a subcritical fold bifurcation of 40-periodic solutions:
$\lambda = -0.2$ (left): Stable 40-periodic solution (blue) and unstable 40-periodic solution (red)
$\lambda = -0.1$ (middle): Stable 40-periodic solution (blue) and unstable 40-periodic solution (red)
$\lambda = 0$ (right): Semistable fixed point $x_+^* = 1$ in the autonomous situation

5.1.2. Autonomous Ricker equation with immigration

In this subsection, we aim to illustrate the global results given in Thms. 3.3, 4.3 and 4.7. Thereto, consider the autonomous *Ricker equation*

$$x_{k+1} = f(x_k, \lambda) := x_k e^{r(1-\lambda x_k)} \tag{5.2}$$

with the intrinsic growth rate $r > 0$ and the carrying capacity $\lambda^{-1} \in \mathbb{R}$. Omitting this biological interpretation, we allow arbitrary parameters $\lambda \in \mathbb{R}$ in (5.2). It has two branches of 1-periodic, i.e. constant solutions

$$\phi_1(\lambda)_k \equiv 0, \qquad\qquad \phi_2(\lambda)_k \equiv \begin{cases} \frac{1}{\lambda}, & \lambda \neq 0, \\ 0, & \lambda = 0 \end{cases} \quad \text{on } \mathbb{Z},$$

which coincide for the parameter value $\lambda = 0$. Moreover, for the derivative it is $D_1 f(\phi_1(\lambda)_k, \lambda) \equiv e^r$ and $D_1 f(\phi_2(\lambda)_k, \lambda) \equiv 1 - r$. Consequently, both fixed point branches turn out to fulfill the weak hyperbolicity condition (3.1) and Thm. 3.3(a) applies for every parameter $\lambda^* \in \mathbb{R}$:

- For the trivial branch $\phi_1(\lambda) = 0$ one has

$$C^- = \{(0, \lambda) \in \mathbb{R} \times \mathbb{R} : \lambda < \lambda^*\}, \quad C^+ = \{(0, \lambda) \in \mathbb{R} \times \mathbb{R} : \lambda > \lambda^*\}.$$

- For the nontrivial branch $\phi_2(\lambda)$, $\lambda^* \neq 0$, it is

$$C^- = \begin{cases} \left\{(\lambda^{-1}, \lambda) \in \mathbb{R} \times \mathbb{R} : \lambda \in (0, \lambda^*)\right\}, & \lambda^* > 0, \\ \left\{(\lambda^{-1}, \lambda) \in \mathbb{R} \times \mathbb{R} : \lambda < \lambda^*\right\}, & \lambda^* < 0, \end{cases}$$

$$C^+ = \begin{cases} \left\{(\lambda^{-1}, \lambda) \in \mathbb{R} \times \mathbb{R} : \lambda > \lambda^*\right\}, & \lambda^* > 0, \\ \left\{(\lambda^{-1}, \lambda) \in \mathbb{R} \times \mathbb{R} : \lambda \in (\lambda^*, 0)\right\}, & \lambda^* < 0. \end{cases}$$

Note that for a Ricker difference eqn. (5.2) both components C^-, C^+ are unbounded (cf. Fig. 11(left)). This situation changes, if we add an immigration term to (5.2) and are interested in the global structure of the solutions with periods $\omega > 1$. More precisely, consider the Ricker equation with immigration

$$x_{k+1} = f(x_k, \lambda) := x_k e^{\lambda(1-\frac{x_k}{K})} + \iota, \tag{5.3}$$

where the intrinsic growth rate $\lambda > 0$ can vary, while the carrying capacity $K > 0$ and the immigration term $\iota > 0$ are kept fixed. The equilibria to (5.3) cannot be expressed as elementary functions of λ, K, ι. Thus, we numerically computed the ω-periodic solutions to (5.3) from the fixed point eqn. $\varphi_\lambda(\omega, 0, x) = x$ and displayed the results in Fig. 11(middle, right).

This illustrates that solutions with minimal period $\omega = 4$ (respectively $\omega \in \{4, 8\}$) form a bounded connected set (cf. Thm. 3.3(b)) in $\ell_\omega \times \mathbb{R}$. Also the global statements of Thm. 4.7 (or Thm. 4.3(c) and (d)) are illuminated:

- Two unbounded branches of 2-periodic solutions to (5.3) bifurcate from the fixed point branch given by $x = f(x, \lambda)$ in a pitchfork way.

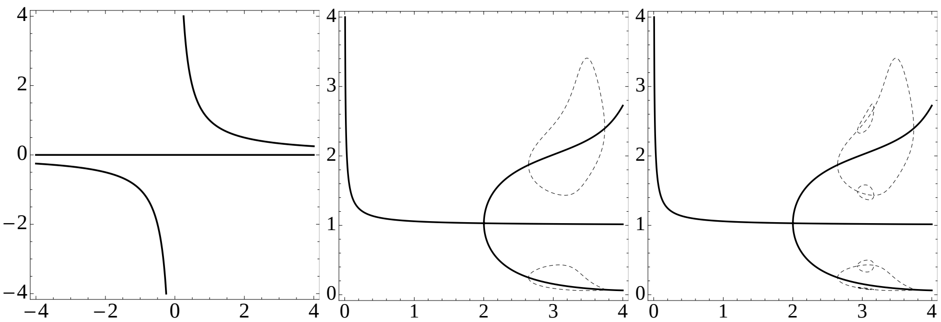

Figure 11: λ-x-plane to illustrate the solution set to the fixed-point eqn. $\varphi_\lambda(\omega, 0, x) = x$ for the Ricker difference eqns. (5.2) (left) and (5.3) (middle, right) with $K = 1$ and immigration $\iota = 0.06$:
Left: Unbounded components of fixed points for (5.2) under variation of λ
Middle: Fixed points for $\omega = 2$ (solid) and $\omega = 4$ (dashed)
Right: Fixed points for $\omega = 6$ (solid) and $\omega = 8$ (dashed)

- As a secondary bifurcation, bounded branches of 4-periodic solutions to (5.3) bifurcate off from the 2-periodic solutions. Fig. 11(right) even indicates a tertiary bifurcation of a bounded branch of 8-periodic solutions. All of them appear of pitchfork-type.

5.1.3. Periodic Ricker equation

Let $\lambda, K > 0$ and $(r_k)_{k\in\mathbb{Z}}$ be an ω_0-periodic sequence with positive values. We consider the ω_0-periodic *Ricker equation*

$$x_{k+1} = f(x_k, \lambda) := x_k e^{\lambda r_k (1 - \frac{x_k}{K})}, \tag{5.4}$$

where the growth rate $\lambda r_k > 0$ is allowed to vary periodically. It has the constant solution branches $\phi_0(\lambda) \equiv 0$ and $\phi(\lambda) \equiv K$ for all $\lambda > 0$. Both can be interpreted as ω_1-periodic with an ambient $\omega_1 \in \mathbb{N}$. Due to

$$D_1 f_k(0, \lambda) = e^{\lambda r_k} > 1, \qquad D_1 f_k(K, \lambda) = 1 - \lambda r_k \quad \text{for all } k \in \mathbb{Z},\ \lambda \in \Lambda,$$

we see that the trivial solution is always unstable. Moreover, for later use we record that linearization along the nontrivial branch $\phi(\lambda)$ yields

$$\Phi_\lambda(0, j) = \prod_{i=0}^{j-1} \frac{1}{1 - \lambda r_i}, \qquad \Phi_\lambda(j-1, 0) = \begin{cases} \frac{1}{1-\lambda r_{-1}}, & j = 0, \\ \prod_{i=0}^{j-2}(1 - \lambda r_i), & j > 0. \end{cases} \tag{5.5}$$

By Thm. 2.3 the stability of $\phi(\lambda)$ depends whether the absolute value of

$$\Xi_{\omega_0}(\lambda) \overset{(5.5)}{=} \prod_{j=0}^{\omega_0 - 1} (1 - \lambda r_j)$$

is < 1 (asymptotic stability) or > 1 (instability). Examples of critical values $\lambda^* > 0$ for the parameter λ such that $\Xi_{\omega_0}(\lambda^*) = \pm 1$ are summarized in the following table[1]

λ^*	$\Xi_{\omega_0}(\lambda^*) = 1$	$\Xi_{\omega_0}(\lambda^*) = -1$
$\omega_0 = 1$	$-$	$\frac{2}{r_0}$
$\omega_0 = 2$	$\frac{r_0+r_1}{r_0 r_1}$	$-$

and let us distinguish two cases: If there exists a $\lambda^* > 0$ such that

(I) $\Xi_{\omega_0}(\lambda^*) = 1$, we choose $\omega_1 = 1$ (i.e. interpret $\phi(\lambda)$ as 1-periodic solutions) and obtain $\omega := \operatorname{lcm}(\omega_0, \omega_1) = \omega_0$.

(II) $\Xi_{\omega_0}(\lambda^*) = -1$, choose $\omega_1 = 2$ (i.e. interpret $\phi(\lambda)$ as 2-periodic solutions) and set $\omega := 2\omega_0$. Hence, it is $\Xi_\omega(\lambda^*) = 1$.

In both cases, thanks to $D_1 D_2 f_k(K, \lambda) = -r_k$ for all $k \in \mathbb{Z}$, $\lambda > 0$, the transversality condition (4.7) becomes

$$g_{11} \overset{(5.5)}{=} -\sum_{j=1}^{\omega} \frac{r_j}{1 - \lambda^* r_j}$$

(we have chosen $\xi_1 = \xi_1' = 1$). Thus, in the generic situation $g_{11} \neq 0$ our Thm. 4.3 shows that in every neighborhood of a critical constant solution (K, λ^*) to eqn. (5.4) there exists another ω-periodic solution. In case $\Xi_{\omega_0}(\lambda^*) = -1$, bifurcating solutions have twice the period of (5.4) — we have a flip bifurcation. Thanks to Prop. 4.6 and the principle of exchange of stability, the stability of the bifurcating solution branch is determined by

$$\Xi_\omega'(\lambda^*) = -\sum_{j=1}^{\omega} \frac{r_j}{1 - \lambda^* r_j}.$$

In order to classify the kind of bifurcation, we get

$$D_1^2 f_k(K, \lambda) = \tfrac{\lambda}{K} r_k (\lambda r_k - 2)$$

for all $\lambda > 0$ and $k \in \mathbb{Z}$. For a critical parameter value $\lambda^* > 0$ this yields

$$g_{20} = -\frac{\lambda^*}{K} \sum_{j=1}^{\omega} \frac{2 - \lambda^* r_{j-1}}{1 - \lambda^* r_{j-1}} r_{j-1} \prod_{i=0}^{j-2} (1 - \lambda^* r_i)$$

and by Cor. 4.4 the generic condition $g_{20} \neq 0$ ensures a transcritical bifurcation of a solution branch consisting of ω-periodic solutions. To determine whether resonance or attenuation is on hand, we employ (4.9) and obtain

$$m'(\lambda^*) = -\frac{g_{20}}{\omega g_{11}} \sum_{k=0}^{\omega-1} \prod_{j=0}^{k-1} (1 - \lambda^* r_j)$$

[1] Using computer algebra it is also possible on a symbolic level to obtain $\Xi_{\omega_0}^{-1}(\{-1\})$ for periods $\omega_0 \in \{3, 4, 5\}$ and the preimage $\Xi_{\omega_0}^{-1}(\{-1\})$ for $\omega_0 \in \{3, 4\}$. For $\omega_0 = 2$ there are no positive values of r_0, r_1 such that $\Xi_{\omega_0}(\lambda^*) = -1$.

indicating resonance for $m'(\lambda^*) > 0$ and attenuation for $m'(\lambda^*) < 0$.

Finally, we consider the degenerate case $g_{20} = 0$. For the partial derivative we obtain

$$D_1^3 f_k(K, \lambda) = \left(\tfrac{\lambda}{K}\right)^2 r_k^2(3 - \lambda r_k)$$

for $\lambda > 0$, $k \in \mathbb{Z}$ and with a critical value $\lambda^* > 0$ it is

$$g_{30} = \left(\frac{\lambda^*}{K}\right)^2 \sum_{j=1}^{\omega} \frac{3 - \lambda^* r_{j-1}}{1 - \lambda^* r_{j-1}} r_{j-1}^2 \prod_{i=0}^{j-2} (1 - \lambda^* r_i)^2.$$

Then Cor. 4.5 and $g_{30} \neq 0$ guarantees a pitchfork bifurcation of a solution branch consisting of ω-periodic solutions.

Note that our bifurcation formulas for g_{11}, g_{20}, g_{30} and $\Xi_\omega'(\lambda^*), m'(\lambda^*)$ are explicit and can be evaluated directly, if the critical parameter value $\lambda^* > 0$ is known. This, however, requires to solve the ωth order polynomial eqn. $\Xi_\omega(\lambda) = \pm 1$ and is only possible numerically (for $\omega > 4$). For the example of an ω_0-periodic sequence

$$r_k := \begin{cases} 2, & k \mod \omega_0 = 0, \\ 1, & k \mod \omega_0 \neq 0 \end{cases} \tag{5.6}$$

we have illustrated such a numerical approach in Figs. 12 and 13.

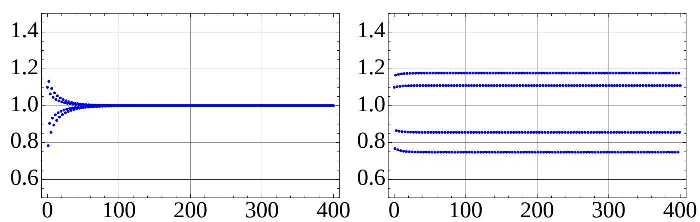

Figure 12: Solution sequences illustrating a transcritical bifurcation in the 4-periodic Ricker eqn. (5.4) with $(r_k)_{k\in\mathbb{Z}}$ given in (5.6) and $K = 1$. The critical parameter value $\lambda^* = 1.74$ yields $\Xi_4(\lambda^*) = 1$ and moreover, the bifurcation indicators $\Xi_4'(\lambda^*) = 4.87$, $g_{11} = -4.87$, $g_{20} = 3.30$
$\lambda = 1.7$ (left): Asymptotically stable constant solution $K = 1$
$\lambda = 1.8$ (right): Asymptotically stable 4-periodic solution

Nevertheless, the 2-periodic case allows explicit computations:

Example 5.1. In the special case of a 2-periodic Ricker difference eqn. (5.4) one has $\Xi_2(\lambda^*) = 1$ for $\lambda^* = \frac{r_0+r_1}{r_0 r_1}$. The bifurcation indicators become

$$\Xi_\omega'(\lambda^*) = r_0 + r_1 > 0, \qquad g_{11} = r_0 + r_1 > 0,$$
$$g_{20} = \frac{(r_1 - r_0)(r_0 + r_1)^2}{K r_0 r_1^2}, \qquad g_{30} = -\frac{2(r_0^2 - r_0 r_1 + r_1^2)(r_0 + r_1)^2}{K^2 r_0 r_1^3} < 0,$$

of whom only g_{20} can change sign: For $\lambda < \frac{r_0+r_1}{r_0 r_1}$ the constant solution K to (5.4) is asymptotically stable, while it becomes unstable for $\lambda > \frac{r_0+r_1}{r_0 r_1}$.

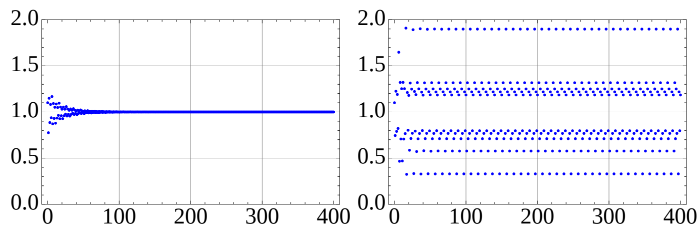

Figure 13: Solution sequences illustrating a period-doubling pitchfork bifurcation in the 5-periodic Ricker eqn. (5.4) with $(r_k)_{k\in\mathbb{Z}}$ given in (5.6) and $K = 1$. The parameter value $\lambda^* = 1.79$ yields $\Xi_5(\lambda^*) = -1$ and with $\omega = 10$ the bifurcation indicators $\Xi'_{10}(\lambda^*) = 11.69$, $g_{11} = -11.69$, $g_{20} = 0$ and $g_{30} = 141.33$
$\lambda = 1.75$ (left): Asymptotically stable constant solution $K = 1$
$\lambda = 1.95$ (right): Asymptotically stable 10-periodic solution

$r_0 \neq r_1$: Transcritical bifurcation: For $\lambda \neq \frac{r_0+r_1}{r_0 r_1}$ there is exactly one non-constant 2-periodic solution. It is unstable for $\lambda < \frac{r_0+r_1}{r_0 r_1}$ and asymptotically stable for $\lambda > \frac{r_0+r_1}{r_0 r_1}$. By (4.9) it is $m'(\lambda^*) = \frac{r_0^2-r_1^2}{2Kr_0r_1}\left(1 - \frac{r_0}{r_1}\right) < 0$ and therefore one locally always has attenuation due to the bifurcation.

$r_0 = r_1$: Supercritical pitchfork bifurcation: The critical parameter becomes $\lambda^* = \frac{2}{r_0}$ and (5.4) is autonomous. For $\lambda < \frac{2}{r_0}$ the constant solution K is locally the unique 2-periodic solution. For $\lambda > \frac{2}{r_0}$ there are precisely two 2-periodic solutions distinct from K, which are asymptotically stable. This is the known period doubling bifurcation in the autonomous Ricker model.

5.1.4. Ricker equation with proportional harvesting

We consider a scalar Ricker model under proportional harvesting

$$x_{k+1} = f_k(x_k, \lambda) := x_k e^{r\left(1-\frac{x_k}{K}\right)} - h_k(\lambda)x_k \tag{5.7}$$

with constant intrinsic growth rate $r > 0$ and carrying capacity $K > 0$. The harvesting function $h_k(\lambda) > 0$ is ω_0-periodic with $\omega_0 > 1$ and of the form

$$h_k(\lambda) := \begin{cases} \lambda, & k \mod \omega_0 = \omega_0 - 1, \\ \gamma, & k \mod \omega_0 \neq \omega_0 - 1 \end{cases}$$

with a fixed $\gamma \in \left[0, e^r - e^{\frac{r}{1-\omega_0}}\right)$ and a bifurcation parameter $\lambda > 0$. For $\gamma = 0$ this means that harvesting takes place at the end of the periodicity interval. For the trivial solution $\phi^* = 0$ to (5.7) we have the transition mapping

$$\Phi_\lambda(k, 0) = \begin{cases} (e^r - \gamma)^k, & 0 \leq k < \omega_0, \\ (e^r - \gamma)^{\omega_0-1}(e^r - \lambda), & k = \omega_0 \end{cases}$$

and in particular the period map $\Xi_{\omega_0}(\lambda) = (e^r - \gamma)^{\omega_0-1}(e^r - \lambda)$ having the derivative $\Xi'_{\omega_0}(\lambda) \equiv -(e^r - \gamma)^{\omega_0-1} < 0$. The situation $\Xi_{\omega_0}(\lambda^*) = 1$ is given for the critical parameter value $\lambda^* = e^r - (e^r - \gamma)^{1-\omega_0} > 0$ and we deduce the bifurcation indicators

$$\begin{aligned} g_{11} &= D_1 D_2 f_{\omega_0 - 1}(0, \lambda^*) \Phi_{\lambda^*}(\omega_0 - 1, 0) + \sum_{j=1}^{\omega_0 - 1} D_1 D_2 f_{j-1}(0, \lambda^*) D_1 f_{j-1}(0, \lambda^*) \\ &= -(e^r - \gamma)^{\omega_0 - 1} < 0, \\ g_{20} &= D_1^2 f_{\omega_0 - 1}(0, \lambda^*) \Phi_{\lambda^*}(\omega_0 - 1, 0)^2 + \sum_{j=1}^{\omega_0 - 1} D_1^2 f_{j-1}(0, \lambda^*) \Phi_{\lambda^*}(j, 0) \\ &= -\frac{2re^r}{K} \left((e^r - \gamma)^{2\omega_0 - 2} + (e^r - \gamma) \frac{1 - (e^r - \gamma)^{\omega_0 - 1}}{1 - e^r + \gamma} \right) < 0. \end{aligned}$$

By Thm. 4.3 and its Cor. 4.4 this ensures that the trivial solution ϕ^* to (5.7) bifurcates into an ω_0-periodic solution in a transcritical fashion. Referring to Prop. 4.6, ϕ^* is unstable for $\lambda < \lambda^*$ and becomes asymptotically stable as the harvesting rate λ surpasses λ^*; during this process, the ω_0-periodic solution looses asymptotic stability and becomes unstable.

This result agrees with biological intuition: Since λ^* is strictly increasing in ω_0 (with limit e^r as $\omega_0 \to \infty$), the harvesting rate $\lambda < \lambda^*$ can be larger, the less often harvesting takes place, without forcing the population to vanish.

5.2. *Higher-dimensional models*

Our techniques are applicable to difference eqns. (Δ_λ) in $\mathbb{R}^d$ with $d > 1$.

5.2.1. *Planar Ricker model*

Let us consider a planar Ricker competition model suggested in [13]. A detailed investigation for the autonomous case was given in [30], so that we proceed to a periodic situation. However, by no means it is the goal of this section to give a rigorous stability analysis in the most general setting. We rather aim to demonstrate our bifurcation results and retreat to a simple special case. Hence, let us consider the planar difference equation

$$\begin{pmatrix} x_{k+1}^1 \\ x_{k+1}^2 \end{pmatrix} = f_k(x_k, \lambda) := \begin{pmatrix} x_k^1 e^{3\lambda r_k (1 - \frac{2}{3} x_k^1 - \frac{1}{3} x_k^2)} \\ x_k^2 e^{3 r_k (1 - \frac{1}{3} x_k^2 - \frac{2}{3} x_k^2)} \end{pmatrix} \tag{5.8}$$

with coordinates $x_k = (x_k^1, x_k^2)$ and an ω_0-periodic sequence $(r_k)_{k \in \mathbb{Z}}$ in $(0, \infty)$. Its constant solutions are independent of the bifurcation parameter $\lambda > 0$:

- First, the trivial equilibrium $(0, 0)$ is always unstable, since it has the Floquet spectrum $\left\{ e^{3\lambda \sum_{j=0}^{\omega_0 - 1} r_j}, e^{3 \sum_{j=0}^{\omega_0 - 1} r_j} \right\} \subset (1, \infty)$.
- Due to the Floquet spectrum $\left\{ \sqrt{e^{3\lambda \sum_{j=0}^{\omega_0 - 1} r_j}}, \prod_{j=0}^{\omega_0 - 1} (1 - 3r_j) \right\}$ also the exclusion equilibrium $(0, \frac{3}{2})$ is unstable.

Yet, the other equilibria $\phi_1^* = (\frac{3}{2}, 0)$ and $\phi_2^* = (1, 1)$ remain interesting for a bifurcation analysis. We know that ϕ_1^* is always unstable, due to its Floquet spectrum

$$\sigma_{\omega_0}(\lambda) = \left\{ \prod_{j=0}^{\omega_0 - 1} (1 - 3\lambda r_j), \sqrt{e^{3\sum_{j=0}^{\omega_0-1} r_j}} \right\}.$$

An expression for the Floquet spectrum of the coexistence equilibrium ϕ_2^* for general ω_0 is more involved, and for the sake of an explicit and parallel presentation for both ϕ_1^* and ϕ_2^*, we restrict to the simplest nonautonomous situation $\omega_0 = 2$ and abbreviate $\rho := \frac{r_0 + r_1}{r_0 r_1}$:

- $\phi_1^* = (\frac{3}{2}, 0)$: The critical bifurcation value $\lambda_1^* = \frac{\rho}{3}$ yields the Floquet spectrum $\sigma_2(\lambda_1^*) = \left\{1, \sqrt{e^{3(r_0+r_1)}}\right\}$ and moreover

  $$N(I_2 - \Xi_2(\lambda_1^*)) = \mathbb{R}e_1, \quad N(I_2 - \Xi_2(\lambda_1^*)^T) = \mathbb{R} \begin{pmatrix} 2\left(\sqrt{e^{3(r_0+r_1)}} - 1\right) r_0 \\ \left(\sqrt{e^{3r_0}} - 1\right)(r_0 + r_1) \end{pmatrix}.$$

  With this information, the bifurcation indicators become

  $$g_{11} = 6r_0 \frac{\sqrt{e^{3(r_0+r_1)}} - 1}{\sqrt{e^{3r_0}} - 1} \neq 0, \qquad g_{20} = \frac{\sqrt{e^{3(r_0+r_1)}} - 1}{\sqrt{e^{3r_0}} - 1} \frac{4r_0}{3r_1}(r_1 - r_0)\rho$$

  and Thm. 4.3 guarantees that 2-periodic solutions bifurcate from the extinction equilibrium ϕ_1^* at $\lambda = \frac{\rho}{3}$. Provided one is in a nonautonomous situation (i.e. it is $r_0 \neq r_1$), our Cor. 4.4 implies a transcritical bifurcation.

- $\phi_2^* = (1, 1)$: Above all, the Floquet spectrum reads as

  $$\begin{aligned} \sigma_2(\lambda) = \{ & r_0\left(2\lambda^2 r_1 + \lambda(r_1 - 1) + 2r_1 - 1\right) \\ & \pm w \left| r_0(1 - 2(\lambda+1)r_1) + r_1 \right| - \lambda r_1 - r_1 + 1\} \end{aligned}$$

  with a positive real $w := \sqrt{\lambda^2 - \lambda + 1}$. The corresponding characteristic polynomial from (4.10) has the coefficients

  $$\begin{aligned} p_1(\lambda) &= 2\left[r_0\left(-2\lambda^2 r_1 - \lambda r_1 + \lambda - 2r_1 + 1\right) + (\lambda+1)r_1 - 1\right], \\ p_0(\lambda) &= \left[3\lambda r_0^2 - 2(\lambda+1)r_0 + 1\right]\left[3\lambda r_1^2 - 2(\lambda+1)r_1 + 1\right]. \end{aligned}$$

  For $\lambda = \frac{\rho}{2} - 1$ both elements of $\sigma_2(\lambda)$ have the same value $1 + \frac{3}{2}r_0 r_1(2 - \rho)$. This Floquet multiplier of multiplicity 2 can never be 1, since this requires $\rho = 2$ and thus a biologically irrelevant parameter $\lambda = 0$. On the other hand, the value $\lambda_2^* = \rho\frac{2-\rho}{3-2\rho}$ guarantees the inclusion $1 \in \sigma_2(\lambda_2^*)$, while the remaining Floquet multiplier is

  $$\prod_{j=0}^{1} \frac{6r_j - 3\frac{r_j}{r_{j+1}} - 6 + 2\rho}{3 - 2\rho}$$

(note there that $\operatorname{tr}\Xi_2(\lambda_2^*)$ is the sum of the Floquet multipliers). The ambient parameter pairs (r_0, r_1) yielding this uncritical Floquet multiplier to be inside the interval $(-1, 1)$ are illustrated in Fig. 14(left). In addition, we have to require $\rho \notin \left[\frac{3}{2}, 2\right]$ in order to enforce $\lambda_2^* > 0$. Under these conditions it is

$$N(I_2 - \Xi_2(\lambda_2^*)) = \mathbb{R}\begin{pmatrix}\rho - 2\\ 1\end{pmatrix}, \qquad N(I_2 - \Xi_2(\lambda_2^*)^T) = \mathbb{R}\begin{pmatrix}2\rho - 3\\ \rho\end{pmatrix}$$

and our bifurcation indicators become

$$\begin{aligned} g_{11} &= r_0 r_1 (3 - 2\rho)^2 \neq 0 \quad \text{since } \rho \neq \frac{3}{2}, \\ g_{20} &= \rho(1-\rho)(r_0 - r_1)\frac{12r_1^2r_0^2 - 9r_1r_0^2 + 2r_0^2 - 9r_1^2r_0 + 4r_1r_0 + 2r_1^2}{r_0r_1^2}. \end{aligned}$$

Thus, in any case Thm. 4.3 ensures that the coexistence equilibrium ϕ_2^* bifurcates into a branch of 2-periodic solutions at $\lambda = \rho\frac{2-\rho}{3-2\rho}$. Provided $\rho \neq 1$ and $r_1 \neq r_0$, by Cor. 4.4 this happens in a transcritical way. A complement of the parameters (r_0, r_1) for which our results hold, is illustrated in Fig. 14(left).

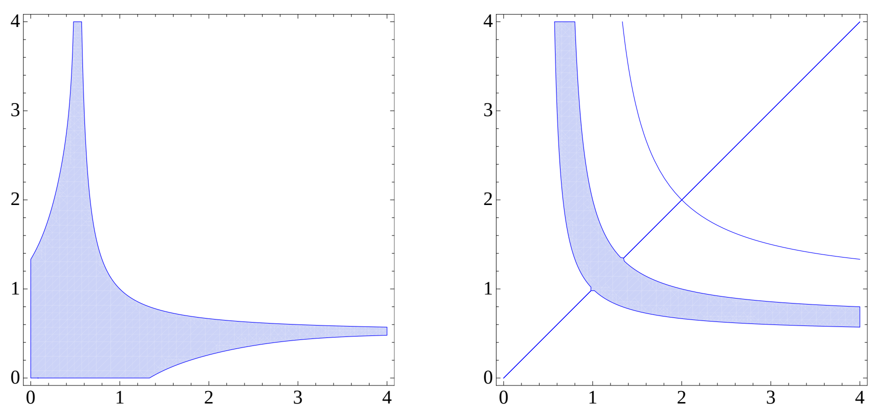

Figure 14: Admissible pairs $(r_0, r_1) \in (0, \infty)^2$ for the planar Ricker map (5.8): The shaded region consists of parameter pairs (r_0, r_1) such that $\sigma_2(\lambda_2^*) \setminus \{1\} \subseteq (-1, 1)$ (left), and $\lambda_2^* \leq 0$ or $g_{20} = 0$ (right)

Finally, thanks to Prop. 4.6, the stability of the bifurcating branch can be obtained from the expression $\frac{p_0'(\lambda_2^*)+p_1'(\lambda_2^*)}{2+p_1(\lambda_2^*)} = -\frac{(r_0+r_1)(3-2\rho)^2}{2(3-3\rho+\rho^2)} < 0$ and we see that the coexistence equilibrium ϕ_2^* looses its asymptotic stability as the parameter λ grows through the value $\rho\frac{2-\rho}{3-2\rho}$, while the bifurcating branch is unstable for $\lambda < \rho\frac{2-\rho}{3-2\rho}$ and becomes asymptotically stable for $\lambda > \rho\frac{2-\rho}{3-2\rho}$.

The symmetry in Fig. 14 illustrates the expected invariance of our results under permutation of the sequence values r_0 and r_1.

5.2.2. A juvenile/adult Ricker model

Our final example is a particular periodic generalization of the Ricker-like competition model from [13], where individuals from one of two species x and y under consideration can be characterized by their reproductive maturity. It reads as

$$\begin{pmatrix} x_{k+1}^1 \\ x_{k+1}^2 \\ x_{k+1}^3 \end{pmatrix} = f_k(x_k, \lambda) := \begin{pmatrix} \lambda x_k^2 e^{-c_{11}x_k^2 - c_{12}x_k^3} \\ \mu_k x_k^1 \\ b x_k^3 e^{-c_{21}x_k^1 - c_{22}x_k^3} \end{pmatrix} \tag{5.9}$$

and x_k^1, x_k^2 denote the numbers of juveniles resp. adults of a species x at time k, while species y with population size x_k^3 remains unstructured. All occurring parameters $c_{11}, c_{12}, c_{21}, c_{22}$ are assumed to be positive, $b > 1$, $(\mu_k)_{k\in\mathbb{Z}}$ is an ω_0-periodic sequence in $(0,1)$ and $\lambda > 0$ will be our bifurcation parameter.

In [13] it is shown that (5.9) has the exclusion equilibrium $\phi^* = (0, 0, \frac{\ln b}{c_{22}})$ with corresponding linearization

$$D_1 f_k(\phi^*, \lambda) = \begin{pmatrix} 0 & \lambda b^{-\frac{c_{12}}{c_{22}}} & 0 \\ \mu_k & 0 & 0 \\ -\frac{c_{21}}{c_{22}} \ln b & 0 & 1 - \ln b \end{pmatrix}.$$

Rather than giving an extensive stability analysis as in [13], where also other equilibria in the autonomous case are considered, we restrict to ϕ^*. From the form of the period matrix $\Xi_{\omega_0}(\lambda) \in \mathbb{R}^{3\times 3}$ (e.g. for even ω_0 it is lower triangular), we deduce the Floquet spectrum

$$\sigma_{\omega_0}(\lambda) = \{(1 - \ln b)^{\omega_0}, \nu_-(\lambda), \nu_+(\lambda)\}$$

with the Floquet multipliers

$$\nu_-(\lambda) := b^{-\frac{\omega_0 c_{12}}{2c_{22}}} \lambda^{\frac{\omega_0}{2}} \begin{cases} -\sqrt{\prod_{j=0}^{\omega_0 - 1} \mu_j}, & \omega_0 \text{ is odd}, \\ \prod_{j=0}^{\frac{\omega_0}{2}} \mu_{2j}, & \omega_0 \text{ is even}, \end{cases}$$

$$\nu_+(\lambda) := b^{-\frac{\omega_0 c_{12}}{2c_{22}}} \lambda^{\frac{\omega_0}{2}} \begin{cases} \sqrt{\prod_{j=0}^{\omega_0 - 1} \mu_j}, & \omega_0 \text{ is odd}, \\ \prod_{j=0}^{\frac{\omega_0}{2}} \mu_{2j+1}, & \omega_0 \text{ is even}. \end{cases}$$

This explicit form of $\sigma_{\omega_0}(\lambda)$ easily yields sufficient conditions for the stability of ϕ^* depending on λ. To give an impression of a bifurcation analysis, we restrict the case $\omega_0 = 3$ and make the simplifying assumptions $c := c_{11} = c_{12} = c_{21} = c_{22}$ and $b = \sqrt{e}$. Then the period matrix becomes

$$\Xi_3(\lambda) = \begin{pmatrix} 0 & \frac{\lambda^2 \mu_1}{e} & 0 \\ \frac{\lambda \mu_0 \mu_2}{\sqrt{e}} & 0 & 0 \\ -\frac{\lambda \mu_0}{2\sqrt{e}} - \frac{1}{8} & -\frac{\lambda}{4\sqrt{e}} & \frac{1}{8} \end{pmatrix}$$

and the Floquet spectrum reduces to

$$\sigma_3(\lambda) = \left\{ \frac{1}{8}, -\frac{\lambda^{3/2}\sqrt{\mu}}{e^{3/4}}, \frac{\lambda^{3/2}\sqrt{\mu}}{e^{3/4}} \right\}, \tag{5.10}$$

were we abbreviated $\mu := \mu_0\mu_1\mu_2$. Thus, we get a Floquet multiplier 1 for the critical parameter $\lambda^* = \frac{\sqrt{e}}{\sqrt[3]{\mu}}$ and particularly $\sigma_3(\lambda^*) = \left\{\frac{1}{8}, -1, 1\right\}$. Moreover,

$$\begin{aligned}
N(I_3 - \Xi_3(\lambda^*)) &= \mathbb{R}\begin{pmatrix} 7\mu_1\sqrt[3]{\mu} \\ 7\mu \\ -4\mu_0\mu_1 - \sqrt[3]{\mu}\mu_1 - 2\sqrt[3]{\mu}^2 \end{pmatrix}, \\
N(I_3 - \Xi_3(\lambda^*)^T) &= \mathbb{R}\begin{pmatrix} \sqrt[3]{\mu}^2 \\ \mu_1 \\ 0 \end{pmatrix}
\end{aligned}$$

and we arrive at the bifurcation indicators

$$\begin{aligned}
g_{11} &= \frac{21\mu^{4/3}}{\sqrt{e}\big(4\mu_0\mu_1 + \sqrt[3]{\mu}\mu_1 + 2\sqrt[3]{\mu}^2\big)} > 0, \\
g_{20} &= 14c\mu\frac{7\sqrt[3]{\mu}^2(1-\mu_2) + 2\sqrt[3]{\mu}\mu_2 + \mu_1(\sqrt[3]{\mu} + \mu_2) - \mu_0\big(\mu_1(7\mu_2 - 4) - \mu_2(2 - 7\sqrt[3]{\mu}) - 4\sqrt[3]{\mu}\big)}{\big(4\mu_0\mu_1 + \sqrt[3]{\mu}\mu_1 + 2\sqrt[3]{\mu}^2\big)^2}.
\end{aligned}$$

Since the transversality condition (4.7) is always fulfilled, Thm. 4.3 shows that a branch of 3-periodic solutions bifurcates at $\lambda = \frac{\sqrt{e}}{\sqrt[3]{\mu}}$ from the constant solution ϕ^*. Generically this bifurcation is transcritical (cf. Cor. 4.4) and we have plotted the triples (μ_0, μ_1, μ_2) for which the necessary condition $g_{20} = 0$ is violated in Fig. 15 (left). Furthermore, from (5.10) one sees that ϕ^* looses its asymptotic stability at $\lambda^* = \frac{\sqrt{e}}{\sqrt[3]{\mu}}$

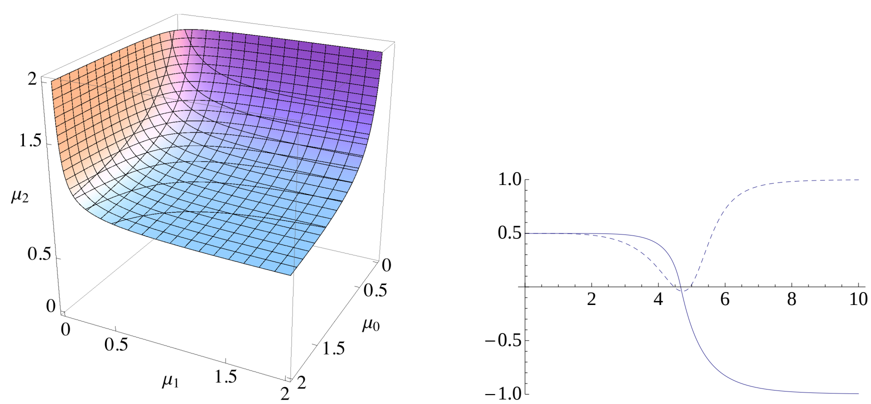

Figure 15: Triples $(\mu_0, \mu_1, \mu_2) \in (0, \infty)^3$ where the condition g_{20} is violated (left) and zeros of the function δ_ω from Thm. 4.7 for $\omega_0 = 7$ (solid) and $\omega_0 = 8$ (dashed) (right)

and becomes unstable.

To detect bifurcation values of λ for periods $\omega_0 > 0$ one can make use of Thm. 4.7. We illustrate this by means of the brief

Example 5.2. Choosing the ω_0-periodic sequence

$$\mu_k := \begin{cases} 1/2, & k \mod \omega_0 = 0, \\ 1/3, & k \mod \omega_0 \neq 0, \end{cases}$$

in Fig. 15(right) we plotted the graphs of $\frac{\delta_{\omega_0}}{1+|\delta_{\omega_0}|}$ in order to indicate bifurcation values, where δ_{ω_0} is the function (4.11) from Thm. 4.7 corresponding to the constant solution branch ϕ^*. Here, the special case $\omega_0 = 7$ yields one bifurcation value, while $\omega_0 = 8$ guarantees two bifurcation values for ω_0-periodic solutions.

6. Perspectives

Our global continuation results can be generalized to periodic difference eqns. (Δ_λ) with infinite-dimensional state spaces. Since their proofs are based on the Leray-Schauder degree, it suffices to additionally assume that the right-hand side of (Δ_λ) is completely continuous — a situation often met for integro-difference equations.

Although our bifurcation criteria are basically applications of more abstract and meanwhile classical, as well as celebrated results due to Crandall-Rabinowitz (our versions hail from [25, 46]), they deserve certain remarks:

- A situation, where the generic assumption $g_{10} \neq 0$ of our fold bifurcation Thm. 4.1 does not hold, can be tackled using [29, Thm. 2.1].
- In infinite-dimensional spaces (and for completely continuous right-hand sides of (Δ_λ)), the global statements of Thm. 4.3 require an adjustment: The assertion (c), as well as the alternatives (e_1) and (e_2) have to be replaced by "C (resp. the branch $C^\pm$) is not compact in $\ell_\omega(\Omega) \times \Lambda$" (see [45, Rem. 4.2]).
- For a violated transversality condition $g_{11} \neq 0$, the structure of the bifurcating solutions can be determined using a Newton polygon technique (see [25, pp. 112, Sect. I.15] for details).
- In applications one rarely knows a given solution branch in advance. Thus, it might be desirable to obtain transcritical- or pitchfork-like bifurcation patterns using local information at the critical parameter value λ^* only. This can be done using [29, Cor. 2.3].

The examples from Sect. 5 had essentially a demonstrative character and were kept short due to the overall length of the paper. Nevertheless, without doubt a more comprehensive analysis of them might be of interest.

As a concluding aspect we briefly discuss the situation of continuous time models of parameter-dependent periodic ordinary differential equations

$$\boxed{\dot{x} = F(t, x, \lambda)} \qquad (D_\lambda)$$

with a sufficiently smooth $F : \mathbb{R} \times \Omega \times \Lambda \to \mathbb{R}^d$ and $F(t + \omega_0, \cdot) \equiv F(t, \cdot)$ on the real axis $\mathbb{R}$ for some basic period $\omega_0 > 0$. On the one hand, its periodic solutions can be detected using fixed or periodic points of the corresponding ω_0-map, i.e. the autonomous difference equation

$$x_{k+1} = f(x_k, \lambda), \qquad f(x, \lambda) := \phi_\lambda(\omega_0; 0, x),$$

where ϕ_λ is the general solution to (D_λ). This transition allows to apply our above time-discrete results to deduce continuation and bifurcation properties for (D_λ). On the other hand, one can also directly carry over the proofs given below in Sect. 7.1 to continuous time problems (D_λ). Basically, the resulting conditions contain integrals $\int_\tau^{\tau+\omega}$ rather than sums $\sum_{j=\kappa}^{\kappa+\omega}$ and the complex unit circle has to be replaced by the imaginary axis.

7. Appendix

These appendices contain rigorous proofs for our results in Sects. 2–4, as well as their mathematical background and the required functional-analytical machinery.

7.1. Proofs — Periodic difference equations

Although we proceed according to the previous numeration, the proof of Prop. 2.2 is postponed to the end of this section and we begin with

Proof of Prop. 2.1. Let $\phi(\eta)$, $\eta \in U$, solve eqn. (g_η) with minimal period ω.

(I) We begin with a preparation on the robustness of injectivity: Given $\lambda, \bar{\lambda} \in \Lambda$ with $g(\phi_k^*, \lambda) = g(\phi_k^*, \bar{\lambda})$, due to the mean value theorem one has the representation

$$\int_0^1 D_2 g(\phi_k^*, \lambda + t(\bar{\lambda} - \lambda))\, dt(\bar{\lambda} - \lambda) = g(\phi_k^*, \bar{\lambda}) - g(\phi_k^*, \lambda) = 0$$

and our assumption (2.3) immediately leads to $\lambda - \bar{\lambda} = 0$. Thus, the linear mappings $\int_0^1 D_2 g(\phi_k^*, \lambda + t(\bar{\lambda} - \lambda))\, dt \in L(\mathbb{R}^p, \mathbb{R}^d)$ are one-to-one and as finite-dimensional operators also bounded below by [1, p. 70, Thm. 2.5]. Thanks to the continuity of the branch ϕ, [1, p. 70, Lemma. 2.4(2)] allows us to deduce that $\int_0^1 D_2 g(\phi(\eta)_k, \lambda + t(\bar{\lambda} - \lambda))\, dt$ is bounded below (and injective) for every sequence $\nu \in B_\rho(\nu^*)$ with a sufficiently small $\rho > 0$. Therefore, since we have

$$\int_0^1 D_2 g(\phi(\eta)_k, \lambda + t(\bar{\lambda} - \lambda))\, dt(\bar{\lambda} - \lambda) = g(\phi(\eta)_k, \lambda) - g(\phi(\eta)_k, \bar{\lambda}) \text{ for all } k \in \mathbb{Z},$$

the assumption $g(\phi(\eta)_k, \lambda) = g(\phi(\eta)_k, \bar{\lambda})$ enforces $\lambda = \bar{\lambda}$.

(II) Due to the limit relation $\lim_{\nu \to \nu^*} \phi(\nu) = \phi^*$ also ϕ^* is ω-periodic and thus ω is a multiple of ω_1; this shows (a). On the other hand, we have

$$g(\phi(\eta)_k, \eta_k) = \phi(\eta)_{k+1} = \phi(\eta)_{k+\omega+1} = g(\phi(\eta)_{k+\omega}, \eta_{k+\omega}) = g(\phi(\eta)_k, \eta_{k+\omega})$$

for all $k \in \mathbb{Z}$, and step (I) guarantees $\nu_{k+\omega} = \nu_k$ for parameter sequences $\nu \in B_\rho(\nu^*)$. Since ν has the minimal period ω_0, we conclude that ω must be a multiple of ω_0. In particular, $\phi(\nu)$ is $\operatorname{lcm}\{\omega_0, \omega_1\}$-periodic and we have $\omega = \operatorname{lcm}\{\omega_0, \omega_1\}$. $\square$

Proof of Thm. 2.3. Let $\lambda \in \Lambda$ be fixed. Since the dichotomy spectrum of (V_λ) is

$$\Sigma(\lambda) = \left\{ \sqrt[\omega]{|\nu|} : \nu \in \sigma_\omega(\lambda) \setminus \{0\} \right\}$$

(see [38, Ex. 2.8]), we obtain:

(a) It is $\Sigma(\lambda) \subseteq (0, 1)$ and the claim follows from [37, Prop. 3.9(b)].

(b) Here $\Sigma(\lambda)$ contains an element with modulus greater than 1 and thus the assertion results by [37, Prop. 3.10(a)]. $\square$

The essential tool for our approach is an equivalent formulation of an ω_0-periodic difference eqn. (Δ_λ) depending on $\lambda \in \Lambda$ as abstract equation in a space $\ell_\omega(\Omega)$ resp. ℓ_ω of sequences with ambient period $\omega \geq \omega_0$. We equip ℓ_ω with the inner product

$$\langle\!\langle \phi, \psi \rangle\!\rangle := \sum_{k=0}^{\omega-1} \langle \phi_k, \psi_k \rangle \quad \text{for all } \phi, \psi \in \ell_\omega,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product on $\mathbb{R}^d$ and observe

$$\langle\!\langle \phi, \psi \rangle\!\rangle = \sum_{k=\kappa}^{\kappa+\omega-1} \langle \phi_k, \psi_k \rangle \quad \text{for all } \kappa \in \mathbb{Z},\ \phi, \psi \in \ell_\omega. \tag{7.1}$$

This makes ℓ_ω a $d\omega$-dimensional real Hilbert space. Moreover, one has the continuous embedding $\ell_\omega \hookrightarrow \ell_{\omega_1}$ for every multiple ω_1 of $\omega \in \mathbb{N}$. The finite dimensionality of ℓ_ω simplifies various of our later global (and topological) arguments drastically.

In the following, suppose $\Omega \subseteq \mathbb{R}^d$, $\Lambda \subseteq \mathbb{R}^p$ are open and $f_k : \Omega \times \Lambda \to \mathbb{R}^d$ is of class C^m, $m \in \mathbb{N}$. For our abstract calculus it is worth to point out (and not difficult to see) that $\ell_\omega(\Omega)$ is an open subset of ℓ_ω. Moreover, if ω is a multiple of both ω_0 and ω_1, then the mapping $G : \ell_{\omega_1}(\Omega) \times \Lambda \to \ell_\omega$, pointwise given as

$$G(\phi, \lambda)_k := \phi_k - f_{k-1}(\phi_{k-1}, \lambda) \quad \text{for all } k \in \mathbb{Z}$$

is well-defined and of class C^m. One easily computes the partial derivatives

$$[D_1 G(\phi, \lambda)\psi]_k = \psi_k - D_1 f_{k-1}(\phi_{k-1}, \lambda)\psi_{k-1}, \tag{7.2}$$

$$[D_1^{n_1} D_2^{n_2} G(\phi, \lambda)\psi^{n_1}\eta^{n_2}]_k = -D_1^{n_1} D_2^{n_2} f_{k-1}(\phi_{k-1}, \lambda)\psi_{k-1}^{n_1}\eta^{n_2} \tag{7.3}$$

for all $\phi \in \ell_{\omega_1}(\Omega)$, $\lambda \in \Lambda$, $\psi \in \ell_{\omega_1}$, $\eta \in \mathbb{R}^p$. The integers $n_1, n_2 \in \mathbb{N}_0$ in (7.3) have to fulfill $(n_1, n_2) \notin \{(0, 0), (1, 0)\}$ and $n_1 + n_2 \leq m$. A precise verification of these facts follows along the lines of [38, Prop. 2.3], where the corresponding situation with the space ℓ_∞ of bounded sequences is considered. Being finite-dimensional operators, for $\omega = \omega_1$ the partial derivatives $D_1 G(\phi, \lambda) \in L(\ell_\omega)$ are Fredholm with index 0.

The next result is merely an observation, but crucial for our overall approach. We leave its straight forward proof to the interested reader.

Theorem 7.1. *Given $\lambda \in \Lambda$, let $\omega_1 \in \mathbb{N}$ and ω be a multiple of ω_0 and ω_1.*

(a) If $\phi \in \ell_{\omega_1}(\Omega)$ solves the ω_0-periodic difference eqn. (Δ_λ), then

$$G(\phi, \lambda) = 0. \tag{O_λ}$$

(b) Conversely, if $\phi \in \ell_\omega(\Omega)$ solves (O_λ), then ϕ is an ω-periodic solution of (Δ_λ).

From now on let us suppose that $\phi(\lambda) \in \ell_\omega(\Omega)$, $\lambda \in \Lambda$, denotes a branch of periodic solutions to (Δ_λ). Then there exists a close relationship between the Floquet spectrum $\sigma_\omega(\lambda)$ of (V_λ) and the eigenvalues of the derivative $D_1G(\phi(\lambda), \lambda)$, when $G(\cdot, \lambda)$ is considered as a mapping between sequence spaces of equal period ω:

Proposition 7.2. *Let $\kappa \in \mathbb{Z}$, $\lambda \in \Lambda$. For $\upsilon \neq 1$ the assertions are equivalent:*

(a) ψ is a nontrivial ω-periodic solution of

$$x_{k+1} = \tfrac{1}{1-\upsilon} D_1 f_k(\phi(\lambda)_k, \lambda) x_k, \tag{7.4}$$

(b) $\psi \in \ell_\omega$ is an eigenvector of $D_1G(\phi(\lambda), \lambda) \in L(\ell_\omega)$ with eigenvalue υ,

(c) $\psi_\kappa \in \mathbb{R}^d$ is an eigenvector of $\Xi_\omega(\lambda) \in \mathbb{R}^{d \times d}$ with eigenvalue $(1-\upsilon)^\omega$.

In particular, one has the spectral mapping relation

$$[1 - \sigma(D_1G(\phi(\lambda), \lambda))]^\omega = \sigma(\Xi_\omega(\lambda)). \tag{7.5}$$

Remark 7.1. The kernel of $D_1G(\phi(\lambda), \lambda)$ consists of ω-periodic solutions to the variational eqn. (V_λ) and consequently allows the representation

$$N(D_1G(\phi(\lambda), \lambda)) = J_\kappa^{-1}\big(\Phi_\lambda(k, \kappa) N(I_d - \Xi_\omega(\lambda))\big)_{k=\kappa}^{\kappa+\omega-1}. \tag{7.6}$$

If all the matrices $D_1 f_k(\phi(\lambda)_k, \lambda) \in \mathbb{R}^{d \times d}$, $\kappa \leq k < \kappa + \omega$, are invertible, then

$$N(D_1G(\phi(\lambda), \lambda)) = \{\Phi_\lambda(\cdot, \kappa)\xi \in \ell_\omega : \xi \in N(I_d - \Xi_\omega(\lambda))\}.$$

Proof of Prop. 7.2. Let $\kappa \in \mathbb{Z}$, $\lambda \in \Lambda$ and $\upsilon \neq 1$.

$(a) \Rightarrow (b)$ Given a solution $\psi \in \ell_\omega \setminus \{0\}$ of (7.4) we get

$$\upsilon \psi_k = \psi_k - D_1 f_{k-1}(\phi(\lambda)_{k-1}, \lambda)\psi_{k-1} \overset{(7.2)}{=} [D_1G(\phi(\lambda), \lambda)\psi]_k \quad \text{for all } k \in \mathbb{Z}$$

and thus υ is an eigenvalue of $D_1G(\phi(\lambda), \lambda)$ with eigenvector ψ.

$(b) \Rightarrow (c)$ The eigenvector identity implies $\psi_{k+1} \equiv \frac{1}{1-\upsilon} D_1 f_k(\phi(\lambda)_k, \lambda)\psi_k$ on the whole integer axis $\mathbb{Z}$ and thus $\psi_{\kappa+\omega} = (1-\upsilon)^{-\omega}\,\Phi_\lambda(\kappa+\omega, \kappa)\psi_\kappa$. Due to the periodicity $\psi \in \ell_\omega$ this shows $(1-\upsilon)^\omega \psi_\kappa = \Xi_\omega(\lambda)\psi_\kappa$ (cf. (2.4)). If we assume $\psi_\kappa = 0$, then the eigenvalue-eigenvector identity $D_1G(\phi(\lambda), \lambda)\psi = \upsilon\psi$ and (7.2) imply $\psi = 0$ and thus ψ cannot be an eigenvector. Therefore, $\psi_\kappa \neq 0$ and (c) follows.

$(c) \Rightarrow (a)$ results from [35, p. 109, Prop. 3.2.3] applied to (7.4). □

Besides the derivative $D_1G(\phi^*, \lambda^*)$ also its adjoint plays an important role:

Lemma 7.3. *Let $\phi^* \in \ell_{\omega_1}(\Omega)$, $\lambda^* \in \Lambda$ and ω be a multiple of ω_0 and ω_1. The adjoint of the partial derivative $D_1G(\phi^*, \lambda^*) \in L(\ell_\omega)$ to $G : \ell_\omega(\Omega) \times \Lambda \to \ell_\omega$ is given by $[D_1G(\phi^*, \lambda^*)'\psi]_k = \psi_k - D_1 f_k(\phi_k^*, \lambda^*)^T \psi_{k+1}$ for all $k \in \mathbb{Z}$.*

Proof of Lemma 7.3. For arbitrary sequences $\phi, \psi \in \ell_\omega$ we obtain

$$\begin{aligned}
\langle\!\langle D_1 G(\phi^*, \lambda^*)\phi, \psi \rangle\!\rangle &\overset{(7.2)}{=} \sum_{k=0}^{\omega-1} \langle \phi_k - D_1 f_{k-1}(\phi^*_{k-1}, \lambda^*)\phi_{k-1}, \psi_k \rangle \\
&\overset{(7.1)}{=} \sum_{k=0}^{\omega-1} \langle \phi_k, \psi_k \rangle - \sum_{k=0}^{\omega-1} \langle D_1 f_k(\phi^*_k, \lambda^*)\phi_k, \psi_{k+1} \rangle \\
&= \sum_{k=0}^{\omega-1} \langle \phi_k, \psi_k - D_1 f_k(\phi^*_k, \lambda^*)^T \psi_{k+1} \rangle \\
&= \langle\!\langle \phi, D_1 G(\phi^*, \lambda^*)' \psi \rangle\!\rangle
\end{aligned}$$

and consequently the claim. □

In addition to (V_λ), we introduce the *dual variational equation*

$$\boxed{x_k = D_1 f_k(\phi(\lambda)_k, \lambda)^T x_{k+1}} \tag{V'_λ}$$

of the branch $\phi(\lambda) \in \ell_{\omega_1}(\Omega)$ for (Δ_λ), and the *dual transition operator*

$$\Phi'_\lambda(k, l) := \begin{cases} D_1 f_k(\phi(\lambda)_k, \lambda)^T \cdots D_1 f_{l-1}(\phi(\lambda)_{l-1}, \lambda)^T, & k < l, \\ I_d, & k = l; \end{cases}$$

thus, $\Phi'_\lambda(\cdot, \kappa)\xi : \mathbb{Z}^-_\kappa \to \mathbb{R}^d$, $\kappa \in \mathbb{Z}$, $\xi \in \mathbb{R}^d$ with $\mathbb{Z}^-_\kappa := \{n \in \mathbb{Z} : n \le \kappa\}$, is the general backward solution of (V'_λ). Moreover, one has the relation

$$\Phi'_\lambda(k, l) = \Phi_\lambda(l, k)^T \quad \text{for all } k \le l \tag{7.7}$$

and we arrive at a dual version of Prop. 7.2:

Proposition 7.4. *Let $\kappa \in \mathbb{Z}$, $\lambda \in \Lambda$. For $\upsilon \neq 1$ the assertions are equivalent:*

(a) ψ is a nontrivial ω-periodic solution of

$$x_k = \tfrac{1}{1-\upsilon} D_1 f_k(\phi(\lambda)_k, \lambda)^T x_{k+1}, \tag{7.8}$$

(b) $\psi \in \ell_\omega$ is an eigenvector of $D_1 G(\phi(\lambda), \lambda)' \in L(\ell_\omega)$ with eigenvalue υ,

(c) $\psi_\kappa \in \mathbb{R}^d$ is an eigenvector of $\Xi_\omega(\lambda)^T \in \mathbb{R}^{d \times d}$ with eigenvalue $(1-\upsilon)^\omega$.

Remark 7.2. Using the notation introduced in (4.5), due to Lemma 7.3 and (7.7) the kernel of $D_1 G(\phi(\lambda), \lambda)'$ allows the representation

$$N(D_1 G(\phi(\lambda), \lambda)') = \hat{\Phi}_{\lambda^*}(\kappa, \cdot)^T N(I_d - \Xi_\omega(\lambda)^T). \tag{7.9}$$

For the invertible special case $D_1 f_k(\phi(\lambda)_k, \lambda) \in GL(\mathbb{R}^d)$, $\kappa \le k < \kappa + \omega$, this simplifies to $N(D_1 G(\phi(\lambda), \lambda)') = \left\{\Phi_\lambda(\kappa, \cdot)^T \eta \in \ell_\omega : \eta \in N(I_d - \Xi_\omega(\lambda)^T)\right\}$.

Proof of Prop. 7.4. With given $\kappa \in \mathbb{Z}$, $\lambda \in \Lambda$, $\upsilon \neq 1$ it is not surprising that the proof resembles the one of Prop. 7.2.

$(a) \Rightarrow (b)$ By Lemma 7.3, every solution $\psi \in \ell_\omega \setminus \{0\}$ to (7.8) satisfies

$$\upsilon\psi_k = \psi_k - D_1 f_k(\phi(\lambda)_k, \lambda)^T \psi_{k+1} = [D_1 G(\phi(\lambda), \lambda)'\psi]_k \quad \text{for all } k \in \mathbb{Z},$$

which implies that ψ is an eigenvector of $D_1 G(\phi(\lambda), \lambda)'$ with eigenvalue υ.

$(b) \Rightarrow (c)$ By the eigenvector relation, $\psi_k = \frac{1}{1-\upsilon} D_1 f_k(\phi(\lambda)_k, \lambda)^T \psi_{k+1}$ and

$$\psi_\kappa = (1-\upsilon)^{-\omega}\, \Phi'_\lambda(\kappa, \kappa+\omega)\psi_{\kappa+\omega} \stackrel{(7.7)}{=} (1-\upsilon)^{-\omega}\, \Phi_\lambda(\kappa+\omega, \kappa)^T \psi_\kappa.$$

This relation, in turn, ensures $(1-\upsilon)^\omega \psi_\kappa = \Xi_\omega(\lambda)^T \psi_\kappa$ and therefore (c).

$(c) \Rightarrow (a)$ By assumption we have the relation

$$(1-\upsilon)^\omega \psi_\kappa = \Xi_\omega(\lambda)^T \psi_\kappa \stackrel{(7.7)}{=} \Phi'_\lambda(\kappa, \kappa+\omega)\psi_\kappa = \Phi'_\lambda(\kappa-\omega, \kappa)\psi_\kappa$$

and define $\psi_k := (1-\upsilon)^{k-\kappa}\Phi'_\lambda(k, \kappa)\psi_\kappa$ for all $k \in \mathbb{Z}_\kappa^-$. Hence, we obtain

$$\begin{aligned} \psi_{k-\omega} &= (1-\upsilon)^{k-\kappa-\omega}\Phi'_\lambda(k-\omega, \kappa-\omega)\Phi'_\lambda(\kappa-\omega, \omega)\psi_\kappa \\ &= (1-\upsilon)^{k-\kappa}\Phi'_\lambda(k-\omega, \kappa-\omega)\psi_\kappa = \psi_k \quad \text{for all } k \in \mathbb{Z}_\kappa^-; \end{aligned}$$

thus, the sequence ψ is ω-periodic on $\mathbb{Z}_\kappa^-$ and solves (7.8) by definition. Moreover, due to $\psi_\kappa \neq 0$ we have that ψ is nontrivial and the sequence ψ can be extended ω-periodically to an entire solution of (7.8) in ℓ_ω. □

It remains to provide the promised

Proof of Prop. 2.2. First, the relation (7.5) shows that $\sigma_\omega(\lambda) = \sigma(\Xi_\omega(\lambda))$ does not depend on the initial time $\kappa \in \mathbb{Z}$. Given $n \in \mathbb{N}$, $\lambda \in \Lambda$ one also has

$$\sigma_{n\omega}(\lambda) = \sigma(\Phi_\lambda(\kappa+n\omega, \kappa)) \stackrel{(2.1)}{=} \sigma(\Xi_\omega(\lambda)^n) = \sigma(\Xi_\omega(\lambda))^n = \sigma_\omega(\lambda)^n$$

using the spectral mapping theorem (cf., e.g., [24, p. 45]). □

7.2. Proofs — Continuation of periodic solutions

Proof of Thm. 3.1. We solve eqn. (O_λ) in ℓ_ω with the implicit function theorem (cf., e.g., [46, pp. 150–151, Thm. 4.B]). First, by Thm. 7.1(a) we have $G(\phi^*, \lambda^*) = 0$. On the other hand, by virtue of [35, p. 113, Prop. 3.2.10] our assumption $1 \notin \sigma_\omega(\lambda^*)$ implies that for every $h \in \ell_\omega$ there exists a unique solution $\psi \in \ell_\omega$ of the inhomogeneous equation $x_{k+1} = D_1 f_k(\phi_k^*, \lambda^*)x_k + h_k$, i.e. it is

$$[D_1 G(\phi^*, \lambda^*)\psi]_k = \psi_k - D_1 f_{k-1}(\phi_{k-1}^*, \lambda^*)\psi_{k-1} = h_{k-1} \quad \text{for all } k \in \mathbb{Z}$$

(cf. (7.2)). This, in turn, means that $D_1 G(\phi^*, \lambda^*) \in L(\ell_\omega)$ is invertible and consequently there exists a unique branch $\phi(\lambda) \in \ell_\omega(\Omega)$ of solutions with $G(\phi(\lambda), \lambda) \equiv 0$ on $B_\rho(\lambda^*)$. Using Thm. 7.1(b) we obtain the assertions (a) and (b), except the formula (3.2) for $\phi'(\lambda^*)$: Concerning this, differentiating the identity $\phi(\lambda)_{k+1} = f_k(\phi(\lambda)_k, \lambda)$

on $B_\rho(\lambda^*)$ immediately yields that the derivative $\phi'(\lambda^*)$ solves the linear inhomogeneous and ω-periodic difference eqn. $x_{k+1} = D_1 f_k(\phi_k^*, \lambda)x_k + D_2 f_k(\phi_k^*, \lambda)$. Thanks to our assumption (3.1), from [35, p. 113, Prop. 3.2.10] we obtain a unique solution in ℓ_ω. Its value at time κ is precisely ξ_κ and the claimed formula (3.2) follows from the variation of constants formula.

For assertion (c), we observe that the period matrix $\Xi_\omega : \Lambda \to \mathbb{R}^{d\times d}$ is continuous. Also the eigenvalues of $\Xi_\omega(\lambda)$ depend continuously on λ (see [24, pp. 107–108]), as well as the Floquet spectrum $\sigma_\omega(\lambda)$. Hence, for a hyperbolic solution ϕ^* to (Δ_{λ^*}), i.e. $\sigma_\omega(\lambda^*) \cap \mathbb{S}^1 = \emptyset$, we deduce $\sigma_\omega(\lambda) \cap \mathbb{S}^1 = \emptyset$ for λ in a whole neighborhood $B_\rho(\lambda^*)$ and thus also the $\phi(\lambda)$ are hyperbolic. $\square$

Proof of Cor. 3.2. From Thm. 3.1 we get a unique branch $\phi(\lambda) \in \ell_\omega(\Omega)$ consisting of $n\omega$-periodic solutions. By Prop. 2.2 it is $1 \notin \sigma_\omega(\lambda)^n = \sigma_{n\omega}(\lambda)$ and thus one shows as in the proof of Thm. 3.1 that (Δ_λ) has a uniquely determined solution branch in $\ell_{n\omega}(\Omega)$, which has to coincide with $\phi(\lambda)$. $\square$

Proof of Thm. 3.3. We make use of the global implicit function theorem [25, p. 210, Thm. II.6.1] to solve the operator eqn. (O_λ). First of all, the mapping G is of the form $G(\phi, \lambda) = \phi - F(\phi, \lambda)$ with the continuous substitution operator $F : \ell_\omega \times \Lambda \to \ell_\omega$, $F(\phi, \lambda)_k := f_{k-1}(\phi_{k-1}, \lambda)$. Since ℓ_ω is finite-dimensional, $F(\cdot, \lambda)$ is completely continuous and degree theory due to Leray-Schauder (even Brouwer!) applies.

Thanks to Thm. 7.1(a) we know that $G(\phi^*, \lambda^*) = 0$ holds. Moreover, as in the proof of Thm. 3.1 one shows $D_1 G(\phi^*, \lambda^*) \in GL(\ell_\omega)$. Then [25, p. 210, Thm. II.6.1] applies directly and the claim follows using Thm. 7.1(b). $\square$

7.3. *Proofs — Bifurcations of periodic solutions*

In the nonhyperbolic situation $1 \in \sigma_\omega(\lambda^*)$ we tackle the problem (Δ_λ) or equivalently (O_λ) using the Lyapunov-Schmidt method (see, e.g., [46, 25]). This classical reduction principle allows an algorithmic formulation:

1. Given the orthonormal vectors $\xi_1, \ldots, \xi_n \in \mathbb{R}^d$ defined in (4.3), we introduce sequences $\phi^i := \Phi_{\lambda^*}(\kappa + (\cdot - \kappa \mod \omega), \kappa)\xi_i$. Referring to Prop. 7.4, each ϕ^i defines a nontrivial ω-periodic solution to the variational eqn. (V_{λ^*}); moreover, $\phi_1, \ldots, \phi_n \in \ell_\omega$ are linearly independent. Furthermore, with the ω-periodic sequences $\psi_j := \delta_{\cdot-\kappa \mod \omega,0}\xi_j$ one obtains the orthonormality relations
$$\langle \phi^i, \psi_j \rangle \overset{(7.1)}{=} \sum_{k=\kappa}^{\kappa+\omega-1} \langle \Phi_{\lambda^*}(k,\kappa)\xi_i, \psi_{j,k} \rangle = \langle \xi_i, \xi_j \rangle = \delta_{i,j} \text{ for all } 1 \le i, j \le n,$$
where $\delta_{i,j}$ stands for the Kronecker symbol.

2. With the orthonormal vectors $\xi_1', \ldots, \xi_r' \in \mathbb{R}^d$ from (4.4) and the notation introduced in (4.5), we define the ω-periodic sequences
$$\phi_i' := \hat{\Phi}_{\lambda^*}(\kappa, \cdot)^T \xi_i', \qquad \psi^j := \delta_{\cdot-\kappa \mod \omega,0}\xi_j'. \tag{7.10}$$

Using Prop. 7.4 we see that $\phi'_1, \dots, \phi'_r \in \ell_\omega$ are linearly independent ω-periodic solutions to the dual variational eqn. (V'_{λ^*}) satisfying

$$\langle\!\langle \phi'_i, \psi^j \rangle\!\rangle = \sum_{k=\kappa}^{\kappa+\omega-1} \langle \hat{\Phi}_{\lambda^*}(\kappa,k)^T \xi'_i, \psi^j_k \rangle = \langle \xi'_i, \xi'_j \rangle = \delta_{i,j} \quad \text{for all } 1 \le i,j \le r$$

and, in addition, the set $\{\psi^1, \dots, \psi^r\} \subseteq \ell_\omega$ is orthonormal.

Our next result allows a convenient representation for $R(D_1G(\phi^*, \lambda^*))$.

Lemma 7.5. *Let $\phi^* \in \ell_\omega(\Omega)$ and $\lambda^* \in \Lambda$. With the linear functionals*

$$\mu_i : \ell_\omega \to \mathbb{R}, \qquad \mu_i(\chi) := \sum_{j=\kappa}^{\kappa+\omega-1} \langle \hat{\Phi}_{\lambda^*}(\kappa,j)^T \xi'_i, \chi_j \rangle \quad \textit{for all } 1 \le i \le r,$$

one has $R(D_1G(\phi^*, \lambda^*)) = \bigcap_{i=1}^r N(\mu_i)$.

Proof. Using [46, p. 366, Prop. 8.14(2)] we obtain the equivalences

$$\begin{aligned}
\chi \in R(D_1G(\phi^*, \lambda^*)) \quad &\Leftrightarrow \quad \chi \in N(D_1G(\phi^*, \lambda^*)')^\perp \\
&\Leftrightarrow \quad \langle\!\langle \psi', \chi \rangle\!\rangle = 0 \quad \text{for all } \psi' \in N(D_1G(\phi^*, \lambda^*)') \\
&\overset{(7.9)}{\Leftrightarrow} \quad \langle\!\langle \hat{\Phi}_{\lambda^*}(\kappa,\cdot)^T \xi'_i, \chi \rangle\!\rangle = 0 \quad \text{for all } 1 \le i \le r \\
&\overset{(7.1)}{\Leftrightarrow} \quad \sum_{j=\kappa}^{\kappa+\omega-1} \langle \hat{\Phi}_{\lambda^*}(\kappa,j)^T \xi'_i, \chi_j \rangle = 0 \quad \text{for all } 1 \le i \le r \\
&\Leftrightarrow \quad \mu_i(\chi) = 0 \quad \text{for all } 1 \le i \le r \Leftrightarrow \chi \in \bigcap_{i=1}^r N(\mu_i),
\end{aligned}$$

which verify our claim. □

Proof of Thm. 4.1. We determine the zeros of $G : \ell_\omega(\Omega) \times \Lambda \to \ell_\omega$. Because ϕ^* is an ω-periodic solution of (Δ_{λ^*}), we obtain $G(\phi^*, \lambda^*) = 0$ from Thm. 7.1(a). Moreover, since 1 is a simple eigenvalue of $\Xi_\omega(\lambda^*)$, referring to the representation (7.6) we see that $N(D_1G(\phi^*, \lambda^*)) = \operatorname{span}\{\phi^1\}$ with $\phi^1 = \Phi_{\lambda^*}(\cdot,\kappa)\xi_1$ (see above). Using the linear functional $\mu = \mu_1$ from Lemma 7.5 our assumptions guarantee

$$\mu(D_2G(\phi^*, \lambda^*)) \overset{(7.3)}{=} - \sum_{j=\kappa}^{\kappa+\omega-1} \langle\!\langle \hat{\Phi}_{\lambda^*}(\kappa,j)^T \xi'_i, D_2 f_{j-1}(\phi^*_{j-1}, \lambda^*) \rangle\!\rangle \neq 0.$$

This makes the abstract fold bifurcation result [25, p. 12, Thm. I.4.1] applicable (see also [36, Thm. A.2]) in the Hilbert space ℓ_ω. The claim follows using Thm. 7.1(b). □

For later use we formulate a technical tool of independent interest. It addresses the smooth dependence of real eigenvalues and -vectors on parameters. Thereto, suppose that $\Xi_\omega : \Lambda \to \mathbb{R}^{d \times d}$ is a matrix function of class C^m, for instance a period matrix.

Theorem 7.6 (perturbation of simple eigenvalues)**.** *Let $\lambda^* \in \Lambda$. If $\nu^* \in \mathbb{R}$ is a simple eigenvalue for $\Xi_\omega(\lambda^*) \in \mathbb{R}^{d\times d}$ with corresponding eigenvector $x^* \in \mathbb{R}^d$ of norm 1, then there exist open neighborhoods $\Lambda_0 \subseteq \Lambda$ of λ^*, $U \subseteq \mathbb{R}$ of ν^* and unique C^m-functions $\nu : \Lambda_0 \to U$, $x : \Lambda_0 \to \mathbb{R}^d$ such that*

(a) $(\nu(\lambda^*), x(\lambda^*)) = (\nu^*, x^*)$,

(b) $\Xi_\omega(\lambda)x(\lambda) = \nu(\lambda)x(\lambda)$ *for all* $\lambda \in \Lambda_0$,

(c) $|x(\lambda)| \equiv 1$ *on* Λ_0.

Proof of Thm. 7.6. We define the C^m-mapping $F : \mathbb{R}^d \times \mathbb{R} \times \Lambda \to \mathbb{R}^d \times \mathbb{R}$ by

$$F(x, \nu; \lambda) := \begin{pmatrix} \Xi_\omega(\lambda)x - \nu x \\ \langle x, x\rangle - 1 \end{pmatrix}.$$

It satisfies $F(x^*, \nu^*; \lambda^*) = (\Xi_\omega(\lambda^*)x^* - \nu^* x^*, |x^*|^2 - 1) = (0, 0)$ and moreover

$$D_{(1,2)}F(x^*, \nu^*; \lambda^*) \begin{pmatrix} x \\ \nu \end{pmatrix} = \begin{pmatrix} \Xi_\omega(\lambda^*)x - \nu^* x - \nu x^* \\ 2\langle x, x^*\rangle \end{pmatrix}$$

for all $x \in \mathbb{R}^d$, $\nu \in \mathbb{R}$. Hence, $D_{(1,2)}F(x^*, \nu^*; \lambda^*)\left(\begin{smallmatrix} x \\ \nu \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ is equivalent to

$$[\Xi_\omega(\lambda^*) - \nu^* I_d]x = \nu x^* \in N, \qquad x \in N^\perp, \tag{7.11}$$

where we have abbreviated $N := N(\Xi_\omega(\lambda^*) - \nu^* I_d) = \operatorname{span}\{x^*\}$. Since ν^* is a simple eigenvalue of $\Xi_\omega(\lambda^*)$, the subspace N reduces $\Xi_\omega(\lambda^*) - \nu^* I_d$, i.e.,

$$[\Xi_\omega(\lambda^*) - \nu^* I_d]N \subseteq N, \qquad [\Xi_\omega(\lambda^*) - \nu^* I_d]N^\perp \subseteq N^\perp$$

and so $[\Xi_\omega(\lambda^*) - \nu^* I_d]x \in N \cap N^\perp = \{0\}$ holds. Hence, we arrive at the inclusion $x \in N$, the right relation in (7.11) leads to $x = 0$ and with the left relation in (7.11) we also get $\nu = 0$. Furthermore, it is $N(D_{(1,2)}F(x^*, \nu^*; \lambda^*)) = \{(0, 0)\}$ and

$$D_{(1,2)}F(x^*, \nu^*; \lambda^*) \in GL(\mathbb{R}^d \times \mathbb{R}).$$

Hence, the implicit mapping theorem (cf., e.g., [46, pp. 150–151, Thm. 4.B]) implies the existence of C^m-functions $x(\lambda), \nu(\lambda)$ with $F(x(\lambda), \nu(\lambda); \lambda) \equiv 0$ satisfying the assertions. □

By Thm. 7.6 we know that a real eigenvalue of, for instance, the period matrix $\Xi_\omega(\lambda)$ perturbs as a real number under variation of the parameter λ and inherits its smoothness from the above matrix function Ξ_ω.

Corollary 7.7. *Suppose that $\nu(\lambda^*) = \pm 1$ is a simple eigenvalue of $\Xi_\omega(\lambda^*)$. If the characteristic polynomial has the representation* (4.10)*, then*

$$\nu'(\lambda^*) = \mp \frac{\sum_{j=0}^d (\pm 1)^j p_j'(\lambda^*)}{\sum_{j=1}^d j(\pm 1)^j p_j(\lambda^*)}.$$

Proof. By Thm. 7.6 we have $\sum_{j=0}^{d} p_j(\lambda)\nu(\lambda)^j \equiv \det(\Xi_\omega(\lambda) - \nu(\lambda)I_d) \equiv 0$ on Λ, where we set $p_d(\lambda) :\equiv 1$. Differentiation w.r.t. λ yields

$$\sum_{j=0}^{d-1} p_j'(\lambda)\nu(\lambda)^j + \nu'(\lambda)\sum_{j=1}^{d} jp_j(\lambda)\nu(\lambda)^{j-1} \equiv 0.$$

The claim follows, if we insert the condition $\nu(\lambda^*) = \pm 1$. □

Proof of Cor. 4.2. Denote the period matrix to the variational equation

$$x_{k+1} = D_1 f_k(\psi(s)_k, \lambda(s))x_k$$

along Γ by $\Xi_\omega(s)$ and its corresponding Floquet spectrum by $\sigma_\omega(s)$.

By assumption the period matrix $\Xi_\omega(\lambda^*)$ has a simple eigenvalue 1 and using Thm. 7.6 we see that 1 is contained in a unique C^{m-1}-curve $\nu(s)$ of eigenvalues to $\Xi_\omega(s)$ with $\nu(0) = 1$. Referring to Prop. 7.2 one knows that $D_1G(\psi^*, \lambda^*)$ has the simple eigenvalue 0. Analogously, in [25, p. 22, Thm. I.7.2] it is shown that 0 is located on a unique C^{m-1}-curve $\upsilon(s)$ of eigenvalues to $D_1G(\psi(s), \lambda(s))$ with $\upsilon(0) = 0$ and moreover, [25, p. 26, (I.7.30)] shows $\upsilon'(0) = -g_{20}$. By Prop. 7.2(c) we have the relation $\nu(s) = (1 - \upsilon(s))^\omega$ yielding $\nu'(s) = -\omega(1 - \upsilon(s))^{\omega-1}\upsilon'(s)$ and thus $\nu'(0) = \omega g_{20}$. After these preparations, the stability assertions for the ω-periodic solutions to (Δ_λ) on Γ yield as follows: Referring to (4.6) and the continuous dependence of the spectrum under perturbations (cf. [24, pp. 107–108]), the disjoint splitting $\sigma_\omega(s) = \{\nu(s)\} \dot\cup \Sigma$ with $\Sigma \subseteq B_1(0)$ persists for $|s|$ near 0. Hence, the location of the dominant eigenvalue $\nu(s)$ implies stability.

(a) If $g_{20} > 0$, then $\nu(s)$ leaves the stability interval $(-1, 1)$ at 1 for increasing parameters s. So, the solutions $\psi(s)$ for $s > 0$ become unstable.

(b) For $g_{20} < 0$ a dual argument applies. □

Proof of Thm. 4.3. We can apply [25, p. 15, Thm. I.5.1] (or, in our notation, [36, Thm. A.3]) to the mapping $G : \ell_\omega(\Omega) \times \Lambda \to \ell_\omega$. Thereto, assumption (O') and Thm. 7.1(a) guarantee a constant solution branch $G(\phi^*, \lambda) \equiv 0$ on Λ and with (7.6) the kernel $N(D_1G(\phi^*, \lambda^*))$ is spanned by the sequence $\phi^1 := \Phi_{\lambda^*}(\cdot, \kappa)\xi_1 \in \ell_\omega$. By (4.7) and (7.3) it is $\mu(D_1D_2G(\phi^*, \lambda^*)\phi^1) \neq 0$ with the functional $\mu = \mu_1$ from Lemma 7.5. Consequently, the abstract branching results [25, p. 15, Thm. I.5.1] or [36, Thm. A.3] imply a further solution curve $\gamma = (\gamma_1, \gamma_2)$ for the abstract eqn. (O_λ), we set $\psi := \gamma_1$, $\lambda := \gamma_2$ and each $\psi(s)$ is an ω-periodic solution of $(\Delta_{\lambda(s)})$.

The global assertions (c) and (d) follow directly from [45, Thm. 4.3]. In order to verify our statements on the structure of the nontrivial solution branches C^+ and C^-, we apply [45, Thm. 4.4]. As complement to $\operatorname{span}\{\phi^1\}$ in ℓ_ω we use the orthogonal complement yielding the condition (4.8). □

Proof of Cor. 4.4. The formula (7.3) for the partial derivatives of G yields $g_{20} \neq 0$ and the claim is immediately implied by [36, Thm. A.3].

It remains to establish the stability assertions, where we argue as in the proof of Cor. 4.2. The simple eigenvalue 0 of $D_1G(\phi^*, \lambda^*)$ allows a continuation as a uniquely determined smooth branch of eigenvalues $\upsilon(\lambda)$ to $D_1G(\psi(\lambda), \lambda)$ satisfying $\upsilon'(\lambda^*) =$

$-g_{11}$ (cf. [25, p. 26, (I.7.34)]). Thus, the corresponding continuation $\nu(\lambda)$ of the simple eigenvalue 1 for $\Xi_\omega(\lambda^*)$ fulfills $\nu'(\lambda^*) = \omega g_{11}$ and we obtain:

(a) If $g_{11} > 0$, then $\nu(\lambda)$ leaves $(-1, 1)$ at λ^* for growing parameters λ and $\chi(\lambda)$ becomes unstable. Thanks to the stability exchange principle from [25, p. 29, Thm. I.7.4], the solution ϕ^* has inverse stability properties.

(b) For $g_{11} < 0$ the solution $\chi(\lambda)$ becomes asymptotically stable, as λ grows through the value λ^* and the claim follows dually to (a). $\square$

Proof of Cor. 4.5. The formula (7.3) for the derivatives of $G : \ell_\omega(\Omega) \times \Lambda \to \ell_\omega$ guarantees $\mu(D_1^2 G(\phi^*, \lambda^*)(\phi^1)^2) = 0$ and $g_{30} = -\mu(D_1^3 G(\phi^*, \lambda^*)(\phi^1)^3) \neq 0$ holds. Our claim follows from [36, Thm. A.4] or [25, I.6]. $\square$

Proof of Prop. 4.6. We define the stability indicator $\theta := \frac{\sum_{j=0}^{d-1} p_j'(\lambda^*)}{d + \sum_{j=1}^{d-1} j p_j(\lambda^*)}$.

(a) For $\theta < 0$ we derive from Cor. 7.7 that the simple eigenvalue $\nu(\lambda)$ of $\Xi_\omega(\lambda)$ leaves the stability interval $(-1, 1)$ at $\nu(\lambda^*) = 1$ as λ grows through the critical value λ^*, due to $\nu'(\lambda^*) > 0$. Thus, the solution ϕ^* to (Δ_λ) becomes unstable for $\lambda > \lambda^*$ and using the stability exchange principle from [46, pp. 663, Sect. 15.5] or [25, p. 29, Thm. I.7.4], stability properties get transferred from ϕ^* to the nonconstant branch.

(b) Here, $\theta > 0$ implies that $\nu(\lambda)$ enters $(-1, 1)$ at $\nu(\lambda^*) = 1$ for a growing parameter λ, since $\nu'(\lambda^*) < 0$ and the proof follows analogously to (a). $\square$

Before we are in a position to finally prove the global bifurcation criterion Thm. 4.7, one needs the following result on the determinant of block matrices:

Lemma 7.8. *For $\omega \in \mathbb{N} \setminus \{1\}$ and $A_0, \ldots, A_{\omega-1} \in \mathbb{R}^{d \times d}$ one has*

$$\det \begin{pmatrix} I_d & & & & & & A_{\omega-1} \\ A_0 & I_d & & & & & \\ & A_1 & I_d & & & & \\ & & \ddots & \ddots & & & \\ & & & & A_{\omega-3} & I_d & \\ & & & & & A_{\omega-2} & I_d \end{pmatrix} = \det\left(I_d + (-1)^{\omega-1} A_{\omega-1} \cdots A_0\right). \quad (7.12)$$

Proof. We begin with some preparations on 2×2-block matrices. With $A \in \mathbb{R}^{n \times n}$, $D \in \mathbb{R}^{d \times d}$, $B \in \mathbb{R}^{n \times d}$ and $0 \in \mathbb{R}^{d \times n}$, using the Laplace expansion theorem choosing the first n columns, one shows $\det\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) = \det A \det D$ and with an appropriate factorization into block-triangular matrices one arrives at

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det D \det(A - BD^{-1}C) \quad \text{for all } C \in \mathbb{R}^{d \times n} \quad (7.13)$$

and $D \in GL(\mathbb{R}^d)$. We verify (7.12) using induction over ω. For $\omega = 2$ one has

$$\det \begin{pmatrix} I_d & A_1 \\ A_0 & I_d \end{pmatrix} \overset{(7.13)}{=} \det I_d \det(I_d - A_1 I_d^{-1} A_2) = \det(I_d - A_1 A_0)$$

and in the induction step $\omega \to \omega + 1$ we suppose that (7.12) holds. Then

$$
\begin{aligned}
&\det \begin{pmatrix} I_d & & & & A_\omega \\ A_0 & I_d & & & \\ & \ddots & \ddots & & \\ & & A_{\omega-2} & I_d & \\ & & & A_{\omega-1} & I_d \end{pmatrix} \\
&\stackrel{(7.13)}{=} \det I_d \det \left(\begin{pmatrix} I_d & & & \\ A_0 & I_d & & \\ & \ddots & \ddots & \\ & & A_{\omega-2} & I_d \end{pmatrix} - \begin{pmatrix} A_\omega \\ 0 \\ \vdots \\ 0 \end{pmatrix} \left(0, \ldots, 0, A_{\omega-1}\right) \right) \\
&= \det \begin{pmatrix} I_d & & & -A_\omega A_{\omega-1} \\ A_0 & I_d & & \\ & \ddots & \ddots & \\ & & A_{\omega-2} & I_d \end{pmatrix} \stackrel{(7.12)}{=} \det\left(I_d + (-1)^\omega A_\omega \cdots A_0\right),
\end{aligned}
$$

which concludes the proof. □

We remind the reader about the isomorphism $J_\kappa : \ell_\omega \to \mathbb{R}^{d\omega}$ from (2.2) and introduce the continuous function $\delta : \mathbb{R} \to \mathbb{R}$,

$$\delta(\lambda) := \det J_\kappa D_1 G(\phi(\lambda), \lambda) J_\kappa^{-1}. \tag{7.14}$$

Proof of Thm. 4.7. Let $\lambda \in \Lambda$. A sequence $\phi \in \ell_\omega(\Omega)$ solves (O_λ) and thus the eqn. (Δ_λ), if and only if $x := J_\kappa \phi \in \mathbb{R}^{d\omega}$ solves $\hat{G}(x, \lambda) = 0$ with the mapping $\hat{G} : \mathbb{R}^{d\omega} \times \mathbb{R} \to \mathbb{R}^{d\omega}$, $\hat{G}(x, \lambda) := J_\kappa G(J_\kappa^{-1} x, \lambda)$. Our claim follows, if we can apply the global bifurcation result [46, p. 658, Prop. 15.1] to

$$\hat{G}(x, \lambda) = 0. \tag{7.15}$$

Thereto, it is clear that also the mapping $\hat{G}$ is of class C^1 and possesses the solution branch $\hat{\phi}(\lambda) := J_\kappa \phi(\lambda) \in \mathbb{R}^{d\omega}$, $\lambda \in \mathbb{R}$. It remains to show that the function δ defined above has a sign change. By the chain rule, $\hat{G}$ has the partial derivative $D_1\hat{G}(x, \lambda) = J_\kappa D_1 G(J_\kappa^{-1} x, \lambda) J_\kappa^{-1}$ and using (7.2) together with the definition of J_κ, we arrive at the explicit block matrix representation

$$
D_1\hat{G}(\hat{\phi}(\lambda), \lambda) = \begin{pmatrix} I_d & & & & & & A_{\omega-1}(\lambda) \\ A_0(\lambda) & I_d & & & & & \\ & A_1(\lambda) & I_d & & & & \\ & & \ddots & \ddots & & & \\ & & & & A_{\omega-3}(\lambda) & I_d & \\ & & & & & A_{\omega-2}(\lambda) & I_d \end{pmatrix},
$$

with $A_j(\lambda) := -D_1 f_{\kappa+j}(\phi(\lambda)_{\kappa+j}, \lambda) \in \mathbb{R}^{d\times d}$ for all $0 \le j < \omega$. Referring to Lemma 7.8 the determinant of $D_1\hat{G}(\hat{\phi}(\lambda), \lambda)$ can be computed as

$$\det D_1\hat{G}(\hat{\phi}(\lambda), \lambda) \stackrel{(7.12)}{=} \det\left(I_d - D_1 f_{\kappa+\omega-1}(\phi(\lambda)_{\kappa+\omega-1}, \lambda) \cdots D_1 f_\kappa(\phi(\lambda)_\kappa, \lambda)\right)$$

and by (2.4) the functions δ_ω defined in (4.11) and δ from (7.14) coincide on $\mathbb{R}$. Our assumption guarantees a sign change in δ and the claims follow. $\square$

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