NONAUTONOMOUS BIFURCATION OF BOUNDED SOLUTIONS III: CROSSING CURVE SITUATIONS

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ABSTRACT. We follow an approach to a bifurcation theory for nonautonomous differential equations based on a change in the structure of bounded complete solutions. In the framework of Carathéodory differential equations, we provide sufficient criteria for such a nonhyperbolic solution to bifurcate into two branches of solutions. This scenario is somewhat typical in the sense that (a) transcritical and pitchfork bifurcations are special cases, but (b) no branch of trivial solutions is supposed to exist. In particular, we discuss a degenerate fold bifurcation pattern, where the transversality assumption is replaced by a nondegeneracy condition on the second order derivative. Both bifurcation patterns are intrinsically nonautonomous and do not occur for time-invariant equations.

Our notion of a nonhyperbolic solution is based on the fact that the associate variational equation possesses exponential dichotomies on both semiaxes with compatible projectors. The resulting Fredholm theory allows to apply recent abstract bifurcation results due to Liu, Shi and Wang (2007).

1. INTRODUCTION

Nonautonomous difference or differential equations, whose right hand side depends explicitly on time, naturally occur in a large variety of applications where seasonal, adaption, modulation, control or random influences on a particular model cannot be neglected. In order to tackle such problems mathematically, it turned out that the classical theory of dynamical systems has to be extended. In fact, one needs suitable notions of hyperbolicity, invariance and attractors. Such tools have been developed in the last decades in form of dichotomies (see, e.g., [Cop78]), nonautonomous sets, integral manifolds or pullback attraction (see [Klo03]).

Yet, in this context it appears that a bifurcation theory for nonautonomous evolutionary equations is a quite recent field of research. The initial reason for this might be the fact that problems with arbitrary time dependence lack equilibria or periodic solutions — the prime examples of bifurcating objects in the classical autonomous theory. Therefore, one had to establish appropriate invariant objects for a nonautonomous setting first, which serve as bifurcating sets. This led to dynamical systems-driven approaches like attractor bifurcation (cf., e.g. [LRS06, Ras07b]) or, within a skew-product setting, a change in the dynamics on the base space (see [NO08]). For a more comprehensive survey on the relevant literature dealing with nonautonomous bifurcations or transitions, the interested reader is referred to [Ras07a, Pöt10a] or to [KS05] for a phenomenological approach.

As opposed to a somewhat crude attractor bifurcation, we aim to obtain a more detailed insight into the dynamical behavior of differential equations depending on a parameter λ. It is based on the observation that equilibria of autonomous equations generically persist

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as globally bounded complete solutions under small perturbations fluctuating in time (cf., e.g. [BM03, Pöt10b]). Keeping this in mind, we roughly understand a nonautonomous bifurcation as a local change in the number (or topological structure) of bounded complete solutions to a given evolutionary equation under variation of λ. This goes hand in hand with a lack of hyperbolicity for a complete solution ϕ^*, i.e. the fact that the corresponding variational equation for $\lambda = \lambda^*$ loses its exponential dichotomy on the whole time axis. Instead, we investigate variational equations with dichotomies on both semiaxes, whose projectors fulfill a compatibility condition (see Hypothesis (H_2) for details). The resulting Fredholm theory (see [Pal84, Pal88]) allows us to tackle two bifurcation scenarios:

- The *crossing curve bifurcation* from Thm. 4.1 resembles a transcritical bifurcation. Yet, the structurally unstable assumption of a trivial solution branch is replaced by a second order nondegeneracy condition. We obtain that the bifurcating solution ϕ^* is the intersection of two transversal branches of bounded complete solutions. This pattern occurs under weaker assumptions than the usual transcritical bifurcation, and a pitchfork-like pattern appears as a degenerate situation.
- On the other hand, we also obtain a *degenerate fold bifurcation* in Thm. 4.2, where the associate transversality condition is violated (cf. (4.8) or (A.4)). Instead of having a nonhyperbolic solution ϕ^* being embedded as fold of a smooth curve of complete bounded solutions, a different scenario occurs: In the bifurcation diagram, either the pair (ϕ^*, λ^*) is isolated, or the intersection point of two transversal branches of bounded complete solutions.

It seems worth to point out that (cf. Exs. 4.1 and 4.2) the crossing curve bifurcation cannot be detected using the dichotomy spectrum (see [SS78, Sie02]) alone.

Since (global) attractors consist of bounded complete solutions, in a way we complement attractor bifurcations. Yet, differing from earlier contributions [LRS06, Ras07b, NO08] dealing with scalar equations, the present approach applies to problems of arbitrary finite dimension $d \geq 2$. Indeed, our bifurcation scenarios follow essentially from recent abstract analytical results due to [Shi99, LSW07]. Rather surprisingly, until now tools from analytical bifurcation theory (for example, [Zei93, Chapt. 8], [Kie04]) have rarely been used in order to approach general nonautonomous branching problems, although the essential Fredholm theory was established in [Pal84, Pal88] already. Following this leitmotiv, in [Pöt10a] we applied classical analytical bifurcation results from [CR71, CR73] to deduce fold, transcritical and pitchfork bifurcation patterns for nonautonomous difference and differential equations.

Our present contribution extends these earlier results from [Pöt10a] in three aspects: On the one hand, in form of the *crossing curve bifurcation* from Thm. 4.1 we obtain a more general scenario than [Pöt10a, Thm. 3.14], since a trivial solution branch is replaced by local conditions on the second order derivative. In this sense, [Pöt10a] dealt with *primary bifurcations* while we address *secondary bifurcations* now. Beyond that, in Thm. 4.2 we complement the fold bifurcation studied in [Pöt10a, Thm. 3.13] by investigating a degenerate version violating the usual transversality condition (cf. (4.8)). The latter means that an integral over $\mathbb{R}$ vanishes, an assumption satisfied for a large class of functions containing e.g. odd ones. Hence, the treated degenerate situation is not that exceptional. This requires a detailed linear solution theory in form of Cor. 3.3. Finally, in this paper we deal with Carathéodory differential equations, whose right hand sides need to be only measurable in time. The required basics on such equations can be found in [Kur86, pp. 315ff] or [AW96]. Such a broader setting is motivated from possible applications to control theory, where the controls are typically L^∞-functions (see [CK99]), or continuous random dynamical systems, whose pathwise realization gives rise to Carathéodory equations (see [Arn98]).

Our presentation splits into three parts: In Section 2 we provide the essential preliminaries on Carathéodory differential equations and introduce an ambient spatial setting in order to formulate them as operator equations. It follows the necessary Fredholm theory for linear differential equations and its close connection to exponential dichotomies. The final Section 4 contains a discussion of our promised bifurcation scenarios and illustrates it using basic examples. Lastly, for the reader's convenience, and in order to be consistent with our earlier contributions, we reproduce the required and essential bifurcation theory of [Shi99, LSW07] in an appendix.

Notation: Banach spaces are denoted by X, Y and equipped with norm $|\cdot|$; when dealing with function spaces, we denote the norm by $\|\cdot\|$. The *interior* of a set $\Omega \subseteq X$ is denoted by Ω° and $B_\varepsilon(x)$ is the open ball with center x and radius $\varepsilon > 0$. The space of bounded linear operators between spaces X and Y is $L(X, Y)$, $L(X) := L(X, X)$, for the corresponding toplinear endomorphisms we write $GL(X)$ and $\mathrm{id} \in GL(X)$ is the identity mapping on X. Given $T \in L(X, Y)$, we write $R(T) := TX$ for the *range* and $N(T) := T^{-1}(0)$ for the *kernel*. The *dual space* of X is X', $\langle x', x\rangle := x'(x)$ the duality product and $T' \in L(Y', X')$ is the *dual operator* to T. For a given subspace $X_0 \subseteq X$ the *annihilator* is defined as the set of functionals

$$X_0^\perp := \{x' \in X' : \langle x', x_0\rangle = 0 \text{ for all } x_0 \in X_0\}.$$

We interpret $\mathbb{R}^d$ as Euclidean space, i.e. it is equipped with the canonical inner product $\langle x, y\rangle := \sum_{j=1}^d x_j y_j$ (dot product) for all $x, y \in \mathbb{R}^d$ and the induced norm $|\cdot|$. In this context, T' is the transpose of $T \in L(\mathbb{R}^d)$. For a given subspace $X_0 \subseteq \mathbb{R}^d$ the *orthogonal complement* is defined as the subspace $X_0^\perp := \left\{y \in \mathbb{R}^d : \langle y, x_0\rangle = 0 \text{ for all } x_0 \in X_0\right\}$. Finally, suppose that the operator $T^\dagger \in L(\mathbb{R}^d)$ denotes the *pseudo-inverse* of T (see [CM79, p. 8, Def. 1.1.1]) given by linear extension using

$$T^\dagger x := \begin{cases} 0, & x \in R(T)^\perp, \\ T|_{R(T')}^{-1} x, & x \in R(T); \end{cases}$$

this is consistent with the Moore-Penrose axioms (cf. [CM79, p. 9, Thm. 1.1.1]).

2. Carathéodory differential equations

Let us suppose throughout the paper that $\Omega \subseteq \mathbb{R}^d$ and the parameter space $\Lambda \subseteq Y$ are nonempty open convex sets. Subsets $\mathcal{V} \subseteq \mathbb{R} \times \Omega$ are called *nonautonomous sets* and $\mathcal{V}(t) := \{x \in \mathbb{R}^d : (t, x) \in \mathcal{V}\}$ as the *t-fiber*, $t \in \mathbb{R}$, of $\mathcal{V}$.

In the following, notions of measurability and integrability are always understood in the Lebesgue sense. Suppose that $f : \mathbb{R} \times \Omega \times \Lambda \to \mathbb{R}^d$ is a *Carathéodory function*, i.e.

- for almost every $t \in \mathbb{R}$ the mapping $f(t, \cdot, \lambda)$, $\lambda \in \Lambda$, is continuous,
- for every $(x, \lambda) \in \Omega \times \Lambda$ the mapping $f(\cdot, x, \lambda)$ is measurable

(cf., e.g., [AW96, Def. 2.1]). Then a *Carathéodory differential equation* (CDE for short) depending on a parameter $\lambda \in \Lambda$ reads as

$$\boxed{\dot{x} = f(t, x, \lambda).} \qquad (D)_\lambda$$

Given an interval $I \subseteq \mathbb{R}$, a *solution* to equation $(D)_\lambda$ is a function $\phi : I \to \Omega$ such that $f(\cdot, \phi(\cdot), \lambda)$ is locally integrable and one has identity

$$\phi(t) - \phi(\tau) = \int_\tau^t f(s, \phi(s), \lambda)\, ds \quad \text{for all } \tau, t \in I. \tag{2.1}$$

From [Kur86, p. 325, 18.2.3] we know that ϕ is (locally) absolutely continuous and therefore differentiable a.e. (see [Kur86, p. 323, 18.1.22]).

For a solution satisfying the initial condition $x(\tau) = \xi$ with given $\tau \in \mathbb{R}$, $\xi \in \Omega$, we write $\varphi_\lambda(\cdot; \tau, \xi)$ and denote φ_λ as *general solution* to $(D)_\lambda$, provided existence and uniqueness is given (for this, see [Kur86, pp. 331ff]). An *entire* or *complete solution* of equation $(D)_\lambda$ exists on $\mathbb{R}$. Moreover, in case $0 \in \Omega$ a complete solution satisfying $\lim_{t \to \pm\infty} \phi(t) = 0$ is called *homoclinic* to 0 and we speak of a *permanent solution*, if

$$\inf_{t \in \mathbb{R}} \operatorname{dist}(\phi(t), \Omega) > 0.$$

The next assumptions hold for right hand sides of equation $(D)_\lambda$, which are C^m-smooth in the second and third variable, and whose derivatives map bounded sets into bounded sets uniformly in time. In particular, this guarantees that the solutions of $(D)_\lambda$ exist and are uniquely determined by [Kur86, p. 331, 18.4.2].

Hypothesis. *Let $m \in \mathbb{N}$, suppose $f : \mathbb{R} \times \Omega \times \Lambda \to \mathbb{R}^d$ is a Carathéodory function and that almost every $f(t, \cdot) : \Omega \times \Lambda \to \mathbb{R}^d$, $t \in \mathbb{R}$, is a C^m-function such that the following holds for $0 \le j \le m$:*

(H_0) *For all bounded $B \subseteq \Omega$ one has*

$$\operatorname*{ess\,sup}_{t \in \mathbb{R}} \sup_{x \in B} \left| D^j_{(2,3)} f(t, x, \lambda) \right| < \infty \quad \textit{for all } \lambda \in \Lambda \tag{2.2}$$

(well-definedness) *and for all $\lambda^* \in \Lambda$ and $\varepsilon > 0$ there exists a $\delta > 0$ with*

$$|x - y| < \delta \quad \Rightarrow \quad \operatorname*{ess\,sup}_{t \in \mathbb{R}} \left| D^j_{(2,3)} f(t, x, \lambda) - D^j_{(2,3)} f(t, y, \lambda) \right| < \varepsilon \tag{2.3}$$

for all $x, y \in \Omega$ and $\lambda \in B_\delta(\lambda^*)$ (uniform continuity).

(H_1) *We have $0 \in \Omega$ and $\lim_{t \to \pm\infty} f(t, 0, \lambda) = 0$ for all $\lambda \in \Lambda$.*

For the sake of finding bounded complete solutions for a CDE $(D)_\lambda$ we need an ambient functional-analytical setting. This requires the spaces $AC(\Omega)$ of locally absolutely continuous, $L^\infty(\Omega)$ of essentially bounded and $W^{1,\infty}(\Omega)$ of bounded functions $\phi : \mathbb{R} \to \Omega$ with essentially bounded (weak) derivative. For $\Omega = \mathbb{R}^d$ we write $AC := AC(\mathbb{R}^d)$ and proceed similarly with the other spaces. It is clear that L^∞ is a Banach space w.r.t. the norm

$$\|\phi\|_0 := \operatorname*{ess\,sup}_{t \in \mathbb{R}} |\phi(t)| < \infty.$$

Moreover, every $\phi \in W^{1,\infty}$ possesses a bounded Lipschitz-continuous representative (see [Leo09, p. 224, Thm. 7.17]) and by Rademachers theorem the (strong) derivative $\dot\phi$ exists almost everywhere in $\mathbb{R}$. From [Leo09, p. 224, Ex. 7.18] we know that $W^{1,\infty}$ is a Banach space under the norm $\|\phi\|_1 := \max\left\{ \|\phi\|_0, \|\dot\phi\|_0 \right\}$; we have the continuous embedding

$$W^{1,\infty} \hookrightarrow L^\infty, \qquad \|\phi\|_0 \le \|\phi\|_1. \tag{2.4}$$

As closed subspaces of L^∞ and $W^{1,\infty}$ we also introduce the corresponding spaces

$$L^\infty_0 := \left\{ \phi \in L^\infty : \lim_{t \to \pm\infty} \phi(t) = 0 \right\}, \qquad W^{1,\infty}_0 := \left\{ \phi \in W^{1,\infty} : \phi, \dot\phi \in L^\infty_0 \right\}$$

of limit 0 functions. Under the above assumptions, we introduce *substitution operators*

$$F(\phi,\lambda)(t) := f(t,\phi(t),\lambda), \quad F^{\upsilon}(\phi,\lambda)(t) := D_2^{\upsilon_1} D_3^{\upsilon_2} f(t,\phi(t),\lambda) \quad \text{for all } t \in \mathbb{R}$$

and pairs $\upsilon = (\upsilon_1,\upsilon_2) \in \mathbb{N}_0^2$ such that $\upsilon_1 + \upsilon_2 \leq m$.

Proposition 2.1. *Under (H_0) the operator $F : L^\infty(\Omega) \times \Lambda \to L^\infty$ is well-defined and m-times continuously differentiable on $L^\infty(\Omega)^\circ \times \Lambda$ with partial derivatives*

$$D^{\upsilon} F(\phi,\lambda) = F^{\upsilon}(\phi,\lambda) \quad \textit{for all } \phi \in L^\infty(\Omega)^\circ,\ \lambda \in \Lambda.$$

If (H_0) and (H_1) are satisfied, then the same holds for $F : L_0^\infty(\Omega) \times \Lambda \to L_0^\infty$.

Proof. Given $\phi \in L^\infty(\Omega)$ there exists a bounded $B \subseteq \Omega$ such that $\phi(t) \in B$ a.e. for $t \in \mathbb{R}$. Thus, (2.2) implies $\operatorname{ess\,sup}_{t\in\mathbb{R}} \left| D_{(2,3)}^j f(t,\phi(t),\lambda) \right| < \infty$ for all $\lambda \in \Lambda$, $0 \leq j \leq m$ and the substitution operators $F : L^\infty \times \Lambda \to L^\infty$ and F^{υ} are well-defined. The smoothness assertion for F can be shown as in [Pöt10b, Prop. 3.4], where we considered continuous right hand sides f on the space BC of bounded continuous functions instead of L^∞. The interested reader is invited to verify that the present situation of functions L^∞ causes no additional difficulties. □

Corollary 2.2. *Under (H_0) the operator $G : W^{1,\infty}(\Omega) \times \Lambda \to L^\infty$,*

$$G(\phi,\lambda) = \dot\phi - F(\phi,\lambda) \tag{2.5}$$

is well-defined and m-times continuously differentiable on $W^{1,\infty}(\Omega)^\circ \times \Lambda$. If (H_0) and (H_1) are satisfied, then the same holds for $G : W_0^{1,\infty}(\Omega) \times \Lambda \to L_0^\infty$.

Proof. First of all, $\phi \mapsto \dot\phi$ is a bounded linear mapping from $W^{1,\infty}$ to L^∞, and hence of class C^∞. Thus, the claim follows with Prop. 2.1 using the embedding (2.4). □

As we will see next, the appropriate function space to solve (2.5) is $W^{1,\infty}$, because L^∞-solutions to $(D)_\lambda$ are actually Lipschitzian and not only in AC. Then the crucial tool for our overall analysis is:

Theorem 2.3. *For every parameter $\lambda \in \Lambda$ the following holds under (H_0):*

(a) *If $\phi \in L^\infty(\Omega)$ has a (strong) derivative a.e. in $\mathbb{R}$ and is a complete solution of $(D)_\lambda$, then $\phi \in W^{1,\infty}(\Omega)$ and*

$$G(\phi,\lambda) = 0; \tag*{$(O)_\lambda$}$$

conversely, if $\phi \in L^\infty(\Omega)$ has a (strong) derivative a.e. in $\mathbb{R}$ and solves $(O)_\lambda$, then ϕ is a complete bounded solution of $(D)_\lambda$ in $W^{1,\infty}(\Omega)$.

(b) *Under additionally (H_1), if $\phi \in L_0^\infty(\Omega)$ is a complete solution of $(D)_\lambda$, then $\phi \in W_0^{1,\infty}(\Omega)$ and $(O)_\lambda$ holds; conversely, provided $\phi \in L_0^\infty(\Omega)$ has a (strong) derivative a.e. in $\mathbb{R}$ and solves $(O)_\lambda$, then ϕ is a complete bounded solution of $(D)_\lambda$ in $W_0^{1,\infty}(\Omega)$.*

Proof. We suppress the dependence on the fixed parameter $\lambda \in \Lambda$.

(a) Given a complete solution $\phi \in L^\infty(\Omega)$ of $(D)_\lambda$ and a bounded set $B \subseteq \Omega$ with $\phi(t) \in B$ a.e. for $t \in \mathbb{R}$ we obtain from (2.2) that $c := \operatorname{ess\,sup}_{t\in\mathbb{R}} |f(t,\phi(t))| < \infty$. Using [Kur86, p. 327, 18.2.7 and p. 325, 18.2.1] we know that the derivative $\dot\phi$ exists a.e. in $\mathbb{R}$. Moreover, $|\dot\phi(t)| \leq \operatorname{ess\,sup}_{t\in\mathbb{R}} |f(t,\phi(t))| =: c < \infty$ a.e. for $t \in \mathbb{R}$ and we deduce

$\dot{\phi} \in L^\infty$. In order to establish the inclusion $\phi \in W^{1,\infty}$ it remains to show that ϕ is Lipschitz. This, however, follows from

$$|\phi(t) - \phi(\tau)| \stackrel{(2.1)}{=} \left| \int_\tau^t f(s, \phi(s))\, ds \right| \leq \left| \int_\tau^t c\, ds \right| = c\,|t - \tau|$$

a.e. for $t \in \mathbb{R}$. In addition, since ϕ is a solution to $(D)_\lambda$, relation $(O)_\lambda$ holds a.e. for $t \in \mathbb{R}$.

Conversely, if the derivative of $\phi \in L^\infty(\Omega)$ exists and fulfills $\dot{\phi}(t) = f(t, \phi(t))$ a.e. in the reals $\mathbb{R}$, we deduce as above that $\phi \in W^{1,\infty}(\Omega)$ and that ϕ solves $(D)_\lambda$.

(b) Now suppose that also (H_1) holds. For a homoclinic solution $\phi \in L_0^\infty(\Omega)$ we have the identity $\dot{\phi}(t) = f(t, \phi(t))$ and (H_2) ensures $|D_2 f(t, h\phi(t))| \leq C$ for $h \in [0, 1]$ a.e. in $\mathbb{R}$, with some $C > 0$. The mean value inequality (see [Lan93, p. 342, Cor. 4.3]) implies

$$\begin{aligned}\left|\dot{\phi}(t)\right| &\leq |f(t, \phi(t)) - f(t, 0)| + |f(t, 0)| \\ &\leq \operatorname*{ess\,sup}_{t \in \mathbb{R}} \sup_{h \in [0,1]} |D_2 f(t, h\phi(t))|\, |\phi(t)| + |f(t, 0)| \leq C|\phi(t)| + |f(t, 0)|\end{aligned}$$

a.e. for $t \in \mathbb{R}$. Passing over to the limit $t \to \pm\infty$ yields the inclusion $\dot{\phi} \in L_0^\infty$ and the remaining assertion follows as in (a). $\square$

3. Fredholm theory for linear equations

In order to solve equation $(O)_\lambda$ using tools from Appendix A, it is essential that the derivative $D_2 G$ is Fredholm. This can be characterized in terms of exponential dichotomies (see [Sac79, Pal84, Pal88]) and we present the required theory for CDEs in the following:

For this, let $I \subseteq \mathbb{R}$ be an interval and suppose that $A : I \to L(\mathbb{R}^d)$ is measurable, locally integrable and essentially bounded:

$$\operatorname*{ess\,sup}_{t \in I} |A(t)| < \infty. \tag{3.1}$$

We consider a linear CDE

$$\boxed{\dot{x} = A(t)x,} \tag{L}$$

whose right hand side defines a Carathéodory function satisfying (H_0). This assumption allows us to define the general solution φ to (L) and the associated *transition operator* $\Phi(t, s) \in GL(\mathbb{R}^d)$ via $\Phi(t, s)\xi := \varphi(t; s, \xi)$, $t, s \in I$ (cf. [AW96, Lemma. 2.9]).

We say a family of projections $P_t \in L(\mathbb{R}^d)$, $t \in I$, is an *invariant projector* for (L), provided the mapping $t \mapsto P_t$ is continuous and

$$P_t \Phi(t, s) = \Phi(t, s) P_s \quad \text{for all } t, s \in I. \tag{3.2}$$

From this commutativity relation we conclude that the nonautonomous sets

$$\mathcal{V}^+ := \left\{(\tau, \xi) \in I \times \mathbb{R}^d : \xi \in R(P_\tau)\right\}, \quad \mathcal{V}^- := \left\{(\tau, \xi) \in I \times \mathbb{R}^d : \xi \in N(P_\tau)\right\}$$

form invariant vector bundles for equation (L). This means that the fibers $\mathcal{V}^\pm(t)$ are linear spaces satisfying the invariance relation $\Phi(t, s)\mathcal{V}^\pm(s) = \mathcal{V}^\pm(t)\Phi(t, s)$ for all $s, t \in I$. In addition, we can introduce *Green's function* as

$$G_P(t, s) := \begin{cases} \Phi(t, s) P_s & \text{for } t \geq s, \\ -\Phi(t, s)[\mathrm{id} - P_s] & \text{for } s > t. \end{cases} \tag{3.3}$$

A linear CDE (L) is said to have an *exponential dichotomy* (ED for short) on I, if there exist an invariant projector P_t and reals $K \geq 1$, $\alpha > 0$ with

$$|\Phi(t, s) P_s| \leq K e^{-\alpha(t-s)} \quad \text{for all } s \leq t, \quad |\Phi(t, s)[\mathrm{id} - P_s]| \leq K e^{\alpha(t-s)} \quad \text{for all } t \leq s$$

and $t, s \in I$. For equations with an ED, the set $\mathcal{V}^+$ is called the *stable vector bundle* (if I is unbounded above), while $\mathcal{V}^-$ is the *unstable vector bundle* (if I is unbounded below).

Along with the linear CDE (L) we also need the *dual* equation

$$\boxed{\dot{x} = -A(t)'x,} \tag{L'}$$

with the transition operator $\Phi'(t,s) = \Phi(s,t)'$ for all $s, t \in I$. Also (L') admits an exponential dichotomy on I with data K, α and the invariant projector $Q_t := \mathrm{id} - P_t'$.

For the sake of a convenient notation, let us assume that X denotes the space $W^{1,\infty}$ (resp. $W_0^{1,\infty}$), while Z stands for L^∞ (resp. L_0^∞). Essential for our approach are Fredholm properties for the Frechét derivative of the operator G defined in Cor. 2.2. This derivative has the form of a differential operator

$$L : X \to Z, \qquad (L\phi)(t) := \dot{\phi}(t) - A(t)\phi(t) \quad \text{for almost every } t \in \mathbb{R}, \tag{3.4}$$

which is well-defined and continuous under (3.1). Let us restrict to the case where (L) has EDs on both semiaxes $[\tau, \infty)$ and $(-\infty, \tau]$, $\tau \in \mathbb{R}$. For the corresponding situation of continuous ODEs rather than CDEs we refer to [Pal84, Boi01].

Proposition 3.1. *Let $\tau \in \mathbb{R}$, $n, r \in \mathbb{N}_0$ and suppose a linear CDE (L) admits an ED both on $[\tau, \infty)$ (with projector P_t^+) and on $(-\infty, \tau]$ (with projector P_t^-). If there exist linearly independent vectors $\xi_1, \dots, \xi_n \in \mathbb{R}^d$, $\xi_1', \dots, \xi_r' \in \mathbb{R}^d$ satisfying*

$$\begin{aligned} R(P_\tau^+) \cap N(P_\tau^-) &= \operatorname{span}\{\xi_1, \dots, \xi_n\}, \\ (R(P_\tau^+) + N(P_\tau^-))^\perp &= \operatorname{span}\{\xi_1', \dots, \xi_r'\}, \end{aligned} \tag{3.5}$$

then $L \in L(X, Z)$ is Fredholm with kernel $N(L) = \operatorname{span}\{\Phi(\cdot, \tau)\xi_i \in X : 1 \leq i \leq n\}$ and $\operatorname{codim} R(L) = r$. *In particular, we have* $\operatorname{ind} L = n - r$.

Remark 3.1. If $R(P_\tau^+) \cap N(P_\tau^-) = \{0\}$ and $R(P_\tau^+) + N(P_\tau^-) = \mathbb{R}^d$ holds, one argues as [Pal84, Prop. 2.1] to show that (L) has an ED on $\mathbb{R}$.

Proof. For linear ODEs (L) with a continuous coefficient matrix A the proof can be found in [Pal84, Lemma 4.2], where $X = BC^1$ and $Z = BC$. The interested reader might verify that the same arguments also hold for linear CDEs, with our appropriately modified spatial setting $X = W^{1,\infty}$, $Z = L^\infty$ or $X = W_0^{1,\infty}$, $Z = L_0^\infty$. □

Corollary 3.2. *The linear functionals*

$$\mu_i : Z \to \mathbb{R}, \qquad \mu_i(\psi) := \int_{\mathbb{R}} \langle \Phi(\tau, s)'\xi_i', \psi(s)\rangle \, ds \quad \textit{for all } 1 \leq i \leq r$$

are continuous with $|\mu_i| \leq \frac{2K}{\alpha} |\xi_i'|$ and $R(L) = \bigcap_{i=1}^r N(\mu_i)$.

Proof. Let $Q_t^\pm \in L(\mathbb{R}^d)$ denote the invariant projectors associated to the ED of the dual equation (L'). Then the vectors $\xi_i' \in \mathbb{R}^d$ from (3.5) satisfy $\xi_i' \in (R(P_\tau^-) + N(P_\tau^+))^\perp = R((Q_\tau^+)') \cap N((Q_\tau^-)')$ and thus, using the assumed dichotomy estimates, we can estimate the functionals $\mu_i(\psi)$ as follows

$$\begin{aligned} |\mu_i(\psi)| &\leq \int_{-\infty}^\tau \left|\langle \Phi(\tau, s)'P_\tau^-\xi_i', \psi(s)\rangle\right| ds + \int_\tau^\infty \left|\langle \Phi(\tau, s)'(\mathrm{id} - P_\tau^+)\xi_i', \psi(s)\rangle\right| ds \\ &\leq \int_{-\infty}^\tau \left|\Phi(\tau, s)P_s^-\right| |\xi_i'| \, |\psi(s)| \, ds + \int_\tau^\infty \left|\Phi(\tau, s)(\mathrm{id} - P_s^+)\right| |\xi_i'| \, |\psi(s)| \, ds \\ &\leq K \, |\xi_i'| \, \|\psi\|_0 \left(\int_{-\infty}^\tau e^{\alpha(s-\tau)} \, ds + \int_\tau^\infty e^{\alpha(\tau - s)} \, ds\right) \quad \text{for all } 1 \leq i \leq r \end{aligned}$$

and functions $\psi \in Z$. This implies the given bound for $|\mu_i|$. From Prop. 3.1 we get that the operator L is Fredholm and [Zei93, p. 366, Prop. 8.14(2)] guarantees the equivalences

$$\begin{aligned}\psi \in R(L) \quad &\Leftrightarrow \quad \psi \in N(L')^\perp \Leftrightarrow \int_{\mathbb{R}} \langle \Phi(\tau,s)'\xi_i', \psi(s)\rangle\, ds = 0 \quad \text{for all } 1 \le i \le r\\ &\Leftrightarrow \quad \mu_i(\psi) = 0 \quad \text{for all } 1 \le i \le r \Leftrightarrow \psi \in \bigcap_{i=1}^r N(\mu_i),\end{aligned}$$

which leads to our assertion. □

Corollary 3.3. *Let $\psi \in R(L)$. If $X_0 \subseteq X$ denotes a complement of $N(L)$, then the inverse of the restriction $L|_{X_0} : X_0 \to R(L)$ is given by $L|_{X_0}^{-1}\psi = \overline{\psi}$ with*

$$\overline{\psi}(t) := \begin{cases} \Phi(t,\tau)P_\tau^+\xi_\tau^* + \int_\tau^\infty G_{P^+}(t,s)\psi(s)\,ds, & t \ge \tau,\\ \Phi(t,\tau)[\mathrm{id} - P_\tau^-]\xi_\tau^* + \int_{-\infty}^\tau G_{P^-}(t,s)\psi(s)\,ds, & t \le \tau,\end{cases} \tag{3.6}$$

$$\xi_\tau^* := [P_\tau^+ + P_\tau^- - \mathrm{id}]^\dagger \left(\int_{-\infty}^\tau \Phi(\tau,s)P_s^-\psi(s)\,ds + \int_\tau^\infty \Phi(\tau,s)[\mathrm{id} - P_s^+]\psi(s)\,ds \right).$$

Proof. Let $\psi \in R(L) \subseteq Z$ and $\xi \in \mathbb{R}^d$. Thanks to the dichotomy assumptions on both semiaxes we know that the bounded (resp. limit zero) forward solutions $\phi^+ : [\tau,\infty) \to \mathbb{R}^d$ to the linear inhomogeneous system

$$\dot{x} = A(t)x + \psi(t) \tag{3.7}$$

are $\phi^+(t) = \Phi(t,\tau)P_\tau^+\xi + \int_\tau^\infty G_{P^+}(t,s)\psi(s)\,ds$, while the corresponding backward solutions $\phi^- : (-\infty,\tau] \to \mathbb{R}^d$ are $\phi^-(t) = \Phi(t,\tau)[\mathrm{id} - P_\tau^-]\xi + \int_{-\infty}^\tau G_{P^-}(t,s)\psi(s)\,ds$. Consequently, the initial values ξ for complete solutions to (3.7) in X can be deduced from the condition $\phi^+(\tau) = \phi^-(\tau)$. This, in turn, is equivalent to

$$\begin{aligned}(P_\tau^+ + P_\tau^- - \mathrm{id})\xi \quad &= \quad \int_{-\infty}^\tau G_{P^-}(\tau,s)\psi(s)\,ds - \int_\tau^\infty G_{P^+}(\tau,s)\psi(s)\,ds\\ &\overset{(3.3)}{=} \quad \int_{-\infty}^\tau \Phi(\tau,s)P_s^-\psi(s)\,ds + \int_\tau^\infty \Phi(\tau,s)(\mathrm{id} - P_s^+)\psi(s)\,ds\end{aligned}$$

and the general solution $\xi \in \mathbb{R}^d$ to this finite-dimensional linear equation is given by (see [CM79, p. 29, Thm. 2.1.2])

$$\begin{aligned}\xi \quad = \quad &[\mathrm{id} - (P_\tau^+ + P_\tau^- - \mathrm{id})^\dagger(P_\tau^+ + P_\tau^- - \mathrm{id})]\eta\\ &+ (P_\tau^+ + P_\tau^- - \mathrm{id})^\dagger \left(\int_{-\infty}^\tau \Phi(\tau,s)P_s^-\psi(s)\,ds + \int_\tau^\infty \Phi(\tau,s)(\mathrm{id} - P_s^+)\psi(s)\,ds \right)\end{aligned}$$

with any $\eta \in \mathbb{R}^d$. Thanks to [CM79, p. 12, Thm. 1.2.2] we have

$$R(\mathrm{id} - (P_\tau^+ + P_\tau^- - \mathrm{id})^\dagger(P_\tau^+ + P_\tau^- - \mathrm{id})) = N(P_\tau^+ + P_\tau^- - \mathrm{id})$$

and consequently $\xi = \sum_{i=1}^n \gamma_i\xi_i + \xi_\tau^*$ with arbitrary coefficients $\gamma_1, \dots, \gamma_n \in \mathbb{R}$. Thus, (3.7) has an n-parameter family of bounded (resp. homoclinic) solutions

$$\phi = \phi_\gamma + \overline{\psi} \quad \text{with } \phi_\gamma := \Phi(\cdot,\tau)\sum_{i=1}^n \gamma_i\xi_i \in N(L)$$

using Prop. 3.1. Due to the direct decomposition $X = N(L) \oplus X_0$ the unique solution in the complement X_0 is given by (3.6). □

We close this section with a prototype example illustrating various situations covered by the above Prop. 3.1 and Cors. 3.2, 3.3 with the aid of a diagonal system in $\mathbb{R}^2$:

Example 3.1. Let $\gamma_-, \beta_-, \gamma_+, \beta_+ \in \mathbb{R}$ be given and suppose $d = 2$. We define a piecewise constant coefficient matrix for the linear equation (L) by

$$A(t) := \begin{pmatrix} b(t) & 0 \\ 0 & c(t) \end{pmatrix}, \quad b(t) := \begin{cases} \beta_-, & t < 0, \\ \beta_+, & t \geq 0, \end{cases} \quad c(t) := \begin{cases} \gamma_-, & t < 0, \\ \gamma_+, & t \geq 0 \end{cases}$$

and easily deduce the transition operator

$$\Phi(t,s) := \begin{cases} \operatorname{diag}(e^{\beta_+(t-s)}, e^{\gamma_+(t-s)}), & t \geq s \geq 0, \\ \operatorname{diag}(e^{\beta_+ t - \beta_- s}, e^{\gamma_+ t - \gamma_- s}), & t \geq 0 > s, \\ \operatorname{diag}(e^{\beta_-(t-s)}, e^{\gamma_-(t-s)}), & 0 > t \geq s; \end{cases}$$

for $t < s$ one extends Φ to $\mathbb{R}^2$ by virtue of $\Phi(t,s) := \Phi(s,t)^{-1}$. We distinguish several cases to describe dichotomy and Fredholm properties of (L) resp. (3.4). In each case, (L) admits an ED on $[0,\infty)$ and $(-\infty, 0]$ with constant projectors P_t^- resp. P_t^+. In the spatial set-up of Prop. 3.1 the operator $L : X \to Z$ is Fredholm and we arrive at:

(a) $\beta_+, \gamma_+ < 0$: $P_t^+ \equiv \mathrm{id}$ on $[0,\infty)$, thus $(P_0^+ + P_0^- - \mathrm{id})^\dagger = (P_0^-)^\dagger$ and L is onto.
  - (a_1) $\beta_-, \gamma_- < 0$: $P_t^- \equiv \mathrm{id}$ on $(-\infty, 0]$, L is invertible, (L) has an ED on $\mathbb{R}$ with projector $P_t \equiv \mathrm{id}$ on $\mathbb{R}$ and (L) is uniformly asymptotically stable
  - (a_2) $\beta_- < 0 < \gamma_-$: $P_t^- \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ on $(-\infty, 0]$ and $\dim N(L) = 1$
  - (a_3) $\gamma_- < 0 < \beta_-$: $P_t^- \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ on $(-\infty, 0]$ and $\dim N(L) = 1$
  - (a_4) $0 < \beta_-, \gamma_-$: $P_t^- \equiv 0$ on $(-\infty, 0]$ and $\dim N(L) = 2$

  In any case, for every $\psi \in Z$ there exists a $\dim N(L)$-parameter family of solutions $\phi \in X$ to the inhomogeneous equation (3.7) and $\operatorname{ind} L = \dim N(L)$

(b) $\beta_+ < 0 < \gamma_+$: (L) admits an ED on $[0,\infty)$ with projector $P_t^+ \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$
  - (b_1) $\beta_-, \gamma_- < 0$: $P_t^- \equiv \mathrm{id}$ on $(-\infty, 0]$, L has 0-dimensional kernel, index -1 and $(P_0^+ + P_0^- - \mathrm{id})^\dagger = P_0^+$
  - (b_2) $\beta_- < 0 < \gamma_-$: $P_t^- \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ on $(-\infty, 0]$, L is invertible, (L) has an ED on $\mathbb{R}$ with $P_t \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $(P_0^+ + P_0^- - \mathrm{id})^\dagger = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$
  - (b_3) $\gamma_- < 0 < \beta_-$: $P_t^- \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ on $(-\infty, 0]$, $\dim N(L) = 1$, $\operatorname{ind} L = 0$ and $(P_0^+ + P_0^- - \mathrm{id})^\dagger = 0$. One can choose $\xi_1 = (1,0)$, $\xi_1' = (0,1)$ and the functional $\mu_1 : Z \to \mathbb{R}$ from Cor. 3.2 reads as

  $$\mu_1(\psi) = \int_{-\infty}^0 e^{-\gamma_- s} \psi_2(s)\, ds + \int_0^\infty e^{-\gamma_+ s} \psi_2(s)\, ds.$$

  Thus, for inhomogeneities $\psi \in Z$ satisfying $\mu_1(\psi) = 0$ the inverse of $L|_{X_0}$ from Cor. 3.3 reads as

  $$\overline{\psi}(t) = \begin{cases} \int_0^t e^{\beta_+(t-s)} \left(\begin{smallmatrix} \psi_1(s) \\ 0 \end{smallmatrix}\right) ds - \int_t^\infty e^{\gamma_+(t-s)} \left(\begin{smallmatrix} 0 \\ \psi_2(s) \end{smallmatrix}\right) ds, & t \geq 0, \\ \int_{-\infty}^t e^{\gamma_-(t-s)} \left(\begin{smallmatrix} 0 \\ \psi_2(s) \end{smallmatrix}\right) ds - \int_t^0 e^{\beta_-(t-s)} \left(\begin{smallmatrix} \psi_1(s) \\ 0 \end{smallmatrix}\right) ds, & t \leq 0 \end{cases}$$

  - (b_4) $0 < \beta_-, \gamma_-$: $P_t^- \equiv 0$ on $(-\infty, 0]$, $\dim N(L) = 1$, $\operatorname{ind} L = 1$ and furthermore $(P_0^+ + P_0^- - \mathrm{id})^\dagger = \left(\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right)$

(c) $\gamma_+ < 0 < \beta_+$: (L) admits an ED on $[0,\infty)$ with projector $P_t^+ = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$
  - (c_1) $\beta_-, \gamma_- < 0$: $P_t^- \equiv \mathrm{id}$ on $(-\infty, 0]$, $\dim N(L) = 0$, $\operatorname{ind} L = -1$ and $(P_0^+ + P_0^- - \mathrm{id})^\dagger = P_0^+$

(c_2) $\beta_- < 0 < \gamma_-$: $P_t^- \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ on $(-\infty, 0]$, $\dim N(L) = 1$, $\operatorname{ind} L = 0$ and $(P_0^+ + P_0^- - \mathrm{id})^\dagger = 0$. The situation is dual to (b_3) and one chooses $\xi_1 = (0, 1)$, $\xi_1' = (1, 0)$. The functional $\mu_1 : Z \to \mathbb{R}$ from Cor. 3.2 reads as

$$\mu_1(\psi) = \int_{-\infty}^{0} e^{-\beta_- s} \psi_1(s)\, ds + \int_0^\infty e^{-\beta_+ s} \psi_1(s)\, ds.$$

With $\psi \in Z$ satisfying $\mu_1(\psi) = 0$ the inverse of $L|_{X_0}$ from Cor. 3.3 is

$$\overline{\psi}(t) = \begin{cases} \int_0^t e^{\gamma_+(t-s)} \left(\begin{smallmatrix} 0 \\ \psi_2(s) \end{smallmatrix}\right) ds - \int_t^\infty e^{\beta_+(t-s)} \left(\begin{smallmatrix} \psi_1(s) \\ 0 \end{smallmatrix}\right) ds, & t \geq 0, \\ \int_{-\infty}^t e^{\beta_-(t-s)} \left(\begin{smallmatrix} \psi_1(s) \\ 0 \end{smallmatrix}\right) ds - \int_t^0 e^{\gamma_-(t-s)} \left(\begin{smallmatrix} 0 \\ \psi_2(s) \end{smallmatrix}\right) ds, & t \leq 0 \end{cases}$$

(c_3) $\gamma_- < 0 < \beta_-$: $P_t^- \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ on $(-\infty, 0]$, L is invertible, (L) has an ED on $\mathbb{R}$ with $P_t \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $(P_0^+ + P_0^- - \mathrm{id})^\dagger = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$

(c_4) $0 < \beta_-, \gamma_-$: $P_t^- \equiv 0$ on $(-\infty, 0]$, $\dim N(L) = 1$, $\operatorname{ind} L = 1$ and furthermore $(P_0^+ + P_0^- - \mathrm{id})^\dagger = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$

(d) $0 < \beta_+, \gamma_+$: $P_t^- \equiv 0$ on $[0, \infty)$, thus $(P_0^+ + P_0^- - \mathrm{id})^\dagger = (P_0^- - \mathrm{id})^\dagger$ and L is one-to-one

(d_1) $\beta_-, \gamma_- < 0$: $P_t^- \equiv \mathrm{id}$ on $(-\infty, 0]$, L has 0-dimensional kernel, index -2 and $(P_0^+ + P_0^- - \mathrm{id})^\dagger = 0$

(d_2) $\beta_- < 0 < \gamma_-$: $P_t^- \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ on $(-\infty, 0]$, $\operatorname{ind} L = -1$ and furthermore $(P_0^+ + P_0^- - \mathrm{id})^\dagger = \left(\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right)$

(d_3) $\gamma_- < 0 < \beta_-$: $P_t^- \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ on $(-\infty, 0]$, $\operatorname{ind} L = -1$ and furthermore $(P_0^+ + P_0^- - \mathrm{id})^\dagger = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$

(d_4) $0 < \beta_-, \gamma_-$: $P_t^- \equiv 0$ on $(-\infty, 0]$, L is invertible, (L) has an ED on $\mathbb{R}$ with $P_t \equiv 0$ and $(P_0^+ + P_0^- - \mathrm{id})^\dagger = -\mathrm{id}$

4. Bifurcation of bounded solutions

Suppose that for a fixed parameter $\lambda^* \in \Lambda$ the CDE $(D)_{\lambda^*}$ has a complete reference solution $\phi^* = \phi(\lambda^*) \in L^\infty(\Omega)$. We say $(D)_\lambda$ undergoes a *bifurcation* at $\lambda = \lambda^*$ along ϕ^*, or ϕ^* *bifurcates* at λ^*, if there exist a convergent parameter sequence $(\lambda_n)_{n \in \mathbb{N}}$ in Λ with limit λ^* so that $(D)_{\lambda_n}$ has two distinct complete solutions $\phi_{\lambda_n}^1, \phi_{\lambda_n}^2 \in L^\infty$ both satisfying

$$\lim_{n \to \infty} \phi_{\lambda_n}^1 = \lim_{n \to \infty} \phi_{\lambda_n}^2 = \phi^*.$$

In other words, the pair (ϕ^*, λ^*) is a *bifurcation point* of the abstract nonlinear equation $(O)_\lambda$ in $W^{1,\infty}(\Omega)$ (cf. [Zei93, p. 358, Defn. 8.1]).

Bifurcation properties of a solution ϕ^* crucially depend on the *variational equation*

$$\dot{x} = D_2 f(t, \phi^*(t), \lambda^*) x \tag{V}$$

with associated dichotomy spectrum $\Sigma(\lambda^*)$ (cf. [SS78, Sie02]) and transition operator Φ.

In this context, we say the solution ϕ^* is *hyperbolic*, provided (V) has an ED on the whole axis $\mathbb{R}$ or equivalently $0 \notin \Sigma(\lambda^*)$. On the other hand, nonhyperbolicity is a necessary condition for bifurcation, which also applies to complete solutions in $W_0^{1,\infty}(\Omega)$, if (H_1) holds — a proof follows the lines of [Pöt10a, Prop. 3.6]. Next we will also give sufficient conditions for bifurcations, where we restrict to real parameters $\lambda \in \mathbb{R}$.

Hypothesis. *Let $\tau \in \mathbb{R}$, $\lambda^* \in \Lambda \subseteq \mathbb{R}$ be given and suppose $(D)_{\lambda^*}$ admits a complete permanent solution $\phi^* \in L^\infty(\Omega)$ with*

(H_2) *the variational equation* (V) *admits an ED both on* $[\tau,\infty)$ *and* $(-\infty,\tau]$ *with respective projectors* P_t^+ *and* P_t^- *so that there exist* $\xi_1,\xi_1'\in\mathbb{R}^d$ *with*

$$R(P_\tau^+)\cap N(P_\tau^-)=\operatorname{span}\{\xi_1\}, \qquad (R(P_\tau^+)+N(P_\tau^-))^\perp=\operatorname{span}\{\xi_1'\}. \tag{4.1}$$

In order to fulfill (H_2) the state space $\Omega\subseteq\mathbb{R}^d$ has to be at least 2-dimensional. Moreover, since the projectors P_τ^-, P_τ^+ need to be nontrivial so that (4.1) holds, our reference solution ϕ^* is unstable a priori. Note that one cannot fulfill (H_2) for almost periodic variational equations, since in this class an ED on $[\tau,\infty)$ already implies that ϕ^* is hyperbolic (see [Cop78, p. 70, Prop. 3]).

The above assumption allows a local dynamical insight into the abstract results from Thm. A.1 and A.2 applied to $(O)_\lambda$ resp. $(D)_\lambda$: Because (V) admits an ED on $[\tau,\infty)$ there exists a stable integral manifold $\phi^*+\mathcal{W}_{\lambda^*}^+$ consisting of all solutions to $(D)_{\lambda^*}$ approaching ϕ^* in forward time, and $\mathcal{W}_{\lambda^*}^+$ is locally a graph over the stable vector bundle $\mathcal{V}_{\lambda^*}^+$. Analogously, the ED on $(-\infty,\tau]$ guarantees an unstable integral manifold $\phi^*+\mathcal{W}_{\lambda^*}^-$ consisting of solutions decaying to ϕ^* in backward time. Both integral manifolds persist under variation of λ near λ^* as $\phi^*+\mathcal{W}_\lambda^+$ and $\phi^*+\mathcal{W}_\lambda^-$ (cf. [Pöt10b, Cor. 3.11]). Then the bounded complete solutions to $(D)_\lambda$ are contained in $(\phi^*+\mathcal{W}_\lambda^+)\cap(\phi^*+\mathcal{W}_\lambda^-)$. We conclude that the intersection of the corresponding fibers

$$S(\lambda):=\left(\phi^*(\tau)+\mathcal{W}_\lambda^+(\tau)\right)\cap\left(\phi^*(\tau)+\mathcal{W}_\lambda^-(\tau)\right)\subseteq\Omega$$

yields initial values for bounded complete solutions (see Figure 1). Here, a bifurcation means a (topological) change in the intersection $S(\lambda)$ for λ near λ^*.

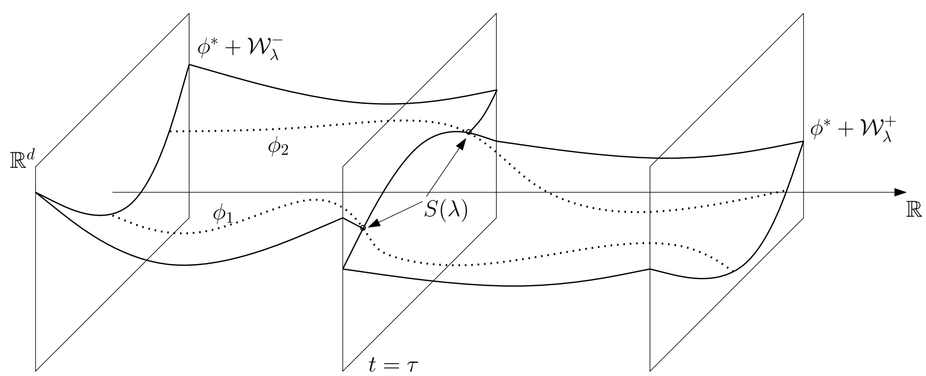

FIGURE 1. Intersection $S(\lambda)\subseteq\Omega$ of the stable integral manifold $\phi^*+\mathcal{W}_\lambda^+\subseteq[\tau,\infty)\times\mathbb{R}^d$ with the unstable integral manifold $\phi^*+\mathcal{W}_\lambda^-\subseteq(-\infty,\tau]\times\mathbb{R}^d$ at $t=\tau$ yields two bounded complete solutions ϕ_1,ϕ_2 to $(D)_\lambda$ indicated as dotted lines

Using this geometric intuition we arrive at our first bifurcation result. It replaces the trivial solution branch typically imposed for transcritical or pitchfork bifurcations by a local condition on the partial derivatives w.r.t. the parameter:

Theorem 4.1 (crossing curve bifurcation). *Let* $m\geq 2$, $\phi^*\in X=W^{1,\infty}$ *and suppose that* (H_0), (H_2) *are fulfilled with* $D_3f(t,\phi^*(t),\lambda^*)\equiv 0$ *on* $\mathbb{R}$,

$$g_{02}:=\int_{\mathbb{R}}\langle\Phi(\tau,s)'\xi_1', D_3^2f(s,\phi^*(s),\lambda^*)\rangle\,ds=0. \tag{4.2}$$

If the transversality condition

$$g_{11} := \int_{\mathbb{R}} \langle \Phi(\tau,s)'\xi_1', D_2D_3f(s,\phi^*(s),\lambda^*)\Phi(s,\tau)\xi_1\rangle\, ds \neq 0 \tag{4.3}$$

holds, then the complete solution ϕ^* of $(D)_{\lambda^*}$ bifurcates at λ^*. In detail, there exist open convex neighborhoods $S \subseteq \mathbb{R}$ of 0, $U_1 \times U_2 \subseteq X(\Omega) \times \Lambda$ of (ϕ^*, λ^*) and C^{m-1}-curves $\gamma_1, \gamma_2 : S \to U_1 \times U_2$ with the following properties:

(a) *The set of bounded complete solutions for $(D)_\lambda$ in U_1 is given by the set intersection $(\gamma_1(S) \cup \gamma_2(S)) \cap X \times \{\lambda\}$ (see Fig. 2).*

(b) *$\gamma_1(s) = (\gamma(s), \lambda^* + s)$ with $\gamma_1(0) = (\phi^*, \lambda^*)$, $\dot\gamma(0) = 0$ and*

$$\gamma_2(0) = \begin{pmatrix}\phi^*\\ \lambda^*\end{pmatrix}, \qquad \dot\gamma_2(0) = \begin{pmatrix}\Phi(\cdot,\tau)\xi_1\\ -\frac{h_{22}}{2g_{11}}\end{pmatrix}$$

where $h_{22} := \int_{\mathbb{R}} \langle \Phi(\tau,s)'\xi_1', D_2^2f(s,\phi^*(s),\lambda^*)[\Phi(s,\tau)\xi_1]^2\rangle\, ds$.

If (H_0) to (H_2) are satisfied, then the same holds with $X = W_0^{1,\infty}$.

Remark 4.1. If the complete solution ϕ^* is embedded into a branch of trivial solutions to $(D)_\lambda$, then (4.2) is automatically fulfilled and γ_1 resp. γ represents the zero branch. In this sense, Thm. 4.1 generalizes [Pöt10a, Thm. 3.14 and Cors. 3.15, 3.16]. Moreover, the direction of the crossing curve bifurcation from Thm. 4.1 is given by the coefficient $\frac{h_{22}}{2g_{11}}$:

(1) For $h_{22} \neq 0$ there are locally exactly two complete solutions to $(D)_\lambda$ in $W^{1,\infty}(\Omega)$ for $\lambda \neq \lambda^*$; these solutions are in $X(\Omega)$. This yields a transcritical pattern (see Figure 2 (left) and Cor. A.3).

(2) In the degenerate case $h_{22} = 0$ we assume a higher order condition

$$\begin{aligned} g_{30} := &\int_{\mathbb{R}} \langle \Phi(\tau,s)'\xi_1', D_2^3f(s,\phi^*(s),\lambda^*)[\Phi(s,\tau)\xi_1]^3\rangle\, ds \\ &- 3\int_{\mathbb{R}} \langle \Phi(\tau,s)'\xi_1', D_2^2f(s,\phi^*(s),\lambda^*)\Phi(s,\tau)\xi_1\overline{D_2^2f(s,\phi^*(s),\lambda^*)[\Phi(s,\tau)\xi_1]^2}\rangle\, ds \end{aligned}$$

with $\overline{\psi(s)}$ defined in Cor. 3.3, yielding a pitchfork pattern (see Figure 2 (right) and Cor. A.4):

(a) For $g_{30}/g_{11} < 0$ (*supercritical case*) there is a unique complete solution of $(D)_\lambda$ in $W^{1,\infty}(\Omega)$ for parameters $\lambda \leq \lambda^*$ and $(D)_\lambda$ has exactly three complete solutions in $W^{1,\infty}(\Omega)$ for $\lambda > \lambda^*$.

(b) For $g_{30}/g_{11} > 0$ (*subcritical case*) there is a unique complete solution of $(D)_\lambda$ in $W^{1,\infty}(\Omega)$ for parameters $\lambda \geq \lambda^*$ and $(D)_\lambda$ has exactly three complete solutions in $W^{1,\infty}(\Omega)$ for $\lambda < \lambda^*$.

The complete solutions are in $X(\Omega)$.

Proof. Since the solution ϕ^* is permanent, ϕ^* is an interior point of $X(\Omega)$. Thanks to (H_0) the coefficient matrix in (V) fulfills (3.1).

(I) We use Thm. A.2 with $X = W^{1,\infty}$, $Z := L^\infty$ applied to the abstract equation $(O)_\lambda$. Since $\phi^* \in L^\infty(\Omega)$ solves $(D)_{\lambda^*}$ we know from Thm. 2.3(a) that $G(\phi^*, \lambda^*) = 0$, i.e. (A.1) holds with $x_0 = \phi^*$ and $\lambda_0 = \lambda^*$. The assumption (H_2) guarantees that Prop. 3.1 can be applied to the variational equation (V) with $n = r = 1$. Due to the identity

$$[D_1G(\phi^*,\lambda^*)\psi](t) \equiv \dot\psi(t) - [F^{(1,0)}(\phi^*,\lambda^*)\psi](t) \equiv \dot\psi(t) - D_2f(t,\phi^*(t),\lambda^*)\psi(t)$$

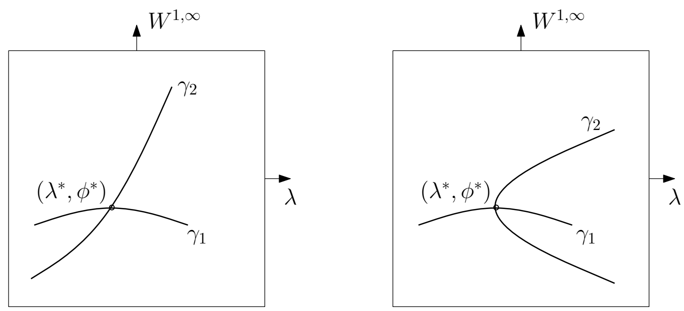

FIGURE 2. Crossing curve bifurcation of Thm. 4.1: Generic case of a transcritical situation (left) and the degenerate case of a supercritical pitchfork situation (right)

a.e. for $t \in \mathbb{R}$ (cf. Prop. 2.1) we therefore obtain that also (A.2) holds with $x_1 = \Phi(0, \cdot)\xi_1$. In addition,

$$[D_2G(\phi^*, \lambda^*)](t) = -D_3f(t, \phi^*(t), \lambda^*) = 0 \quad \text{a.e. in } \mathbb{R}.$$

Thanks to our Cor. 3.2 on the functional $\mu_1 : Z \to \mathbb{R}$, the condition (A.8) takes the form (4.2) and (4.3). Then the claim for $X = W^{1,\infty}$ follows using both Thm. A.2 and 2.3(a).

(II) When dealing with a homoclinic solution ϕ^*, i.e. for spaces $X = W_0^{1,\infty}$, $Z := L_0^\infty$, the proof follows along the same lines using Thm. 2.3(b). Yet, we can apply Thm. A.2 twice: Step (I) shows that the bifurcating solutions are unique in the large space $W^{1,\infty}$ of bounded functions, while itself being in $W_0^{1,\infty}$. □

We illustrate Thm. 4.1 and our further results using a class of examples, whose linear parts are nonhyperbolic in the sense that they match the assumptions of Exam. 3.1(b_3).

Example 4.1 (quadratic perturbations). For $\alpha > 0$ we consider the planar CDE

$$\dot{x} = f(t, x, \lambda) = \begin{pmatrix} b(t) & \\ & c(t) \end{pmatrix} x + g(t, x, \lambda) \tag{4.4}$$

depending on $\lambda \in \mathbb{R}$ as bifurcation parameter, measurable functions

$$b(t) := \begin{cases} \alpha, & t < 0, \\ -\alpha, & t \geq 0, \end{cases} \qquad c(t) := \begin{cases} -\alpha, & t < 0, \\ \alpha, & t \geq 0 \end{cases} \tag{4.5}$$

and a quadratic nonlinearity $g : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ given by

$$g(t, x, \lambda) := \lambda \begin{pmatrix} b_1(t) & b_2(t) \\ c_1(t) & c_2(t) \end{pmatrix} x + \begin{pmatrix} b_{20}(t)x_1^2 + b_{11}(t)x_1x_2 + b_{02}(t)x_2^2 \\ c_{20}(t)x_1^2 + c_{11}(t)x_1x_2 + c_{02}(t)x_2^2 \end{pmatrix} + \lambda^2 \begin{pmatrix} b_0(t) \\ c_0(t) \end{pmatrix}$$

with coefficients $b_i, c_i \in L^\infty$ for $i = 0, 1$ and $b_{ij}, c_{ij} \in L^\infty$ for $i, j = 0, 1, 2$ satisfying

$$\int_{\mathbb{R}} e^{-\alpha|s|} c_0(s)\, ds = 0, \qquad \int_{\mathbb{R}} e^{-2\alpha|s|} c_1(s)\, ds \neq 0; \tag{4.6}$$

note that the first condition in (4.6) holds for odd functions c_0. We conclude that (4.4) fulfills (H_0) with $\Omega = \mathbb{R}^2$ but not necessarily (H_1); so choose $X = W^{1,\infty}$ and $Z = L^\infty$.

We observe that the linear part of (4.4) is nonhyperbolic for $\lambda = 0$, since the dichotomy spectrum is $\Sigma(0) = [-\alpha, \alpha]$. Indeed, from Exam. 3.1(b_3) we see that (4.4) satisfies (H_2) with $\tau = 0$, $\phi^*(t) \equiv 0$ on $\mathbb{R}$, $\lambda^* = 0$ and dichotomy projectors $P_t^+ \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $P_t^- \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Thus, we choose $\xi_1 = \binom{1}{0}$ and $\xi_1' = \binom{0}{1}$ and the functional $\mu_1 : L^\infty \to \mathbb{R}$ from Cor. 3.2, which characterizes $R(L)$, is

$$\mu_1(\psi) = \int_{\mathbb{R}} e^{-\alpha|s|}\psi_2(s)\,ds.$$

Now we compute the derivatives required to apply Thm. 4.1 and obtain

$$D_3 f(s,0,0) = \begin{pmatrix}0\\0\end{pmatrix}, \qquad D_3^2 f(s,0,0) = 2\begin{pmatrix}b_0(s)\\c_0(s)\end{pmatrix},$$

$$D_2D_3 f(s,0,0)[\Phi(s,0)\xi_1] = e^{-\alpha|s|}\begin{pmatrix}b_1(s)\\c_1(s)\end{pmatrix},$$

$$D_2^2 f(s,0,0)[\Phi(s,0)\xi_1]^2 = 2e^{-2\alpha|s|}\begin{pmatrix}b_{20}(s)\\c_{20}(s)\end{pmatrix},$$

which in turn yields the coefficients

$$g_{02} = 2\int_{\mathbb{R}} e^{-\alpha|s|}c_0(s)\,ds, \quad g_{11} = \int_{\mathbb{R}} e^{-2\alpha|s|}c_1(s)\,ds, \quad h_{22} = \int_{\mathbb{R}} e^{-3\alpha|s|}c_{20}(s)\,ds.$$

Thanks to the assumptions (4.6) our Thm. 4.1 is applicable and yields that the zero solution to (4.4) for $\lambda = 0$ is at the intersection of two branches of bounded complete solutions. More precisely, there exists a neighborhood $S \subseteq \mathbb{R}$ of 0 such that (4.4) for $\lambda = s$ has the complete solution $\phi_1(s) = \gamma(s) \in W^{1,\infty}$, while for parameters $\lambda = -\frac{h_{22}}{2g_{11}}s + o(s)$ the equation (4.4) admits complete solutions of the form

$$\phi_2(s) = s\begin{pmatrix}e^{-\alpha|\cdot|}\\0\end{pmatrix} + o(s) \in W^{1,\infty} \quad \text{for all } s \in S.$$

In addition, every bounded complete solution to (4.4) in a neighborhood of 0 is either $\phi_1(s)$ or $\phi_2(s)$. Finally, in case $b_0, c_0 \in L_0^\infty$ our assumption (H_1) is fulfilled and the bifurcating solutions $\phi_1(s), \phi_2(s)$, $s \in S$, are homoclinic.

Being a quadratic system, the CDE from Exam. 4.1 cannot fulfill the assumptions for a pitchfork bifurcation as stated in Rem. 4.1(2). Yet we discuss a minimal cubic system exhibiting this pattern:

Example 4.2 (minimal pitchfork bifurcation). In this example, let $\alpha > 0$ and $\gamma, \delta, \varepsilon \in \mathbb{R}$ with $\gamma, \delta \neq 0$ be real parameters. We consider the planar asymptotically autonomous CDE

$$\dot{x} = f(t,x,\lambda) = \begin{pmatrix}b(t) & 0\\ \gamma\lambda & c(t)\end{pmatrix}x + \delta\begin{pmatrix}0\\x_1^3\end{pmatrix} + \varepsilon\begin{pmatrix}0\\\lambda^3\end{pmatrix} \tag{4.7}$$

depending on a bifurcation parameter $\lambda \in \mathbb{R}$ and coefficients c, d defined in (4.5). The right hand side of (4.7) fulfills (H_0) with $\Omega = \mathbb{R}^2$ but not (H_1); we also see that (H_2) holds with the same data as in Exam. 4.1. In order to apply Thm. 4.1 we observe $D_2 f(s,0,0) \equiv 0$,

$$D_3^2 f(s,0,0) = \begin{pmatrix}0\\0\end{pmatrix}, \qquad D_2D_3 f(s,0,0)[\Phi(s,0)\xi_1] = \gamma\begin{pmatrix}0\\e^{-\alpha|s|}\end{pmatrix},$$

$$D_2^2 f(s,0,0)[\Phi(s,0)\xi_1]^2 = \begin{pmatrix}0\\0\end{pmatrix}, \qquad D_2^3 f(s,0,0)[\Phi(s,0)\xi_1]^3 = 6\delta\begin{pmatrix}0\\e^{-3\alpha|s|}\end{pmatrix}$$

a.e. for $s \in \mathbb{R}$. Note that Exam. 3.1(b_3) ensures $\overline{D_2^2 f(s,0,0)[\Phi(s,0)\xi_1]^2} = 0$ using the notation of Cor. 3.3. This yields the relations

$$g_{02} = 0, \qquad g_{11} = \tfrac{\gamma}{\alpha} \neq 0, \qquad h_{22} = 0, \qquad g_{30} = \tfrac{3\delta}{\alpha}$$

and so there is a crossing curve bifurcation pattern near the trivial solution ϕ^* of (4.7) for $\lambda = 0$. Therefore, ϕ^* is at the intersection of two branches of bounded complete solutions

$$\phi_1(s) = \gamma(s) \quad \text{for all } \lambda = s, \qquad \phi_2(s) = s\begin{pmatrix} e^{-\alpha|\cdot|} \\ 0 \end{pmatrix} + o(s) \quad \text{for all } \lambda = o(s)$$

with $s \in S$. More precisely, following Rem. 4.1(2) there is a pitchfork bifurcation of $\phi^* = 0$ with $\frac{g_{30}}{g_{11}} = \frac{3\delta}{\gamma}$; in particular, 0 is the unique bounded complete solution to (4.7) for $\lambda = 0$. If δ, γ have the same sign, the bifurcation is subcritical and otherwise supercritical.

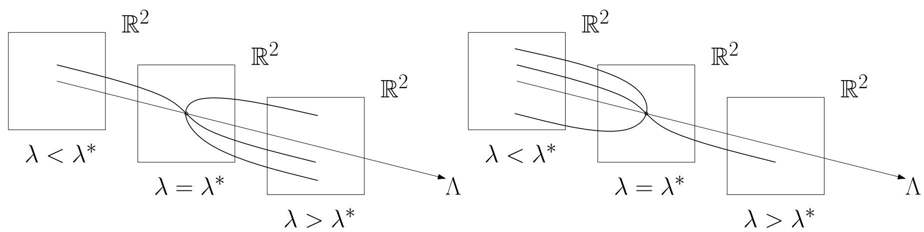

FIGURE 3. Left ($\frac{\gamma}{\delta} > 0$): Initial values for a subcritical pitchfork bifurcation of a bounded complete solution of (4.9) at $\lambda = \lambda^*$.
Right ($\frac{\gamma}{\delta} < 0$): Initial values for a supercritical pitchfork bifurcation of a bounded complete solution of (4.9) at $\lambda = \lambda^*$.

On the other hand, equation (4.7) also allows an explicit solution: The general solution φ_λ to (4.7) has the first component $\varphi_\lambda^1(t; 0, \eta) = e^{-\alpha|t|}\eta_1$ for all $t \in \mathbb{R}$. The second component follows by the variation of constants formula (cf. [AW96, Thm. 2.10]) applied to the scalar equation $\dot{x}_2 = c(t)x_2 + \gamma\lambda\varphi_\lambda^1(t; 0, \eta) + \delta\varphi_\lambda^1(t; 0, \eta)^3 + \varepsilon\lambda^3$ as:

$$\varphi_\lambda^2(t; 0, \eta) = e^{\alpha|t|}\eta_2 + \int_0^t e^{\alpha|t-s|}\left(\gamma\lambda e^{-\alpha|s|}\eta_1 + \delta e^{-3\alpha|s|}\eta_1^3 + \varepsilon\lambda^3\right) ds \quad \text{for all } t \in \mathbb{R}.$$

This implies the asymptotic representation

$$\varphi_\lambda^2(t; 0, \eta) = \begin{cases} e^{\alpha t}\left(\eta_2 + \frac{\lambda\gamma}{2\alpha}\eta_1 + \frac{\delta}{4\alpha}\eta_1^3 + \frac{\varepsilon\lambda^3}{\alpha}\right) + O(t) \text{ as } t \to \infty, \\ e^{-\alpha t}\left(\eta_2 - \frac{\lambda\gamma}{2\alpha}\eta_1 - \frac{\delta}{4\alpha}\eta_1^3 - \frac{\varepsilon\lambda^3}{\alpha}\right) + O(t) \text{ as } t \to -\infty \end{cases}$$

from which we derive the 0-fibers

$$\mathcal{W}_\lambda^+(0) := \left\{\left(\eta_1, -\tfrac{\lambda\gamma}{2\alpha}\eta_1 - \tfrac{\delta}{4\alpha}\eta_1^3 - \tfrac{\varepsilon\lambda^3}{\alpha}\right) : \eta_1 \in \mathbb{R}\right\},$$
$$\mathcal{W}_\lambda^-(0) := \left\{\left(\eta_1, \tfrac{\lambda\gamma}{2\alpha}\eta_1 + \tfrac{\delta}{4\alpha}\eta_1^3 + \tfrac{\varepsilon\lambda^3}{\alpha}\right) : \eta_1 \in \mathbb{R}\right\}$$

of the stable resp. unstable integral manifolds of (4.7). In order to determine their intersection, i.e. initial values $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$ for globally bounded solutions, one solves the nonlinear equations $\eta_2 = -\frac{\lambda\gamma}{2\alpha}\eta_1 - \frac{\delta}{4\alpha}\eta_1^3 - \frac{\varepsilon\lambda^3}{\alpha}$, $\eta_2 = \frac{\lambda\gamma}{2\alpha}\eta_1 + \frac{\delta}{4\alpha}\eta_1^3 + \frac{\varepsilon\lambda^3}{\alpha}$, or equivalently

$$\frac{\lambda\gamma}{2}\eta_1 + \frac{\delta}{4}\eta_1^3 + \varepsilon\lambda^3 = 0, \qquad \eta_2 = 0.$$

The cubic equation for η_1 has the discriminant $\Delta := -\left(\frac{\delta\gamma\lambda}{4}\right)^2\left[27\left(\frac{\varepsilon\lambda^2}{\gamma}\right)^2 + 2\lambda\frac{\gamma}{\delta}\right]$ and for parameters λ close to the critical value $\lambda^* = 0$ one has $\operatorname{sgn}\Delta = -\operatorname{sgn}(\lambda\frac{\gamma}{\delta})$. Thus, in case δ, γ have the same sign, it follows $\Delta > 0$ for $\lambda < 0$ guaranteeing three distinct initial value pairs η yielding bounded complete solutions (subcritical case). On the other hand, in case

δ, γ having different signs, one has $\Delta > 0$ for $\lambda > 0$ and the bifurcation is supercritical. This corresponds to the conclusion of Thm. 4.1 and we refer to Fig. 3 for an illustration.

An interesting observation can be made for $\varepsilon = 0$. Then one of the crossing curves is the trivial solution branch and the corresponding linearization $\dot{x} = \left(\begin{smallmatrix} b(t) & 0 \\ \gamma\lambda & c(t) \end{smallmatrix}\right)x$ has constant dichotomy spectrum $\Sigma(\lambda) = [-\alpha, \alpha]$ for all $\lambda \in \mathbb{R}$. Hence, the pitchfork bifurcation occurs without a change in $\Sigma(\lambda)$ alone.

Next we tackle a nongeneric fold bifurcation, where the second order degeneracy condition (4.2) is replaced by a first order one:

Theorem 4.2 (degenerate fold bifurcation)**.** *Let $m \geq 2$, $\phi^* \in X = W^{1,\infty}$, suppose that (H_0), (H_2) are fulfilled with*

$$g_{01} := \int_{\mathbb{R}} \langle \Phi(\tau, s)'\xi_1', D_3 f(s, \phi^*(s), \lambda^*)\rangle \, ds = 0 \tag{4.8}$$

and define

$$\begin{aligned} h_{11} &:= \int_{\mathbb{R}} \langle \Phi(\tau, s)'\xi_1', D_3^2 f(s, \phi^*(s), \lambda^*) - 2D_2 D_3 f(s, \phi^*(s), \lambda^*)\overline{D_3 f(s, \phi^*(s), \lambda^*)} \\ &\qquad + D_2^2 f(s, \phi^*(s), \lambda^*)[\overline{D_3 f(s, \phi^*(s), \lambda^*)}]^2\rangle \, ds, \\ h_{12} &:= \int_{\mathbb{R}} \langle \Phi(\tau, s)'\xi_1', D_2 D_3 f(s, \phi^*(s), \lambda^*)\Phi(s, \tau)\xi_1 \\ &\qquad + D_2^2 f(s, \phi^*(s), \lambda^*)\Phi(s, \tau)\xi_1 \overline{D_3 f(s, \phi^*(s), \lambda^*)}\rangle \, ds, \end{aligned}$$

with the function $\overline{\psi(s)}$ defined in Cor. 3.3. If $h_{11}h_{22} \neq h_{12}^2$, then the complete solution $\phi^* \in X$ of $(D)_{\lambda^*}$ bifurcates at λ^*. In detail, there exist open convex neighborhoods $S \subseteq \mathbb{R}$ of 0 and $U_1 \times U_2 \subseteq X(\Omega) \times \Lambda$ of (ϕ^*, λ^*) such that the following holds:

(a) *For $h_{11}h_{22} > h_{12}^2$ the unique complete solution of $(D)_{\lambda^*}$ in $X(\Omega)$ is ϕ^*, whereas equation $(D)_\lambda$ has no solution in $X(\Omega)$ for $\lambda \neq \lambda^*$ (see Fig. 4 (left)),*

(b) *for $h_{11}h_{22} < h_{12}^2$ there exist two C^{m-1}-curves $\gamma_1, \gamma_2 : S \to U_1 \times U_2$ such that the set of complete solutions for $(D)_\lambda$ in $W^{1,\infty}(\Omega)$ is given by the intersection $\big((\gamma_1(S) \cup \gamma_2(S)) \cap X\big) \times \{\lambda\}$ (see Fig. 4 (right)). One has the representation*

$$\gamma_i(0) = \begin{pmatrix} \phi^* \\ \lambda^* \end{pmatrix}, \qquad \dot{\gamma}_i(0) = \begin{pmatrix} \rho_i \Phi(\cdot, \tau)\xi_1 \\ \nu_i \end{pmatrix} \quad \text{for all } i \in \{1, 2\},$$

where $(\rho_1, \nu_1), (\rho_2, \nu_2) \in \mathbb{R}^2$ are nonzero linearly independent solutions of

$$h_{11}\nu^2 + 2h_{12}\rho\nu + h_{22}\rho^2 = 0.$$

If (H_0) to (H_2) are satisfied, then the same holds with $X = W_0^{1,\infty}$.

Proof. The proof is very similar to the one of Thm. 4.1 and thus kept shorter. One applies Thm. A.1 with $X = W^{1,\infty}$, $Z := L^\infty$ to (2.3). Our assumption (H_2) implies (A.1), (A.2) and (A.4) is guaranteed by (4.8). Due to Prop. 2.1 the abstract equation (A.5) reads as

$$\dot{x} = D_2 f(t, \phi^*(t), \lambda^*)x + D_3 f(t, \phi^*(t), \lambda^*)$$

and by Cor. 3.3 its unique solution in Z is given by $\overline{D_3 f(\cdot, \phi^*(\cdot), \lambda^*)}$. Having this at hand, the matrix from Thm. A.1 reads as $H(\phi^*, \lambda^*) = \left(\begin{smallmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{smallmatrix}\right)$. Hence, Thm. A.1 is applicable and implies the claim. $\square$

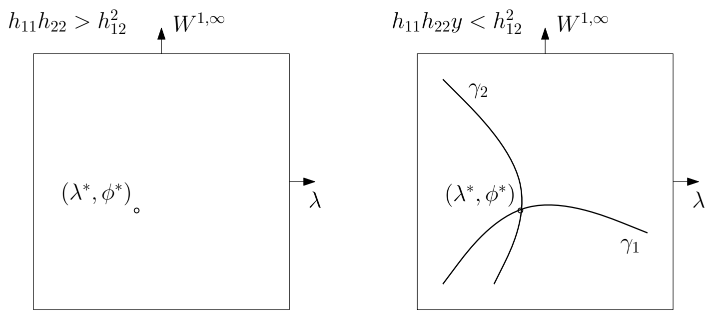

FIGURE 4. Degenerate fold bifurcation of Thm. 4.2:
Definite case $h_{11}h_{22} > h_{12}^2$ (left), where the pair (ϕ^*, λ^*) is isolated.
Indefinite case $h_{11}h_{22} < h_{12}^2$ (right), where (ϕ^*, λ^*) is a transversal intersection for two branches of bounded complete solutions in $W^{1,\infty}$

Example 4.3 (quadratic perturbation). The situation is similar to Exam. 4.1. We perturb its linear part by a general quadratic system. Thereto, given $\alpha > 0$ consider a planar CDE

$$\dot{x} = f(t, x, \lambda) = \begin{pmatrix} b(t) & 0 \\ 0 & c(t) \end{pmatrix} x + \lambda g(t, x) \tag{4.9}$$

depending on a bifurcation parameter $\lambda \in \mathbb{R}$ and functions $b, c : \mathbb{R} \to \mathbb{R}$ given by (4.5). Exactly as in Exam. 4.1 our Hypothesis (H_1) is satisfied with $\phi^* = 0$, $\lambda^* = 0$ and dichotomy projectors $P_t^+ \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $P_t^- \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. It is $\left[P_0^+ + P_0^- - \mathrm{id}\right]^\dagger = 0^\dagger = 0$ and $\xi_0^* = 0$ for the vector from (3.6).

We study (4.9) with general quadratic perturbations $g : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, whose coefficient functions are essentially bounded, $b_i, c_i \in L^\infty$ for $i = 0, 1$ and $b_{ij}, c_{ij} \in L^\infty$ for $i, j = 0, 1, 2$. As consequence, the right hand side of (4.9) fulfills (H_0) and not (H_1). In addition, suppose an inhomogeneity in g acts only on the first component, i.e. we have

$$g(t, x) := \begin{pmatrix} b_1(t)x_1 + b_2(t)x_2 + b_{20}(t)x_1^2 + b_{11}(t)x_1x_2 + b_{02}(t)x_2^2 \\ c_1(t)x_1 + c_2(t)x_2 + c_{20}(t)x_1^2 + c_{11}(t)x_1x_2 + c_{02}(t)x_2^2 \end{pmatrix} + \begin{pmatrix} b_0(t) \\ 0 \end{pmatrix}$$

and we assume that

$$\int_{\mathbb{R}} e^{-\alpha|s|} c_1(s)\, ds \neq 0; \tag{4.10}$$

for instance, c_1 is not an odd function. From the explicit form $D_3 f(s, 0, 0) = \left(\begin{smallmatrix} b_0(s) \\ 0 \end{smallmatrix}\right)$ and Cor. 3.2 we obtain $D_3 f(\cdot, 0, 0) \in R(L)$. By Exam. 3.1$(b_3)$ the relation (3.6) reduces to

$$\overline{D_2 f(s, 0, 0)} = \begin{cases} \begin{pmatrix} \int_0^t e^{-\alpha(t-s)} b_0(s)\, ds \\ 0 \end{pmatrix}, & t \geq 0, \\ \begin{pmatrix} -\int_t^0 e^{\alpha(t-s)} b_0(s)\, ds \\ 0 \end{pmatrix}, & t < 0 \end{cases} = \begin{pmatrix} \int_0^t e^{-\alpha|t-s|} b_0(s)\, ds \\ 0 \end{pmatrix}.$$

Furthermore, we compute the derivatives $D_3^2 f(s, 0, 0) = 0$, $D_2^2 f(s, 0, 0) = 0$ and

$$D_2 D_3 f(s, 0, 0)\zeta = \begin{pmatrix} b_1(s)\zeta_1 + b_2(s)\zeta_2 \\ c_1(s)\zeta_1 + c_2(s)\zeta_2 \end{pmatrix} \quad \text{for all } \zeta \in \mathbb{R}^2,$$

consequently,

$$D_2 D_3 f(s, 0, 0)\overline{D_3 f(s, 0, 0)} = \int_0^s e^{-\alpha|t-r|} b_0(r)\, dr \begin{pmatrix} b_1(s) \\ c_1(s) \end{pmatrix},$$

$$D_2D_3f(s,0,0)\Phi(s,0)\xi_1 = e^{-\alpha|s|}\begin{pmatrix} b_1(s) \\ c_1(s) \end{pmatrix},$$

which implies $g_{01} = 0$, $h_{12} = \int_{\mathbb{R}} e^{-2|s|}c_1(s)\,ds$, $h_{22} = 0$ and

$$h_{11} = -2\int_{\mathbb{R}} e^{-\alpha|s|}c_1(s)\int_0^s e^{-\alpha|s-r|}b_0(r)\,dr\,ds.$$

Due to our generic assumption (4.10) we are in the scope of Thm. 4.2(b) and choose pairs $(\rho_1,\nu_1) = (1,0)$, $(\rho_2,\nu_2) = (-\frac{h_{11}}{2h_{12}},1)$. Thus, there exists a neighborhood $S \subseteq \mathbb{R}$ of 0 such that the solutions of (4.9)

- with $\lambda = o(s)$ are of the form $\phi_1(s) = s\begin{pmatrix} e^{-\alpha|\cdot|} \\ 0 \end{pmatrix} + o(s)$
- with $\lambda = s + o(s)$ are of the form $\phi_2(s) = -\frac{h_{11}}{2h_{12}}\begin{pmatrix} e^{-\alpha|\cdot|} \\ 0 \end{pmatrix} + o(s)$.

In case of inhomogeneities $b_0 \in L_0^\infty$ these solutions $\phi_1(s), \phi_2(s)$ are homoclinic.

The previous Exam. 4.3 was not able to cover the full scope of Thm. 4.2, since it only reproduced the case (b). For the other alternative (a) we refer to

Example 4.4. With given $\alpha > 0$ and $\delta, \varepsilon \in \mathbb{R}$ with $\delta, \varepsilon \neq 0$ we consider a planar CDE

$$\dot{x} = f(x,\lambda) = \begin{pmatrix} b(t) & 0 \\ 0 & c(t) \end{pmatrix} x + \delta\begin{pmatrix} 0 \\ x_1^2 \end{pmatrix} + \varepsilon\begin{pmatrix} 0 \\ \lambda^2 \end{pmatrix} \tag{4.11}$$

depending on a bifurcation parameter $\lambda \in \mathbb{R}$ and functions $b, c : \mathbb{R} \to \mathbb{R}$ from (4.5). Again, the right hand side of (4.11) fulfills (H_0), not (H_1) and as in Exam. 4.1 the assumption (H_2) holds with the same data. The derivatives of the right hand side of (4.11) are

$$D_3f(s,0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad D_3^2f(s,0,0) = \begin{pmatrix} 0 \\ 2\varepsilon \end{pmatrix}, \qquad D_3^3f(s,0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and consequently we see from (3.6) and Exam. 3.1(b_3) that $\overline{D_3f(s,0,0)}$ is the zero function. Thus, it remains to compute the derivatives

$$D_2D_3f(s,0,0)\Phi(s,0)\xi_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad D_2^2f(s,0,0)[\Phi(s,0)\xi_1]^2 = 2\delta\begin{pmatrix} 0 \\ e^{-2\alpha|s|} \end{pmatrix}$$

a.e. for $s \in \mathbb{R}$. This yields the relations

$$g_{01} = 0, \qquad h_{11} = \tfrac{2\varepsilon}{\alpha} \neq 0, \qquad h_{12} = 0, \qquad h_{22} = \tfrac{4\delta}{3\alpha} \neq 0$$

and Thm. 4.2 is applicable with $h_{11}h_{22} = \frac{8\varepsilon\delta}{3\alpha^2}$. This guarantees

- for $\varepsilon\delta > 0$ that bounded solutions of (4.11) exist for $\lambda = 0$ only (see Fig. 5 (left)),
- for $\varepsilon\delta < 0$ that the trivial solution ϕ^* is the intersection of two branches of bounded solutions to (4.11) (see Fig. 5 (right)).

The decoupled structure of (4.11) allows us to verify this statement explicitly. Indeed, as indicated the general solution φ_λ of (4.11) has the first component $\varphi_\lambda^1(t;0,\eta) = e^{-\alpha|t|}\eta_1$ for all $t \in \mathbb{R}$ and for the second component one deduces an asymptotic representation

$$\varphi_\lambda^2(t;0,\eta) = \begin{cases} e^{\alpha t}\left(\eta_2 + \frac{\delta}{\alpha}\eta_1^2 + \frac{\varepsilon\lambda^2}{\alpha}\right) + O(t) \text{ as } t \to \infty, \\ e^{-\alpha t}\left(\eta_2 - \frac{\delta}{\alpha}\eta_1^2 - \frac{\varepsilon\lambda^2}{\alpha}\right) + O(t) \text{ as } t \to -\infty. \end{cases}$$

From this we see that $\varphi_\lambda(\cdot;0,\eta) \in L^\infty$ holds if and only if

$$\eta_2 = -\frac{\delta}{\alpha}\eta_1^2 - \frac{\varepsilon\lambda^2}{\alpha}, \qquad \eta_2 = \frac{\delta}{\alpha}\eta_1^2 + \frac{\varepsilon\lambda^2}{\alpha}$$

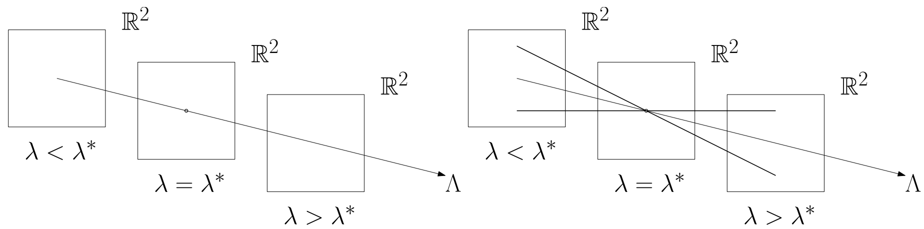

FIGURE 5. Left ($\varepsilon\delta > 0$): Isolated initial value yielding a bounded complete solution of (4.11) for $\lambda = \lambda^*$.
Right ($\varepsilon\delta < 0$): Intersection of two branches of initial values yielding bounded complete solutions of (4.11)

and we conclude:

- If $\frac{\varepsilon}{\delta} > 0$, then the initial value $\eta = (0, 0)$ yields the unique bounded solution to (4.11) for $\lambda = 0$, while there are no bounded solutions for $\lambda \neq 0$,
- if $\frac{\varepsilon}{\delta} < 0$, then initial values for bounded solution to (4.11) are $\eta = \big(0, \pm|\lambda|\sqrt{-\frac{\varepsilon}{\delta}}\big)$.

5. PERSPECTIVES

As shown in [LSW07] the crossing curve bifurcation from Thm. 4.1 (see Thm. A.2 for the abstract form) is actually a corollary of the degenerate fold scenario in Thm. 4.2 (resp. Thm. A.1). Further results of [Shi99, LSW07] also allow to investigate degenerate versions of the transcritical and pitchfork bifurcation patterns for bounded complete solutions of nonautonomous difference and differential equations, whose generic (and primary) versions have been formulated in [Pöt10a]. We think this paper contains the essential ingredients (for instance, Cor. 3.3) and techniques, so that the interested reader is able to deduce such corresponding results on secondary bifurcations.

Furthermore, as a concluding remark we point out that our whole theory clearly applies to nonautonomous differential equations with continuous time dependence. Under this assumption, one replaces $W^{1,\infty}$ by the space of bounded C^1-functions BC^1 and L^∞ by the bounded functions BC; then the present theory resembles the previous situation considered in [Pöt10a, Sect. 3]. With obvious modifications, quite similar results also hold for differential equations with a right hand side being piecewise continuous in time, or in the discrete case of nonautonomous difference equations.

APPENDIX A. ABSTRACT BIFURCATION RESULTS

We review the results of [Shi99, LSW07] in a notation appropriate for our purposes and already established in [Pöt10b]. Throughout, assume that X, Z are real Banach spaces and $\Omega \subseteq X$, $\Lambda \subseteq \mathbb{R}$ denote nonempty open neighborhoods of $x_0 \in X$, $\lambda \in \mathbb{R}$ in the respective spaces. We consider C^m-mappings $G : \Omega \times \Lambda \to Z$, $m \geq 2$, vanishing at (x_0, λ_0),

$$G(x_0, \lambda_0) = 0 \tag{A.1}$$

and with a partial Fréchet derivative $D_1G(x_0, \lambda_0) \in L(X, Z)$ satisfying

$$\begin{aligned} \dim N(D_1G(x_0, \lambda_0)) = \operatorname{codim} R(D_1G(x_0, \lambda_0)) = 1, \\ N(D_1G(x_0, \lambda_0)) = \operatorname{span}\{x_1\} \end{aligned} \tag{A.2}$$

for some nonzero $x_1 \in X$. Hence, the derivative $D_1G(x_0, \lambda_0)$ is a Fredholm operator of index 0. Thus, from the Hahn-Banach theorem (cf. [Lan93, p. 69, Thm. 1.1]) we get the existence of a continuous functional $\mu \in L(Z, \mathbb{R})$ such that $N(\mu) = R(D_1G(x_0, \lambda_0))$. The Fredholm property (A.2) yields that the kernel $N(D_1G(x_0, \lambda_0))$, as well as the range $R(D_1G(x_0, \lambda_0))$ split the respective space X and Z, i.e. there exist two closed subspaces $X_0 \subseteq X$, $Z_0 \subseteq Z$ with

$$X = N(D_1G(x_0, \lambda_0)) \oplus X_0, \qquad Z = Z_0 \oplus R(D_1G(x_0, \lambda_0)). \tag{A.3}$$

We conclude that the restriction $D_1G(x_0, \lambda_0)|_{X_0} : X_0 \to R(D_1G(x_0, \lambda_0))$ is a toplinear isomorphism and under the assumption

$$D_2G(x_0, \lambda_0) \in R(D_1G(x_0, \lambda_0)) \tag{A.4}$$

there exists a unique solution $\bar{x} \in X_0$ to the linear equation

$$D_2G(x_0, \lambda_0) + D_1G(x_0, \lambda_0)\bar{x} = 0. \tag{A.5}$$

For the sake of a convenient notation we abbreviate $G_{ij} := D_1^i D_2^j G(x_0, \lambda_0)$ for $i, j \in \mathbb{N}_0$ with $i + j \le m$, and formulate the basic bifurcation results:

First of all, locally near (x_0, λ_0) the solution set to $G(x, \lambda) = 0$ is either an isolated point or the transversal intersection of two smooth curves (cf. Fig. 4).

Theorem A.1 (abstract degenerate fold bifurcation). *Let $m \ge 2$ and suppose that the mapping $G : \Omega \times \Lambda \to Z$ satisfies* (A.1), (A.2) *and* (A.4). *If the matrix*

$$H(x_0, \lambda_0) := \begin{pmatrix} \mu(G_{02} + 2G_{11}\bar{x} + G_{20}\bar{x}^2) & \mu(G_{11}x_1 + G_{20}x_1\bar{x}) \\ \mu(G_{11}x_1 + G_{20}x_1\bar{x}) & \mu(G_{20}x_1^2) \end{pmatrix}$$

is invertible, then there exist open convex neighborhoods $S \subseteq \mathbb{R}$ of 0 and $U_1 \times U_2 \subseteq \Omega \times \Lambda$ of (x_0, λ_0) such that the following holds:

(a) *For $\det H(x_0, \lambda_0) > 0$ one has*

$$\{(x, \lambda) \in U_1 \times U_2 : \, G(x, \lambda) = 0\} = \{(x_0, \lambda_0)\},$$

(b) *for $\det H(x_0, \lambda_0) < 0$ there exist two C^{m-1}-curves $\gamma_1, \gamma_2 : S \to U_1 \times U_2$ with*

$$\{(x, \lambda) \in U_1 \times U_2 : \, G(x, \lambda) = 0\} = \gamma_1(S) \cup \gamma_2(S)$$

and the branches $\gamma_1(S), \gamma_2(S)$ have the explicit representation

$$\gamma_i(s) = \begin{pmatrix} x_0 + s\rho_i x_1 + s y_i(s) \\ \lambda_0 + s\nu_i + s\theta_i(s) \end{pmatrix} \quad \text{for all } s \in S,$$

where $(\rho_1, \nu_1), (\rho_2, \nu_2) \in \mathbb{R}^2$ are nonzero linearly independent solutions of

$$\mu(G_{02} + 2G_{11}\bar{x} + G_{20}\bar{x}^2)\nu^2 + 2\mu(G_{11}x_1 + G_{20}x_1\bar{x})\rho\nu + \mu(G_{20}x_1^2)\rho^2 = 0 \tag{A.6}$$

and $\theta_i : S \to \mathbb{R}$, $y_i : S \to X_0$ are functions satisfying

$$\theta_i(0) = \dot{\theta}_i(0) = 0, \qquad y_i(0) = \dot{y}_i(0) = 0 \quad \text{for all } i = 1, 2.$$

Remark A.1. Geometrically, the set of all pairs $(\nu, \rho) \in \mathbb{R}^2$ satisfying (A.6) is given by two straight lines in the plane $\mathbb{R}^2$ intersecting at the origin.

Proof. The case of C^2-smoothness is formulated in [LSW07, Thm. 2.1], but a closer look to [LSW07, Lemma 2.6] yields the claimed C^{m-1}-regularity of the curves γ_i, $i = 1, 2$. □

For the classical result of Crandall-Rabinowitz [CR71, Thm. 17] it is assumed that a trivial solution branch $G(x_0, \lambda) \equiv 0$ on Λ exists. Here, we weaken this to

$$D_2G(x_0, \lambda_0) = 0 \tag{A.7}$$

and a transversality condition. This guarantees that one of the intersecting curves $\gamma_i(S)$ is tangential to the λ-axis in (x_0, λ_0) (cf. Fig. 2):

Theorem A.2 (abstract crossing curve bifurcation)**.** *Let $m \geq 2$ and suppose that the mapping $G : \Omega \times \Lambda \to Z$ satisfies* (A.1)*,* (A.2) *and* (A.7)*. If the* transversality condition

$$g_{11} := \mu(G_{11}x_1) \neq 0, \qquad g_{02} := \mu(G_{02}) = 0 \tag{A.8}$$

holds, then there exist open convex neighborhoods $S \subseteq \mathbb{R}$ of 0 and $U_1 \times U_2 \subseteq \Omega \times \Lambda$ of (x_0, λ_0) such that

$$\{(x, \lambda) \in U_1 \times U_2 : G(x, \lambda) = 0\} = \Gamma_1 \cup \Gamma_2,$$

where the branches Γ_1, Γ_2 have the following properties:

(a) $\Gamma_1 = \gamma_1(S)$ *with a C^{m-1}-curve $\gamma_1 : S \to U_1 \times U_2$, $\gamma_1(s) = (\gamma(s), \lambda_0 + s)$ and*

$$\gamma_1(0) = (x_0, \lambda_0), \qquad \dot{\gamma}(0) = 0,$$

(b) $\Gamma_2 = \gamma_2(S)$ *with a C^{m-1}-curve $\gamma_2 : S \to U_1 \times U_2$ and*

$$\gamma_2(s) = \begin{pmatrix} x_0 + sx_1 + sy_2(s) \\ \lambda_0 + s\mu_2 + s\theta_2(s) \end{pmatrix}, \qquad \mu_2 := -\frac{\mu(G_{20}x_1^2)}{2g_{11}},$$

where the properties of y_2, θ_2 are given in Thm. A.1.

Proof. See [LSW07, Cor. 2.3]. □

Corollary A.3 (abstract transcritical bifurcation)**.** *If $\mu(G_{20}x_1^2) \neq 0$, then*

$$\#\{x \in U_1 : G(x, \lambda) = 0\} = \begin{cases} 1, & \lambda = \lambda_0, \\ 2, & \lambda \neq \lambda_0. \end{cases}$$

Proof. Due to $\mu(G_{20}x_1^2) \neq 0$ one has $\mu_2 \neq 0$ and the claim follows from Thm. A.2. □

Concerning the degenerate case $\mu(G_{20}x_1^2) = 0$ we point out that the linear equation $G_{20}x_1^2 + G_{10}\bar{x}_1 = 0$ has a unique solution $\bar{x}_1 \in X_0$, which yields

Corollary A.4 (abstract pitchfork bifurcation)**.** *If $m \geq 3$ and*

$$\mu(G_{20}x_1^2) = 0, \qquad \mu(G_{30}x_1^3) + 3\mu(G_{20}x_1\bar{x}_1) \neq 0$$

then $\ddot{\gamma}_2^2(0) = -\frac{\mu(G_{30}x_1^3)+3\mu(G_{20}x_1\bar{x}_1)}{3g_{11}}$ holds true and

(a) *If $\ddot{\gamma}_2^2(0) > 0$, then* $\#\{x \in U_1 : G(x, \lambda) = 0\} = \begin{cases} 1, & \lambda \leq \lambda_0, \\ 3, & \lambda > \lambda_0, \end{cases}$

(b) *If $\ddot{\gamma}_2^2(0) < 0$, then* $\#\{x \in U_1 : G(x, \lambda) = 0\} = \begin{cases} 3, & \lambda < \lambda_0, \\ 1, & \lambda \geq \lambda_0. \end{cases}$

Proof. The formula determining $\ddot{\gamma}_2(0)$ has been given in [Shi99, (4.6)] and then Thm. A.2 yields the assertion. □

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