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# NONAUTONOMOUS BIFURCATION OF BOUNDED SOLUTIONS I: A LYAPUNOV-SCHMIDT APPROACH

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ABSTRACT. We investigate local bifurcation properties for nonautonomous difference and ordinary differential equations. Extending a well-established autonomous theory, due to our arbitrary time dependence, equilibria or periodic solutions typically do not exist and are replaced by bounded complete solutions as possible bifurcating objects.

Under this premise, appropriate exponential dichotomies in the variational equation along a nonhyperbolic solution on both time axes provide the necessary Fredholm theory in order to employ a Lyapunov-Schmidt reduction. Among other results, this yields nonautonomous versions of the classical fold, transcritical and pitchfork bifurcation patterns.

*Dedicated to Peter E. Kloeden on the occasion of his 60th birthday.*

## 1. **Nonautonomous equations and bifurcations.**

Evolutionary equations modeling dynamic phenomena in physics, biology or other applied sciences depend on parameters, which might be natural constants but also variables influenced by the environment. Such magnitudes are responsible for the characteristic asymptotics or further typical features of a system. Thus, it is of eminent importance to understand the behavior of these properties under parameter variation. As a matter of course, already during the last century such questions became a well-investigated and -understood topic with an abundant literature — as long as stationary, periodic or homo-/heteroclinic solutions of autonomous (or periodic) equations are addressed. Indeed, the above battery of questions splits into two subareas, namely continuation and bifurcation problems.

Continuation problems deal with the question of finding conditions, yielding that a solution of an evolutionary equation persists under varying system parameters, without loosing its stability properties. This is strongly related to the concept of structural stability implying that hyperbolic equilibria, orbits or more general objects are robust under perturbations.

The opposite situation is covered in the framework of bifurcation methods, which allow two philosophically different approaches:

- *Dynamic bifurcations*, as part of dynamical systems theory, ask for conditions under which a solution of an evolutionary equation loses its (structural) stability in a super-, sub- or transcritical direction. This is intimately connected to an exchange of stability properties with newly generated solutions. Typical tools in this field are normal forms (simplifying the right-hand side) or center manifolds (lowering the dimension), and well-known monographs such as, e.g., [19, 32, 54] provide comprehensive introductions.

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2000 *Mathematics Subject Classification.* 37B55, 39A11, 34A34, 34C23, 37G99, 39A28, 47J15.
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- On the other hand, in *static bifurcation* (or *branching*) *theory*, the bifurcating (or branching) objects are solutions of abstract operator equations in function spaces. Hence, this approach has a wide applicability and is not restricted to the area of dynamical systems. Essential for this approach is to deal with Fredholm mappings in order to employ a Lyapunov-Schmidt reduction and we refer to monographs, like e.g. [9, 28, 55] or the appendix for further details.

In the paper at hand, we focus on a less classical situation of *nonautonomous dynamic equations*, which attracted a certain popularity over the recent years. More precisely, our interest is centered around nonautonomous difference equations (also called mappings), as well as ordinary differential equations (ODEs for short) in general Banach spaces. Here, the right-hand sides are explicitly time-dependent and thus fail to generate 1-parameter semigroups fitting in the standard dynamical systems theory. Such a more flexible framework is interesting from a mathematical perspective, but additionally strongly motivated from applications in order to include external temporal influences into realistic models. Accordingly, various basic tools from dynamic bifurcation theory have already been extended to the time-variant situation, like normal forms (cf. [50, 51]) or a center-manifold reduction (cf. [40, 44, 45]).

However, due to their aperiodic time-dependence, nonautonomous equations can feature a very complex dynamical behavior and usually do not possess constant (equilibria) or periodic solutions and one has to broaden the scope. Consequently, it makes little sense to look for such solutions as bifurcating objects. Moreover the notion of structural stability is still in its infancy in a nonautonomous setup. Yet, it proved very fruitful to describe bifurcation patterns in terms of *attractor bifurcation*, i.e. the scenario that an appropriate nonautonomous attractor becomes nontrivial (or topologically different) under variation of the system parameters. An illuminating survey of such topics is given in [31]. A systematic treatment of nonautonomous attractor (and repeller) transitions and bifurcations is due to [47]. Furthermore, in this spirit, the classical transcritical and pitchfork bifurcation patterns have been extended to nonautonomous equations in [46, 48].

We also briefly survey further approaches to a nonautonomous bifurcation theory: Basic elements for a theory of Hopf bifurcation from non-periodic solutions of ODEs have been developed in [24, 25]. The authors of [34, 35] introduce stability and instability notions for solutions of scalar ODEs based on the concept of pullback convergence. Relying on these notions, nonautonomous counterparts to the saddle-node, the transcritical and the pitchfork bifurcation are established. Averaging techniques have been used in order to obtain time-variant versions of saddle-node and transcritical patterns in [23, 15]. The contribution [36] discusses a bifurcation theory based on the variation of the number and attraction properties of minimal sets for the corresponding skew product dynamical system; this yields the above bifurcation patterns for scalar differential equations. Finally, using Conley index theory the bifurcation of control sets is investigated in [8].

All the above approaches are driven by dynamic bifurcation theory, since they are based on a dynamical interpretation, and go hand in hand with loss of stability in the super- or subcritical direction. In contrast, this paper aims to make use of static bifurcation theory, which seems to be largely overlooked when dealing with nonautonomous questions — hence, a different angle of the corresponding bifurcation theory is illuminated. Indeed, it has been established in [41] (see also [20, Lemma 3] for the discrete case) that generically e.g. equilibria persist as bounded complete solutions under arbitrary bounded, but small, temporal perturbation of the parameters. We refer to Fig. 1 (left) for an illustration of this fact in case of differential equations. A similar statement holds for almost periodic, asymptotically autonomous or periodic (discrete) equations (cf. [42]).

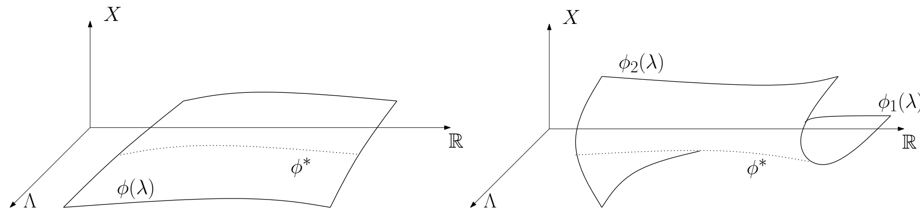


FIGURE 1. Let $\phi^* = \phi(\lambda^*)$ (dotted line) be a bounded complete solution of an ODE $\dot{u} = f(t, u, \lambda)$ in $X$ depending on a parameter $\lambda \in \Lambda$.
Left: $\phi^*$ persists as a complete bounded solution $\phi(\lambda)$ under variation of $\lambda$ near $\lambda^*$.
Right: $\phi^*$ vanishes for $\lambda > \lambda^*$ and branches into two complete bounded solutions $\phi_1(\lambda), \phi_2(\lambda)$ for $\lambda < \lambda^*$

For this reason, it is natural to search for bounded complete solutions as bifurcating objects in general nonautonomous equations. Similarly, in order to detect homoclinic bifurcations, one looks for solutions decaying to zero. Consequently, in this paper a bifurcation is roughly understood as a change in the number of bounded complete solutions under variation of the parameter; see Figs. 1 (right) and 2 for an illustration.

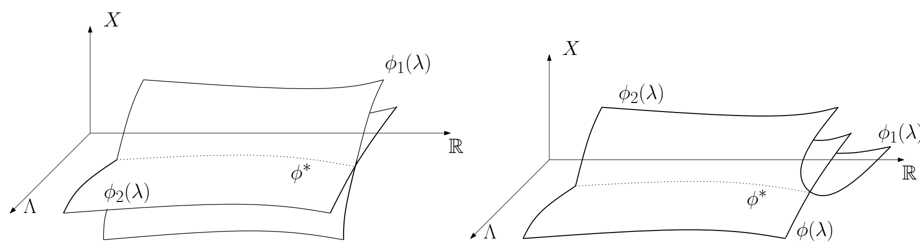


FIGURE 2. Let $\phi^* = \phi(\lambda^*)$ (dotted line) be a bounded complete solution of an ODE $\dot{u} = f(t, u, \lambda)$ in $X$ depending on a parameter $\lambda \in \Lambda$.
Left: For each parameter $\lambda \neq \lambda^*$ there exist two branches of bounded complete solutions $\phi_1(\lambda), \phi_2(\lambda)$.
Right: For $\lambda > \lambda^*$ there is a unique bounded complete solution, while there exist three such solutions $\phi(\lambda), \phi_1(\lambda), \phi_2(\lambda)$ for $\lambda < \lambda^*$

A link to the concept of attractor bifurcation is as follows: Global attractors consist of complete bounded solutions and consequently a change in the number of such solutions results in a variation of the attractor vice versa. In particular, as recently observed in [30], positively (or negatively) invariant compact sets always contain a maximal invariant set which in turn consists of complete solutions.

Our approach relies on the idea that difference (or differential) equations allow a formulation as operator equations in appropriate sequence (or function) spaces. Once such a spatial setting is established, the nonautonomous character of the underlying evolution equation is of minor importance. This observation enables us to use well-established tools from branching theory as discussed in the above mentioned monographs [9, 28, 55], as well as in more recent research papers [16]. It yields explicit conditions for the bifurcation of bounded complete solutions — independent of the dimension of the given discrete or continuous evolutionary equation.

Starting with the time-discrete case of difference equations in Sect. 2, our presentation splits into several parts. Understanding such problems as operator equations in the space of bounded or zero sequences requires to deduce certain differentiability properties of substitution operators. We continue by introducing the necessary Fredholm theory for linear equations or related difference operators. Here, the notion of an exponential dichotomy is crucial, since it provides the adequate hyperbolicity concept in our nonautonomous framework (cf. [18, 38, 41, 42]) and, correspondingly, nonhyperbolicity will be formulated via assuming dichotomies on both semiaxes. Such an assumption requires the state space to be at least 2-dimensional and has the consequence that only unstable solutions can bifurcate. This form of nonhyperbolicity is necessary for bifurcation, but the converse is more subtle and depends on properties of the nonlinear terms. With the aid of the Fredholm theory developed in Subsect. 2.1 we apply the Lyapunov-Schmidt method to arrive at a finite-dimensional branching equation. This yields sufficient criteria for bifurcations with odd-dimensional kernel, as well as for multiparameter bifurcations. Finally, in case the linearization admits a 1-dimensional space of bounded complete solutions, we deduce nonautonomous counterparts of the classical fold, transcritical and pitchfork bifurcation patterns, where equilibria are replaced by bounded complete solutions. Simple quantitative examples illustrate this. In doing so, we obtain local bifurcation results, in the sense that the number of bounded complete solutions changes in a neighborhood, when a parameter is varied, whereas stability issues are not discussed. Related work on difference equations can be found in [17], who investigates the bifurcation of almost periodic solutions.

The Sect. 3 presents the analogous theory for nonautonomous ordinary differential equations in (possibly infinite-dimensional) Banach spaces $X$ (cf. [13, 3]). Their theory is barely more complex than the classical case $X = \mathbb{R}^N$, but at least in principle allows applications to certain integro-differential equations, to infinite systems of ODEs or to pseudo-parabolic equations (cf. [14]). Yet, there are differences to the discrete case from Sect. 2: First, some arguments are simpler since solutions exist in backward time yielding invertible transition operators. Second, symmetry properties of the derivative $\dot{\phi}(t)$ as opposed to the forward difference operator $\phi_{k+1}$ lead to structurally different adjoint operators. And finally, the Lyapunov-Schmidt projectors have a slightly modified form (compare Lemma 2.8 and 3.8). A related Fredholm theory has been developed in [49, 37] for ODEs in $\mathbb{R}^N$. Extensions to parabolic evolution equations are due to [6, 56]. A different approach to the bifurcation of bounded solutions in almost periodic ODEs using Conley's index theory for skew-product flows can be found in [53]. Moreover, [22] use topological arguments to investigate bifurcations of bounded solutions in autonomous ODEs. Bifurcations of almost periodic solutions for homogenous nonlinearities are investigated in [29].

Finally, in order to keep the paper self-contained, we moved the necessary functional-analytical tools like Lyapunov-Schmidt reduction or bifurcation results into an appendix.

**Notation**: We use the *Kronecker symbol* $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$.

Generic real Banach spaces are denoted by $X, Y$ and equipped with norm $|\cdot|$. The *interior* of a set $\Omega \subseteq X$ is denoted by $\Omega^\circ$ and $B_\varepsilon(x)$ is the open ball with center $x$ and radius $\varepsilon > 0$. The complete vector space of bounded linear operators between spaces $X$ and $Y$ is $L(X,Y)$, $L(X) := L(X,X)$ and for the corresponding toplinear endomorphisms we write $GL(X,Y)$. Given $T \in L(X,Y)$, we write $R(T) := TX$ for the *range* and $N(T) := T^{-1}(0)$ for the *kernel*. The *dual space* of $X$ is $X'$, $\langle x', x\rangle := x'(x)$ the duality product and $T' \in L(Y', X')$ is the *dual operator* to $T$. For a given subspace $X_0 \subseteq X$ the *annihilator* is defined as the set of functionals

$$X_0^\perp := \{x' \in X' : \langle x', x_0\rangle = 0 \text{ for all } x_0 \in X_0\}.$$

2. **Difference equations.** As usual, $\mathbb{Z}$ denotes the ring of integers, $\mathbb{N}$ are the positive integers and a *discrete interval* $\mathbb{I}$ is the intersection of a real interval with $\mathbb{Z}$; sometimes it is convenient to introduce the shifted interval $\mathbb{I}' := \{k \in \mathbb{I} : k+1 \in \mathbb{I}\}$. Given an integer $\kappa \in \mathbb{Z}$ we define the discrete intervals $\mathbb{Z}_\kappa^+ := \{k \in \mathbb{Z} : \kappa \leq k\}$ and $\mathbb{Z}_\kappa^- := \{k \in \mathbb{Z} : \kappa \geq k\}$.

The idea behind our overall strategy is to rephrase difference equations as operator equations in suitable sequence spaces (cf. Thm. 2.1) in order to detect their globally defined solutions. In such a functional-analytical approach, ambient spaces are indispensable. For this, suppose throughout that $\Omega \subseteq X$ and $\Lambda \subseteq Y$ are nonempty open convex sets. The set of bounded sequences $\phi = (\phi_k)_{k\in\mathbb{Z}}$ with $\phi_k \in \Omega$ is denoted by $\ell^\infty(\Omega)$ and in case $0 \in \Omega$ we write $\ell_0(\Omega)$ for the space of all such sequences converging to 0. Convexity of $\Omega$ carries over to the spaces $\ell^\infty(\Omega), \ell_0(\Omega)$. We briefly write $\ell^\infty := \ell^\infty(X)$, $\ell_0 := \ell_0(X)$ or simply $\ell$ for one of these two spaces, which both are Banach spaces canonically equipped with the natural norm $\|\phi\| := \sup_{k\in\mathbb{I}} |\phi_k|$.

We consider functions $f_k : \Omega \times \Lambda \to X$, $k \in \mathbb{Z}$, which are the right-hand sides of nonautonomous parameter-dependent difference equations

$$\boxed{x_{k+1} = f_k(x_k, \lambda).} \tag{$\Delta$}_\lambda$$

For a fixed parameter $\lambda \in \Lambda$, a *complete* or *entire solution* of the difference equation $(\Delta)_\lambda$ is a sequence $\phi = (\phi_k)_{k\in\mathbb{Z}}$ with $\phi_k \in \Omega$ satisfying the recursion $(\Delta)_\lambda$ on the whole integer axis $\mathbb{Z}$. In order to emphasize the dependence on $\lambda$, we sometimes write $\phi(\lambda)$. Provided $0 \in \Omega$, a complete solution satisfying $\lim_{k\to\pm\infty} \phi_k = 0$ is called *homoclinic* to 0 and we speak of a *permanent solution*, if

$$\inf_{k\in\mathbb{Z}} \operatorname{dist}(\phi_k, \Omega) > 0.$$

Finally, the *general solution* $\varphi_\lambda(\cdot; \kappa, \eta)$ is the solution to $(\Delta)_\lambda$ satisfying $x_\kappa = \eta \in \Omega$.

The following assumptions hold for $C^m$-smooth right-hand sides of $(\Delta)_\lambda$, whose derivatives map bounded into bounded sets uniformly in time.

**Hypothesis.** *Let $m \in \mathbb{N}$ and suppose each $f_k : \Omega \times \Lambda \to X$, $k \in \mathbb{Z}$, is a $C^m$-function such that the following holds for $0 \leq j \leq m$:*

$(H_0)$ *For all bounded $B \subseteq \Omega$ one has*

$$\sup_{k\in\mathbb{Z}} \sup_{x\in B} \left|D^j f_k(x, \lambda)\right| < \infty \quad \textit{for all } \lambda \in \Lambda$$

(well-definedness) *and for all $\lambda^* \in \Lambda$ and $\varepsilon > 0$ there exists a $\delta > 0$ with*

$$|x - y| < \delta \quad \Rightarrow \quad \sup_{k\in\mathbb{Z}} \left|D^j f_k(x, \lambda) - D^j f_k(y, \lambda)\right| < \varepsilon \tag{2.1}$$

*for all $x, y \in \Omega$ and $\lambda \in B_\delta(\lambda^*)$* (uniform continuity).

$(H_1)$ *We have $0 \in \Omega$ and $\lim_{k\to\pm\infty} f_k(0, \lambda) = 0$ for all $\lambda \in \Lambda$.*

Having this available, the crucial tool for our whole analysis is given in

**Theorem 2.1.** *For every parameter $\lambda \in \Lambda$ a sequence $\phi$ in $\Omega$ is a solution of the difference equation $(\Delta)_\lambda$, if and only if $\phi$ solves the nonlinear equation*

$$G(\phi, \lambda) = 0 \tag{2.2}$$

*with a formally defined operator $G(\phi, \lambda) = S\phi - F(\phi, \lambda)$, where*

$$(S\phi)_k := \phi_{k+1}, \qquad\qquad (F(\phi, \lambda))_k := f_k(\phi_k, \lambda).$$

*Moreover, under $(H_0)$ the mapping $G$ fulfills:*

*(a) $G : \ell^\infty(\Omega) \times \Lambda \to \ell^\infty$ is well-defined and of class $C^m$ on $\ell^\infty(\Omega)^\circ \times \Lambda$,*

*(b) If $(H_0)$ and $(H_1)$ hold, then $G : \ell_0(\Omega) \times \Lambda \to \ell_0$ is well-defined and of class $C^m$.*

*Proof.* See [41, Thm. 2.4 and Prop. 2.3]. □

2.1. **Linear difference equations.** Let $\mathbb{I}$ be a discrete interval. For a given operator sequence $A_k \in L(X)$, $k \in \mathbb{Z}$, linear difference equations are of the form

$$\boxed{x_{k+1} = A_k x_k} \tag{$L$}$$

with associated *transition operator* $\Phi(k,l) \in L(X)$, $k,l \in \mathbb{Z}$, defined by

$$\Phi(k,l) := \begin{cases} I_X & \text{for } k = l, \\ A_{k-1} \cdots A_l & \text{for } k > l; \end{cases}$$

if every $A_k$ is invertible, we additionally set $\Phi(k,l) := A_k^{-1} \cdots A_{l-1}^{-1}$ for $k < l$. We say a sequence of projections $P_k \in L(X)$, $k \in \mathbb{I}$, is an *invariant projector*, provided

$$A_k P_k = P_{k+1} A_k \quad \text{for all } k \in \mathbb{I}' \tag{2.3}$$

and we speak of a *regular projector*, if the restriction $A_k : N(P_k) \to N(P_{k+1})$ is an isomorphism. Thus, the restricted transition operator $\Phi(k,l) : N(P_l) \to N(P_k)$, $k \le l$, is well-defined with a bounded inverse $\Phi(l,k)$. A linear difference equation $(L)$ is said to have an *exponential dichotomy* (ED for short) on $\mathbb{I}$, if there exist reals $K \ge 1$, $\alpha \in (0,1)$ such that

$$|\Phi(k,l)P_l| \le K\alpha^{k-l}, \qquad |\Phi(l,k)[I - P_k]| \le K\alpha^{k-l} \quad \text{for all } l \le k,\ k,l \in \mathbb{I}$$

with some regular invariant projector $P_k$ (cf., e.g. [18]). Dynamically this means:

- For $\mathbb{I}$ unbounded above, the *stable vector bundle* $\{(\kappa, x) \in \mathbb{I} \times X : x \in R(P_\kappa)\}$ contains the solutions to $(L)$ decaying to 0 in forward time; in particular,
$$R(P_\kappa) = \{\xi \in X : \Phi(\cdot,\kappa)\xi \in \ell^\infty\} \quad \text{for all } \kappa \in \mathbb{I} \tag{2.4}$$
and the ranges $R(P_\kappa)$ are uniquely determined.
- For $\mathbb{I}$ unbounded below, the *unstable vector bundle* $\{(\kappa, x) \in \mathbb{I} \times X : x \in N(P_\kappa)\}$ consists of solutions to $(L)$ which exist and decay exponentially to 0 in backward time; in particular
$$N(P_\kappa) = \left\{ \xi \in X : \begin{array}{l} \text{there exists a solution } \phi : \mathbb{Z}_\kappa^- \to X \\ \text{of } (L) \text{ with } \phi(\kappa) = \xi \text{ and } \phi \in \ell^\infty \end{array} \right\} \quad \text{for all } \kappa \in \mathbb{I} \tag{2.5}$$
and the kernels $N(P_\kappa)$ are uniquely determined.

A proof of the dynamical characterizations (2.4), (2.5) has been given in [38, pp. 268–269, Prop. 2.3] for the invertible finite-dimensional situation, and a generalization to our setting is due to [26, p. 22, Satz 2.3.2]. The stable and unstable vector bundles generalize the stable resp. unstable subspaces known from the autonomous theory of hyperbolic linear operators (see, e.g. [19, pp. 445ff] or [21, p. 6, Technical lemma 1]).

We define the *dichotomy spectrum* of $(L)$ by

$$\Sigma(A) := \left\{\gamma > 0 : x_{k+1} = \gamma^{-1} A_k x_k \text{ does not have an ED on } \mathbb{Z}\right\}.$$

Conditions guaranteeing an ED on $\mathbb{Z}$ and explicit forms of the dichotomy spectrum $\Sigma(A)$ are summarized in [41, Examples 2.2–2.5] for various linear difference equations.

Essential for our approach are Fredholm properties for the derivative of the operator $G$ defined in Thm. 2.1. It has the form of a difference operator

$$L : \ell \to \ell, \qquad (L\phi)_k := \phi_{k+1} - A_k \phi_k \quad \text{for all } k \in \mathbb{Z}, \tag{2.6}$$

which is well-defined and continuous under *bounded forward growth* of $(L)$, i.e.

$$\sup_{k\in\mathbb{Z}} |A_k| < \infty.$$

For our further strategy we also need the *dual difference equation* to $(L)$ given by

$$\boxed{x_k = A'_{k+1}x_{k+1},} \tag{$L'$}$$

which has variables in the dual space $X'$. While the existence of forward solutions for $(L)$ is trivially given, $(L')$ has backward solutions and its transition operator $\Phi'$ reads as

$$\Phi'(k,l) = \Phi(l+1,k+1)' \quad \text{for all } k \le l. \tag{2.7}$$

Moreover, an exponential dichotomy carries over from $(L)$ to $(L')$ as follows:

**Lemma 2.2.** *If a linear equation $(L)$ has an exponential dichotomy with $\alpha, K$ and invariant projector $P_k$ on $\mathbb{I}$, then the dual equation $(L')$ admits an exponential dichotomy with $\alpha, K$ on the shifted interval $\mathbb{I}'$, whose invariant projectors are $P_k^* := I - P'_{k+1}$ and*

$$R(P_k^*) = R(P_{k+1})^\perp, \qquad\qquad N(P_k^*) = N(P_{k+1})^\perp. \tag{2.8}$$

*Proof.* To verify an exponential dichotomy for $(L')$ is rather straight forward and left to the interested reader. The assertion concerning range and kernel of $P_k^*$ can be found, for instance, in [27, p. 156]. □

Next, we introduce the dual operator

$$L' : \ell' \to \ell', \qquad\qquad (L'\psi)_k := \psi_k - A'_{k+1}\psi_{k+1} \quad \text{for all } k \in \mathbb{Z}; \tag{2.9}$$

it is well-defined and continuous under bounded forward growth of $(L)$, since boundedness of the sequence $A_k$ carries over to $A'_k$ (cf. [27, p. 154]). Referring to [55, pp. 366–367, Prop. 8.14(4)]) we know that Fredholm properties of $L$ are inherited by $L'$ with

$$\dim N(L') = \operatorname{codim} R(L), \qquad\qquad \operatorname{codim} R(L') = \dim N(L).$$

In order to study possible Fredholm properties of the operator $L$ itself, we benefit from the quite detailed discussion in [4]. Here, it is of particular importance to investigate systems, which are dichotomous on both a positive and a negative semiaxis.

**Proposition 2.3** (nodal operator)**.** *Let $\underline{\kappa}, \overline{\kappa} \in \mathbb{Z}$ with $\underline{\kappa} < \overline{\kappa}$. Suppose a linear equation $(L)$ admits an ED both on $\mathbb{Z}^+_{\overline{\kappa}}$ (with projector $P_k^+$) and on $\mathbb{Z}^-_{\underline{\kappa}}$ (with projector $P_k^-$). Then the operator $L : \ell \to \ell$ is Fredholm, if and only if the* nodal operator

$$\Xi(\overline{\kappa}, \underline{\kappa}) := (I - P^+_{\overline{\kappa}})\Phi(\overline{\kappa}, \underline{\kappa})(I - P^-_{\underline{\kappa}}) : \, N(P^-_{\underline{\kappa}}) \to N(P^+_{\overline{\kappa}})$$

*is Fredholm. Both operators have the same Fredholm index, which for finite-dimensional kernels $N(P^-_{\underline{\kappa}}), N(P^+_{\overline{\kappa}})$ is given by* $\dim N(P^-_{\underline{\kappa}}) - \dim N(P^+_{\overline{\kappa}})$.

*Proof.* See [4, Thm. 8] and [55, p. 367, Example 8.15] for the index. □

In a parallel fashion to Prop. 2.3 we now consider the situation where $(L)$ admits EDs on positive and negative semiaxes with nonempty intersection. For the corresponding finite-dimensional situation we refer to [5, 7].

**Proposition 2.4.** *Let $\kappa \in \mathbb{Z}$. Suppose a linear equation $(L)$ admits an ED both on $\mathbb{Z}^+_\kappa$ (with projector $P_k^+$) and on $\mathbb{Z}^-_\kappa$ (with projector $P_k^-$). Then the operator $L : \ell \to \ell$ is Fredholm, if the spaces*

$$X_1 := R(P^+_\kappa) \cap N(P^-_\kappa), \qquad\qquad X_2 := R(P^+_\kappa) + N(P^-_\kappa)$$

*have finite dimension resp. codimension. The index is* $\dim X_1 - \operatorname{codim} X_2$.

*Proof.* For $\kappa \in \mathbb{Z}$ we introduce the homomorphism $T : X_1 \to N(L)$, $T\xi := \Phi(\cdot,\kappa)\xi$. Using the dynamical characterizations (2.4), (2.5) it is easily seen that $T$ is well-defined and an isomorphism, thus $\dim N(L) < \infty$.

From Lemma 2.2 we deduce that the dual equation $(L')$ admits EDs on both semiaxes $\mathbb{Z}^-_{\kappa-1}$ and $\mathbb{Z}^+_{\kappa-1}$ with respective invariant projectors $(I-P^-_{k+1})'$, $(I-P^+_{k+1})'$. Similarly to the above, a corresponding dynamical characterization yields that the linear mapping $\xi' \mapsto \Phi'(\cdot,\kappa-1)\xi'$ is an isomorphism from

$$R((I-P^+_\kappa)') \cap N((I-P^-_\kappa)') \overset{(2.8)}{=} R(P^+_\kappa)^\perp \cap N(P^-_\kappa)^\perp = (R(P^+_\kappa)+N(P^-_\kappa))^\perp$$

onto the kernel $N(L') = R(L)^\perp$ (cf. [27, p. 168, (5.10)]). Thus, by assumption the operator $L$ is Fredholm. $\square$

**Corollary 2.5.** *If the assumptions of Prop. 2.4 are satisfied, then for every $\kappa \in \mathbb{Z}$ one has*

$$N(L) = \left\{\Phi(\cdot,\kappa)\xi \in \ell : \xi \in R(P^+_\kappa) \cap N(P^-_\kappa)\right\},$$
$$N(L') = \left\{\Phi(\kappa,\cdot+1)'\xi' \in \ell' : \xi' \in (R(P^+_\kappa)+N(P^-_\kappa))^\perp\right\}$$

*and furthermore*

$$\dim N(L) = \dim R(P^+_\kappa) \cap N(P^-_\kappa), \qquad \dim N(L') = \operatorname{codim} R(P^+_\kappa) + N(P^-_\kappa).$$

*Proof.* Concerning the kernel $N(L)$ the assertion directly follows from the fact that the operator $T$ introduced in the proof of Prop. 2.4 is an isomorphism; so does the claim on the adjoint $N(L')$, if we keep (2.7) in mind. $\square$

We close this subsection with a prototype example illustrating the above concepts:

*Example* 2.1. Let $\gamma_-, \beta_-, \gamma_+, \beta_+ \in \mathbb{R} \setminus \{0\}$ be given and suppose $X = \mathbb{R}^2$. We define a piecewise constant coefficient matrix for the linear equation $(L)$ by

$$A_k := \begin{pmatrix} b_k & 0 \\ 0 & c_k \end{pmatrix}, \qquad b_k := \begin{cases} \beta_-, & k<0, \\ \beta_+, & k \ge 0, \end{cases} \qquad c_k := \begin{cases} \gamma_-, & k<0, \\ \gamma_+, & k \ge 0 \end{cases}$$

and easily deduce at the transition matrix

$$\Phi(k,l) := \begin{cases} \operatorname{diag}(\beta_+^{k-l}, \gamma_+^{k-l}), & k \ge l \ge 0, \\ \operatorname{diag}(\beta_+^k\beta_-^{-l}, \gamma_+^k\gamma_-^{-l}), & k \ge 0 > l, \\ \operatorname{diag}(\beta_-^{k-l}, \gamma_-^{k-l}), & 0 > k \ge l; \end{cases} \tag{2.10}$$

due to $A_k \in GL(\mathbb{R}^2)$ one sets $\Phi(k,l) := \Phi(l,k)^{-1}$ for $k<l$. We distinguish several cases to describe the dichotomy and Fredholm properties of $(L)$. In each case, $(L)$ admits an ED on $\mathbb{Z}^+_0$ and $\mathbb{Z}^-_{-1}$ with constant projectors $P^-_k$ resp. $P^+_k$; it is easy to see that the ED on $\mathbb{Z}^-_{-1}$ extends to $\mathbb{Z}^-_0$. By Prop. 2.4 the operator $L : \ell \to \ell$ is Fredholm and we arrive at:

- (a) $|\beta_+|, |\gamma_+| < 1$: $P^+_k \equiv I$
  - $(a_1)$ $|\beta_-|, |\gamma_-| < 1$: $P^-_k \equiv I$, $L$ is invertible, $(L)$ has an ED on $\mathbb{Z}$ and 0 is uniformly asymptotically stable
  - $(a_2)$ $|\beta_-| < 1 < |\gamma_-|$: $P^-_k \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $L$ has 1-dimensional kernel, index 1 and 0 is asymptotically stable
  - $(a_3)$ $|\gamma_-| < 1 < |\beta_-|$: $P^-_k \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $L$ has 1-dimensional kernel, index 1 and 0 is asymptotically stable
  - $(a_4)$ $1 < |\beta_-|, |\gamma_-|$: $P^-_k \equiv 0$, $L$ has 2-dimensional kernel, index 2 and 0 is asymptotically stable
- (b) $|\beta_+| < 1 < |\gamma_+|$: $(L)$ admits an ED on $\mathbb{Z}^+_0$ with projector $P^+_k \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$
  - $(b_1)$ $|\beta_-|, |\gamma_-| < 1$: $P^-_k \equiv I$, $L$ has 0-dimensional kernel and index $-1$

($b_2$) $|\beta_-| < 1 < |\gamma_-|$: $P_k^- \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $L$ is invertible and $(L)$ has an ED on $\mathbb{Z}$
($b_3$) $|\gamma_-| < 1 < |\beta_-|$: $P_k^- \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $L$ has 1-dimensional kernel and index 0
($b_4$) $1 < |\beta_-|, |\gamma_-|$: $P_k^- \equiv 0$, $L$ has 1-dimensional kernel and index 1

(c) $|\gamma_+| < 1 < |\beta_+|$: $(L)$ admits an ED on $\mathbb{Z}_0^+$ with projector $P_k^+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
($c_1$) $|\beta_-|, |\gamma_-| < 1$: $P_k^- \equiv I$, $L$ has 0-dimensional kernel and index $-1$
($c_2$) $|\beta_-| < 1 < |\gamma_-|$: $P_k^- \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $L$ has 1-dimensional kernel and index 0
($c_3$) $|\gamma_-| < 1 < |\beta_-|$: $P_k^- \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $L$ is invertible and $(L)$ has an ED on $\mathbb{Z}$
($c_4$) $1 < |\beta_-|, |\gamma_-|$: $P_k^- \equiv 0$, $L$ has 1-dimensional kernel and index 1

(d) $1 < |\beta_+|, |\gamma_+|$: $(L)$ admits an ED on $\mathbb{Z}_0^+$ with projector $P_k^+ = 0$
($d_1$) $|\beta_-|, |\gamma_-| < 1$: $P_k^- \equiv I$, $L$ has 0-dimensional kernel and index $-2$
($d_2$) $|\beta_-| < 1 < |\gamma_-|$: $P_k^- \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $L$ has 0-dimensional kernel and index $-1$
($d_3$) $|\gamma_-| < 1 < |\beta_-|$: $P_k^- \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $L$ has 0-dimensional kernel and index $-1$
($d_4$) $1 < |\beta_-|, |\gamma_-|$: $P_k^- \equiv 0$, $L$ is invertible and $(L)$ has an ED on $\mathbb{Z}$.

2.2. **Bifurcation of bounded solutions.** As indicated in the introduction, up to the present point the concept of structural stability is not truly developed for nonautonomous systems. Likewise, it is subtle to define a notion of bifurcation. We bypass such deficits by simply adopting the corresponding well-established terminology from branching theory applied to the abstract problem (2.2).

For this, let us assume that for some parameter $\lambda^* \in \Lambda$ the nonautonomous difference equation $(\Delta)_{\lambda^*}$ possesses a bounded complete reference solution $\phi^* = \phi(\lambda^*)$. We say that $(\Delta)_\lambda$ undergoes a *bifurcation* at $\lambda = \lambda^*$ along $\phi^*$, or $\phi^*$ *bifurcates* at $\lambda^*$, if there exists a convergent sequence of parameters $(\lambda_n)_{n\in\mathbb{N}}$ in $\Lambda$ with limit $\lambda^*$ so that $(\Delta)_{\lambda_n}$ has two distinct bounded complete solutions $\phi^1(\lambda_n), \phi^2(\lambda_n)$ satisfying

$$\lim_{n\to\infty} \phi^1(\lambda_n) = \lim_{n\to\infty} \phi^2(\lambda_n) = \phi^*.$$

Given a parameter space $\Lambda \subseteq \mathbb{R}$, one speaks of a *subcritical* or a *supercritical bifurcation*, if the sequence $(\lambda_n)_{n\in\mathbb{N}}$ can be chosen according to $\lambda_n < \lambda^*$ or $\lambda_n > \lambda^*$, respectively. In other words, the pair $(\phi^*, \lambda^*)$ is a *bifurcation* or *branching point* of the abstract nonlinear equation (2.2) in $\ell^\infty(\Omega)$ (cf. [55, p. 358, Definition 8.1]).

Bifurcation properties of $\phi^*$ crucially depend on the *variational equation*

$$x_{k+1} = D_1 f_k(\phi_k^*, \lambda^*) x_k \tag{2.11}$$

with associated dichotomy spectrum $\Sigma(\phi^*, \lambda^*)$ and a transition operator denoted by $\Phi_{\lambda^*}$.

In this context, we say the solution $\phi^*$ is *hyperbolic*, if (2.11) has an ED on $\mathbb{Z}$ or equivalently $1 \notin \Sigma(\phi^*, \lambda^*)$. Nonhyperbolicity yields the subsequent necessary condition for bifurcation, which also applies to complete solutions in $\ell_0(\Omega)$, provided $(H_1)$ holds.

**Proposition 2.6.** *Let $\lambda^* \in \Lambda$ and suppose $(H_0)$ holds. If a complete permanent solution $\phi^* \in \ell^\infty(\Omega)$ of $(\Delta)_{\lambda^*}$ bifurcates at $\lambda^*$, then $\phi^*$ is nonhyperbolic.*

*Proof.* We proceed indirectly and suppose that $1 \notin \Sigma(\phi^*, \lambda^*)$. Then our [41, Thm. 2.11] guarantees that neighborhoods $\Lambda_0 \subseteq \Lambda$ for $\lambda^*$, $U \subseteq \ell^\infty(\Omega)$ for $\phi^*$ exist, so that $(\Delta)_\lambda$ has a unique complete solution $\phi(\lambda) \in U$ for $\lambda \in \Lambda_0$. Hence, $\phi^*$ cannot bifurcate at $\lambda^*$. □

Hence, in order to deduce bifurcation results for complete solutions, we have to assume weaker concepts than an exponential dichotomy on the whole integer axis. One possibility, namely dichotomies on both semiaxes, will be discussed in the following. The alternative concept of an exponential trichotomy, and its consequences for bifurcation phenomena, is postponed to [43].

**Hypothesis.** *Let $n, r \in \mathbb{N}$, $\kappa \in \mathbb{Z}$, $\lambda^* \in \Lambda$ be given, suppose $X$ is reflexive and $(\Delta)_{\lambda^*}$ admits a complete permanent solution $\phi^* \in \ell^\infty(\Omega)$ with*

$(H_2)$ *the variational equation (2.11) admits an ED both on $\mathbb{Z}_\kappa^+$ and $\mathbb{Z}_\kappa^-$ with respective projectors $P_k^+$ and $P_k^-$ satisfying*

$$R(P_\kappa^+) \cap N(P_\kappa^-) = \operatorname{span}\{\xi_1, \dots, \xi_n\},$$
$$(R(P_\kappa^+) + N(P_\kappa^-))^\perp = \operatorname{span}\{\xi_1', \dots, \xi_r'\}$$

*and linearly independent vectors $\xi_1, \dots, \xi_n \in X$, resp. $\xi_1', \dots, \xi_r' \in X'$. Moreover, we choose $\eta_1, \dots, \eta_r \in X$, resp. $\eta_1', \dots, \eta_n' \in X'$ such that*

$$\langle \eta_i', \xi_j \rangle = \delta_{i,j} \text{ for } 1 \le i, j \le n, \qquad \langle \xi_i', \eta_j \rangle = \delta_{i,j} \text{ for } 1 \le i, j \le r. \tag{2.12}$$

*Remark* 2.1. (1) The permanence assumption on the complete solution $\phi^*$ guarantees that the sequence $\phi^*$ is an interior point of $\ell^\infty(\Omega)$.

(2) We have the orthogonality relation $\langle \xi_i', \xi_j \rangle = 0$ for $1 \le i \le n$, $1 \le j \le r$.

(3) In order to satisfy $(H_2)$ it is necessary to require $\dim X > 1$ and furthermore the conditions $n, r \in \mathbb{N}$ demand $P_\kappa^+ \neq I$, $P_\kappa^- \neq 0$ and $P_\kappa^+ \neq 0$, $P_\kappa^- \neq I$. Thus, the existence of an ED on $\mathbb{Z}_\kappa^-$ with nontrivial unstable vector bundle yields that there is an unstable fiber bundle corresponding to $\phi^*$. An unstable fiber bundle of a solution $\phi^*$ is a bundle of submanifolds and the nonautonomous counterpart to unstable manifolds through equilibria; their construction has been described in, for instance, [44]. In conclusion, a priori $\phi^*$ is unstable.

(4) Since $X$ is assumed to be a reflexive Banach space (and this is exactly where we need reflexivity), the existence of $\eta_1, \dots, \eta_r \in X$ and $\eta_1', \dots, \eta_n' \in X'$ satisfying (2.12) results from the Hahn-Banach theorem (cf. [27, p. 135, Thm. 1.22]).

*Example* 2.2 (almost periodic case). Suppose the variational equation (2.11) is almost periodic and admits an exponential dichotomy on one of the semiaxes $\mathbb{Z}_\kappa^+$ or $\mathbb{Z}_\kappa^-$. The dichotomy extends to the whole axis $\mathbb{Z}$ (see [2, Prop. 3.2]) and thus $P_\kappa^+ = P_\kappa^-$, since in this situation invariant projectors are uniquely determined (cf. [38, pp. 268–269, Prop. 2.3(iii)] or [26, pp. 24–25, Satz 2.4.2(iv)]). This implies $R(P_\kappa^+) \cap N(P_\kappa^-) = \{0\}$, $R(P_\kappa^+) + N(P_\kappa^-) = X$ and $(H_2)$ cannot hold for almost periodic (or periodic, or autonomous) variational equations.

Next we apply the Fredholm theory from Subsect. 2.1 to the variational equation (2.11) with the coefficient operator $A_k = D_1 f_k(\phi_k^*, \lambda^*)$.

**Lemma 2.7.** *If $(H_0)$, $(H_2)$ hold, then the linear operator $L : \ell \to \ell$ is Fredholm of index $n - r$ and with the dual operator $L'$ defined in (2.9) one has*

$$N(L) = \operatorname{span}\{\Phi_{\lambda^*}(\cdot, \kappa)\xi_1, \dots, \Phi_{\lambda^*}(\cdot, \kappa)\xi_n\},$$
$$N(L') = \operatorname{span}\{\Phi_{\lambda^*}(\kappa, \cdot + 1)'\xi_1', \dots, \Phi_{\lambda^*}(\kappa, \cdot + 1)'\xi_r'\}, \tag{2.13}$$

*where $\Phi_{\lambda^*}(\cdot, \kappa)\xi_i$, $1 \le i \le n$, resp. $\Phi_{\lambda^*}(\kappa, \cdot + 1)'\xi_j'$, $1 \le j \le r$, are linearly independent.*

*Proof.* Using Prop. 2.4 our assumptions immediately imply that $L$ is Fredholm with index $n - r$ and $\dim N(L) = n$. Indeed, referring to Cor. 2.5, the kernel of $L$ consists of bounded complete solutions of (2.11), which due to the dichotomy assumptions are linear combinations of the linearly independent vectors $\Phi_{\lambda^*}(\cdot, \kappa)\xi_i$ for $1 \le i \le n$. Using the same argument, bounded complete solutions of $(L')$ allow a representation as linear combinations of the functionals $\Phi_{\lambda^*}(\kappa, \cdot + 1)'\xi_j'$, $1 \le j \le r$. □

**Lemma 2.8.** *If $(H_0)$, $(H_2)$ hold, then the mappings $P, Q \in L(\ell)$,*

$$Px := \sum_{i=1}^{n} \langle \eta_i', x_\kappa \rangle \Phi_{\lambda^*}(\cdot, \kappa)\xi_i, \qquad Qx := x - E_\kappa \sum_{i=1}^{r} \sum_{j \in \mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)'\xi_i', x_j \rangle \eta_i$$

*are bounded projections onto $N(L)$ and $R(L)$, resp., where $E_\kappa : X \to \ell$ reads as*

$$(E_\kappa \xi)_k := \delta_{k,\kappa}\xi.$$

*Proof.* Following the procedure described in Subsect. A.1 we introduce a bilinear form

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle : \ell^\beta \times \ell \to \mathbb{R}, \qquad \langle\!\langle \phi', \psi \rangle\!\rangle := \sum_{j \in \mathbb{Z}} \langle \phi_j', \psi_j \rangle, \tag{2.14}$$

where $\ell^\beta$ is the $\beta$-dual of $\ell$, i.e. the set of all sequences $\phi' = (\phi_j')_{j \in \mathbb{Z}}$ in $X'$ such that $\langle\!\langle \phi', \cdot \rangle\!\rangle$ defines a continuous linear form on the sequence space $\ell$. Now, from Lemma 2.7 we know that the linearly independent vectors $\phi^i := \Phi_{\lambda^*}(\cdot, \kappa)\xi_i$ in $\ell$ span the kernel $N(L)$. Thus, if we define sequences $\psi_i' \in \ell^\beta$ by $\psi_i' := \delta_{\cdot,\kappa}\eta_i'$ our assumption (2.12) implies $\langle\!\langle \psi_i', \phi^j \rangle\!\rangle = \langle \psi_{i,\kappa}', \phi_\kappa^j \rangle = \langle \eta_i', \xi_j \rangle = \delta_{i,j}$ and $\{\psi_i', \phi_j\}$ forms a biorthogonal system. Then

$$Px := \sum_{i=1}^{n} \langle\!\langle \psi_i', x \rangle\!\rangle \phi^i = \sum_{i=1}^{n} \langle \eta_i', x_\kappa \rangle \phi^i = \sum_{i=1}^{n} \langle \eta_i', x_\kappa \rangle \Phi_{\lambda^*}(\cdot, \kappa)\xi_i$$

is a projection $P \in L(\ell)$ onto $N(L)$. Defining sequences $\phi_i' \in \ell^\beta$, $\psi^i \in \ell$ by

$$\phi_i' := \Phi_{\lambda^*}(\kappa, \cdot)'\xi_i', \qquad \psi^i := E_\kappa \eta_i \quad \text{for all } 1 \le i \le r$$

we immediately obtain from (2.12) that $\langle\!\langle \phi_i', \psi^j \rangle\!\rangle = \langle \xi_i', \eta_j \rangle = \delta_{i,j}$; thus, also $\{\phi_i', \psi^j\}$ forms a biorthogonal system. In addition, we define a projection $I - Q \in L(\ell)$ given by

$$(I - Q)y := \sum_{i=1}^{r} \langle\!\langle \phi_i', y \rangle\!\rangle \psi^i = E_\kappa \sum_{i=1}^{r} \sum_{j \in \mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)'\xi_i', y_j \rangle \eta_i,$$

whose complementary projection $Q$ maps onto $R(L)$. $\square$

**Proposition 2.9** (branching equation)**.** *Suppose that $(H_0)$, $(H_2)$ hold. If $\ell = \ell^\infty$, then there exist open convex neighborhoods $S \subseteq \mathbb{R}^n$ of 0, $\Lambda_0 \subseteq \Lambda$ of $\lambda^*$ and a $C^m$-function $\vartheta : S \times \Lambda_0 \to \ell$ satisfying $\vartheta(0, \lambda^*) = 0$, $D_1\vartheta(0, \lambda^*) = 0$ and*

$$\begin{aligned} &\phi_{\kappa+1}^* + \sum_{l=1}^{n} s_l A_\kappa \xi_l + \vartheta(s, \lambda)_{\kappa+1} - H_\kappa(s, \lambda) \\ &\qquad - \sum_{i=1}^{r} \sum_{j \in \mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)'\xi_i', \phi_{j+1}^* + \vartheta(s, \lambda)_{j+1} - H_j(s, \lambda) \rangle = 0, \end{aligned} \tag{2.15}$$

$$\phi_{k+1}^* + \sum_{l=1}^{n} s_l \Phi_{\lambda^*}(k+1, \kappa)\xi_l + \vartheta(s, \lambda)_{k+1} - H_k(s, \lambda) = 0 \tag{2.16}$$

*for all $k \neq \kappa$ with the function $H_k(s, \lambda) = f_k\left(\phi_k^* + \sum_{l=1}^{n} s_l \Phi_{\lambda^*}(k, \kappa)\xi_l + \vartheta(s, \lambda)_k, \lambda\right)$. Moreover, the branching equation for (2.2) reads as $g(s, \lambda) = 0$, where $g : S \times \Lambda_0 \to \mathbb{R}^r$*

*is a $C^m$-function whose components $g_1, \dots, g_r$ read as*

$$\begin{aligned} g_l(s,\lambda) :=& \sum_{j\in\mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)'\xi_l', \phi_{j+1}^* + \vartheta(s,\lambda)_{j+1}\rangle \\ & - \sum_{j\in\mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)'\xi_l', f_j(\phi_j^* + \sum_{i=1}^n s_i \Phi_{\lambda^*}(j,\kappa)\xi_i + \vartheta(s,\lambda)_j, \lambda)\rangle. \end{aligned} \tag{2.17}$$

*Given $\phi^* \in \ell_0(\Omega)$ with $(H_0)$ to $(H_2)$, the assertion holds with $\ell = \ell_0$.*

*Proof.* We apply the machinery presented in Sect. A.2 to the problem $G(\phi,\lambda) = 0$ with $G : \ell^\infty(\Omega) \times \Lambda \to \ell^\infty$ defined in Thm. 2.1 by $G(\phi,\lambda) = S\phi - F(\phi,\lambda)$. Above all, we have $G(\phi^*,\lambda^*) = 0$ and due to the assumed permanence we know that $\phi^* \in \ell^\infty(\Omega)$ is an interior point. In addition, Thm. 2.1(a) (see also [41, Prop. 2.3 and Thm. 2.4] for the explicit form of the derivatives) yields that $G$ is $m$-times continuously differentiable in the interior point $\phi^* \in \ell^\infty(\Omega)$ with partial derivative

$$D_1G(\phi^*,\lambda^*)\psi = S\psi - D_1F(\phi^*,\lambda^*)\psi = L\psi,$$

where $(L\psi)_k = \psi_{k+1} - D_1 f_k(\phi_k^*,\lambda^*)\psi_k$. Due to Lemma 2.7 the operator $L$ is Fredholm with index $n-r$ and $n$-dimensional kernel. Hence, Lemma A.1 provides a function $\vartheta$ as above satisfying the abstract equation (A.4), which in our setup of projections given in Lemma 2.8 has the concrete representation (2.16) for $k \neq \kappa$ and

$$\begin{aligned} \phi_{\kappa+1}^* + \sum_{l=1}^n s_l A_\kappa \xi_l + \vartheta(s,\lambda)_{\kappa+1} - f_\kappa\left(\phi_\kappa^* + \sum_{l=1}^n s_l\xi_l + \vartheta(s,\lambda)_\kappa, \lambda\right) \\ - \sum_{i=1}^r \sum_{j\in\mathbb{Z}} \Bigg\langle \Phi_{\lambda^*}(\kappa,j+1)'\xi_i', \phi_{j+1}^* + \Phi_{\lambda^*}(\kappa,j+1)'\xi_i' + \vartheta(s,\lambda)_{j+1} \\ - f_j\left(\phi_j^* + \sum_{l=1}^n s_l \Phi_{\lambda^*}(j,\kappa)\xi_l + \vartheta(s,\lambda)_j, \lambda\right)\Bigg\rangle = 0 \quad \text{for } k = \kappa. \end{aligned}$$

The above relation simplifies to (2.15), since we deduce from Rem. 2.1(2) that

$$\langle \Phi_{\lambda^*}(\kappa,j+1)'\xi_l', \Phi_{\lambda^*}(j+1,\kappa)\xi_l\rangle = \langle \xi_i', \xi_l\rangle = 0.$$

Similarly, for $1 \le l \le r$, in our situation the branching equation (A.5) has components

$$\begin{aligned} g_l(s,\lambda) \overset{(A.6)}{=} & \sum_{j\in\mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa,j+1)'\xi_l', \phi_{j+1}^* + \sum_{i=1}^n s_i \Phi_{\lambda^*}(j+1,\kappa)\xi + \psi(s,\lambda)_{j+1}\rangle \\ & - \sum_{j\in\mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa,j+1)'\xi_l', f_j(\phi_j^* + \sum_{i=1}^n s_i\Phi_{\lambda^*}(j,\kappa)\xi + \psi(s,\lambda)_j,\lambda)\rangle, \end{aligned}$$

which reduces to (2.17), by Rem. 2.1(2). If we replace $\ell^\infty(\Omega)$ by $\ell_0(\Omega)$, then $(H_1)$ yields that $G : \ell_0(\Omega) \times \Lambda \to \ell_0(\Omega)$ is well-defined and the claim follows analogously. $\square$

Before we present bifurcation criteria, it is useful to introduce certain functionals:

**Lemma 2.10.** *If $(H_0)$, $(H_2)$ hold, then the linear functionals*

$$\mu_i : \ell \to \mathbb{R}, \qquad \mu_i(\phi) := \sum_{j\in\mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa,j+1)'\xi_i', \phi_j\rangle \quad \textit{for all } 1 \le i \le r$$

*are continuous with $|\mu_i| \le K\frac{1+\alpha}{1-\alpha}|\xi_i'|$ and one has $R(L) = \bigcap_{i=1}^r N(\mu_i)$.*

*Proof.* The functionals $\xi_i' \in X'$ from $(H_2)$ satisfy

$$\xi_i' \in (R(P_\kappa^-) + N(P_\kappa^+))^\perp = R((I - P_\kappa^+)') \cap N((I - P_\kappa^-)')$$

and thus, using the assumed dichotomy estimates, we can estimate $\mu_i(\phi)$ as follows

$$\begin{aligned}
|\mu_i(\phi)| &\leq \sum_{j=-\infty}^{\kappa-1} \left|\langle \Phi_{\lambda^*}(\kappa, j+1)' P_\kappa^- \xi_i', \phi_j \rangle\right| + \sum_{j=\kappa}^{\infty} \left|\langle \Phi_{\lambda^*}(\kappa, j+1)'(I - P_\kappa^+)\xi_i', \phi_j\rangle\right| \\
&\leq \sum_{j=-\infty}^{\kappa-1} \left|\Phi_{\lambda^*}(\kappa, j+1) P_{j+1}^-\right| |\xi_i'| \, |\phi_j| + \sum_{j=\kappa}^{\infty} \left|\Phi_{\lambda^*}(\kappa, j+1)(I - P_{j+1}^+)\right| |\xi_i'| \, |\phi_j| \\
&\leq K \, |\xi_i'| \, \|\phi\| \left( \sum_{j=-\infty}^{\kappa-1} \alpha^{\kappa-j-1} + \sum_{j=\kappa}^{\infty} \alpha^{j+1-\kappa} \right) \quad \text{for all } 1 \leq i \leq r.
\end{aligned}$$

This implies the given bound for $|\mu_i|$. From Lemma 2.7 we get that the operator $L$ is Fredholm and [55, p. 366, Prop. 8.14(2)] guarantees the following equivalences

$$\begin{aligned}
\phi \in R(L) \quad &\Leftrightarrow \quad \phi \in N(L')^\perp \Leftrightarrow \langle\!\langle \psi', \phi \rangle\!\rangle = 0 \quad \text{for all } \psi' \in N(L') \\
&\overset{(2.13)}{\Leftrightarrow} \quad \langle\!\langle \Phi_{\lambda^*}(\kappa, \cdot + 1)'\xi_i', \phi \rangle\!\rangle = 0 \quad \text{for all } 1 \leq i \leq r \\
&\Leftrightarrow \quad \sum_{j \in \mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)'\xi_i', \phi \rangle = 0 \quad \text{for all } 1 \leq i \leq r \\
&\Leftrightarrow \quad \mu_i(\phi) = 0 \quad \text{for all } 1 \leq i \leq r \Leftrightarrow \phi \in \bigcap_{i=1}^{r} N(\mu_i),
\end{aligned}$$

which leads to our assertion. □

We now investigate the situation where a family of complete bounded solutions $\phi(\lambda)$, $\lambda \in \Lambda$, of a nonautonomous difference equation $(\Delta)_\lambda$ is known — one speaks of a *solution branch*. Then the graph $\Gamma := \{(\lambda, \phi(\lambda)) \in Y \times \ell^\infty(\Omega) : \lambda \in \Lambda\}$ is a submanifold of $\Lambda \times \ell^\infty(\Omega)$ and geometrically $(\Delta)_\lambda$ bifurcates at the point $\lambda = \lambda^*$, if in every neighborhood of $(\lambda^*, \phi(\lambda^*))$ there exists a solution $\phi^* \in \ell^\infty(\Omega)$ of $(\Delta)_\lambda$ such that $(\lambda, \phi^*) \notin \Gamma$.

In order to provide sufficient criteria for a bounded complete solution $\phi(\lambda^*)$ of $(\Delta)_{\lambda^*}$ to bifurcate, the following simplification is helpful. Namely, having such a reference solution at hand, there exists a one-to-one relation between $\phi(\lambda^*)$ and the trivial solution of the corresponding *equation of perturbed motion*

$$x_{k+1} = \hat{f}_k(x_k, \lambda) \tag{2.18}$$

for $\lambda = \lambda^*$, whose right-hand side $\hat{f}_k(x, \lambda) := f_k(x + \phi(\lambda)_k, \lambda) - f_k(\phi(\lambda)_k, \lambda)$ satisfies

$$\hat{f}_k(0, \lambda) = 0 \quad \text{for all } k \in \mathbb{Z},\ \lambda \in \Lambda.$$

Thus, the above solution manifold reduces to $\Gamma = \Lambda \times \{0\}$ and referring to Thm. 2.1 this yields the identity $G(0, \lambda) \equiv 0$ on $\Lambda$. Indeed, the complete solution $\phi(\lambda^*)$ of $(\Delta)_{\lambda^*}$ bifurcates at $\lambda^*$, if and only if the zero solution of (2.18) bifurcates at $\lambda^*$. Nevertheless, in order to circumvent the technical problem of imposing conditions on the derivatives $D^n\phi$ such that $\hat{f}_k$ fulfills $(H_0)$, we retreat to the following simplification:

**Hypothesis.** *Let* $0 \in \Omega$ *and suppose*

$(H_3)$ $f_k(0, \lambda) \equiv 0$ *on* $\mathbb{Z} \times \Lambda$.

*Remark* 2.2. Obviously, $(H_3)$ implies $(H_1)$ and since the trivial solution branch consists of sequences in $\ell_0(\Omega)$, it is reasonable to search in $\ell_0(\Omega)$ (instead of $\ell^\infty(\Omega)$) for bifurcating solutions of $(\Delta)_\lambda$, i.e. we are interested in the branching of solutions heteroclinic to $0$.

At this point we can apply the abstract bifurcation results from Subsect. A.3 to (2.2). We first address the situation of Fredholm operators with odd-dimensional kernel.

**Theorem 2.11** (bifurcation with odd-dimensional kernel)**.** *Let $\Lambda \subseteq \mathbb{R}$ and $m \geq 2$. If $(H_0)$ to $(H_3)$ hold with $n = r$ and $\phi^* = 0$, then the trivial solution of a difference equation $(\Delta)_\lambda$ bifurcates at $\lambda^*$, provided $n$ is odd and*

$$\det\Big(\sum_{j\in\mathbb{Z}}\langle \Phi_{\lambda^*}(\kappa, j+1)'\xi_l', D_1D_2f_j(0,\lambda^*)\Phi_{\lambda^*}(j,\kappa)\xi_i\rangle\Big)_{1\leq i,l\leq n} \neq 0.$$

*Proof.* We will apply Thm. A.5 to the operator equation (2.2). Thereto, we choose an element $\phi \in N(L)$ with representation $\phi = \sum_{i=1}^n s_i\Phi_{\lambda^*}(\cdot,\kappa)\xi_i$ for reals $s_i$ (cf. Lemma 2.7). Due to [41, Prop. 2.3 and Thm. 2.4] one has

$$(D_1D_2G(0,\lambda^*)\phi)_k = -D_1D_2f_k(0,\lambda^*)\phi_k = -D_1D_2f_k(0,\lambda^*)\sum_{i=1}^n s_i\Phi_{\lambda^*}(k,\kappa)\xi_i$$

for all $k \in \mathbb{Z}$. By Lemma 2.10, this sequence is contained in $R(L)$, if and only if

$$\begin{aligned} 0 &= \sum_{j\in\mathbb{Z}}\langle \Phi_{\lambda^*}(\kappa, j+1)'\xi_l', D_1D_2f_j(0,\lambda^*)\sum_{i=1}^n s_i\Phi_{\lambda^*}(j,\kappa)\xi_i\rangle \\ &= \sum_{i=1}^n s_i \sum_{j\in\mathbb{Z}}\langle \Phi_{\lambda^*}(\kappa, j+1)'\xi_l', D_1D_2f_j(0,\lambda^*)\Phi_{\lambda^*}(j,\kappa)\xi_i\rangle \quad \text{for all } 1\leq l\leq n \end{aligned}$$

holds. Thanks to our assumptions, this linear-homogeneous algebraic equation is uniquely solvable yielding $s_1 = \ldots = s_n = 0$. $\square$

We will illuminate our bifurcation results using various pairs of examples. They have a parameter space $\Lambda = \mathbb{R}$ and a 2-dimensional state space $X = \Omega = \mathbb{R}^2$ in common; equipped with the dot product, $X$ becomes a Hilbert space, it is therefore reflexive, the adjoint is simply the transpose and the annihilator the orthogonal complement. These pairs of examples begin with a “minimal” one allowing an explicit solution and a quantitative understanding of their behavior. Interestingly, in each of these Examples 2.3, 2.4, 2.6 and 2.7 the dichotomy spectrum of the linear part is independent of the bifurcation parameter.

*Example* 2.3 (linear homogeneous equation). Suppose that $\alpha \in (-1, 1)$ and $\gamma$ are fixed nonzero reals. We consider the linear homogeneous difference equation

$$x_{k+1} = f_k(x_k,\lambda) := \begin{pmatrix} b_k & 0 \\ \lambda\gamma & c_k \end{pmatrix} x_k \tag{2.19}$$

depending on a parameter $\lambda \in \mathbb{R}$ with asymptotically constant sequences

$$b_k := \begin{cases} \alpha^{-1}, & k < 0, \\ \alpha, & k \geq 0, \end{cases} \qquad c_k := \begin{cases} \alpha, & k < 0, \\ \alpha^{-1}, & k \geq 0. \end{cases} \tag{2.20}$$

Since (2.19) is triangular, the dichotomy spectrum reads as $\Sigma(0,\lambda) = [\alpha, \frac{1}{\alpha}]$. It is easily seen that equation (2.19) fulfills $(H_0)$, $(H_1)$, $(H_3)$ and we set $\kappa = 0$, $\phi^* = 0$, $\lambda^* = 0$.

Furthermore, we can deduce from Exam. 2.1$(b_3)$ that also $(H_2)$ holds with $n = r = 1$ and the invariant projectors $P_k^+ \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $P_k^- \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. This yields

$$R(P_0^+) \cap N(P_0^-) = \mathbb{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad R(P_0^+) + N(P_0^-) = \mathbb{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and we can choose the vectors $\xi_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $\xi_1' = (0, 1)$. After these observations the linear functional $\mu_1 : \ell_0 \to \mathbb{R}$ from Lemma 2.10 computes as

$$\begin{aligned} \mu_1(\phi) &= \sum_{j\in\mathbb{Z}} \langle \xi_1' \Phi_0(0, j+1)', \phi_j \rangle \\ &= \sum_{j=-\infty}^{-2} \langle \xi_1' \Phi_0(0, j+1)', \phi_j \rangle + \langle \xi_1', \phi_{-1} \rangle + \sum_{j=0}^{\infty} \langle \xi_1' \Phi_0(0, j+1)', \phi_j \rangle \\ &\overset{(2.10)}{=} \sum_{j\in\mathbb{Z}} \alpha^{|j+1|} \phi_j^2, \end{aligned} \tag{2.21}$$

where $\phi = (\phi^1, \phi^2) \in \ell_0$. On the other hand we have

$$D_1 D_2 f_j(0,0)\Phi_0(j,0)\xi_1 = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} \Phi_0(j,0)\xi_1 \overset{(2.10)}{=} \gamma \begin{pmatrix} 0 \\ \alpha^{|j|} \end{pmatrix} \quad \text{for all } j \in \mathbb{Z}$$

and consequently arrive at

$$\sum_{j\in\mathbb{Z}} \langle \Phi_0(0, j+1)'\xi_1', D_1 D_2 f_j(0,0)\Phi_0(j,0)\xi_1 \rangle = \gamma \sum_{j\in\mathbb{Z}} \alpha^{|j+1|}\alpha^{|j|} = \frac{2\gamma\alpha}{1-\alpha^2} \neq 0.$$

Thus, Thm. 2.11 shows that the trivial solution bifurcates at $\lambda = 0$.

In order to illustrate this bifurcation scenario, we exploit the simple triangular structure of (2.19) and compute its general solution $\varphi_\lambda(\cdot; 0, \eta)$ for arbitrary initial values $\eta \in \mathbb{R}^2$. For the first component $\varphi_\lambda^1$ we obtain from (2.10) that

$$\varphi_\lambda^1(k; 0, \eta) = \alpha^{|k|}\eta_1 \quad \text{for all } k \in \mathbb{Z},\ \eta \in \mathbb{R}^2 \tag{2.22}$$

and thus $\varphi_\lambda^1(\cdot; 0, \eta) \in \ell_0$. The second component can be tackled using the variation of constants formula (cf. [1, p. 59]). Before doing this, for later use we establish the following elementary summation formulas

$$\begin{aligned} \sum_{n=0}^{k-1} \frac{\alpha^{mn}}{\alpha^{k-n-1}} &= \alpha \frac{\alpha^{mk} - \alpha^{-k}}{\alpha^{m+1} - 1} \quad \text{for all } m \in \mathbb{N}_0,\ k \in \mathbb{N}, \\ \sum_{n=k}^{-1} \frac{\alpha^{k-n-1}}{\alpha^{mn}} &= \alpha^m \frac{\alpha^{-mk} - \alpha^k}{\alpha^{m+1} - 1} \quad \text{for all } m \in \mathbb{N}_0,\ k < 0. \end{aligned} \tag{2.23}$$

Then the second component of $\varphi_\lambda$ reads as

$$\varphi_\lambda^2(k; 0, \eta) = \alpha^{-|k|}\eta_2 + \lambda\gamma \begin{cases} \sum_{n=0}^{k-1} \frac{1}{\alpha^{k-n-1}} \alpha^n \eta_1, & k \geq 0, \\ -\sum_{n=k}^{-1} \alpha^{k-n-1} \alpha^{-n} \eta_1, & k < 0 \end{cases}$$

and together with (2.23) we arrive at the asymptotic relation

$$\varphi_\lambda^2(k; 0, \eta) = \begin{cases} \alpha^{-k}\left(\eta_2 - \frac{\lambda\alpha\gamma}{\alpha^2-1}\eta_1\right) + o(1), & k \to \infty, \\ \alpha^k \left(\eta_2 + \frac{\lambda\alpha\gamma}{\alpha^2-1}\eta_1\right) + o(1), & k \to -\infty. \end{cases}$$

Consequently, for parameters $\lambda \neq 0$ the inclusion $\varphi_\lambda(\cdot; 0, \eta) \in \ell_0$ holds if and only if $\eta_2 = \frac{\lambda\alpha\gamma}{\alpha^2-1}\eta_1$ and $\eta_2 = -\frac{\lambda\alpha\gamma}{\alpha^2-1}\eta_1$, i.e. $\eta = (0, 0)$. In conclusion, 0 is the unique homoclinic solution to (2.19) for $\lambda \neq 0$, while in case $\lambda = 0$ the trivial solution $\phi^* = 0$ is embedded into a whole 1-parameter family of homoclinic solutions $s\Phi_0(\cdot, 0)\xi_1$ (see Fig. 3 (left)).

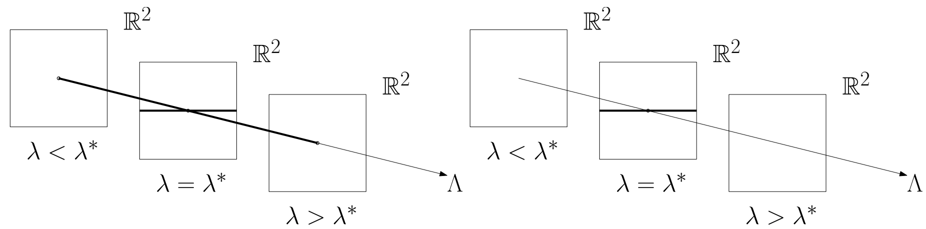


FIGURE 3. Left: Initial values $\eta \in \mathbb{R}^2$ yielding a homoclinic solution $\varphi_\lambda(\cdot; 0, \eta)$ of (2.19) for different parameter values $\lambda$.
Right: Initial values $\eta \in \mathbb{R}^2$ yielding a bounded solution $\varphi_\lambda(\cdot; 0, \eta)$ of (2.24) for different parameter values $\lambda$

*Remark* 2.3 (bifurcation diagrams). One should not understand the Figs. 3, 4 and 5 as bifurcation diagrams, i.e. subset of $\Lambda \times \ell$ indicating bounded solutions to $(\Delta)_\lambda$. Indeed, they are subsets of $\Lambda \times \mathbb{R}^2$ indicating initial values $\eta$ yielding bounded solutions $\varphi_\lambda(\cdot; 0, \eta)$ to the respective difference equations treated in the Examples 2.3, 2.4, 2.6 and 2.7.

**Theorem 2.12** (multiparameter bifurcation)**.** *Let $\Lambda \subseteq \mathbb{R}^n$ and $m \geq 2$. If $(H_0)$ to $(H_3)$ hold with $n = r$ and $\phi^* = 0$, then the trivial solution of a difference equation $(\Delta)_\lambda$ bifurcates at $\lambda^*$, provided there exists a $\hat{\xi} \in R(P_\kappa^+) \cap N(P_\kappa^-)$ such that*

$$\det\Big(\sum_{j\in\mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)'\xi_i', D_{\lambda_l} D_1 f_k(0, \lambda^*) \Phi_{\lambda^*}(j, \kappa)\hat{\xi}\rangle\Big)_{1\leq i,l\leq n} \neq 0.$$

*More precisely, there exist a $\rho > 0$ and open convex neighborhoods $U \subseteq \ell^\infty(\Omega)$ of 0, $\Lambda_0 \subseteq \Lambda$ of $\lambda^*$ and $C^{m-1}$-functions $\phi : (-\rho, \rho) \to U$, $\lambda : (-\rho, \rho) \to \Lambda_0$ with*

*(a) $\phi(0) = 0$, $\lambda(0) = \lambda^*$, $\dot{\phi}(0) = \Phi_{\lambda^*}(\cdot, \kappa)\hat{\xi}$,*
*(b) each $\phi(s)$ is a nontrivial complete solution of $(\Delta)_{\lambda(s)}$ in $\ell_0(\Omega)$ with*

$$\sum_{i=1}^n \langle \xi_i, \phi(s)_\kappa\rangle \Phi_{\lambda^*}(\cdot, \kappa)\xi_i = s\Phi_{\lambda^*}(\cdot, \kappa)\hat{\xi}.$$

*Proof.* First of all, we choose $\hat{x}_1 = \Phi_{\lambda^*}(\cdot, \kappa)\hat{\xi} \in \ell_0$ and thanks to Lemma 2.7 one obtains $\hat{x}_1 \in N(D_1G(0, \lambda^*))$. Also from Lemma 2.7 we know that $\Phi_{\lambda^*}(\kappa, \cdot + 1)'\xi_i'$, $1 \leq i \leq n$, is a basis of $N(L')$. Then the claim follows from Thm. A.6, which applies to the abstract operator equation (2.2) with $G : \ell(\Omega) \times \Lambda \to \ell$ for both $\ell = \ell^\infty$ and $\ell = \ell_0$. In case $\ell = \ell^\infty$ we obtain a neighborhood $U \subseteq \ell^\infty(\Omega)$ and for $\ell = \ell_0$ we see that the complete solutions $\phi(s)$, $s \in S$, are homoclinic to 0, i.e. $\phi(s) \in \ell_0(\Omega)$. Concerning assertion (b), the explicit form of the projection $P$ is given in Lemma 2.8. □

More specific results can be obtained for index 0 Fredholm operators with 1-dimensional kernel. For our first result we need not to impose a trivial solution branch as in $(H_3)$.

**Theorem 2.13** (fold bifurcation)**.** *Let $\Lambda \subseteq \mathbb{R}$ and also suppose that $(H_0)$, $(H_2)$ hold with $n = r = 1$. If $\ell = \ell^\infty$ and*

$$g_{01} := \sum_{j\in\mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)'\xi_1', D_2 f_j(\phi_j^*, \lambda^*)\rangle \neq 0,$$

*then there exists a $\rho > 0$, open convex neighborhoods $U \subseteq \ell^\infty(\Omega)$ of $\phi^*$, $\Lambda_0 \subseteq \Lambda$ of $\lambda^*$ and $C^m$-functions $\phi : (-\rho, \rho) \to U$, $\lambda : (-\rho, \rho) \to \Lambda_0$ such that*

*(a) $\phi(0) = \phi^*$, $\lambda(0) = \lambda^*$ and $\dot{\phi}(0) = \Phi_{\lambda^*}(\cdot, \kappa)\xi_1$, $\dot{\lambda}(0) = 0$,*

*(b) each $\phi(s)$ is a complete solution of $(\Delta)_{\lambda(s)}$ in $\ell(\Omega)$.*

*Moreover, in case $m \geq 2$ and under the additional assumption*

$$g_{20} := \sum_{j\in\mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)'\xi_1', D_1^2 f_j(\phi_j^*, \lambda^*)[\Phi_{\lambda^*}(j,\kappa)\xi_1]^2\rangle \neq 0,$$

*the solution $\phi^* \in \ell(\Omega)$ of $(\Delta)_{\lambda^*}$ bifurcates at $\lambda^*$, one has $\ddot{\lambda}(0) = -\frac{g_{20}}{g_{01}}$ and the following holds locally in $U \times \Lambda_0$:*

*(c)* Subcritical case*: If $g_{20}/g_{01} > 0$, then $(\Delta)_\lambda$ has no complete solution in $\ell^\infty(\Omega)$ for $\lambda > \lambda^*$, $\phi^*$ is the unique complete solution of $(\Delta)_{\lambda^*}$ in $\ell^\infty(\Omega)$ and $(\Delta)_\lambda$ has exactly two distinct complete bounded solutions for $\lambda < \lambda^*$; they are in $\ell(\Omega)$.*

*(d)* Supercritical case*: If $g_{20}/g_{01} < 0$, then $(\Delta)_\lambda$ has no complete solution in $\ell^\infty(\Omega)$ for $\lambda < \lambda^*$, $\phi^*$ is the unique complete solution of $(\Delta)_{\lambda^*}$ in $\ell^\infty(\Omega)$ and $(\Delta)_\lambda$ has exactly two distinct complete bounded solutions for $\lambda > \lambda^*$; they are in $\ell(\Omega)$.*

*If $(H_0)$ to $(H_2)$ are satisfied, then the same holds with $\ell = \ell_0$.*

*Proof.* Our strategy is to apply Thm. A.2 with $x_1 = \Phi_{\lambda^*}(\cdot, \kappa)\xi_1$ to (2.2). By Thm. 2.1(a) the mapping $G : \ell^\infty(\Omega)^\circ \times \Lambda \to \ell^\infty$ is of class $C^m$ and we have $G(\phi^*, \lambda^*) = 0$. With the functional $\mu = \mu_1$ from Lemma 2.10 we calculate $-\mu(D_2 G(\phi^*, \lambda^*)) = g_{01}$ and Thm. A.2 guarantees a solution curve $\gamma = (\gamma_1, \gamma_2)$ for (2.2); we define $\phi := \gamma_1$ and $\lambda := \gamma_2$. For at least $C^2$-smooth right-hand sides $f_k$ we also get $-\mu(D_1^2 G(x_0, \lambda_0)x_1^2) = g_{20}$ and the claim follows in case $\ell = \ell^\infty$. Yet, under $(H_1)$ the above arguments also apply to $G : \ell_0(\Omega) \times \Lambda \to \ell_0$ and the bounded bifurcating solutions are actually in $\ell_0(\Omega)$. □

*Example* 2.4 (inhomogeneous equations). Let $\alpha \in (-1, 1)$ and $\gamma, \delta$ be fixed nonzero reals. We initially consider the linear inhomogeneous difference equation

$$x_{k+1} = f_k(x_k, \lambda) := \begin{pmatrix} b_k & 0 \\ 0 & c_k \end{pmatrix} x_k + \lambda \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \tag{2.24}$$

depending on a bifurcation parameter $\lambda \in \mathbb{R}$ and sequences $b_k, c_k$ defined in (2.20). As in the previous Exam. 2.3 we see that (2.24) fulfills $(H_0)$ and $(H_2)$ with $\kappa = 0$, $\phi^* = 0$, $\lambda^* = 0$ and $\xi_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Moreover, due to $D_2 f_j(0,0) \equiv \left(\begin{smallmatrix} 0 \\ \gamma \end{smallmatrix}\right)$ one obtains from (2.21) that

$$g_{01} = \sum_{j\in\mathbb{Z}} \alpha^{|j+1|}\gamma = \gamma\frac{1+\alpha}{1-\alpha} \neq 0.$$

A detailed picture can be obtained using the general solution $\varphi_\lambda(\cdot; 0, \eta)$ for (2.24). Its first component $\varphi_\lambda^1$ is given by (2.22), while the second component follows from the variation of constants formula (cf. [1, p. 59]). Using the relations (2.23) one deduces

$$\varphi_\lambda^2(k; 0, \eta) = \begin{cases} \alpha^{-k}\left(\eta_2 - \frac{\lambda\alpha\gamma}{\alpha-1}\right) + O(1), & k \to \infty, \\ \alpha^k\left(\eta_2 + \frac{\lambda\gamma}{\alpha-1}\right) + O(1), & k \to -\infty, \end{cases}$$

and thus the inclusion $\varphi_\lambda(\cdot; 0, \eta) \in \ell^\infty$ holds if and only if both conditions $\eta_2 = \frac{\lambda\alpha\gamma}{\alpha-1}$ and $\eta_2 = -\frac{\lambda\gamma}{\alpha-1}$ are satisfied. This, however, is not possible unless $\lambda = 0$. In conclusion, there exists no bounded complete solution of (2.24) for $\lambda \neq 0$, while there is a 1-parameter family of bounded solutions in case $\lambda = 0$. In the terminology introduced in Thm. 2.13 this means $\phi(s) = s\Phi_0(\cdot, 0)\xi_1 = s\left(\begin{smallmatrix} \alpha^{|\cdot|} \\ 0 \end{smallmatrix}\right)$ and $\lambda(s) \equiv 0$ for all $s \in \mathbb{R}$. Also the zero solution to (2.24) bifurcates at $\lambda = 0$.

It is understood that the linear equation (2.24) does not fulfill the condition $g_{20} \neq 0$ yielding a fold. In fact, the situation becomes more interesting, if we consider the following nonlinear perturbation of (2.24) defined as

$$x_{k+1} = f_k(x_k, \lambda) := \begin{pmatrix} b_k & 0 \\ 0 & c_k \end{pmatrix} x_k + \delta \begin{pmatrix} 0 \\ (x_k^1)^2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \tag{2.25}$$

with $x_k = (x_k^1, x_k^2)$. Our hypotheses $(H_0)$ and $(H_2)$ are fulfilled with the data given above. Moreover, the relation $D_1^2 f_j(0,0)\zeta^2 = \begin{pmatrix} 0 \\ 2\delta\zeta_1^2 \end{pmatrix}$ for all $j \in \mathbb{Z}$, $\zeta \in \mathbb{R}^2$ and (2.21) ensure

$$g_{20} = 2\delta \frac{\alpha + \alpha^2}{1 - \alpha^3} \neq 0.$$

Hence, Thm. 2.13 implies that the trivial solution to (2.25) has a fold bifurcation at $\lambda = 0$. Due to $\frac{g_{20}}{g_{01}} = 2\frac{\delta}{\gamma}\frac{\alpha}{1+\alpha+\alpha^2}$ it is subcritical for $\alpha\frac{\delta}{\gamma} > 0$ and supercritical for $\alpha\frac{\delta}{\gamma} < 0$.

We are able to verify this statement explicitly using the general solution $\varphi_\lambda(\cdot; 0, \eta)$ to (2.25). Its first component $\varphi_\lambda^1$ has been computed in (2.22), while the variation of constants formula (cf. [1, p. 59]) and (2.23) can be used to deduce the asymptotic representation

$$\varphi_\lambda^2(k; 0, \eta) = \begin{cases} \alpha^{-k}\left(\eta_2 - \frac{\delta\alpha}{\alpha^3-1}\eta_1^2 - \frac{\lambda\alpha\gamma}{\alpha-1}\right) + O(1), & k \to \infty, \\ \alpha^{k}\left(\eta_2 + \frac{\delta\alpha^2}{\alpha^3-1}\eta_1^2 + \frac{\lambda\gamma}{\alpha-1}\right) + O(1), & k \to -\infty. \end{cases}$$

Therefore, the sequence $\varphi_\lambda(\cdot; 0, \eta)$ is bounded if and only if $\eta_2 = \frac{\delta\alpha}{\alpha^3-1}\eta_1^2 + \frac{\lambda\alpha\gamma}{\alpha-1}$ and $\eta_2 = -\frac{\delta\alpha^2}{\alpha^3-1}\eta_1^2 - \frac{\lambda\gamma}{\alpha-1}$ holds, i.e. $\eta_1^2 = -\frac{\alpha^2+\alpha+1}{\alpha}\frac{\gamma}{\delta}\lambda$, $\eta_2 = \gamma\lambda$. From the first relation we see that there exist two bounded solutions, if $\alpha\frac{\delta}{\gamma}\lambda < 0$, the trivial solution is the unique bounded solution for $\lambda = 0$ and there are no bounded solutions for $\alpha\frac{\delta}{\gamma}\lambda > 0$. This perfectly corresponds to the above fold bifurcation pattern deduced from Thm. 2.13 and we refer to Fig. 4 (left) for an illustration.

The method of explicit solutions can also be applied to the nonlinear difference equation

$$x_{k+1} = f_k(x_k, \lambda) := \begin{pmatrix} b_k & 0 \\ 0 & c_k \end{pmatrix} x_k + \delta \begin{pmatrix} 0 \\ (x_k^1)^3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ \gamma \end{pmatrix}, \tag{2.26}$$

where the condition $g_{20} \neq 0$ is violated. However, using the variation of constants formula (cf. [1, p. 59]) and (2.23) we can show that the crucial second component of the general solution $\varphi_\lambda(\cdot; 0, \eta)$ for (2.26) fulfills

$$\varphi_\lambda^2(k; 0, \eta) = \begin{cases} \alpha^{-k}\left(\eta_2 - \frac{\delta\alpha}{\alpha^4-1}\eta_1^3 - \frac{\lambda\alpha\gamma}{\alpha-1}\right) + O(1), & k \to \infty, \\ \alpha^{k}\left(\eta_2 + \frac{\delta\alpha^3}{\alpha^4-1}\eta_1^3 + \frac{\lambda\gamma}{\alpha-1}\right) + O(1), & k \to -\infty. \end{cases}$$

Since the first component is given in (2.22), we see that $\varphi_\lambda(\cdot; 0, \eta)$ is bounded if and only if $\eta_2 = \frac{\delta\alpha}{\alpha^4-1}\eta_1^3 + \frac{\lambda\alpha\gamma}{\alpha-1}$ and $\eta_2 = -\frac{\delta\alpha^3}{\alpha^4-1}\eta_1^3 - \frac{\lambda\gamma}{\alpha-1}$, which in turn is equivalent to

$$\eta_1 = \sqrt[3]{-\frac{(\alpha+1)^2}{\alpha}\frac{\gamma}{\delta}\lambda}, \qquad \eta_2 = \frac{\alpha^2+\alpha+1}{\alpha^2+1}\gamma\lambda.$$

Hence, these particular initial values $\eta \in \mathbb{R}^2$ given by the cusp shaped curve depicted in Fig. 4 (right) lead to bounded complete solutions of (2.26). As opposed to the linear equation (2.24), note that the trivial solution of (2.26) does not bifurcate at $\lambda = 0$.

A more general situation is captured in

*Example* 2.5 (perturbed planar equations). Assume that $m \geq 2$. We consider a nonautonomous difference equation $(\Delta)_\lambda$, whose right-hand side $f_k : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is of class $C^m$ and supposed to satisfy $(H_0)$ such that one has the following assumptions:

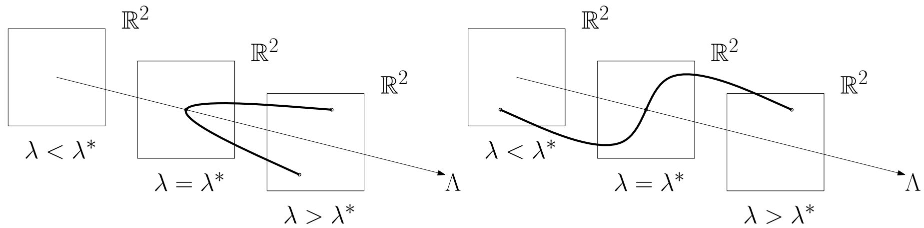


FIGURE 4. Left (supercritical fold): Initial values $\eta \in \mathbb{R}^2$ yielding a bounded solution $\varphi_\lambda(\cdot; 0, \eta)$ of (2.25) for different parameter values $\lambda$. Right (cusp): Initial values $\eta \in \mathbb{R}^2$ yielding a bounded solution $\varphi_\lambda(\cdot; 0, \eta)$ of (2.26) for different parameter values $\lambda$

(i) the homogeneous part $f_k(0, \lambda) = \begin{pmatrix} h_k^+(\lambda) \\ h_k^-(\lambda) \end{pmatrix}$ satisfies $h_k^\pm(0) \equiv 0$ on $\mathbb{Z}$

(ii) the linear part is given by

$$D_1 f_k(0, \lambda) = \begin{pmatrix} b_k(\lambda) & \lambda \\ \lambda & c_k(\lambda) \end{pmatrix}, \quad b_k(0) = \begin{cases} \beta_-, & k < 0, \\ \beta_+, & k \geq 0, \end{cases} \quad c_k(0) = \begin{cases} \gamma_-, & k < 0, \\ \gamma_+, & k \geq 0 \end{cases}$$

(ii) the higher-order Taylor coefficients allow the representation

$$D_1^n f_k(0, 0)\zeta^n = \sum_{i=2}^{n} \sum_{l=0}^{i} \zeta_1^l \zeta_2^{i-l} \begin{pmatrix} F_{l,i-l}^+(k) \\ F_{l,i-l}^-(k) \end{pmatrix} \quad \text{for all } \zeta \in \mathbb{R}^2,\ 2 \leq n \leq m.$$

Consequently, for $\lambda^* = 0$ our equation $(\Delta)_0$ has the trivial solution $\phi^* = 0$. In order to fulfill Hypothesis $(H_2)$ with $\kappa = 0$, we focus on parameters satisfying

$$\begin{aligned} &(b_3) \quad |\beta_+| < 1 < |\gamma_+|, \ |\gamma_-| < 1 < |\beta_-|, \\ &(c_2) \quad |\gamma_+| < 1 < |\beta_+|, \ |\beta_-| < 1 < |\gamma_-|, \end{aligned} \tag{2.27}$$

respectively (cf. Exam. 2.1), and arrive at:

$(b_3)$ We have $P_k^+ \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $P_k^- \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, choose $\xi_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $\xi_1' = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ and get

$$\begin{aligned} g_{01} &= \sum_{j=-\infty}^{-2} \dot{h}_j^-(0)\gamma_-^{-j-1} + \sum_{j=-1}^{\infty} \dot{h}_j^-(0)\gamma_+^{-j-1}, \\ g_{20} &= \frac{1}{\gamma_-} \sum_{j=-\infty}^{-2} F_{2,0}^-(j) \left(\frac{\beta_-^2}{\gamma_-}\right)^j + \frac{F_{2,0}^-(-1)}{\beta_-^2} + \frac{1}{\gamma_+} \sum_{j=0}^{\infty} F_{2,0}^-(j) \left(\frac{\beta_+^2}{\gamma_+}\right)^j \end{aligned} \tag{2.28}$$

$(c_2)$ We have $P_k^+ \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $P_k^- \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, choose $\xi_1 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, $\xi_1' = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and get

$$\begin{aligned} g_{01} &= \sum_{j=-\infty}^{-2} \dot{h}_j^+(0)\beta_-^{-j-1} + \sum_{j=-1}^{\infty} \dot{h}_j^+(0)\beta_+^{-j-1}, \\ g_{20} &= \frac{1}{\beta_-} \sum_{j=-\infty}^{-2} F_{0,2}^+(j) \left(\frac{\gamma_-^2}{\beta_-}\right)^j + \frac{F_{0,2}^+(-1)}{\gamma_-^2} + \frac{1}{\beta_+} \sum_{j=0}^{\infty} F_{0,2}^+(j) \left(\frac{\gamma_+^2}{\beta_+}\right)^j \end{aligned} \tag{2.29}$$

Thus, in case $g_{01} \neq 0$ and $g_{20} \neq 0$ the above Thm. 2.13 yields a family of bounded complete solutions for $(\Delta)_\lambda$ and $\lambda$ close to 0. More detailed,

- *Subcritical case* $(g_{20}/g_{01} > 0)$: Equation $(\Delta)_\lambda$ has no bounded complete solution for $\lambda > 0$, 0 is the unique bounded complete solution for $\lambda = 0$ and $(\Delta)_\lambda$ has two distinct bounded complete solutions for $\lambda < 0$.

- *Supercritical case* ($g_{20}/g_{01} < 0$): Equation $(\Delta)_\lambda$ has no bounded complete solution for $\lambda < 0$, $0$ is the unique bounded complete solution for $\lambda = 0$ and $(\Delta)_\lambda$ has two distinct bounded complete solutions for $\lambda > 0$.

Now we return to the situation under hypothesis $(H_3)$, where $(\Delta)_\lambda$ has a trivial solution branch. Since smoothness of $G$ and Fredholm properties of its derivative $L$ are independent of the space $\ell = \ell^\infty$ or $\ell = \ell_0$, one can apply the abstract bifurcation criteria from Subsect. A.3 twice. This yields uniqueness of the bifurcating solutions in the large sequence space $\ell^\infty$ and existence in the smaller space $\ell_0$.

**Theorem 2.14** (bifurcation from known solutions)**.** *Let $\Lambda \subseteq \mathbb{R}$ and $m \geq 2$. If $(H_0)$ to $(H_3)$ hold with $n = r = 1$, $\phi^* = 0$ and the* transversality condition

$$g_{11} := \sum_{j\in\mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)'\xi_1', D_1D_2f_j(0,\lambda^*)\Phi_{\lambda^*}(j,\kappa)\xi_1\rangle \neq 0 \tag{2.30}$$

*is satisfied, then the trivial solution of a difference equation $(\Delta)_\lambda$ bifurcates at $\lambda^*$. In particular, there exists a $\rho > 0$, open convex neighborhoods $U \subseteq \ell^\infty(\Omega)$ of $0$, $\Lambda_0 \subseteq \Lambda$ of $\lambda^*$ and $C^{m-1}$-functions $\phi : (-\rho, \rho) \to U$, $\lambda : (-\rho,\rho) \to \Lambda_0$ with*

*(a) $\phi(0) = 0$, $\lambda(0) = \lambda^*$ and $\dot\phi(0) = \Phi_{\lambda^*}(\cdot, \kappa)\xi_1$,*
*(b) each $\phi(s)$ is a nontrivial solution of $(\Delta)_{\lambda(s)}$ homoclinic to $0$.*

*Proof.* We will apply the first part of Thm. A.3 or A.4 to the map $G : \ell(\Omega)\times\Lambda \to \ell$. First of all, our assumptions imply the trivial solution branch $G(0,\lambda) \equiv 0$ on $\Lambda$. By Lemma 2.7 the kernel $N(L)$ is spanned by the sequence $\Phi_{\lambda^*}(\cdot,\kappa)\xi_1 \in \ell_0$ and so the transversality condition (2.30) guarantees $\mu(D_1D_2G(0,\lambda^*)\Phi_{\lambda^*}(\cdot,\kappa)\xi_1) \neq 0$ with the functional $\mu = \mu_1$ from Lemma 2.10. Consequently, for $\ell = \ell^\infty$ our Thms. A.3 or A.4 guarantee a nontrivial solution curve $\gamma = (\gamma_1, \gamma_2)$ for (2.2), we set $\phi := \gamma_1$, $\lambda := \gamma_2$ and each $\phi(\lambda)$ is a bounded complete solution of $(\Delta)_\lambda$. Since the above argument is independent, whether one chooses $\ell = \ell_0$ or $\ell = \ell^\infty$, it is possible to take $U$ as a neighborhood in the large space $\ell^\infty(\Omega)$ and to verify $\phi(\lambda)$ as solution in the smaller space $\ell_0(\Omega)$. □

**Corollary 2.15** (transcritical bifurcation)**.** *Under the additional assumption*

$$g_{20} := \sum_{j\in\mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)'\xi_1', D_1^2f_j(0,\lambda^*)[\Phi_{\lambda^*}(j,\kappa)\xi_1]^2\rangle \neq 0$$

*one has $\dot\lambda(0) = -\frac{g_{20}}{2g_{11}}$ and the following holds locally in $U \times \Lambda_0$: A difference equation $(\Delta)_\lambda$ has a unique nontrivial complete bounded solution $\phi_\lambda$ for $\lambda \neq \lambda^*$ and $0$ is the unique complete bounded solution of $(\Delta)_{\lambda^*}$; moreover, $\phi_\lambda \in \ell_0(\Omega)$.*

*Proof.* Thanks to $g_{20} = -\mu(D_1^2F(0,\lambda^*)[\Phi_{\lambda^*}(\cdot,\kappa)\xi_1]^2) \neq 0$ the claim is immediately implied by Thm. A.3. □

*Example* 2.6. Let $\alpha \in (-1, 1)$ and $\gamma, \delta$ be fixed nonzero reals. We consider the nonlinear difference equation

$$x_{k+1} = f_k(x_k, \lambda) := \begin{pmatrix} b_k & 0 \\ \lambda\gamma & c_k \end{pmatrix} x_k + \delta \begin{pmatrix} 0 \\ (x_k^1)^2 \end{pmatrix} \tag{2.31}$$

depending on a bifurcation parameter $\lambda \in \mathbb{R}$ and sequences $b_k, c_k$ defined in (2.20). As in our previous examples we see that $(H_0)$ to $(H_3)$ hold with $\kappa = 0$, $\phi^* = 0$, $\lambda^* = 0$ and

$$g_{11} = \frac{2\gamma\alpha}{1-\alpha^2} \neq 0, \qquad g_{20} = 2\delta\frac{\alpha+\alpha^2}{1-\alpha^3} \neq 0.$$

Hence, we are able to employ Cor. 2.15 in order to see that the trivial solution of (2.31) has a transcritical bifurcation at $\lambda = 0$.

Again, we can describe this bifurcation quantitatively. While the first component of the general solution $\varphi_\lambda(\cdot; 0, \eta)$ given by (2.22) is homoclinic, the second component fulfills

$$\varphi_\lambda^2(k; 0, \eta) = \begin{cases} \alpha^{-k}\left(\eta_2 - \frac{\delta\alpha}{\alpha^3-1}\eta_1^2 - \frac{\lambda\alpha\gamma}{\alpha^2-1}\eta_1\right) + o(1), & k \to \infty, \\ \alpha^k\left(\eta_2 + \frac{\delta\alpha^2}{\alpha^3-1}\eta_1^2 + \frac{\lambda\alpha\gamma}{\alpha^2-1}\eta_1\right) + o(1), & k \to -\infty; \end{cases}$$

in conclusion, from this we see that $\varphi_\lambda(\cdot; 0, \eta)$ is bounded if and only if $\eta = (0, 0)$ or

$$\eta_1 = -2\frac{\alpha^2+\alpha+1}{(\alpha+1)^2}\frac{\gamma}{\delta}\lambda, \qquad \eta_2 = -2\frac{\alpha(\alpha^2+\alpha+1)}{(\alpha+1)^4}\frac{\gamma^2}{\delta}\lambda^2.$$

Hence, besides the zero solution we have a unique nontrivial complete solution passing through the initial point $\eta = (\eta_1, \eta_2)$ at time $k = 0$ for $\lambda \neq 0$. This means the bifurcation pattern sketched in Fig. 5 (left) holds.

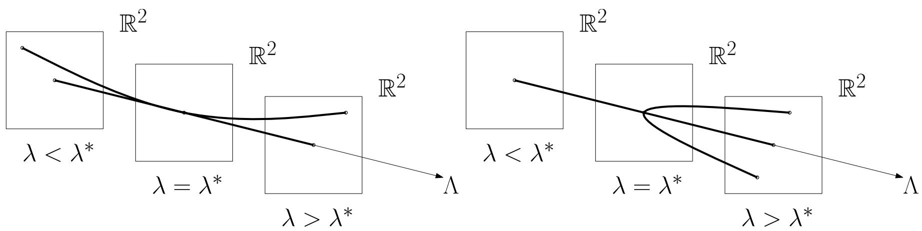


FIGURE 5. Left (transcritical): Initial values $\eta \in \mathbb{R}^2$ yielding a homoclinic solution $\varphi_\lambda(\cdot; 0, \eta)$ of (2.31) for different parameter values $\lambda$. Right (supercritical pitchfork): Initial values $\eta \in \mathbb{R}^2$ yielding a homoclinic solution $\varphi_\lambda(\cdot; 0, \eta)$ of (2.32) for different parameter values $\lambda$

**Corollary 2.16** (pitchfork bifurcation)**.** *For $m \geq 3$ and under the additional assumptions*

$$\sum_{j\in\mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)'\xi_1', D_1^2 f_j(0, \lambda^*)[\Phi_{\lambda^*}(j, \kappa)\xi_1]^2 \rangle = 0,$$

$$g_{30} := \sum_{j\in\mathbb{Z}} \langle \Phi_{\lambda^*}(\kappa, j+1)'\xi_1', D_1^3 f_j(0, \lambda^*)[\Phi_{\lambda^*}(j, \kappa)\xi_1]^3 \rangle \neq 0$$

*one has $\dot\lambda(0) = 0$, $\ddot\lambda(0) = -\frac{g_{30}}{3g_{11}}$ and the following holds locally in $U \times \Lambda_0$:*

*(c)* Subcritical case*: If $g_{30}/g_{11} > 0$, then the unique complete bounded solution of $(\Delta)_\lambda$ is the trivial one for $\lambda \geq \lambda^*$ and $(\Delta)_\lambda$ has exactly two nontrivial complete complete solutions for $\lambda < \lambda^*$; both are homoclinic to $0$.*

*(d)* Supercritical case*: If $g_{30}/g_{11} < 0$, then the unique complete bounded solution of $(\Delta)_\lambda$ is the trivial one for $\lambda \leq \lambda^*$ and $(\Delta)_\lambda$ has exactly two nontrivial complete solutions for $\lambda > \lambda^*$; both are homoclinic to $0$.*

*Proof.* Since our assumptions imply $\mu(D_1^2 F(0, \lambda^*)[\Phi_{\lambda^*}(\cdot, \kappa)\xi_1]^2) = 0$ and the condition $g_{30} = -\mu(D_1^3 F(0, \lambda^*)[\Phi_{\lambda^*}(\cdot, \kappa)\xi_1]^3) \neq 0$ holds, the claim follows from Thm. A.4. □

*Example* 2.7. Let us again suppose that $\alpha \in (-1, 1)$ and $\gamma, \delta$ are fixed nonzero reals. Here, we consider the nonlinear difference equation

$$x_{k+1} = f_k(x_k, \lambda) := \begin{pmatrix} b_k & 0 \\ \lambda\gamma & c_k \end{pmatrix} x_k + \delta \begin{pmatrix} 0 \\ (x_k^1)^3 \end{pmatrix} \tag{2.32}$$

depending on a bifurcation parameter $\lambda \in \mathbb{R}$ and the $b_k, c_k$ defined in (2.20). As in our above Exam. 2.6 the assumptions of Cor. 2.16 are fulfilled with $\kappa = 0$, $\phi^* = 0$ and $\lambda^* = 0$. The transversality condition reads as $g_{11} = \frac{2\alpha\gamma}{1-\alpha^2} \neq 0$. Moreover, $D_1^2 f_j(0,0) \equiv 0$ on $\mathbb{Z}$ implies $g_{20} = 0$, whereas the relation $D_1^3 f_j(0,0)\zeta^3 = \binom{0}{6\delta\zeta_1^3}$ for all $j \in \mathbb{Z}$, $\zeta \in \mathbb{R}^2$ and (2.21) leads to $g_{30} = 6\delta\frac{\alpha}{1-\alpha^2} \neq 0$; having this at our disposal, we arrive at the crucial quotient $\frac{g_{30}}{g_{11}} = 3\frac{\delta}{\gamma}$. By Cor. 2.16 one deduces a subcritical (supercritical) pitchfork bifurcation of the trivial solution to (2.32) at $\lambda = 0$, provided $\frac{\delta}{\gamma} > 0$ (resp. $\frac{\delta}{\gamma} < 0$).

Anew we will illustrate this result using the general solution $\varphi_\lambda(\cdot; 0, \eta)$ to (2.32). As above, the first component is given by (2.22) and the sums (2.23) help us to compute for the second component that

$$\varphi_\lambda^2(k;0,\eta) = \begin{cases} \alpha^{-k}\left(\eta_2 - \frac{\delta\alpha}{\alpha^4-1}\eta_1^3 - \frac{\lambda\alpha\gamma}{\alpha^2-1}\eta_1\right) + o(1), & k \to \infty, \\ \alpha^{k}\left(\eta_2 + \frac{\delta\alpha^3}{\alpha^4-1}\eta_1^3 + \frac{\lambda\alpha\gamma}{\alpha^2-1}\eta_1\right) + o(1), & k \to -\infty. \end{cases}$$

This asymptotic representation shows us that $\varphi_\lambda(\cdot; 0, \eta) \in \ell_0$ holds if and only if $\eta = 0$ or $\eta_1^2 = -2\frac{\gamma}{\delta}\lambda$ and $\eta_2 = -2\frac{\alpha}{\alpha^4-1}\frac{(\delta\alpha^2+4\lambda\gamma+\delta)\gamma^2}{\delta^2}\lambda^2$. Hence, we have a correspondence to the pitchfork bifurcation described in Cor. 2.16. An illustration is given in Fig. 5 (right).

*Example* 2.8 (unperturbed planar equations). Let us return to the planar difference equations $(\Delta)_\lambda$ studied in Exam. 2.5, but now with identically vanishing homogeneous part $h_k^\pm(\lambda) \equiv 0$. Then $(\Delta)_\lambda$ has a branch of trivial solutions, $(H_2)$ holds and we obtain

$$D_1 D_2 f_k(0,0)\zeta = \begin{pmatrix} \dot b_k(0)\zeta_1 + \zeta_2 \\ \zeta_1 + \dot c_k(0)\zeta_2 \end{pmatrix} \quad \text{for all } \zeta \in \mathbb{R}^2;$$

note that $D_1^2 f_k(0,0)[\zeta]^2$ has already been computed in Exam. 2.5. With the respective parameter constellations from (2.27), we get

$(b_3)$ $g_{20}$ is given by (2.28), $g_{11} = \frac{1}{\beta_-}\left(\frac{\gamma_-}{\beta_- - \gamma_-} + 1\right) + \frac{1}{\gamma_+ - \beta_+}$ and

$$g_{30} = \frac{1}{\gamma_-}\sum_{j=-\infty}^{-2} F_{3,0}^-(j)\left(\frac{\beta_-^2}{\gamma_-}\right)^j + \frac{F_{3,0}^-(-1)}{\beta_-^2} + \frac{1}{\gamma_+}\sum_{j=0}^{\infty} F_{3,0}^-(j)\left(\frac{\beta_+^2}{\gamma_+}\right)^j,$$

$(c_2)$ $g_{20}$ is given by (2.29), $g_{11} = \frac{1}{\gamma_-}\left(\frac{\beta_-}{\gamma_- - \beta_-} + 1\right) + \frac{1}{\beta_+ - \gamma_+}$ and

$$g_{30} = \frac{1}{\beta_-}\sum_{j=-\infty}^{-2} F_{3,0}^+(j)\left(\frac{\gamma_-^2}{\beta_-}\right)^j + \frac{F_{3,0}^+(-1)}{\gamma_-^2} + \frac{1}{\beta_+}\sum_{j=0}^{\infty} F_{3,0}^+(j)\left(\frac{\gamma_+^2}{\beta_+}\right)^j$$

and in particular the transversality condition (2.30) is satisfied with $g_{11} > 0$. Thus, the above Thm. 2.14 guarantees that the trivial solution of $(\Delta)_\lambda$ bifurcates at $\lambda^* = 0$.

More precisely, under the assumption $g_{20} \neq 0$ we are able to deduce a transcritical bifurcation from Cor. 2.15, i.e. in a neighborhood of 0, the only complete bounded solution of $(\Delta)_\lambda$ is the trivial one for $\lambda = 0$, and for $\lambda \neq 0$ there exists a unique branch of solutions homoclinic to 0, which depends smoothly on the parameter $\lambda$.

In the nongeneric situation $g_{20} = 0$ and $g_{30} \neq 0$, Cor. 2.16 yields a pitchfork bifurcation of the zero solution, i.e. in a neighborhood of 0 one has:

- *Subcritical case* ($\frac{g_{30}}{g_{11}} > 0$): The unique bounded complete solution of $(\Delta)_\lambda$ is the trivial one for $\lambda \geq 0$ and for $\lambda < 0$ two distinct solutions homoclinic to 0 bifurcate.
- *Supercritical case* ($\frac{g_{30}}{g_{11}} < 0$): The unique bounded complete solution of $(\Delta)_\lambda$ is the trivial one for $\lambda \leq 0$ and for $\lambda > 0$ two distinct solutions homoclinic to 0 bifurcate from the trivial branch.

3. **Ordinary differential equations.** The above results on nonautonomous difference equations are applicable to a large class of evolutionary differential equations depending on parameters $\lambda$, which are well-posed in the sense that they generate a nonlinear 2-parameter semiflow $S_\lambda(t,s)$, $s \leq t$, on a reflexive Banach space $X$ and in particular a Hilbert space. Indeed, fixing a real sequence $(t_k)_{k\in\mathbb{Z}}$ with $t_k < t_{k+1}$ for $k \in \mathbb{Z}$ and $\lim_{k\to\pm\infty} t_k = \pm\infty$ we simply have to define $f_k(x,\lambda) := S_\lambda(t_{k+1}, t_k)x$. Nevertheless, in order to illustrate the continuous time case as well, we now discuss the corresponding theory for ordinary differential equations in Banach spaces.

Suppose that $O \subseteq X$ is a nonempty open set. Our functional-analytical approach is based on the following spaces: The continuous bounded functions $\phi : \mathbb{R} \to O$ are denoted by $BC(\mathbb{R}, O)$, $BC_0(\mathbb{R}, O)$ are such functions satisfying $\lim_{t\to\pm\infty} \phi(t) = 0$ (if $0 \in O$) and we consider these sets as subspaces of $BC := BC(\mathbb{R}, X)$ equipped with the norm

$$\|\phi\|_0 := \sup_{t\in\mathbb{R}} |\phi(t)| .$$

Moreover, we write $BC^1(\mathbb{R}, O)$ for the $C^1$-functions $\phi : \mathbb{R} \to O$ with $\phi, \dot\phi \in BC$ and the set $BC_0^1(\mathbb{R}, O)$ consists of such functions satisfying $\phi, \dot\phi \in BC_0$ (if $0 \in O$). We consider $BC^1(\mathbb{R}, O), BC_0^1(\mathbb{R}, O)$ as subspaces of $BC^1 := BC^1(\mathbb{R}, O)$ equipped with the norm

$$\|\phi\|_1 := \max\left\{ \|\phi\|_0 , \|\dot\phi\|_0 \right\} .$$

Let us assume $\Omega \subseteq X$ and $\Lambda \subseteq Y$ are nonempty open convex subsets. With a right-hand side $f : \mathbb{R} \times \Omega \times \Lambda \to X$ we consider a nonautonomous ordinary differential equation

$$\boxed{\dot u = f(t, u, \lambda)} \qquad (D)_\lambda$$

depending on a parameter $\lambda \in \Lambda$. Our bifurcation notion is based on the concept of a *complete* or *entire solution* to $(D)_\lambda$; this is a $C^1$-function $\phi : \mathbb{R} \to X$ with $\phi(t) \in \Omega$ satisfying the *solution identity* $\dot\phi(t) \equiv f(t, \phi(t), \lambda)$ on the real axis $\mathbb{R}$. In case $\phi \in BC$ we speak of a *bounded complete solution*. Furthermore, as in the discrete case a *permanent solution* of $(D)_\lambda$ is supposed to satisfy

$$\inf_{t\in\mathbb{R}} \operatorname{dist}(\phi(t), \Omega) > 0.$$

For $0 \in \Omega$ a complete solution $\phi$ with $\lim_{t\to\pm\infty} \phi(t) = 0$ is called *homoclinic* to 0.

The subsequent hypotheses guarantee existence and uniqueness of solutions for $(D)_\lambda$.

**Hypothesis.** *Let $m \in \mathbb{N}$, suppose $f : \mathbb{R} \times \Omega \times \Lambda \to X$ is continuous and the partial derivatives $D^j_{(2,3)} f$, $0 \leq j \leq m$ exist, are continuous and satisfy:*

$(H_0)$ *For all bounded $B \subseteq \Omega$ one has*

$$\sup_{t\in\mathbb{R}} \sup_{u\in B} \left| D^j_{(2,3)} f(t, u, \lambda) \right| < \infty \quad \textit{for all } \lambda \in \Lambda$$

(well-definedness) *and for all $\lambda^* \in \Lambda$ and $\varepsilon > 0$ there exists a $\delta > 0$ with*

$$|u - \bar u| < \delta \quad \Rightarrow \quad \sup_{t\in\mathbb{R}} \left| D^j_{(2,3)} f(t, u, \lambda) - D^j_{(2,3)} f(t, \bar u, \lambda) \right| < \varepsilon \tag{3.1}$$

*for all $u, \bar u \in \Omega$ and $\lambda \in B_\delta(\lambda^*)$* (uniform continuity).

$(H_1)$ *We have $0 \in \Omega$ and $\lim_{t\to\pm\infty} f(t, 0, \lambda) = 0$ for all $\lambda \in \Lambda$.*

Having these assumptions satisfied, we introduce the *substitution operators*

$$F(\phi, \lambda)(t) := f(t, \phi(t), \lambda), \qquad F^\upsilon(\phi, \lambda)(t) := D_2^{\upsilon_1} D_3^{\upsilon_2} f(t, \phi(t), \lambda) \quad \text{for all } t \in \mathbb{R}$$

and pairs $\upsilon = (\upsilon_1, \upsilon_2) \in \mathbb{N}_0^2$ such that $\upsilon_1 + \upsilon_2 \leq m$. The following results have been shown in [41] in the context of functional differential equations in finite-dimensional spaces. Modifications to our present setting are rather obvious.

**Proposition 3.1.** *Under $(H_0)$ the operator $F : BC(\mathbb{R}, \Omega) \times \Lambda \to BC$ is well-defined and $m$-times continuously differentiable on $BC(\mathbb{R}, \Omega)^\circ \times \Lambda$ with partial derivatives*

$$D^\upsilon F(\phi, \lambda) = F^\upsilon(\phi, \lambda) \quad \textit{for all } \phi \in BC(\mathbb{R}, \Omega)^\circ,\ \lambda \in \Lambda.$$

*If $(H_0)$ and $(H_1)$ are satisfied, then the same holds for $F : BC_0(\mathbb{R}, \Omega) \times \Lambda \to BC_0$.*

*Proof.* Proceed as in [41, Prop. 3.3]. □

**Corollary 3.2.** *Under $(H_0)$ the operator $G : BC^1(\mathbb{R}, \Omega) \times \Lambda \to BC$,*

$$G(\phi, \lambda) = \dot{\phi} - F(\phi, \lambda)$$

*is well-defined and $m$-times continuously differentiable on $BC^1(\mathbb{R}, \Omega)^\circ \times \Lambda$. If $(H_0)$ and $(H_1)$ are satisfied, then the same holds for $G : BC_0^1(\mathbb{R}, \Omega) \times \Lambda \to BC_0$.*

*Proof.* Proceed as in [41, Cor. 3.4]. □

**Theorem 3.3.** *For $\lambda \in \Lambda$ the following holds under $(H_0)$:*

*(a) If $\phi \in BC(\mathbb{R}, \Omega)$ is a complete solution of $(D)_\lambda$, then $\phi \in BC^1(\mathbb{R}, \Omega)$ and*

$$G(\phi, \lambda) = 0; \tag{3.2}$$

*conversely, if $\phi \in C^1(\mathbb{R}, \Omega) \cap BC$ solves (3.2), then $\phi \in BC^1(\mathbb{R}, \Omega)$ and $\phi$ is a complete bounded solution of $(D)_\lambda$.*

*(b) Under additionally $(H_1)$, if $\phi \in BC_0(\mathbb{R}, \Omega)$ is a complete solution of $(D)_\lambda$, then $\phi \in BC_0^1(\mathbb{R}, \Omega)$ and (3.2) holds; conversely, if $\phi \in C^1(\mathbb{R}, \Omega) \cap BC_0$ solves (3.2), then $\phi \in BC_0^1(\mathbb{R}, \Omega)$ and $\phi$ is a complete bounded solution of $(D)_\lambda$.*

*Proof.* Follow [41, Thm. 3.5]. □

3.1. **Linear ODEs.** In order to study complete solutions of $(D)_\lambda$ we need some notions for linear ODEs. Given a continuous mapping $A : \mathbb{R} \to L(X)$, they are of the form

$$\boxed{\dot{u}(t) = A(t)u.} \tag{$LD$}$$

From standard references (e.g., [13, pp. 96ff, §2] or [3, pp. 136ff]) we know that the general solution $\varphi$ of $(LD)$ exists as a linear function and we define the *transition operator* $\Phi(t, s) \in GL(X)$ of $(LD)$ by

$$\Phi(t, s)\xi := \varphi(t; s, \xi) \quad \text{for all } s, t \in \mathbb{R}.$$

Under the boundedness assumption $b := \sup_{t \in \mathbb{R}} |A(t)| < \infty$, we deduce from Gronwall's lemma that $|\Phi(t, s)| \leq e^{b(t-s)}$ for $s \leq t$.

Due to the invertibility of $\Phi(t, s)$ the following concepts are simpler than in the case of difference equations. We say equation $(LD)$ or the associated transition operator $\Phi$ admits an *exponential dichotomy* (ED for short, see [13, pp. 162ff, §3]) on a subinterval $\mathbb{I} \subseteq \mathbb{R}$, if there exists a continuous projection-valued mapping $P : \mathbb{I} \to L(X)$ and reals $\alpha > 0$, $K \geq 1$ so that $\Phi(t, s)P(s) = P(t)\Phi(t, s)$ for all $s \leq t$, $s, t \in \mathbb{I}$ and

$$|\Phi(t, s)P(s)| \leq Ke^{-\alpha(t-s)}, \qquad |\Phi(s, t)[I - P(t)]| \leq Ke^{\alpha(s-t)} \quad \text{for all } s \leq t.$$

The stable and the unstable vector bundle of $(LD)$ are defined as in the discrete case.

Moreover, we suppose throughout that the fibers $N(P(t))$, $t \in \mathbb{I}$, of the unstable vector bundle are finite-dimensional; this is fulfilled under compactness assumptions on the transition operator $\Phi(t, s)$, $s < t$ (cf. [18, p. 226, Ex. $4\frac{1}{2}$] or [52, p. 196, Lemma 45.3]).

In this framework, the *Bohl* or *dichotomy spectrum* of $\Phi$ or $(LD)$ is defined as (see [10, p. 62, Def. 3.9]) $\Sigma(A) := \{\gamma \in \mathbb{R} : \Phi_\gamma$ has no ED on $\mathbb{R}\}$ with a scaled transition operator $\Phi_\gamma(t,s) := e^{\gamma(s-t)}\Phi(t,s)$.

The *dual* differential equations to $(LD)$ reads as

$$\dot{u}(t) = -A(t)'u; \tag{$LD'$}$$

it is an equation in the dual space $X'$, whose evolution operator $\Phi'(t,s) \in GL(X')$ is given by (cf. [3, pp. 147–148, (11.15)])

$$\Phi'(t,s) = \Phi(s,t)' \quad \text{for all } s,t \in \mathbb{I}. \tag{3.3}$$

An exponential dichotomy carries over from $\Phi$ to $\Phi'$ as follows:

**Lemma 3.4.** *If a linear differential equation $(LD)$ has an exponential dichotomy with $\alpha, K$ and invariant projector $P$ on $\mathbb{I}$, then also the dual transition operator $\Phi'$ admits an exponential dichotomy on $\mathbb{I}$ as follows: $\Phi'(t,s)P^*(s) = P^*(t)\Phi'(t,s)$,*

$$|\Phi'(t,s)P^*(s)| \le Ke^{-\alpha(s-t)}, \quad |\Phi'(s,t)[I - P^*(t)]| \le Ke^{\alpha(t-s)}$$

*for all $t \le s$, with an invariant projector $P^*(t) := I - P(t)'$ and*

$$R(P^*(t)) = N(P(t))^\perp, \qquad N(P^*(t)) = R(P(t))^\perp. \tag{3.4}$$

*Proof.* The claim follows using (3.3), where (3.4) has been shown in [27, p. 156]. □

Unless otherwise noted, the symbol $\mathcal{C}$ stands for one of the spaces $BC$ or $BC_0$ in the following. As counterpart to the difference operator (2.6) in the present framework of linear ODEs, we introduce the obviously well-defined differential operator

$$L : \mathcal{C}^1 \to \mathcal{C}, \qquad (L\phi)(t) := \dot{\phi}(t) - A(t)\phi(t) \quad \text{for all } t \in \mathbb{R}; \tag{3.5}$$

the following Fredholm theory for $L$ is essentially due to [49, 37].

**Proposition 3.5.** *Let $\tau \in \mathbb{R}$. If a linear differential equation $(LD)$ admits an ED both on $[\tau,\infty)$ (with projector $P^+$) and on $(-\infty,\tau]$ (with projector $P^-$), then the operator $L : \mathcal{C}^1 \to \mathcal{C}$ is Fredholm with index $\dim X_1 - \operatorname{codim} X_2$, where*

$$X_1 := R(P^+(\tau)) \cap N(P^-(\tau)), \qquad X_2 := R(P^+(\tau)) + N(P^-(\tau)).$$

*In particular, one has*

$$N(L) = \left\{\Phi(\cdot,\tau)\xi \in \mathcal{C}^1 : \xi \in R(P^+(\tau)) \cap N(P^-(\tau))\right\},$$
$$R(L) = \left\{\phi \in \mathcal{C} \;\middle|\; \begin{array}{l} \int_{\mathbb{R}}\langle \psi'(\sigma), \phi(\sigma)\rangle\, d\sigma = 0 \textit{ for all } \psi' \in BC(\mathbb{R}, X') \\ \textit{solving the dual differential equation } (LD') \end{array}\right\}$$

*and furthermore* $\dim N(L) = \dim X_1$, $\dim R(L) = \operatorname{codim} X_2$.

*Remark* 3.1. A converse to Prop. 3.5 has been shown in [39].

*Proof.* See [6, Lemma 3.2]. □

3.2. **Bifurcation of bounded solutions.** Given a parameter value $\lambda^* \in \Lambda$, the bifurcation concept for bounded complete solutions $\phi^*$ to $(D)_{\lambda^*}$ is defined analogously to the discrete situation treated in Subsect. 2.2. Furthermore, in our present ODE setting, the *variational equation* of $(D)_\lambda$ along $\phi^*$ reads as

$$\dot{u} = D_2 f(t, \phi^*(t), \lambda^*)u \tag{3.6}$$

and its transition operator is denoted by $\Phi_{\lambda^*}$. If equation (3.6) admits an ED on $\mathbb{R}$, then $\phi^*$ is denoted as *hyperbolic* and $\phi^*$ persists under variation of $\lambda$ (see Fig. 1 (left)). Writing $\Sigma(\phi^*, \lambda^*)$ for the dichotomy spectrum of (3.6) this means $0 \notin \Sigma(\phi^*, \lambda^*)$ and we arrive at

**Proposition 3.6.** *Let $\lambda^* \in \Lambda$ be given and suppose $(H_0)$ holds. If a complete permanent solution $\phi^* \in BC(\mathbb{R}, \Omega)$ of $(D)_{\lambda^*}$ bifurcates at $\lambda^*$, then $\phi^*$ is nonhyperbolic.*

*Proof.* Proceed as in the proof of Prop. 2.6 using [41, Thm. 3.9]. □

**Hypothesis.** *Let $n, r \in \mathbb{N}$, $\tau \in \mathbb{R}$, $\lambda^* \in \Lambda$ be given, suppose $X$ is reflexive and $(D)_{\lambda^*}$ admits a complete permanent solution $\phi^* \in BC(\mathbb{R}, \Omega)$ with*

$(H_2)$ *the variational equation (3.6) admits an ED both on $[\tau, \infty)$ and $(-\infty, \tau]$ with respective projectors $P^+$ and $P^-$ satisfying*

$$R(P^+(\tau)) \cap N(P^-(\tau)) = \operatorname{span}\{\xi_1, \ldots, \xi_n\},$$
$$\big(R(P^+(\tau)) + N(P^-(\tau))\big)^\perp = \operatorname{span}\{\xi_1', \ldots, \xi_r'\}$$

*and linearly independent vectors $\xi_1, \ldots, \xi_n \in X$, resp. $\xi_1', \ldots, \xi_r' \in X'$. Moreover, we choose $\eta_1, \ldots, \eta_r \in X$, resp. $\eta_1', \ldots, \eta_n' \in X'$ such that*

$$\langle \eta_i', \xi_j \rangle = \delta_{i,j} \text{ for } 1 \le i, j \le n, \qquad \langle \xi_i', \eta_j \rangle = \delta_{i,j} \text{ for } 1 \le i, j \le r. \tag{3.7}$$

Under these assumptions we make use of the preparations from Subsect. 3.1 applied to the linear equation $(LD)$ with $A(t) := D_2 f(t, \phi^*(t), \lambda^*)$. Note that the above hypothesis is the same as in the discrete case and we desist from formulating the corresponding counterpart to Rem. 2.1. Similarly, a counterpart to Exam. 2.2 is valid.

**Lemma 3.7.** *If $(H_0)$, $(H_2)$ hold, then the linear operator $L : \mathcal{C}^1 \to \mathcal{C}$ is Fredholm of index $n - r$ and one has*

$$N(L) = \operatorname{span}\{\Phi_{\lambda^*}(\cdot, \tau)\xi_1, \ldots, \Phi_{\lambda^*}(\cdot, \tau)\xi_n\},$$
$$N(L') = \operatorname{span}\{\Phi_{\lambda^*}(\tau, \cdot)'\xi_1', \ldots, \Phi_{\lambda^*}(\tau, \cdot)'\xi_r'\}, \tag{3.8}$$

*where $\Phi_{\lambda^*}(\cdot, \tau)\xi_i$, $1 \le i \le n$, resp. $\Phi_{\lambda^*}(\tau, \cdot)'\xi_j'$, $1 \le j \le r$, are linearly independent.*

*Remark* 3.2. If the equation $(D)_{\lambda^*}$ is autonomous, then $\dot\phi^* \in N(L)$.

*Proof.* Referring to Prop. 3.5 our assumptions guarantee that $L$ is Fredholm with index $n - r$ and $\dim N(L) = n$, since the kernel of $L$ consists of bounded complete solutions for (3.6), which due to the dichotomy assumptions are linear combinations of the linearly independent functions $\Phi_{\lambda^*}(\cdot, \tau)\xi_i \in \mathcal{C}$ for $1 \le i \le n$. The assertion for $N(L')$ follows analogously using Lemma 3.4. □

The above bilinear form (2.14) essential to construct the Lyapunov-Schmidt projectors in Lemma 2.8 finds its continuous version in replacing the infinite sum by infinite integrals. However, we have to introduce a normalized positive function $\omega$, which has no influence on the resulting branching equation (A.5).

**Lemma 3.8.** *If $(H_0)$, $(H_2)$ hold, then the mappings $P \in L(\mathcal{C}^1)$, $Q \in L(\mathcal{C})$,*

$$Px := \sum_{i=1}^n \langle \eta_i', x(\tau) \rangle \Phi_{\lambda^*}(\cdot, \tau)\xi_i,$$
$$Qy := y - \omega(\cdot) \sum_{i=1}^r \int_{\mathbb{R}} \langle \Phi_{\lambda^*}(\tau, \sigma)'\xi_i', y(\sigma) \rangle \, d\sigma \, \Phi_{\lambda^*}(\cdot, \tau)\eta_i \tag{3.9}$$

*are bounded projections onto $N(L)$ and $R(L)$, respectively, where $\omega : \mathbb{R} \to (0, \infty)$ is a continuous function satisfying $\int_{\mathbb{R}} \omega = 1$.*

*Proof.* Above all, given $x \in \mathcal{C}^1$, we obtain from Lemma 3.7 that $Px \in N(L)$ holds and (3.7) ensures $P^2 = P$. Thus, $P$ is a bounded projector onto the kernel $N(L)$. On the other hand, we formally introduce the bilinear form

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle : N(L') \times \mathcal{C} \to \mathbb{R}, \qquad \langle\!\langle \phi', \psi \rangle\!\rangle := \int_{\mathbb{R}} \langle \phi'(\sigma), \psi(\sigma) \rangle \, d\sigma;$$

by Lemma 3.7 and 3.4 the dichotomy assumptions guarantee $\langle\!\langle \phi', \psi \rangle\!\rangle < \infty$. We now define functions $\phi_i' \in N(L')$, $\psi^i \in \mathcal{C}$ by

$$\phi_i' := \Phi_{\lambda^*}(\tau, \cdot)' \xi_i', \qquad \psi^i := \omega(\cdot) \Phi_{\lambda^*}(\cdot, \tau) \eta_i \quad \text{for all } 1 \leq i \leq r$$

and consequently arrive at

$$\langle\!\langle \phi_i', \psi^j \rangle\!\rangle = \int_{\mathbb{R}} \omega(\sigma) \langle \Phi_{\lambda^*}(\tau, \sigma)' \xi_i', \Phi_{\lambda^*}(\sigma, \tau) \eta_j \rangle \, d\sigma = \int_{\mathbb{R}} \omega(\sigma) \langle \xi_i', \eta_j \rangle \, d\sigma \overset{(3.7)}{=} \delta_{i,j}$$

for $1 \leq i, j \leq r$. So, $\left\{\phi_i', \psi^j\right\}$ is a biorthogonal system and from Subsect. A.1 we obtain that the linear operator $Q$ defined above is the desired bounded projection onto $R(L)$. $\square$

Our following step is to formulate the finite-dimensional branching equation associated to the abstract problem (3.2). Its explicit form might be helpful when it comes to bifurcation phenomena not covered by the subsequent Thms. 3.11–3.14.

**Convention**: Let $U$ be a set. Dealing with functions $\vartheta : U \to BC$ having values $\vartheta(u)$, $u \in U$, in the function space $BC$, we conveniently write $\vartheta(t; u) := \vartheta(u)(t)$.

**Proposition 3.9** (branching equation)**.** *Suppose that $(H_0)$, $(H_2)$ hold. If $\mathcal{C} = BC$, then there exist open convex neighborhoods $S \subseteq \mathbb{R}^n$ of 0, $\Lambda_0 \subseteq \Lambda$ of $\lambda^*$ and a $C^m$-function $\vartheta : S \times \Lambda_0 \to \mathcal{C}$ satisfying $\vartheta(0, \lambda^*) = 0$, $D_1\vartheta(0, \lambda^*) = 0$ and*

$$\begin{aligned} &\dot{\phi}^* + \sum_{l=1}^n s_l A(\cdot) \Phi_{\lambda^*}(\cdot, \tau) \xi_l + D_1\vartheta(\cdot; s, \lambda) - H(\cdot, s, \lambda) \\ &- \omega(\cdot) \sum_{i=1}^r \int_{\mathbb{R}} \langle \Phi_{\lambda^*}(\tau, \sigma)' \xi_i', \dot{\phi}^*(\sigma) + \sum_{l=1}^n s_l A(\sigma) \Phi_{\lambda^*}(\sigma, \tau) \xi_l + D_1\vartheta(\sigma; s, \lambda) \rangle \, d\sigma \Phi_{\lambda^*}(\cdot, \tau) \eta_i \\ &\qquad + \omega(\cdot) \sum_{i=1}^r \int_{\mathbb{R}} \langle \Phi_{\lambda^*}(\tau, \sigma)' \xi_i', H(\sigma, s, \lambda) \rangle \, d\sigma \Phi_{\lambda^*}(\cdot, \tau) \eta_i = 0 \end{aligned} \tag{3.10}$$

*with the function $H(t, s, \lambda) = f\left(t, \phi^*(t) + \sum_{l=1}^n s_l \Phi_{\lambda^*}(t, \tau) \xi_l + \vartheta(t; s, \lambda), \lambda\right)$. The branching equation (A.5) for (3.2) is equivalent to $g(s, \lambda) = 0$, where $g : S \times \Lambda_0 \to \mathbb{R}^r$ is a $C^m$-function whose components $g_1, \ldots, g_r$ read as*

$$\begin{aligned} g_l(s, \lambda) := &\int_{\mathbb{R}} \langle \Phi_{\lambda^*}(\tau, \sigma)' \xi_l', \dot{\phi}^*(\sigma) + \sum_{i=1}^n s_i A(\sigma) \Phi_{\lambda^*}(\sigma, \tau) \xi_i + D_1\vartheta(\sigma; s, \lambda) \rangle \, d\sigma \\ &- \int_{\mathbb{R}} \langle \Phi_{\lambda^*}(\tau, \sigma)' \xi_l', f(\sigma, \phi^*(\sigma) + \sum_{i=1}^n s_i \Phi_{\lambda^*}(\sigma, \tau) \xi_i + \vartheta(\sigma; s, \lambda), \lambda) \rangle \, d\sigma. \end{aligned} \tag{3.11}$$

*Given $\phi^* \in BC_0(\mathbb{R}, \Omega)$ and $(H_0)$ to $(H_2)$, the assertion holds with $\mathcal{C} = BC_0$.*

*Proof.* The methods from Sect. A.2 apply to the problem $G(\phi, \lambda) = 0$ with a right-hand side $G : BC^1(\mathbb{R}, \Omega) \times \Lambda \to BC$ defined in Cor. 3.2 by $G(\phi, \lambda) = \dot{\phi} - F(\phi, \lambda)$. Above all, from Thm. 3.3 we have $G(\phi^*, \lambda^*) = 0$. Moreover, from Cor. 3.2 (cf. [41, Cor. 3.4 and Thm. 3.5]) we derive that $G$ is $m$-times continuously differentiable with

$$D_1G(\phi^*, \lambda^*)\psi = \dot{\psi} - D_1F(\phi^*, \lambda^*)\psi = L\psi,$$

with $(L\psi)(t) = \dot\psi(t) - D_2 f(t, \phi^*(t), \lambda)\psi(t)$ for all $t \in \mathbb{R}$. Thanks to hypothesis $(H_2)$ and Prop. 3.5 the operator $L$ is Fredholm with index $n - r$ and $n$-dimensional kernel. Hence, Lemma A.1 provides a function $\vartheta$ as above satisfying the abstract equation (A.4). In our setup of projections given in Lemma 3.8 one directly shows that (A.4) has the concrete representation (3.10). Similarly we compute the components (3.11) of the abstract bifurcation equation (A.5) using (3.9). □

**Lemma 3.10.** *If $(H_0)$, $(H_2)$ hold, then the linear functionals*

$$\mu_i : \mathcal{C} \to \mathbb{R}, \qquad \mu_i(\phi) := \int_{\mathbb{R}} \langle \Phi_{\lambda^*}(\tau, \sigma)' \xi_i', \phi(\sigma)\rangle \, d\sigma \quad \text{for all } 1 \le i \le r$$

*are continuous with $|\mu_i| \le \frac{2K}{\alpha} |\xi_i'|$ and one has $R(L) = \bigcap_{i=1}^r N(\mu_i)$.*

*Proof.* Using Lemma 3.7 for the Fredholm properties of $L : \mathcal{C}^1 \to \mathcal{C}$, the proof is analogous to that of Lemma 2.10. □

Having the above functionals $\mu_i$ available, we can apply our abstract bifurcation criteria from Subsects. A.3–A.4 to (3.2), where Thm. 3.3 yields the corresponding interpretation for an ODE $(D)_\lambda$. The parallel structure of Lemma 2.10 and 3.10 ensures that our up-coming results are analogous to the discrete case from Subsect. 2.2 with infinite sums replaced by integrals over $\mathbb{R}$. Nonetheless, for the sake of completeness we give provide a complete formulation.

First we assume that a trivial solution branch is known, with the conclusion that the following assumption $(H_3)$ implies $(H_1)$.

**Hypothesis.** *Let $0 \in \Omega$ and suppose*

$(H_3)$ $f(t, 0, \lambda) \equiv 0$ *on* $\mathbb{R} \times \Lambda$.

**Theorem 3.11** (bifurcation with odd-dimensional kernel)**.** *Let $\Lambda \subseteq \mathbb{R}$ and $m \ge 2$. If $(H_0)$ to $(H_3)$ hold with $n = r$ and $\phi^* = 0$, then the trivial solution of an ODE $(D)_\lambda$ bifurcates at $\lambda^*$, provided $n$ is odd and*

$$\det\Big(\int_{\mathbb{R}} \langle \Phi_{\lambda^*}(\tau, \sigma)' \xi_l', D_2 D_3 f(\sigma, 0, \lambda^*) \Phi_{\lambda^*}(\sigma, \tau)\xi_i\rangle \, d\sigma\Big)_{1 \le i,l \le n} \ne 0.$$

*Proof.* As in Thm. 2.11 we apply Thm. A.5 to equation (3.2), which is possible due to our preparations in Thm. 3.3, Prop. 3.1 and Lemma 3.7. □

**Theorem 3.12** (multiparameter bifurcation)**.** *Let $\Lambda \subseteq \mathbb{R}^n$ and $m \ge 2$. If $(H_0)$ to $(H_3)$ hold with $n = r$ and $\phi^* = 0$, then the trivial solution of an ODE $(D)_\lambda$ bifurcates at $\lambda^*$, provided there exists a $\hat\xi \in R(P^+(\tau)) \cap N(P^-(\tau))$ such that*

$$\det\Big(\int_{\mathbb{R}} \langle \Phi_{\lambda^*}(\tau, \sigma)' \xi_i', D_{\lambda_l} D_2 f(\sigma, 0, \lambda^*) \Phi_{\lambda^*}(\sigma, \tau)\hat\xi\rangle \, d\sigma\Big)_{1 \le i,l \le n} \ne 0.$$

*More precisely, there exist a $\rho > 0$ and open convex neighborhoods $U \subseteq BC^1(\mathbb{R}, \Omega)$ of $0$, $\Lambda_0 \subseteq \Lambda$ of $\lambda^*$ and $C^{m-1}$-functions $\phi : (-\rho, \rho) \to U$, $\lambda : (-\rho, \rho) \to \Lambda_0$ with*

*(a)* $\phi(0) = 0$, $\lambda(0) = \lambda^*$, $\dot\phi(0) = \Phi_{\lambda^*}(\cdot, \tau)\hat\xi$,

*(b)* *each $\phi(s)$ is a nontrivial complete solution of $(D)_{\lambda(s)}$ in $BC_0^1(\mathbb{R}, \Omega)$ with*

$$\sum_{i=1}^n \langle \xi_i, \phi(\tau; s)\rangle \Phi_{\lambda^*}(\cdot, \tau)\xi_i = s\Phi_{\lambda^*}(\cdot, \tau)\hat\xi.$$

*Proof.* Using arguments analogous to the proof of Thm. 2.12, the claim follows from the bunch Thm. A.6 and Lemma 3.8. □

**Theorem 3.13** (fold bifurcation)**.** *Let $\Lambda \subseteq \mathbb{R}$ and also suppose that $(H_0)$, $(H_2)$ hold with $n = r = 1$. If $\mathcal{C} = BC$ and*

$$g_{01} := \int_{\mathbb{R}} \langle \Phi_{\lambda^*}(\tau, \sigma)' \xi_1', D_3 f(\sigma, \phi^*(\sigma), \lambda^*) \rangle \, d\sigma \neq 0,$$

*then there exists a $\rho > 0$, open convex neighborhoods $U \subseteq BC^1(\mathbb{R}, \Omega)$ of $\phi^*$, $\Lambda_0 \subseteq \Lambda$ of $\lambda^*$ and $C^m$-functions $\phi : (-\rho, \rho) \to U$, $\lambda : (-\rho, \rho) \to \Lambda_0$ such that*

*(a) $\phi(0) = \phi^*$, $\lambda(0) = \lambda^*$ and $\dot{\phi}(0) = \Phi_{\lambda^*}(\cdot, \tau)\xi_1$, $\dot{\lambda}(0) = 0$,*
*(b) each $\phi(s)$ is a complete solution of $(D)_{\lambda(s)}$ in $\mathcal{C}(\mathbb{R}, \Omega)$.*

*Moreover, in case $m \geq 2$ and under the additional assumption*

$$g_{20} := \int_{\mathbb{R}} \langle \Phi_{\lambda^*}(\tau, \sigma)' \xi_1', D_2^2 f(\sigma, \phi^*(\sigma), \lambda^*)[\Phi_{\lambda^*}(\sigma, \tau)\xi_1]^2 \rangle \, d\sigma \neq 0,$$

*the solution $\phi^* \in \mathcal{C}(\mathbb{R}, \Omega)$ of $(D)_{\lambda^*}$ bifurcates at $\lambda^*$, one has $\ddot{\lambda}(0) = -\frac{g_{20}}{g_{01}}$ and the following holds locally in $U \times \Lambda_0$:*

*(c)* Subcritical case*: If $g_{20}/g_{01} > 0$, then $(D)_\lambda$ has no complete solution in $BC(\mathbb{R}, \Omega)$ for $\lambda > \lambda^*$, $\phi^*$ is the unique complete solution of $(D)_{\lambda^*}$ in $BC(\mathbb{R}, \Omega)$ and $(D)_\lambda$ has exactly two distinct complete bounded solutions for $\lambda < \lambda^*$; they are in $\mathcal{C}(\mathbb{R}, \Omega)$.*
*(d)* Supercritical case*: If $g_{20}/g_{01} < 0$, then $(D)_\lambda$ has no complete solution in $BC(\mathbb{R}, \Omega)$ for $\lambda < \lambda^*$, $\phi^*$ is the unique complete solution of $(D)_{\lambda^*}$ in $BC(\mathbb{R}, \Omega)$ and $(D)_\lambda$ has exactly two distinct complete bounded solutions for $\lambda > \lambda^*$; they are in $\mathcal{C}(\mathbb{R}, \Omega)$.*

*If $(H_0)$ to $(H_2)$ are satisfied, then the same holds with $\mathcal{C} = BC_0$.*

A subcritical fold bifurcation is schematically depicted in Fig. 1 (right).

*Proof.* We are in the position to apply Thm. A.2 to equation (3.2), since Cor. 3.2 guarantees that $G : BC^1(\mathbb{R}, \Omega)^\circ \times \Lambda \to BC$ is of class $C^m$ and we have $G(\phi^*, \lambda^*) = 0$. For further details one proceeds as in Thm. 2.13 using Lemma 3.10. □

Now we return to the situation, where $(D)_\lambda$ has a trivial solution branch.

**Theorem 3.14** (bifurcation from known solutions)**.** *Let $\Lambda \subseteq \mathbb{R}$ and $m \geq 2$. If $(H_0)$ to $(H_3)$ hold with $n = r = 1$, $\phi^* = 0$ and the* transversality condition

$$g_{11} := \int_{\mathbb{R}} \langle \Phi_{\lambda^*}(\tau, \sigma)' \xi_1', D_2 D_3 f(\sigma, 0, \lambda^*) \Phi_{\lambda^*}(\sigma, \tau)\xi_1 \rangle \, d\sigma \neq 0 \tag{3.12}$$

*is satisfied, then the trivial solution of an ODE $(D)_\lambda$ bifurcates at $\lambda^*$. In particular, there exists a $\rho > 0$, open convex neighborhoods $U \subseteq BC^1(\mathbb{R}, \Omega)$ of $0$, $\Lambda_0 \subseteq \Lambda$ of $\lambda^*$ and $C^{m-1}$-functions $\phi : (-\rho, \rho) \to U$, $\lambda : (-\rho, \rho) \to \Lambda_0$ with*

*(a) $\phi(0) = 0$, $\lambda(0) = \lambda^*$ and $\dot{\phi}(0) = \Phi_{\lambda^*}(\cdot, \tau)\xi_1$,*
*(b) each $\phi(s)$ is a nontrivial solution of $(D)_{\lambda(s)}$ homoclinic to $0$.*

*Proof.* As in the proof of Thm. 2.14 we can apply the first part of Thm. A.3 or A.4 to the mapping $G : \mathcal{C}^1(\mathbb{R}, \Omega) \times \Lambda \to \mathcal{C}$. From Lemma 3.7 we derive that the kernel $N(L)$ is spanned by $\Phi_{\lambda^*}(\cdot, \tau)\xi_1 \in BC_0$ and the transversality condition (3.12) yields $\mu(D_1 D_2 G(0, \lambda^*)\Phi_{\lambda^*}(\cdot, \tau)\xi_1) \neq 0$ with the functional $\mu = \mu_1$ from Lemma 3.10. □

The bifurcation scenarios of the following corollaries have already been illustrated in Sect. 1. While Fig. 2 (left) shows a transcritical bifurcation, a subcritical pitchfork bifurcation is depicted in Fig. 2 (right).

**Corollary 3.15** (transcritical bifurcation)**.** *Under the additional assumption*

$$g_{20} := \int_{\mathbb{R}} \langle \Phi_{\lambda^*}(\tau,\sigma)'\xi_1', D_2^2 f(\sigma,0,\lambda^*)[\Phi_{\lambda^*}(\sigma,\tau)\xi_1]^2\rangle \, d\sigma \neq 0$$

*one has* $\dot\lambda(0) = -\frac{g_{20}}{2g_{11}}$ *and the following holds locally in* $U \times \Lambda_0$*: An ODE* $(D)_\lambda$ *has a unique nontrivial complete bounded solution* $\phi_\lambda$ *for* $\lambda \neq \lambda^*$ *and* $0$ *is the unique complete bounded solution of* $(D)_{\lambda^*}$*; moreover,* $\phi_\lambda \in BC_0(\mathbb{R}, \Omega)$.

*Proof.* As in Cor. 2.15 we deduce the assertion from Thm. A.3. □

**Corollary 3.16** (pitchfork bifurcation)**.** *For* $m \geq 3$ *and under the additional assumptions*

$$\int_{\mathbb{R}} \langle \Phi_{\lambda^*}(\tau,\sigma)'\xi_1', D_2^2 f(\sigma,0,\lambda^*)[\Phi_{\lambda^*}(\sigma,\tau)\xi_1]^2\rangle \, d\sigma = 0,$$

$$g_{30} := \int_{\mathbb{R}} \langle \Phi_{\lambda^*}(\tau,\sigma)'\xi_1', D_1^3 f(\sigma,0,\lambda^*)[\Phi_{\lambda^*}(\sigma,\tau)\xi_1]^3\rangle \, d\sigma \neq 0$$

*one has* $\dot\lambda(0) = 0$, $\ddot\lambda(0) = -\frac{g_{30}}{3g_{11}}$ *and the following holds locally in* $U \times \Lambda_0$*:*

*(c)* Subcritical case*: If* $g_{30}/g_{11} > 0$*, then the unique complete bounded solution of* $(D)_\lambda$ *is the trivial one for* $\lambda \geq \lambda^*$ *and* $(D)_\lambda$ *has exactly two nontrivial complete complete solutions for* $\lambda < \lambda^*$*; both are homoclinic to* $0$.

*(d)* Supercritical case*: If* $g_{30}/g_{11} < 0$*, then the unique complete bounded solution of* $(D)_\lambda$ *is the trivial one for* $\lambda \leq \lambda^*$ *and* $(D)_\lambda$ *has exactly two nontrivial complete solutions for* $\lambda > \lambda^*$*; both are homoclinic to* $0$.

*Proof.* Analogously to Cor. 2.16 the claim results from Thm. A.4. □

**Appendix** A. **Tools from functional analysis.** In this appendix we briefly review essential tools from static local bifurcation theory for Fredholm operators, like Lyapunov-Schmidt reduction and abstract versions of fold, transcritical and pitchfork bifurcations. Most of the results can be found in standard references (cf. e.g. [9, 28, 55]), but since we also made use of the contributions [11, 12, 16], it seems advantageous to present them in a unified fashion.

Suppose throughout that $X, Y, Z$ are real Banach spaces and $\Omega \subseteq X$, $\Lambda \subseteq Y$ denote nonempty open neighborhoods of $x_0 \in X$, $\lambda \in Y$ in the respective spaces. We deal with $C^m$-mappings $G : \Omega \times \Lambda \to Z$, $m \in \mathbb{N}$, vanishing at $(x_0, \lambda_0)$, i.e.

$$G(x_0, \lambda_0) = 0, \tag{A.1}$$

whose partial derivative $D_1 G(x_0, \lambda_0) \in L(X, Z)$ is a Fredholm operator. The pair $(x_0, \lambda_0)$ is called a *bifurcation point* of the abstract equation (A.1), if there exists a convergent parameter sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\Lambda$ with limit $\lambda_0$ and distinct solutions $x_n^1, x_n^2 \in \Omega$, $n \in \mathbb{N}$, to the equation $G(x, \lambda_n) = 0$ with $\lim_{n\to\infty} x_n^1 = \lim_{n\to\infty} x_n^2 = x_0$.

A.1. **Fredholm operators.** A linear operator $T \in L(X, Z)$ is called *Fredholm*, if

$$n := \dim N(T) < \infty, \qquad r := \operatorname{codim} R(T) < \infty$$

and its *index* is defined as $n - r$. This Fredholm property yields that $N(T)$, as well as $R(T)$ split the respective space $X$ and $Z$, i.e. there exist closed subspaces $X_0 \subseteq X$, $Z_0 \subseteq Z$,

$$X = N(T) \oplus X_0, \qquad Z = Z_0 \oplus R(T). \tag{A.2}$$

The associated projection operators $P : X \to N(T)$, $Q : Z \to R(T)$ and linear subspaces $X_0, Z_0$, resp., can be constructed explicitly. To this end, we choose a basis $\{x_1, \dots, x_n\}$ of $N(T)$ and corresponding $y_1', \dots, y_n' \in X'$ such that

$$\langle y_i', x_j\rangle = \delta_{i,j} \quad \text{for all } 1 \le i, j \le n$$

in order to construct a biorthogonal system $\{y_i', x_j\}$. Given such linearly independent vectors $x_i$, by the Hahn-Banach theorem (cf. [33, p. 69, Thm. 1.1]), we can always find corresponding elements $y_i'$ for $1 \le i \le n$. Then we define

$$Px := \sum_{j=1}^{n} \langle y_j', x\rangle x_j,$$

and $I - P$ is a projection from $X$ onto $X_0 = (I - P)X$.

Also the dual operator $T' \in L(Z', X')$ is Fredholm and

$$\dim N(T') = \operatorname{codim} R(T), \qquad \operatorname{codim} R(T') = \dim N(T)$$

(see [55, pp. 366–367, Prop. 8.14(4)]). Analogously to the above construction, we choose a basis $\{x_1', \dots, x_r'\}$ of $N(T')$, complete it to a biorthogonal system $\{x_i', y_j\}$ with $y_j \in Z$, and set

$$Qy := y - \sum_{i=1}^{r} \langle x_i', y\rangle y_i.$$

Then $I - Q$ is a projection from $X$ onto $Z_0 = (I - Q)Z$.

A.2. **Lyapunov-Schmidt method.** Assume the derivative $D_1G(x_0, \lambda_0) \in L(X, Z)$ is Fredholm as above. The *method of Lyapunov-Schmidt* enables us to reduce the possibly infinite-dimensional equation

$$G(x, \lambda) = 0 \tag{A.3}$$

to a finite-dimensional problem. We abbreviate $T := D_1G(x_0, \lambda_0)$ and obtain spaces $X_0, Z_0$ as in (A.2) with associated *Lyapunov-Schmidt projectors* $P \in L(X)$, $Q \in L(Z)$. There exist linearly independent vectors $x_1, \dots, x_n \in X$ and $z_1, \dots, z_r \in Z$ with

$$N(T) = \operatorname{span}\{x_1, \dots, x_n\}, \qquad (I - Q)Z_0 = \operatorname{span}\{z_1, \dots, z_r\}.$$

If we decompose $x \in \Omega$ according to $x = x_0 + v + w$ with $v \in N(T)$ and $w \in X_0$ (see (A.2)), then the key observation is that (A.3) is equivalent to the equations

$$QG(x_0 + v + w, \lambda) = 0, \qquad (I - Q)G(x_0 + v + w, \lambda) = 0,$$

which we are about to solve separately.

**Lemma A.1.** *There exist open convex neighborhoods $S \subseteq \mathbb{R}^n$ of $0$, $\Lambda_0 \subseteq \Lambda$ of $\lambda_0$ and a $C^m$-function $\vartheta : S \times \Lambda_0 \to X_0$ satisfying*

$$QG\left(x_0 + \sum_{i=1}^{n} s_i x_i + \vartheta(s, \lambda), \lambda\right) \equiv 0 \quad \text{on } S \times \Lambda_0 \tag{A.4}$$

*and $\vartheta(0, \lambda_0) = 0$, $D_1\vartheta(0, \lambda_0) = 0$.*

*Proof.* On a small neighborhood $U \subseteq \mathbb{R}^n$ of $0$ we define $\hat{G} : U \times X_0 \times \Lambda \to R(Q)$,

$$\hat{G}(s, w, \lambda) := QG\left(x_0 + \sum_{i=1}^{n} s_i x_i + w, \lambda\right)$$

and observe $D_2\hat{G}(0, 0, \lambda_0) = T|_{X_0} \in GL(X_0, R(Q))$ by assumption. Thus, the implicit function theorem (cf., [55, p. 150, Thm. 4.B]) yields our claim. ∎

Summarizing the previous analysis, it remains to solve an $r$-dimensional system for $n$ real variables, the so-called *bifurcation* or *branching equation*

$$g(s,\lambda)=0, \tag{A.5}$$

whose components are given by

$$\sum_{j=1}^{r} g_j(s,\lambda)z_j := [I-Q]G\left(x_0+\sum_{i=1}^{n} s_ix_i+\vartheta(s,\lambda),\lambda\right), \tag{A.6}$$

where each $g_j : S\times\Lambda_0\to\mathbb{R}$ is a $C^m$-function with derivative $D_1g(0,\lambda_0)=0$. Therefore, in case the Fredholm operator $L$ has positive index, then (A.5) is underdetermined and will possess a whole family of solutions, i.e. we are in the framework of the surjective implicit function theorem (cf. [55, p. 177, Thm. 4.H]). In the converse situation of a negative index, (A.5) is overdetermined and solutions might not exist.

A.3. **Bifurcation with one-dimensional kernel.** In this subsection we suppose the partial derivative $D_1G(x_0,\lambda_0)\in L(X,Z)$ is Fredholm with index 0 and

$$\dim N(D_1G(x_0,\lambda_0))=1. \tag{A.7}$$

Thus, the Hahn-Banach theorem (cf. [33, p. 69, Thm. 1.1]) yields the existence of a continuous functional $\mu\in Z'$ such that

$$N(\mu)=R(D_1G(x_0,\lambda_0))$$

and by virtue of the above Lyapunov-Schmidt method, the branching equation (A.5) reduces to a one-dimensional problem with

$$g(0,\lambda_0)=0, \qquad D_1g(0,\lambda_0)=0.$$

In conclusion, the solution structure for (A.5) can be obtained using standard results for the bifurcation of scalar equations (see, for instance, [32, 54]) and we arrive at an implicit function theorem for one-dimensional kernels (see [12, Thm. 3.2]).

**Theorem A.2** (abstract fold bifurcation)**.** *Let $\Lambda\subseteq\mathbb{R}$. If the assumptions (A.1), (A.7) and*

$$g_{01}:=\mu(D_2G(x_0,\lambda_0))\neq 0$$

*hold, then there exist open convex neighborhoods $S\subseteq\mathbb{R}$ of $0$, $U_1\times U_2\subseteq\Omega\times\Lambda$ of $(x_0,\lambda_0)$ and a $C^m$-function $\gamma=(\gamma_1,\gamma_2): S\to U_1\times U_2$ such that*

$$\gamma(S)=\{(x,\lambda)\in U_1\times U_2 : G(x,\lambda)=0\},$$

*where $\gamma$ satisfies $\gamma(0)=(x_0,\lambda_0)$ and $\dot\gamma(0)=(x_1,0)$. Moreover, in case $m\geq 2$ and*

$$g_{20}:=\mu(D_1^2G(x_0,\lambda_0)x_1^2)\neq 0,$$

*the pair $(x_0,\lambda_0)$ is a bifurcation point of (A.3), one has $\ddot\gamma_2(0)=-\frac{g_{20}}{g_{01}}$ and the following holds:*

*(a) If $g_{20}/g_{01}<0$, then* $\#\{x\in U_1 : G(x,\lambda)=0\}=\begin{cases}0, & \lambda<\lambda_0,\\ 1, & \lambda=\lambda_0,\\ 2, & \lambda>\lambda_0.\end{cases}$

*(b) If $g_{20}/g_{01}>0$, then* $\#\{x\in U_1 : G(x,\lambda)=0\}=\begin{cases}0, & \lambda>\lambda_0,\\ 1, & \lambda=\lambda_0,\\ 2, & \lambda<\lambda_0.\end{cases}$

*Proof.* Apply [54, p. 260ff] to equation (A.5) or use [28, p. 12, Theorem I.4.1] directly. □

Now we deal with bifurcations from known solutions and strengthen (A.1) by assuming a whole branch of trivial solutions, i.e.

$$G(0, \lambda) \equiv 0 \quad \text{on } \Lambda. \tag{A.8}$$

Hence, the function $\vartheta$ from Lemma A.1 satisfies $\vartheta(0, \lambda) \equiv 0$ on $\Lambda$ and we obtain the identity $D_2\psi(0, \lambda) \equiv 0$ on $\Lambda$. Also for the branching equation (A.5) we get $g(0, \lambda) \equiv 0$ and consequently

$$D_1 g(0, \lambda_0) = 0, \qquad\qquad D_2 g(0, \lambda) \equiv 0 \quad \text{on } \Lambda.$$

Moreover, $g$ allows the representation

$$g(v, \lambda) = v h(v, \lambda) \quad \text{for all } v \in V,\ \lambda \in \Lambda_0$$

with a $C^{m-1}$-function $h : V \times \Lambda_0 \to \mathbb{R}$ satisfying the relation $h(0, \lambda_0) = 0$. By the mean value theorem (cf. [33, p. 341, Thm. 4.2]) it reads as

$$h(v, \lambda) = \int_0^1 D_1 g(tv, \lambda)\, dt.$$

The following two bifurcation results are essentially consequences of the celebrated Crandall-Rabinowitz theorem (see [11, Thm. 1.7] or [55, p. 383, Thm. 8.A]).

**Theorem A.3** (abstract transcritical bifurcation)**.** *Suppose $\Lambda \subseteq \mathbb{R}$, $m \geq 2$ and that assumptions (A.7), (A.8) hold. Then $(0, \lambda_0)$ is a bifurcation point of (A.3), provided*

$$g_{11} := \mu(D_1 D_2 G(0, \lambda_0) x_1) \neq 0.$$

*In particular, there exist open convex neighborhoods $S \subseteq \mathbb{R}$ of $0$, $U_1 \times U_2 \subseteq \Omega \times \Lambda$ of $(0, \lambda_0)$ and a nontrivial $C^{m-1}$-function $\gamma = (\gamma_1, \gamma_2) : S \to U_1 \times U_2$ such that*

$$\gamma(S) \setminus \{(0, \lambda_0)\} = \{(x, \lambda) \in U_1 \times U_2 : G(x, \lambda) = 0 \text{ and } x \neq 0\},$$

*where $\gamma$ satisfies $\gamma(0) = (0, \lambda_0)$ and $\dot\gamma_1(0) = x_1$. Moreover, in case*

$$g_{20} := \mu(D_1^2 G(0, \lambda_0) x_1^2) \neq 0$$

*one has $\dot\gamma_2(0) = -\frac{g_{20}}{2g_{11}}$ and the following holds:*

$$\#\{x \in U_1 : G(x, \lambda) = 0\} = \begin{cases} 1, & \lambda = \lambda_0, \\ 2, & \lambda \neq \lambda_0. \end{cases}$$

*Proof.* Apply [54, p. 263] to equation (A.5) or use [28, pp. 18ff, Sect. I.6] directly. □

**Theorem A.4** (abstract pitchfork bifurcation)**.** *Suppose $\Lambda \subseteq \mathbb{R}$, $m \geq 3$ and that assumptions (A.7), (A.8) hold. Then $(0, \lambda_0)$ is a bifurcation point of (A.3), provided*

$$g_{11} := \mu(D_1 D_2 G(0, \lambda_0) x_1) \neq 0.$$

*In particular, there exist open convex neighborhoods $S \subseteq \mathbb{R}$ of $0$, $U_1 \times U_2 \subseteq \Omega \times \Lambda$ of $(0, \lambda_0)$ and a nontrivial $C^{m-1}$-function $\gamma = (\gamma_1, \gamma_2) : S \to U_1 \times U_2$ such that*

$$\gamma(S) \setminus \{(0, \lambda_0)\} = \{(x, \lambda) \in U_1 \times U_2 : G(x, \lambda) = 0 \text{ and } x \neq 0\},$$

*where $\gamma$ satisfies $\gamma(0) = (0, \lambda_0)$ and $\dot\gamma_1(0) = x_1$. Moreover, in case*

$$\mu(D_1^2 G(0, \lambda_0) x_1^2) = 0, \qquad\qquad g_{30} := \mu(D_1^3 G(0, \lambda_0) x_1^3) \neq 0$$

*one has $\dot\gamma_2(0) = 0$, $\ddot\gamma_2(0) = -\frac{g_{30}}{3g_{11}}$ and the following holds:*

*(a) If $g_{30}/g_{11} < 0$, then* $\#\{x \in U_1 : G(x, \lambda) = 0\} = \begin{cases} 1, & \lambda \leq \lambda_0, \\ 3, & \lambda > \lambda_0. \end{cases}$

*(b) If* $g_{30}/g_{11} > 0$*, then* $\#\{x \in U_1 : G(x,\lambda) = 0\} = \begin{cases} 1, & \lambda \geq \lambda_0, \\ 3, & \lambda < \lambda_0. \end{cases}$

*Proof.* Apply [54, pp. 267ff] to equation (A.5) or use [28, pp. 18ff, Sect. 1.6] directly. □

A.4. **Bifurcation with higher dimensional kernel.** To generalize the above setting, suppose $D_1G(0,\lambda_0) \in L(X,Z)$ is Fredholm with index 0 and a higher dimensional kernel.

**Theorem A.5.** *Suppose* $\Lambda \subseteq \mathbb{R}$*,* $m \geq 2$ *and that* (A.8) *holds. Then* $(0,\lambda_0)$ *is a bifurcation point of* (A.3)*, provided* $\dim N(D_1G(0,\lambda_0))$ *is odd and*

$$D_1D_2G(0,\lambda_0)N(D_1G(0,\lambda_0)) \cap R(D_1G(0,\lambda_0)) = \{0\}\,.$$

*Proof.* See [16, Cor.]. □

**Theorem A.6** (bunch theorem)**.** *Suppose* $\Lambda \subseteq \mathbb{R}^p$*,* $m \geq 2$*, that assumption* (A.8) *holds with* $p = \dim N(D_1G(0,\lambda_0))$*. If there exists a* $\hat{x}_1 \in N(D_1G(0,\lambda_0))$ *such that the matrix*

$$(\langle x_j', D_{\lambda_i}D_1G(0,\lambda_0)\hat{x}_1\rangle)_{1\leq i,j\leq p}$$

*is regular, then* $(0,\lambda_0)$ *is a bifurcation point of* (A.3)*. In particular, there exist open convex neighborhoods* $S \subseteq \mathbb{R}$ *of* $0$*,* $U_1 \times U_2 \subseteq \Omega \times \Lambda$ *of* $(0,\lambda_0)$ *and a nontrivial* $C^{m-1}$*-function* $\gamma = (\gamma_1,\gamma_2) : S \to U_1 \times U_2$ *such that*

$$\gamma(S) \setminus \{(0,\lambda_0)\} = \{(x,\lambda) \in U_1 \times U_2 : G(x,\lambda) = 0 \text{ and } x \neq 0\}\,,$$

*where* $\gamma$ *satisfies* $\gamma(0) = (0,\lambda_0)$*,* $\dot{\gamma}_1(0) = \hat{x}_1$ *and* $P\gamma_1(s) = s\hat{x}_1$*.*

*Proof.* See [55, pp. 392–393, Thm. 8.B]. □


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