# Robustness of Hyperbolic Solutions under Parametric Perturbations†

Christian Pötzsche‡

*Centre for Mathematics, Munich University of Technology, Boltzmannstr. 3, D-85748 Garching, Germany*

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We show that bounded and globally defined hyperbolic solutions of difference equations persist under time-dependent parametric perturbations, which are assumed to be uniformly small, but otherwise arbitrarily bounded. Moreover, we demonstrate that similar robustness results hold in the classes of almost periodic, periodic and limit zero solutions. The proof is based on the idea to convert difference equations into operator equations in corresponding sequence spaces, and the implicit function theorem.

**Keywords:** Hyperbolic solution, nonautonomous difference equation, persistence, structural stability, parametric perturbation, almost periodic

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## 1. Introduction and dedication

Difference equations provide a successful deterministic framework to describe evolutionary processes, where time evolves discretely, like e.g. temporal discretizations of differential equations or various examples from biology, economics and other sciences. Since the related models aim to capture real-world phenomena, they naturally involve a set of parameters being responsible for their characteristic dynamical behavior. In general, an explicit difference equation is of the form

$$x_{k+1} = f_p(x_k), \tag{1}$$

where $p$ stands for a corresponding (multi-dimensional) parameter vector and the right-hand side $f_p$ dictates the evolution. Clearly, equation (1) generates a discrete semigroup via the iterates $f_p^k$, $k \geq 0$, and the classical local theory of dynamical systems provides a powerful toolbox with ingredients like Lyapunov functions, invariant manifolds, normal forms and others, to understand the behavior of (1) close to given fixed reference solutions, like typically equilibria or periodic orbits.

Unfortunately, obtaining the precise value of the parameters $p$ from experimental data is an inverse problem and therefore delicate. However, one can encounter this intrinsic modeling difficulty of fuzzy values for $p$ using the concept of *structural stability* which investigates whether qualitative properties of (1) persist under

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†**Dedication**: This paper is dedicated to Professor Robert J. Sacker on the occasion of his 70th birthday. His bifurcation turned out to be my first serious encounter with dynamical systems, while the contributions to skew-product flows paved my way to nonautonomous problems.

‡Email: christian.poetzsche@ma.tum.de URL: http://www-m12.ma.tum.de/poetzsche

perturbation of $p$, or not (see, for instance, [12, pp. 304ff]). Indeed, it is an easy consequence of the implicit function theorem that equilibria (or periodic solutions) of (1) persist generically, if $p$ is varied (cf. Remark 1(a) and Remark 2(a) in Section 3).

Nevertheless, in many applications, equations of the from (1) are too restrictive to provide realistic models. The main reason for this is that parameters in real-world problems are rarely constant over time. This is due to various reasons, like absence of lab conditions, adaption processes or seasonal effects in biology, control strategies or further external influences. Therefore, difference equations

$$x_{k+1} = f_{p_k}(x_k) \tag{2}$$

with parameters $p_k$ that change with time $k$, frequently provide a more honest, flexible and realistic description. As a conceptional discrepancy, (1) is an autonomous, whereas equation (2) is a nonautonomous problem. Here, as a difficulty one is confronted with the fact that equilibria (resp. periodic solutions) for (1) are usually no longer equilibria (resp. periodic solutions) of the so-called *parametrically perturbed* problem (2) with an aperiodic sequence $(p_k)_{k\in\mathbf{Z}}$. Consequently, the question for an appropriate substitute of equilibria (or periodic solutions) arises, i.e., what replaces them in a general nonautonomous setting?

In the present paper we rigorously prove that hyperbolic equilibria (or more general bounded solutions) of (1) persist under bounded time-dependent parametric perturbations as bounded globally defined solutions for (2) — with small $\ell_\infty$-distance to the original equilibria. In this setting, hyperbolicity is formulated in terms of an exponential dichotomy in the variational equation. Actually, beyond bounded perturbations, we also investigate the subclasses of almost periodic, periodic and zero sequences (i.e., sequences $(\phi_k)_{k\in\mathbf{Z}}$ with limit zero for $k \to \pm\infty$). As a byproduct, this yields an approach to the existence of almost periodic solutions different from e.g. [14, 26, 38].

Our main technical tools will be an abstract formulation of difference equation (2) as nonlinear operator equation in ambient sequence spaces, which has already been exploited in [8, 23] for stability issues, as well as the classical implicit function theorem in Banach spaces. This gives our approach a somewhat functional-analytical flavor. Yet, from a technical perspective it has the advantage that arguments are largely parallel for various sequence spaces.

Related robustness theory for equilibria of autonomous evolutionary differential equations via the Lyapunov-Perron method can be found in [39, pp. 481ff]. These results extend to the discrete case. In comparison, our approach for difference equations yields less quantitative perturbation results, but is based on an elegant implicit function argument. Moreover, we are able to deal with nonautonomous perturbations of the above form.

After submitting this paper to JDEA we became aware of the preprint [15] and realized that Theorem 3.4 essentially coincides with [15, Lemma 2, 3]. However, our setting is slightly more general, since we consider not necessarily invertible systems, where state space and parameters can be infinite-dimensional. Beyond that we also treat further perturbation classes of (almost) periodic or zero sequences.

**Dedication**: In the framework of this paper, the appropriate robust hyperbolicity concept for nonautonomous difference equations is an exponential dichotomy. Significant parts of the corresponding linear theory for ordinary and evolutionary differential equations have been developed by Prof. Robert J. Sacker. This was part of a fruitful collaboration on linear skew-product flows with Prof. George R. Sell during a series of papers spanning over two decades [30–34] and [27]. As pointed out in [28], “it became immediately apparent that discrete flows could be handed with little extra effort".

A further crucial ingredient of our analysis is the $\ell_\infty$-roughness of exponential dichotomies. In fact, the contributions of Prof. Sacker mentioned above, already indicated that dichotomies are even more robust. Indeed, as demonstrated in [39, pp. 210ff], [21], one can replace the $\ell_\infty$-topology with a topology of uniform convergence on bounded (in the discrete case, finite) sets. Related results concerning this matter for differential equations can be found in [33], but also in [16, Theorem 3.1], [20, Theorem 2]. Similar investigations in the area of difference equations and equations on general time scales are due to [22].

Also Prof. Sacker's more recent research on the behavior of discrete dynamical systems with a focus on biological applications (cf. [5–7, 29, 35–37]) turned out to be a stimulating input for the paper at hand. We claim that for example results on periodic attractive solutions in discrete population models (cf. [4, 5, 29]) persist at least locally under aperiodically perturbed parameters.

**Notation**: As common in the literature, $\mathbf{Z}$ denotes the ring of integers, $\mathbf{N}$ are the positive integers and a *discrete interval* $\mathbf{I}$ is the intersection of a real interval with $\mathbf{Z}$. Throughout, $X, Y$ are Banach spaces equipped with the norm $\|\cdot\|$ and if necessary, we indicate the norm on $X$ by $\|\cdot\|_X$. For open $\rho$-balls with center $x$ we write $B_\rho(x)$. Moreover, $\Omega^\circ$ is the interior, $\overline{\Omega}$ the closure and $\partial\Omega$ the boundary of a set $\Omega \subseteq X$. Finally, $\operatorname{dist}(x, \Omega)$ is the distance of a point $x \in X$ from $\Omega$.

## 2. Sequence spaces and difference equations

The state space for difference equations under consideration in this paper, is a nonempty open subset $\Omega \subseteq X$. For the set of all sequences $\phi = (\phi_k)_{k\in\mathbf{Z}}$ with values in $\Omega$ we write $\ell(\Omega)$. Among the subsets of $\ell(\Omega)$, we are interested in:

- The set of *bounded sequences*

$$\ell_\infty(\Omega) := \Big\{\phi \in \ell(\Omega) \Big| \sup_{k\in\mathbf{Z}} \|\phi_k\| < \infty\Big\},$$

- the set of *zero sequences* (provided $0 \in \Omega$)

$$\ell_0(\Omega) := \Big\{\phi \in \ell(\Omega) \Big| \lim_{k\to\pm\infty} \phi_k = 0\Big\},$$

- the set of *almost periodic sequences*

$$\ell_{ap}(\Omega) := \{\phi \in \ell(\Omega) \,|\, \phi \text{ is almost periodic}\}\,;$$

  note that a sequence $\phi = (\phi_k)_{k\in\mathbf{Z}}$ in $\Omega$ is called *almost periodic* (for short, *ap*), if for all $\varepsilon > 0$ there exists an *inclusion length* $l(\varepsilon) \in \mathbf{N}$ such that every discrete interval of length $l(\varepsilon)$ contains a *translation number* $n \in \mathbf{Z}$ with

$$\|\phi_{k+n} - \phi_k\| < \varepsilon \quad \text{for all } k \in \mathbf{Z},$$

- the set of *$\theta$-periodic sequences*, $\theta \in \mathbf{N}$,

$$\ell_\theta(\Omega) := \{\phi \in \ell(\Omega) \,|\, \phi_k = \phi_{k+\theta} \text{ for all } k \in \mathbf{Z}\}\,.$$

In situations where $\Omega$ is the whole space $X$, we briefly write $\ell := \ell(X)$, proceed accordingly with our other sequence spaces and equip them with the norm

$$\|\phi\|_\infty := \sup_{k\in\mathbf{Z}} \|\phi_k\| .$$

They can be seen to be Banach spaces satisfying the following continuous embeddings (concerning $\ell_{ap}$, one uses [40, p. 14, Theorem 2.5] for the completeness, and [26, Proposition 5] or [40, p. 7, Theorem 2.1] for the embedding)

$$\ell_0 \hookrightarrow \ell_\infty, \qquad\qquad \ell_\theta \hookrightarrow \ell_{ap} \hookrightarrow \ell_\infty.$$

It is easy to construct examples showing that $\ell_\infty(\Omega)$ is not open. Yet, we have

LEMMA 2.1. *Suppose $\Omega$ is convex. Then also the sequence spaces $\ell_\infty(\Omega), \ell_{ap}(\Omega)$, $\ell_0(\Omega)$, $\ell_\theta(\Omega)$ are nonempty convex, and the latter two sets are open.*

*Proof.* It is clear that the above sequence spaces are nonempty and convex.

Concerning the openness assertion, suppose the symbol $\ell_*$ stands for $\ell_0$ or $\ell_\theta$. Choose $\phi \in \ell_*(\Omega)$ and define $S := \overline{\{\phi_k\}_{k\in\mathbf{Z}}}$. For $\ell_* = \ell_0$ (where $0 \in \Omega$) and $\ell_* = \ell_\theta$ we observe that $S$ is compact and disjoint from $\partial\Omega$. Since the boundary $\partial\Omega$ is closed, one has $\rho := \inf_{k\in\mathbf{Z}} \operatorname{dist}(\phi_k, \partial\Omega) > 0$ and thus $B_\rho(\phi) \subseteq \ell_*(\Omega)$, i.e., $\phi$ is an interior point. Because $\phi$ was arbitrary, $\ell_*(\Omega)$ is open. $\square$

From now on, suppose that $P \subseteq Y$ is an open neighborhood of 0 in a Banach space $Y$. For a given mapping $f : \mathbf{Z}\times\Omega\times P \to \Omega$, we interpret $\Omega$ and $P$ as *state* and *parameter space*, respectively. To mimic notation from the introduction, we write $f_p(k, x) := f(k, x, p)$ and extend the autonomous position of (1) by dealing with nonautonomous equations

$$x_{k+1} = f_0(k, x_k). \tag{3}$$

As a *parametrically perturbed* version of (3) we consider the difference equation

$$x_{k+1} = f_{p_k}(k, x_k), \tag{4}$$

where $(p_k)_{k\in\mathbf{Z}}$ is a sequence in $P$. Due to the parametric perturbation, (4) becomes a nonautonomous difference equation, even if the unperturbed problem (3) is autonomous. With a discrete interval $\mathbf{I}$ unbounded above, a *solution* of (4) is a sequence $\phi = (\phi_k)_{k\in\mathbf{I}}$ in $\Omega$ satisfying the recursion $\phi_{k+1} \equiv f_{p_k}(k, \phi_k)$ on $\mathbf{I}$. A *globally defined* or *complete solution* is a solution defined on $\mathbf{Z}$. Since $f_{p_k}(k, \cdot) : \Omega \to \Omega$ is not assumed to be invertible, backward and complete solutions need not to exist or to be unique.

We propagate the idea to rephrase difference equations like (4) as operator equations in sequence spaces. For fixed initial times $\kappa \in \mathbf{Z}$ and initial values $\xi \in \Omega$, our reformulation of (4) is based on a linear *embedding operator* $E : \Omega \to \ell$, a linear *backward shift operator* $S : \ell \to \ell$,

$$(E\xi)_k := \begin{cases} 0, & \text{for } k \neq \kappa \\ \xi, & \text{for } k = \kappa \end{cases}, \qquad (S\phi)_k := \begin{cases} 0, & \text{for } k = \kappa \\ \phi_{k-1}, & \text{for } k \neq \kappa \end{cases},$$

respectively, as well as a nonlinear *Nemytskii operator* $N : \ell(\Omega\times P) \to \ell(\Omega)$,

$$N(\phi, p) := \big(f(k, \phi_k, p_k)\big)_{k\in\mathbf{Z}}.$$

For simplicity we suppressed the dependence of $E, S$ on the initial time $\kappa \in \mathbf{Z}$ and will identify the two spaces $\ell(\Omega\times P)$ and $\ell(\Omega)\times\ell(P)$. Having this available, the crucial tool for our analysis is given in

THEOREM 2.2. *Suppose $\kappa \in \mathbf{Z}$, $\xi \in \Omega$ and $p \in \ell(P)$. A sequence $\phi \in \ell(\Omega)$ is a globally defined solution of a parametrically perturbed difference equation* (4) *with $\phi_\kappa = \xi$, if and only if $\phi$ solves the nonlinear equation*

$$F(\phi, \xi; p) = 0 \tag{5}$$

*in $\ell(\Omega)$, where the operator $F : \ell(\Omega)\times\Omega\times\ell(P)\to\ell$ is given by*

$$F(\phi, \xi; p) = \phi - SN(\phi, p) - E\xi. \tag{6}$$

*Remark 1.* An alternative formulation of (4) as a problem in a sequence space is the coincidence equation $S^+\phi = N(\phi, p)$ with the forward shift operator $(S^+\phi)_k = \phi_{k+1}$. Nevertheless, in the present paper, we favor the approach via (5) for the following reason: Since it incorporates initial conditions, one can apply the characterization from Theorem 2.2 to deduce attractivity properties for a fixed reference solution of (4), like w.l.o.g. the trivial solution. Thereto, one solves (5) w.r.t. $\phi$ in an appropriate subset of the space of all sequences $(\phi_k)_{k\geq\kappa}$ in $\Omega$ with limit 0 for $k \to \infty$. Although such an endeavor is not in our present scope, we intent to popularize this method and refer to [8, 23] for corresponding results.

*Proof.* See [8, Theorem 3.5] for the straight-forward proof. □

## 3. Robustness of hyperbolic solutions

Let us assume from now on that state and parameter space $\Omega \subseteq X$, $P \subseteq Y$ are nonempty open convex sets with $0 \in P$.

STANDING HYPOTHESIS. Let $m \in \mathbf{N}$, $0 \leq n \leq m$ and suppose the right-hand side $f : \mathbf{Z}\times\Omega\times P\to\Omega$ of (4) is a mapping such that every $f(k, \cdot)$, $k \in \mathbf{Z}$, is $m$-times continuously Fréchet-differentiable with derivatives $D^n_{(2,3)}f : \mathbf{Z}\times\Omega\times P\to L_n(X\times Y, X)$.

Our goal is to solve the abstract operator equation (5) in appropriate sequence spaces by means of the implicit function theorem. This requires conditions to guarantee that the partial derivative of $F$ w.r.t. $(\phi, \xi)$ is invertible. For this purpose, let us suppose $(p_k)_{k\in\mathbf{Z}}$ is a fixed sequence in $P$, $(\phi_k)_{k\in\mathbf{Z}}$ is a globally defined solution of equation (4), and we consider the *variational equation*

$$x_{k+1} = D_2 f(k, \phi_k, p_k)x_k \tag{7}$$

of (4) along $\phi$; here, $D_2 f$ is the partial Fréchet-derivative of $f : \mathbf{Z}\times\Omega\times P\to\Omega$ w.r.t. the second variable. This linear difference equation has the *transition operator*

$$\Phi_p(k, \kappa) := \begin{cases} I_X, & k = \kappa, \\ D_2 f(k-1, \phi_{k-1}, p_{k-1}) \cdots D_2 f(\kappa, \phi_\kappa, p_\kappa), & \kappa \leq k. \end{cases}$$

An *invariant projector* for (7) is a sequence $Q_p : \mathbf{Z}\to L(X)$ with $Q_p(k) = Q_p(k)^2$,

$$D_2 f(k, \phi_k, p_k)Q_p(k) = Q_p(k+1)D_2 f(k, \phi_k, p_k) \quad \text{for all } k \in \mathbf{Z}$$

and the restriction $D_2 f(k, \phi_k, p_k)|_{\ker Q_p(k)}$ is supposed to be an isomorphism from the kernel $\ker Q_p(k)$ onto $\ker Q_p(k+1)$ for all $k \in \mathbf{Z}$. Provided an invariant projector for (7) is given, the restriction $\Phi_p(k, \kappa) : \ker Q_p(\kappa) \to \ker Q_p(k)$, $\kappa \leq k$, is well-defined and has a bounded inverse $\bar{\Phi}_p(\kappa, k)$.

We say the above complete solution $\phi$ is *hyperbolic*, if its associated variational equation (7) admits an *exponential dichotomy*, i.e., there exist real constants $K \geq 1$, $\alpha \in (0, 1)$ and an invariant projector $Q_p$ such that

$$\begin{aligned} \|\Phi_p(k, \kappa) Q_p(\kappa)\| &\leq K \alpha^{k-\kappa} \quad \text{for all } \kappa \leq k, \\ \left\|\bar{\Phi}_p(k, \kappa)[I_X - Q_p(\kappa)]\right\| &\leq K \alpha^{\kappa-k} \quad \text{for all } k \leq \kappa. \end{aligned}$$

*Remark 1.* (a) A $\theta$-periodic, $\theta \in \mathbf{N}$, variational equation (7) admits an exponential dichotomy, if the spectrum of the *monodromy operator* $M := \Phi_p(\kappa + \theta, \kappa) \in L(X)$ does not intersect the unit circle $\mathbf{S}^1$ in the complex plane (see, e.g., [24, Proposition 2.2] for the finite-dimensional situation), i.e.,

$$\sigma(M) \cap \mathbf{S}^1 = \emptyset. \tag{8}$$

In case $\dim X < \infty$ the set of operators $M \in L(X)$ satisfying (8) is open and dense in $L(X)$ w.r.t. the norm topology (cf. [12, pp. 153ff]). Thus, equilibria (or periodic solutions) of autonomous (or periodic) equations are generically hyperbolic.

(b) For general aperiodic time-dependence, the notion of an exponential dichotomy is an open property in the class of linear-homogeneous difference equations

$$x_{k+1} = A(k) x_k \tag{9}$$

with bounded coefficient sequence $A : \mathbf{Z} \to L(X)$. Indeed, using [2, Theorem 2] we can deduce that (9) admits an exponential dichotomy if and only if the weighted shift operator $T : \ell_\infty \to \ell_\infty$, $(T\phi)_k := A(k-1)\phi_{k-1}$ is hyperbolic, i.e., its spectrum $\sigma(T)$ does not intersect $\mathbf{S}^1$. Due to the upper-semicontinuity of $\sigma(T)$ (cf. [17, pp. 208–209, Remark 3.3]) this is an open property. Yet, an exponential dichotomy itself is not generic. A corresponding example for linear ODEs on the half-line — a weaker assumption than an exponential dichotomy on the whole line — can be found in [19].

### 3.1 Parametric bounded perturbations

Beyond our standing hypothesis, we assume the derivatives of the right-hand side $f$ map bounded sets into bounded sets uniformly in time. More detailed:

HYPOTHESIS. Suppose the right-hand side $f : \mathbf{Z} \times \Omega \times P \to \Omega$ satisfies:

$H(\ell_\infty)_1$ There exists a pair $(x_0, p_0) \in \Omega \times P$ such that

$$\sup_{k \in \mathbf{Z}} \left\| D^n_{(2,3)} f(k, x_0, p_0) \right\| < \infty \quad \text{for all } 0 \leq n < m,$$

$H(\ell_\infty)_2$ $D^m_{(2,3)} f$ is *uniformly bounded*, i.e., for every bounded subset $V_0 \subseteq \Omega \times P$ one has

$$\sup_{k \in \mathbf{Z}} \sup_{(x,p) \in V_0} \left\| D^m_{(2,3)} f(k, x, p) \right\| < \infty,$$

$H(\ell_\infty)_3$ $D^n_{(2,3)}f$ is *uniformly continuous*, i.e., for $\varepsilon > 0$, bounded subsets $V_0 \subseteq \Omega\times P$ and pairs $(\bar{x}, \bar{p}) \in V_0$ there exists a $\delta > 0$ such that $(x, p) \in B_\delta(\bar{x}, \bar{p}) \cap \Omega\times P$ implies

$$\left\| D^n_{(2,3)}f(k, x, p) - D^n_{(2,3)}f(k, \bar{x}, \bar{p}) \right\| < \varepsilon \quad \text{for all } k \in \mathbf{Z},\ 0 < n \leq m.$$

Due to the lack of an appropriate reference, we derive differentiability results on substitution operators between spaces of bounded sequences. More precisely, under the above assumptions we consider *Nemytskii operators* point-wise given by

$$N^n(\phi, p)_k := D^n_{(2,3)}f(k, \phi_k, p_k) \in L_n(X\times Y, X) \quad \text{for all } k \in \mathbf{Z},\ 0 < n \leq m \tag{10}$$

and sequences $\phi \in \ell(\Omega)$, $p \in \ell(P)$.

LEMMA 3.1. *The operators $N^n : \ell_\infty(\Omega\times P) \to \ell_\infty(L_n(X\times Y, X))$ are well-defined and continuous on $\ell_\infty(\Omega\times P)^\circ$ for $0 < n \leq m$.*

*Proof.* We proceed in three steps and abbreviate $Z := X\times Y$, $V := \Omega\times P$. A convenient norm on the product space is $\|(x, y)\|_Z := \max\{\|x\|_X, \|y\|_Y\}$.

(I) We show that each derivative $D^n_{(2,3)}f$ is uniformly bounded for $0 \leq n \leq m$. In case $n = m$, this is guaranteed by $H(\ell_\infty)_2$; for $n < m$ suppose $z_0 := (x_0, p_0) \in V$ is as in assumption $H(\ell_\infty)_1$ and let $V_0 \subseteq V$ be a fixed bounded set. The convexity of $V$ implies the inclusion $\{z_0 + t(z - z_0) \in Z : t \in [0, 1]\} \subseteq V$ for $z \in V_0$, we introduce bounded sets

$$W := \bigcup_{z\in V_0} \{z_0 + t(z - z_0) \in Z : t \in [0, 1]\} \subseteq V,$$

and from assumption $H(\ell_\infty)_2$ we get $C_m := \sup_{k\in\mathbf{Z}} \sup_{w\in W} \|D^m_2 f(k, w)\| < \infty$. Then lower triangle and mean value inequality (cf. [18, p. 342, Corollary 4.3]) yield

$$\|D^n_2 f(k, z)\| \leq \|D^n_2 f(k, z_0)\| + \sup_{t\in[0,1]} \left\|D^{n+1}_2 f(k, z_0 + t(z - z_0))\right\| \|z - z_0\| \tag{11}$$

for $0 \leq n < m$ and, in particular, for $n = m - 1$ one has

$$\left\|D^{m-1}_2 f(k, z)\right\| \leq \left\|D^{m-1}_2 f(k, z_0)\right\| + C_m \|z - z_0\| \quad \text{for all } z \in V_0.$$

Therefore, assumption $H(\ell_\infty)_1$ guarantees that also $D^{m-1}_2 f$ is uniformly bounded and we derive $C_{m-1} := \sup_{k\in\mathbf{Z}} \sup_{w\in W} \left\|D^{m-1}_2 f(k, w)\right\| < \infty$. Using (11) we inductively construct real constants $C_n \geq 0$, depending only on $V_0$ such that

$$\|D^n_2 f(k, z)\| \leq \|D^n_2 f(k, z_0)\| + C_n \|z - z_0\| \quad \text{for all } z \in V_0,\ 0 \leq n < m$$

and thus $D^n_2 f$ maps bounded subsets of $V$ into bounded sets, uniformly in $k \in \mathbf{Z}$.

(II) For every $\phi \in \ell_\infty(V)$, $p \in \ell_\infty(P)$ we know from step (I) that $(D^n_2 f(k, \phi_k, p_k))_{k\in\mathbf{Z}}$ is a bounded sequence, i.e., $N^n$ is well-defined for $0 \leq n \leq m$.

(III) In order to verify the continuity of $N^n$ for $0 < n \leq m$, we choose $\varepsilon > 0$, sequences $\bar{\phi} \in \ell_\infty(V)^\circ$, $\bar{p} \in \ell_\infty(P)^\circ$ and set $\bar{\zeta} := (\bar{\phi}, \bar{p})$. Our uniform continuity assumption $H(\ell_\infty)_3$ yields the existence of a $\delta > 0$ such that

$$\left\|D^n_2 f(k, z) - D^n_2 f(k, \bar{\zeta}_k)\right\| < \tfrac{\varepsilon}{2} \quad \text{for all } z \in B_\delta(\bar{\zeta}_k),\ k \in \mathbf{Z}.$$

Consequently, with every sequence $\zeta := (\phi, p) \in B_\delta(\bar{\phi}, \bar{p}) \subseteq \ell_\infty(V)$ one arrives at

$$\sup_{k \in \mathbf{Z}} \left\| D_2^n f(k, \zeta_k) - D_2^n f(k, \bar{\zeta}_k) \right\| \leq \tfrac{\varepsilon}{2},$$

and therefore, $\left\| N^n(\phi, p) - N^n(\bar{\phi}, \bar{p}) \right\|_\infty < \varepsilon$. This was the assertion. □

In order to give an explicit expression for the derivatives of $N$, it is convenient to identify the space $\ell_\infty(L_n(X \times Y, X))$ of bounded sequences in $L_n(X \times Y, X)$ with the $n$-linear mappings $L_n(\ell_\infty(X \times Y), \ell_\infty)$ between $\ell_\infty(X \times Y)$ and $\ell_\infty$. The corresponding continuous isomorphism $J_n : \ell_\infty(L_n(X \times Y, X)) \to L_n(\ell_\infty(X \times Y), \ell_\infty)$ reads as

$$\left((J_n \Lambda)\psi^1 \cdots \psi^n\right)_k := \Lambda(k)\psi_k^1 \cdots \psi_k^n \quad \text{for all } k \in \mathbf{Z},\ \psi^1, \cdots, \psi^n \in \ell_\infty(X \times Y).$$

Proposition 3.2. *The operator $N : \ell_\infty(\Omega \times P) \to \ell_\infty$ is well-defined and $m$-times continuously Fréchet-differentiable on $\ell_\infty(\Omega \times P)^\circ$ with derivatives $D^n N = J_n N^n$ for $0 < n \leq m$.*

*Proof*. We again abbreviate $Z := X \times Y$, $V := \Omega \times P$.

Let $\zeta \in \ell_\infty(V)$ and apply step (I) in the proof of Lemma 3.1 to see that $N$ is well-defined. For the differentiability assertion, choose $\zeta \in \ell_\infty(V)^\circ$, $\psi \in \ell_\infty(Z)$ such that $\zeta_k + \psi_k \in V$ for all $k \in \mathbf{Z}$. Then the mean value theorem (cf. [18, p. 341, Theorem 4.2]) implies

$$\begin{aligned}
&\left\| D_2^n f(k, \zeta_k + \psi_k) - D_2^n f(k, \zeta_k) - D_2^{n+1} f(k, \zeta_k)\psi_k \right\| \\
\leq &\int_0^1 \left\| D_2^{n+1} f(k, \zeta_k + t\psi_k) - D_2^{n+1} f(k, \zeta_k) \right\| \, dt \, \|\psi_k\| \\
\leq &\sup_{t \in [0,1]} \sup_{k \in \mathbf{Z}} \left\| D_2^{n+1} f(k, \zeta_k + t\psi_k) - D_2^{n+1} f(k, \zeta_k) \right\| \|\psi\|_\infty \\
= &\sup_{t \in [0,1]} \left\| N^{n+1}(\zeta + t\psi) - N^{n+1}(\zeta) \right\|_\infty \|\psi\|_\infty \quad \text{for all } 0 \leq n < m
\end{aligned}$$

and the continuity of $N^{n+1}$ from Lemma 3.1 implies

$$\lim_{\psi \to 0} \sup_{t \in [0,1]} \left\| N^{n+1}(\zeta + t\psi) - N^{n+1}(\zeta) \right\|_\infty = 0 \quad \text{for all } 0 \leq n < m.$$

Since $\zeta \in \ell_\infty(\Omega \times P)^\circ$ was arbitrary, each $N^n : \ell_\infty(\Omega \times P)^\circ \to \ell_\infty(L_n(X \times Y, X))$ is Fréchet-differentiable with continuous derivative $N^{n+1}$. As a result, the mapping $N : \ell_\infty(\Omega \times P)^\circ \to \ell_\infty$ is $m$-times continuously differentiable. □

Proposition 3.3. *The operator $F : \ell_\infty(\Omega) \times \Omega \times \ell_\infty(P) \to \ell_\infty$ defined in Theorem 2.2 is well-defined and $m$-times continuously Fréchet-differentiable on the open set $\ell_\infty(\Omega)^\circ \times \Omega \times \ell_\infty(P)^\circ$ with partial derivative*

$$D_{(1,2)} F(\phi, \xi; p) \begin{pmatrix} x \\ \eta \end{pmatrix} = x - S(D_2 f(k, \phi_k, p_k) x_k)_{k \in \boldsymbol{Z}} - E\eta \tag{12}$$

*for all $\phi \in \ell_\infty(\Omega)^\circ$, $\xi \in \Omega$, $p \in \ell_\infty(P)^\circ$ and $x \in \ell_\infty$, $\eta \in X$.*

*Proof*. From Proposition 2.2 and (6) we deduce that $F$ is well-defined. The linear operators $E : X \to \ell_\infty$ and $S : \ell_\infty \to \ell_\infty$ are clearly bounded and Proposition 3.2 with [18, p. 352, Theorem 7.1] imply that $F$ is of class $C^m$. Furthermore, the remaining relation (12) follows, since the joint partial derivative of $F$ w.r.t. the first two arguments can be computed as

$$D_{(1,2)}F(\phi,\xi;p)\begin{pmatrix}x\\ \eta\end{pmatrix}=D_1F(\phi,\xi;p)x+D_2F(\phi,\xi;p)\eta=x-D_1S\circ N(\phi,p)x-E\eta$$

for all $\phi\in\ell_\infty(\Omega)^\circ$, $\xi\in\Omega$, $p\in\ell_\infty(P)^\circ$, $x\in\ell_\infty$, $\eta\in X$. □

Now we can prove the existence of bounded globally defined solutions for the parametrically perturbed difference equation (4). This result, as well as its proof, has prototype character for the remaining paper and further perturbation classes.

THEOREM 3.4 (bounded perturbations) *Suppose $H(\ell_\infty)$ holds. If $\phi^0\in\ell_\infty(\Omega)$ is a hyperbolic solution of the unperturbed equation* (3) *satisfying*

$$\inf_{k\in\boldsymbol{Z}}\operatorname{dist}(\phi_k,\partial\Omega)>0, \tag{13}$$

*then there exist $\delta,\rho>0$ and a unique $C^m$-function $\phi:B_\delta(0)\subseteq\ell_\infty(P)\to B_\rho(\phi^0)$ with $\phi(0)=\phi^0$ such that each $\phi(p)\in\ell_\infty(\Omega)$ is a globally defined hyperbolic solution of the parametrically perturbed difference equation* (4).

**Example 3.5** Already autonomous and scalar difference equations (4) with right-hand side $f_p(k,x)=x+p$ illustrate that one cannot disclaim the hyperbolicity assumption in Theorem 3.4. Indeed, no solution $\phi_k^0\equiv\xi$, $\xi\in\mathbf{R}$, of the unperturbed equation $x_{k+1}=x_k$ persists as bounded solution for parameters $p\neq 0$, since all forward solutions of $x_{k+1}=x_k+p$ are unbounded.

*Remark 2.* (a) Obviously, equilibria (or periodic solutions) of time-periodic unperturbed equations (3) give rise to a periodic variational equation (7). Hence, hyperbolicity of such solutions can be characterized in terms of condition (8) with the monodromy operator $M$ (see Remark 1(a)). Since the set of operators satisfying (8) is open and dense in $L(X)$, equilibria (or periodic solutions) of autonomous or periodic equations generically persists under parametric perturbations.

(b) Information on the size of $\rho>0$ can be obtained using a quantitative version of the implicit function theorem (cf. [13]).

(c) By Theorem 3.4 the saddle point structure consisting of stable and unstable fiber bundles (or manifolds in the autonomous case, cf. [24, 25]) associated to the hyperbolic globally defined solution $\phi^0$ persists under variation of $p\in B_\delta(0)$.

(d) We can immediately deduce (nonlinear) admissibility results from Theorem 3.4: If a difference equation (3) has a hyperbolic globally defined solution, then there exists a $\delta>0$ such that for all $g\in B_\delta(0)\subseteq\ell_\infty(P)$ also the linear-inhomogeneously perturbed system $x_{k+1}=f_0(k,x_k)+g_k$ has a globally defined hyperbolic bounded solution.

*Proof.* We intend to apply the implicit function theorem (cf., e.g., [18, p. 364, Theorem 2.1]) to solve $F(\psi,\xi;p)=0$ for $(\psi,\xi)\in\ell_\infty(\Omega)\times\Omega$. Above all, (13) guarantees $\phi^0\in\ell_\infty(\Omega)^\circ$. Since $(\phi_k^0)_{k\in\mathbf{Z}}$ is a solution of (3) we know from Theorem 2.2 that $F(\phi^0,\phi_\kappa^0;0)=0$ holds and we show that the partial derivative

$$D_{(1,2)}F(\phi^0,\phi_\kappa^0;0)\in L(\ell_\infty\times X,\ell_\infty),$$

which exists by Proposition 3.3, is a toplinear isomorphism. For given $y\in\ell_\infty$ this is equivalent to the existence of a unique pair $(x, \eta) \in \ell_\infty \times X$ such that

$$y \overset{(12)}{=} x - S(D_2 f(k, \phi_k^0, 0)x_k)_{k\in\mathbf{Z}} - E\eta \tag{14}$$

which, in turn, holds if and only if $y_\kappa = x_\kappa - \eta$ and

$$x_{k+1} = D_2 f(k, \phi_k^0, 0)x_k + y_{k+1}, \tag{15}$$

where (15) is supposed to hold for $k \neq \kappa$. From [11, p. 230, Theorem 7.6.5] we know that there exists a unique bounded solution

$$\chi_k = \sum_{n=-\infty}^{\infty} G(k, n+1)y_{n+1} \tag{16}$$

of the inhomogeneous difference equation (15), where $G(k, n) \in L(X)$ is *Green's function* associated with the dichotomy of (7) for $p = 0$ given by

$$G(k, n) := \begin{cases} \Phi_0(k, n)Q_0(n), & n \leq k, \\ -\Phi_0(k, n)[I_X - Q_0(n)], & k < n. \end{cases}$$

Choosing $x_k = \chi_k$ and $\eta = x_\kappa - y_\kappa$ to see that (14) possesses a solution. In order to show its uniqueness, let the pair $(\bar{x}, \bar{\eta}) \in \ell_\infty \times X$ be a further solution. Then the difference $x - \bar{x}$ is a bounded solution of the homogeneous initial value problem

$$x_{k+1} = D_2 f(k, \phi_k^0, 0)x_k, \qquad x_\kappa = \eta - \bar{\eta},$$

whose only bounded solution, due to its exponential dichotomy, is the trivial one. Hence, $x = \bar{x}$, $\eta = \bar{\eta}$ and Banach's theorem (see [18, p. 388, Corollary 1.4]) implies that $D_{(1,2)}F(\phi^0, \phi_\kappa^0; 0) \in L(\ell_\infty \times X, \ell_\infty)$ has a bounded inverse. Thus, the implicit function theorem (cf., [18, p. 364, Theorem 2.1]) yields that there exists a $\bar{\delta} > 0$ and a unique $C^m$-mapping $\Psi = (\Psi_1, \Psi_2) : B_{\bar{\delta}}(0) \to \ell_\infty(\Omega) \times \Omega$ such that $\Psi(0) = (\phi^0, \phi_\kappa^0)$ and $F(\Psi(p); p) \equiv 0$ on $B_{\bar{\delta}}(0) \subseteq \ell_\infty(P)$. We define $\phi := \Psi_1$ and Theorem 2.2 guarantees that $\phi(p) \in \ell_\infty(\Omega)$ is a globally defined solution of (4) satisfying the initial condition $\phi(p)_\kappa = \Psi_2(p)$.

It remains to show the hyperbolicity of $\phi(p)$. To establish this, note that the perturbed difference equation $x_{k+1} = D_2 f(k, \phi(p)_k, p_k)x_k$ can be written as

$$x_{k+1} = D_2 f(k, \phi_k^0, 0)x_k + [D_2 f(k, \phi(p)_k, p_k) - D_2 f(k, \phi_k^0, 0)]x_k. \tag{17}$$

With arbitrarily chosen $\varepsilon > 0$, the continuity of $\Psi$ and hypothesis $H(\ell_\infty)_3$ guarantee that for sufficiently small $\delta \in (0, \bar{\delta})$ one has

$$\left\| D_2 f(k, \phi(p)_k, p_k) - D_2 f(k, \phi_k^0, 0) \right\| < \varepsilon \quad \text{for all } k \in \mathbf{Z}, p \in B_\delta(0).$$

Consequently, the $\ell_\infty$-roughness of exponential dichotomies (see [11, p. 232, Theorem 7.6.7]) implies that (17) admits an dichotomy and $\phi(p)$ is hyperbolic. $\square$

*Remark 3* "*Strong Boundedness Property*". The essential argument carrying the above proof is the admissibility property [11, p. 230, Theorem 7.6.5] ("The Henry Theorem"), characterizing exponential dichotomies. A flexible approach to this result via discrete linear skew-product semiflows can be found in form of the "Strong Boundedness Property" given in [39, p. 213, Theorem 45.8] or [21, Section 4].

Corollary 3.6. *If a bounded globally defined solution $\phi^0$ of (3) is exponentially stable, then there is a $\delta > 0$ such that also $\phi(p)$ , $p \in B_\delta(0)$, is exponentially stable.*

*Proof.* Let $\kappa \in \mathbf{Z}$. Since $(\phi_k^0)_{k\in\mathbf{Z}}$ is an exponentially stable solution of (3), we know that 0 is an exponentially stable equilibrium of the equation of perturbed motion

$$x_{k+1} = f_0(k, x_k + \phi_k^0) - f_0(k, \phi_k^0),$$

whereby the converse of the theorem on stability by first approximation due to Győri and Pituk (see [10, Theorem 4]) implies the existence of constants $K_0 \geq 1$, $\alpha_0 \in (0,1)$ such that $\|\Phi_0(k,\kappa)\| \leq K_0\alpha_0^{k-\kappa}$ for $\kappa \leq k$. Hence, $\phi^0$ is hyperbolic with associated invariant projector $Q_0(k) \equiv I$. We apply Theorem 3.4 to deduce the existence of a $\delta > 0$ and hyperbolic globally defined solutions $\phi(p)$ of (4) for sequences $p \in B_\delta(0) \subseteq \ell_\infty(P)$. Choosing $\delta > 0$ sufficiently small, we know that the invariant projectors $Q_0$ and $Q_p$ associated with the exponential dichotomies of the variational equations for (3) and (4) along $\phi^0$ and $\phi(p)$, resp., are linearly conjugated (cf. [17, pp. 32–34]). This guarantees that there exist real constants $K \geq 1$, $\alpha \in (0,1)$ such that

$$\|\Phi_p(k,\kappa)\| \leq K\alpha^{k-\kappa} \quad \text{for all } \kappa \leq k$$

holds for the corresponding transition operator $\Phi_p(k,\kappa)$ of the variational equation for (4) along $\phi(p)$. The theorem on stability by first approximation (also for this classical case we refer to [10, Theorem 4]) implies that the zero solution of

$$x_{k+1} = f_{p_k}(k, x_k + \phi(p)_k) - f_{p_k}(k, \phi(p)_k)$$

is exponentially stable, i.e., $\phi(p)$ is an exponentially stable solution of (4). $\square$

Our Theorem 3.4 guarantees that bounded solutions of (3) persist under parametric perturbations. Next we discuss several subclasses of the bounded sequences with this property.

### 3.2 Parametric limit zero perturbations

Now we investigate solutions homoclinic to 0, i.e., which converge to 0 as $k \to \pm\infty$.

Hypothesis. Suppose $0 \in \Omega$ and that $f : \mathbf{Z}\times\Omega\times P \to \Omega$ satisfies:

$H(\ell_0)_1$ One has $\lim_{k\to\pm\infty} f(k,0,0) = 0$ and

$$\sup_{k\in\mathbf{Z}} \left\| D^n_{(2,3)} f(k,0,0) \right\| < \infty \quad \text{for all } 1 \leq n < m,$$

$H(\ell_0)_2$ $D^m_{(2,3)} f$ is *uniformly bounded*, i.e., for every bounded subset $V_0 \subseteq \Omega\times P$ one has

$$\sup_{k\in\mathbf{Z}} \sup_{(x,p)\in V_0} \left\| D^m_{(2,3)} f(k,x,p) \right\| < \infty,$$

$H(\ell_0)_3$ $D^n_{(2,3)} f$ is *uniformly continuous*, i.e., for $\varepsilon > 0$, bounded subsets $V_0 \subseteq \Omega\times P$ and pairs $(\bar{x},\bar{p}) \in V_0$ there exists a $\delta > 0$ such that $(x,p) \in B_\delta(\bar{x},\bar{p}) \cap \Omega\times P$ implies

$$\left\| D^n_{(2,3)} f(k,x,p) - D^n_{(2,3)} f(k,\bar{x},\bar{p}) \right\| < \varepsilon \quad \text{for all } k \in \mathbf{Z},\ 0 < n \leq m.$$

From Lemma 2.1 we know that $\ell_0(\Omega\times P)$ is open and Proposition 3.2 reads as

Proposition 3.7. *The operator $N : \ell_0(\Omega\times P)\to\ell_0$ is well-defined and $m$-times continuously Fréchet-differentiable with derivatives $D^n N = J_n N^n$ for $0 < n \leq m$.*

*Proof*. As above, we abbreviate $Z := X\times Y$, $V := \Omega\times P$. Since $\phi \in \ell_0(\Omega)$, $p \in \ell_0(P)$ are bounded sequences, we obtain from the proof of Lemma 3.1 that

$$C := \sup_{k\in\mathbf{Z}} \sup_{h\in[0,1]} \left\| D_{(2,3)} f(k, h\phi_k, hp_k)\right\| < \infty.$$

Since mean value inequality (cf. [18, p. 342, Corollary 4.3]) and $H(\ell_0)_1$ guarantee

$$\|f(k, \phi_k, p_k)\| \leq \|f(k, 0, 0)\| + C\,\|(\phi_k, p_k)\| \xrightarrow[k\to\pm\infty]{} 0,$$

the mapping $N$ is well-defined, i.e., has values in $\ell_0(\Omega)$. For the remainder of the proof, we assume $0 < n \leq m$. Due to Lemma 3.1 and $\ell_0 \subseteq \ell_\infty$ we know that $(D^n_{(2,3)} f(k, \phi_k, p_k))_{k\in\mathbf{Z}}$ is a bounded sequence in $L_n(Z, X)$ and, therefore,

$$\left((J_n N^n(\phi, p))\psi^1 \cdots \psi^n\right)_k = N^n(\phi, p)_k \psi^1_k \cdots \psi^n_k \xrightarrow[k\to\pm\infty]{} 0$$

for all $\psi^1, \ldots, \psi^n \in \ell_0(Z)$. This implies $J_n N^n(\phi, p) \in L_n(\ell_0(Z), \ell_0)$ and as in Proposition 3.2 one shows that $N$ is of class $C^m$ with $J_n N^n$ as derivatives. □

After these preparations we are in the position to prove the following analog to Theorem 3.4 for solutions of (4) in $\ell_0$.

Theorem 3.8 (limit zero perturbations) *Suppose $H(\ell_0)$ holds. If $\phi^0 \in \ell_0(\Omega)$ is a hyperbolic solution of the unperturbed equation* (3)*, then there exist $\delta, \rho > 0$ and a unique $C^m$-function $\phi : B_\delta(0) \subseteq \ell_0(P)\to B_\rho(\phi^0)$ with $\phi(0) = \phi^0$ such that each $\phi(p) \in \ell_0(\Omega)$ is a globally defined hyperbolic solution of the parametrically perturbed difference equation* (4).

*Proof*. Let $\phi^0 \in \ell_0(\Omega)$ and $p \in \ell_0(P)$ be given. Thanks to Proposition 3.7 we know that the mapping $F : \ell_0(\Omega)\times\Omega\times\ell_0(P)\to\ell_0$ is well-defined. Parallel to the proof of Theorem 3.4 we use the implicit function theorem to solve $F(\psi, \xi; p) = 0$ for $(\psi, \xi) \in \ell_0(\Omega)\times\Omega$ in a neighborhood of the point $(\phi^0, \phi^0_\kappa)$. Here, the partial derivative $D_{(1,2)}F(\phi^0, \phi^0_\kappa; 0) \in L(\ell_0\times X, \ell_0)$ exists, and in order to show that it is a toplinear isomorphism, we use the admissibility result in $\ell_0$ from [2, Corollary 3] instead of [11, p. 230, Theorem 7.6.5]. □

### *3.3 Parametric almost periodic perturbations*

Another important perturbation class are the almost periodic sequences. Here, an ap solution of (4) is already hyperbolic, provided its almost periodic variational equation (7) has an exponential dichotomy on a semiaxis (in fact, a large discrete interval is sufficient, cf. [1]), instead of on the whole integer axis.

Conditions guaranteeing the existence of ap solutions for nonautonomous difference equations can be found, for example, in [14, 26, 38]. In this spirit, our perturbation theory implies that periodic solutions persist as ap functions under almost periodic perturbations.

Let $E$ be a further Banach space. We say a function $g : \mathbf{Z}\times\Omega\times P\to E$ is *uniformly almost periodic* (for short, *uniformly ap*), if for all $\varepsilon > 0$ and compact $B \subseteq \Omega\times P$

there exists an inclusion length $l(\varepsilon, B) \in \mathbf{N}$ such that every discrete interval of length $l(\varepsilon, B)$ contains an $n \in \mathbf{Z}$ with

$$\|g(k+n, x, p) - g(k, x, p)\| < \varepsilon \quad \text{for all } k \in \mathbf{Z},\ (x, p) \in B.$$

An illuminating discussion of conditions under which variational equations of ap differential equations are almost periodic can be found in [9]. In our present discrete context, an almost periodic time-dependence enables us to weaken some uniformity and boundedness conditions needed above in $H(\ell_\infty)$ and $H(\ell_0)$.

Hypothesis. Suppose $\Omega = X$, $P = Y$ and that $f : \mathbf{Z}\times X\times Y \to X$ satisfies:

$H(\ell_{ap})_1$ $D^n_{(2,3)} f$ is uniformly ap for all $0 \le n \le m$,

$H(\ell_{ap})_2$ $D^n_{(2,3)} f$ is *uniformly continuous*, i.e., for $\varepsilon > 0$, $k \in \mathbf{Z}$, bounded $V_0 \subseteq X\times Y$ and pairs $(\bar{x}, \bar{p}) \in V_0$ there exists a $\delta > 0$ such that $(x, p) \in B_\delta(\bar{x}, \bar{p})$ implies

$$\left\| D^n_{(2,3)} f(k, x, p) - D^n_{(2,3)} f(k, \bar{x}, \bar{p}) \right\| < \varepsilon \quad \text{for all } 0 < n \le m.$$

Proposition 3.9. *The operator $N : \ell_{ap}(X\times Y) \to \ell_{ap}$ is well-defined and $m$-times continuously Fréchet-differentiable with derivatives $D^n N = J_n N^n$ for $0 < n \le m$.*

*Proof*. We briefly write $Z := X\times Y$. Since the sequences $\phi \in \ell_{ap}$, $p \in \ell_{ap}(Y)$ have a relatively compact image (cf. [3, p. 140, Theorem 6.5]), there exists a compact subset $S \subseteq Z$ such that $(\phi_k, p_k) \in S$ for all $k \in \mathbf{Z}$. Referring to the assumption $H(\ell_{ap})_1$ we can apply [40, p. 16, Theorem 2.7][1] to see that $(f(k, \phi_k, p_k))_{k\in\mathbf{Z}}$ is an almost periodic sequence. Consequently, the mapping $N$ is well-defined.

For every $0 < n \le m$ the very same argument shows that also the $n$th order derivatives $(D^n_{(2,3)} f(k, \phi_k, p_k))_{k\in\mathbf{Z}}$ are ap sequences in $L_n(Z, X)$ and

$$(J_n N^n(\phi, p))\psi^1 \cdots \psi^n = N^n(\phi, p).\psi^1_\cdot \cdots \psi^n_\cdot \in \ell_{ap} \quad \text{for all } \psi^1, \ldots, \psi^n \in \ell_{ap}(Z)$$

yields $J_n N^n(\phi, p) \in L_n(\ell_{ap}(Z), \ell_{ap})$. The continuity of $N^n$ has been shown in Lemma 3.1 under the uniform continuity assumption $H(\ell_\infty)_3$. However, condition $H(\ell_{ap})_2$ and [40, p. 7, Theorem 2.1] guarantee that the partial derivatives $D^n_{(2,3)} f$ satisfy $H(\ell_\infty)_3$ in our present almost periodic setting. Thus, one can proceed as in Proposition 3.2 in order to prove that $N$ is $m$-times continuously differentiable with $J_n N^n$ as derivatives. $\square$

Theorem 3.10 (almost periodic perturbations) *Suppose $H(\ell_{ap})$ holds. If $\phi^0 \in \ell_{ap}$ is a hyperbolic solution of the unperturbed equation* (3)*, then there exist $\delta, \rho > 0$ and a unique $C^m$-function $\phi : B_\delta(0) \subseteq \ell_{ap}(Y) \to B_\rho(\phi^0)$ with $\phi(0) = \phi^0$ such that each $\phi(p) \in \ell_{ap}$ is a globally defined hyperbolic solution of the parametrically perturbed difference equation* (4).

*Proof*. Let $\phi^0 \in \ell_{ap}$ and $p \in \ell_{ap}(Y)$. From the above Proposition 3.9 we deduce that $F : \ell_{ap}\times X\times \ell_{ap}(Y) \to \ell_{ap}$ is well-defined and of class $C^m$. The implicit function theorem is applicable in order to solve $F(\psi, \xi; p) = 0$ for $(\psi, \xi) \in \ell_{ap}\times X$, provided

$$D_{(1,2)} F(\phi^0, \phi^0_\kappa; 0) \in L(\ell_{ap}\times X, \ell_{ap}) \quad \text{is invertible.} \tag{18}$$

[1] Admittedly, this result addresses uniformly almost periodic functions $f : \mathbf{R}\times\Omega \to X$, $X$ finite-dimensional, and guarantees that the composition $t \mapsto g(t, \phi(t))$ is ap for almost-periodic $\phi : \mathbf{R} \to X$ with values in a compact subset $S \subseteq \Omega$. The interested reader may check in detail that the proofs of [3, p. 140, Theorem 6.5], [40, Theorem 2.7] and [40, p. 7, Theorem 2.1] carry over to the case of ap sequences and Banach space-valued functions $f : \mathbf{Z}\times\Omega \to X$ with obvious modifications.

Since the derivative $D_2 f$ is uniformly ap, the variational equation (7) along $\phi^0$ has an almost periodic coefficient operator (cf. [40, p. 16, Theorem 2.7]). Thus, the corresponding admissibility result for almost periodic linear difference equations guaranteeing (18) can be found in [14, Proposition 13]. □

### 3.4 Parametric periodic perturbations

Finally, we round off our investigations and consider the classical situation of periodic solutions $\phi^0$ for time-periodic difference equations (3) with possibly different periods. Here, hyperbolicity of $\phi^0$ can be characterized and verified in terms of Floquet multipliers of the variational equation (7) along $\phi^0$ (see Remark 1(a)). Furthermore, uniformity conditions w.r.t. the time-dependence of $f$ are trivially satisfied and we precisely assume

HYPOTHESIS. Suppose $\theta_0, \theta_1, \theta_2 \in \mathbf{N}$ and $f : \mathbf{Z}\times\Omega\times P\to\Omega$ satisfies:

$H(\ell_\theta)_1$ $f(k, \cdot) = f(k + \theta_0, \cdot)$ for all $k \in \mathbf{Z}$,

$H(\ell_\theta)_2$ $D^n_{(2,3)} f$ is *uniformly continuous*, i.e., for $\varepsilon > 0$, $k \in \mathbf{Z}$, bounded subsets $V_0 \subseteq \Omega\times P$ and pairs $(\bar{x}, \bar{p}) \in V_0$ there exists a $\delta > 0$ such that $(x, p) \in B_\delta(\bar{x}, \bar{p})\cap\Omega\times P$ implies

$$\left\| D^n_{(2,3)} f(k, x, p) - D^n_{(2,3)} f(k, \bar{x}, \bar{p})\right\| < \varepsilon \quad \text{for all } 0 < n \leq m.$$

THEOREM 3.11 (periodic perturbations) *Suppose $H(\ell_\theta)$ holds and define $\theta \in \mathbf{N}$ as the least common multiple* lcm $\{\theta_0, \theta_1, \theta_2\}$. *If $\phi^0 \in \ell_{\theta_1}(\Omega)$ is a hyperbolic solution of the unperturbed equation* (3)*, then there exist $\delta, \rho > 0$ and a unique $C^m$-function $\phi : B_\delta(0) \subseteq \ell_{\theta_2}(P)\to B_\rho(\phi^0)$ with $\phi(0) = \phi^0$ such that each $\phi(p) \in \ell_\theta(\Omega)$ is a globally defined hyperbolic solution of the parametrically perturbed difference equation* (4).

*Proof*. For given $\phi \in \ell_{\theta_1}$ and $p \in \ell_{\theta_2}$ it is not difficult to deduce from assumption $H(\ell_\theta)_1$ that the sequence $(f(k, \phi_k, p_k))_{k\in\mathbf{Z}}$ is $\theta$-periodic with $\theta = \text{lcm}\{\theta_0, \theta_1, \theta_2\}$. Thus, the mapping $F : \ell_{\theta_1}(\Omega)\times\Omega\times\ell_{\theta_2}(P)\to\ell_\theta$ is well-defined and has an open domain of definition (cf. Lemma 2.1). Due to the embeddings $\ell_{\theta_1}, \ell_{\theta_2} \hookrightarrow \ell_\theta$ we can solve $F(\psi, \xi; p) = 0$ for $(\psi, \xi) \in \ell_\theta(\Omega)\times\Omega$. As in the proof of Theorem 3.4, for every inhomogeneity $y \in \ell_\theta$ the unique bounded solution of (15) is given by (16) – a sequence easily seen to be in $\ell_\theta$ by our periodicity assumptions. This finally implies that $D_{(1,2)}F(\phi^0, \phi^0_\kappa; 0) \in L(\ell_\theta\times X, \ell_\theta)$ has a bounded inverse. □

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