Persistence and imperfection of nonautonomous bifurcation patterns

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Abstract

For nonautonomous dynamical systems a bifurcation can be understood as topological change in the set of bounded entire solutions to a given time-dependent evolutionary equation. Following this idea, a Fredholm theory via exponential dichotomies on semiaxes enables us to employ tools from analytical branching theory yielding nonautonomous versions of fold, transcritical and pitchfork patterns. This approach imposes the serious hypothesis that precise quantitative information on the dichotomies is required — an assumption hard to satisfy in applications. Thus, imperfect bifurcations become important.

In this paper, we discuss persistence and changes in the previously mentioned bifurcation scenarios by including an additional perturbation parameter. While the unperturbed case captures the above bifurcation patterns, we obtain their unfolding and therefore the local branching picture in a whole neighborhood of the system. Using an operator formulation of parabolic differential, Carathéodory differential and difference equations, this will be achieved on the basis of recent abstract analytical techniques due to Shi (1999) and Liu, Shi & Wang (2007).

Keywords: nonautonomous bifurcation, imperfect bifurcation, unfolding, semilinear parabolic equation, Carathéodory differential equation, nonautonomous difference equation, exponential dichotomy

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1. Introduction

1.1. Motivation

By definition, a local theory of dynamical systems deals with the behavior of difference or differential equations in the vicinity of invariant sets like equilibria or periodic solutions. As soon as the equations of interest become aperiodically time-dependent, however, usually neither equilibria nor periodic solutions exist. This turns out to be problematic and challenging, since such nonautonomous problems are omnipresent in a multitude of applications where modulation, control or even random effects cannot be

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neglected. Thus, the question arises which invariant objects are appropriate to establish a suitable nonautonomous bifurcation theory?

In this regard, it was observed that equilibria generically persist as bounded entire solutions under small temporally fluctuating perturbations (see, for instance, [6, 36]). More general this behavior holds for so-called *hyperbolic* entire solutions, whose corresponding variational equation admits an exponential dichotomy on the whole time axis. On this basis, it is reasonable to replace equilibria by bounded entire solutions as natural bifurcating objects in a time-varying framework. Furthermore, related nonautonomous problems also occur in a purely autonomous setting when one is interested in the behavior near aperiodic reference solutions and their behavior under varying parameters (or even equations).

We considered this as motivation and starting point to investigate the bifurcation behavior of bounded entire solutions in [34, 35] using tools from analytical branching theory (cf. [18] or [43, Chapter 8]), like Lyapunov-Schmidt reduction. The required Fredholm theory is provided by means of dynamical properties for the variational equation along a *nonhyperbolic* reference solution. This enabled us to derive nonautonomous versions of the classical fold, transcritical and pitchfork bifurcation patterns in [34]. Furthermore, a crossing curve bifurcation (generalizing transcritical and pitchfork patterns) and a degenerate fold bifurcation has been obtained in [35] on the basis of abstract analytical results due to [28]. Keeping in mind that a (global) pullback attractor $\mathcal{A}$ (cf., e.g. [19]) of a nonautonomous dynamical system consists of bounded entire solutions, the foregoing bifurcation concept has also stringent consequences on the structure of $\mathcal{A}$ and the resulting notion of attractor bifurcation as investigated in [37, pp. 42ff, Sect. 2.5] or [20, 25, 38]. Yet, extending these preparations and earlier approaches, the motivation for our present work is two-fold:

First, one problem in our previous approach is that it essentially requires very detailed information on the exponential dichotomy data (the invariant projectors), as well as hypotheses on the whole time axis. In practice, such conditions can be verified only numerically or approximately. Despite yielding very precise information on the local structure of the set of bounded solutions, the results of [34, 35] are therefore somewhat academic. This gives rise to the natural question for the behavior of bifurcation scenarios under perturbation yielding so-called imperfect bifurcations: What is the actual bifurcation diagram for systems in a neighborhood of the bifurcating one. Second, we like to investigate nonautonomous bifurcations under external perturbations, which can be small, but otherwise arbitrary bounded fluctuations. In doing so, we give an accurate description on how the structurally unstable scenario of a nonautonomous bifurcation gets destroyed under perturbation.

Throughout we are interested in the behavior of evolutionary equations depending on a real bifurcation parameter λ, which for given bifurcation value $\lambda = \lambda^*$ possesses a bounded entire reference solution ϕ^*. This solution is supposed to be nonhyperbolic in the sense that the corresponding variational equation has 0 in its dichotomy spectrum (cf. [39, 42]), respectively 1 in the discrete case (see [2]). More precisely, it admits exponential dichotomies on both the positive and the negative semiaxis, whose projectors do not span the whole state space (cf. (2.7)). This is an intrinsically nonautonomous form of nonhyperbolicity and cannot occur for almost-periodic, periodic or autonomous equations. Thus, certain natural nonautonomous bifurcation scenarios are not covered by our abstract approach. Yet we believe to make a valid contribution being complementary to prior nonautonomous bifurcation scenarios of e.g. [20, 25, 38].

Our semiaxes dichotomy assumption requires the evolutionary equations to be at least two dimensional and the bifurcating solution ϕ^* to be unstable. Such one-sided dichotomies guarantee the existence of a stable integral manifold $\mathcal{W}_\lambda^+$ (consisting of forward bounded solutions), and of an unstable integral manifold $\mathcal{W}_\lambda^-$ (which contains the backward bounded solutions). Our bifurcation notion is based on the structure of all entire bounded solutions near ϕ^* and therefore, a bifurcation is a topological change in the intersection $\mathcal{W}_\lambda^+ \cap \mathcal{W}_\lambda^-$ for varying parameters λ. In fact, this intersection yields

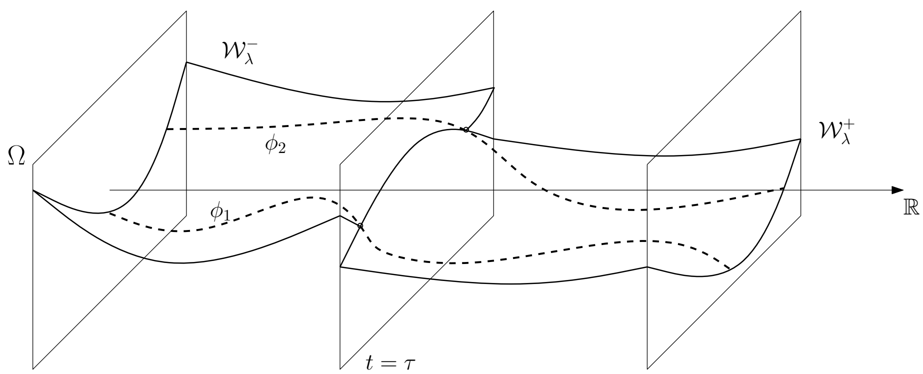

Figure 1: Extended state space $\mathbb{R} \times \Omega$: Intersection of the stable integral manifold $\mathcal{W}_\lambda^+$ with the unstable integral manifold $\mathcal{W}_\lambda^-$ yields two bounded entire solutions ϕ_1, ϕ_2 indicated as dashed lines

initial values for bounded entire solutions (cf. Fig. 1) and allows a vivid illustration of our bifurcation scenarios. Nonetheless, since $\mathcal{W}_\lambda^+$ and $\mathcal{W}_\lambda^-$ are not explicitly known, our approach is purely analytical and the fundamental results of [41, 28] show that

- a fold bifurcation and the fold point are robust (see Fig. A.3 and Thm. A.1),
- transcritical bifurcations break either into two branches of hyperbolic solutions or two folds (see Fig. A.4 and Thm. A.2) and
- pitchfork bifurcations break into a fold and a branch of hyperbolic solutions (see Fig. A.5 and Thm. A.3)

under perturbation. The mentioned intersection of stable and unstable manifolds gives a geometric interpretation of the at first hand abstract Lyapunov-Schmidt reduction.

In order to tackle these branching problems technically, beyond the bifurcation parameter λ, we introduce an additional perturbation (or imperfection) parameter $\varepsilon \in \mathbb{R}$ into our evolutionary equations and investigate the behavior of the above standard bifurcation patterns under variation of ε. After a Lyapunov-Schmidt reduction this yields a finite-dimensional branching equation (see [34, Props. 2.11, 3.9]) depending on two parameters λ, ε. Such a reduced problem can be treated using established methods from singularity theory (see, e.g., [14]) in order to obtain an unfolding of the bifurcation. On the other hand, the previous perturbation problem has been analyzed in [41, 28] on the abstract level of analytical branching theory (with applications to elliptic PDEs). We significantly benefit from their general as well as flexible results and present an alternative application to a wide class of nonautonomous dynamical systems generated by semilinear parabolic PDEs, Carathéodory differential equations in $\mathbb{R}^d$ and difference equations in Hilbert spaces. In addition, our proofs can be kept short.

The presentation of our corresponding results splits into three sections, which are somewhat parallel, and an appendix. We illustrate each of the mentioned bifurcation patterns using a different of the above evolutionary equations supplemented by remarks on the discrepancies between the corresponding cases. Following our preparation, though, the interested reader should be able to deduce the remaining results for each class of equations on his own. For the reader's convenience, the necessary abstract branching theory from [41, 28] is summarized in the appendix. A similar analysis seems possible for nonautonomous functional differential equations (FDEs), where the required Fredholm theory in terms of exponential dichotomies was developed in [27].

Concerning the related literature, it should be noted that our methods (Fredholm theory and Lyapunov-Schmidt reduction) are common tools in the context of transversal homoclinic orbits for autonomous dynamical systems; by way of example we refer to [9, 31] dealing with ODEs, [21] for maps, [7] for parabolic PDEs or [27] for FDEs; Fredholm theory for more general classes of evolutionary equations is due to [13, 26]. On the other hand, there are various results in the framework of random dynamical systems as opposed to our nonautonomous approach. The effect of (additive) noise to bifurcation patterns was studied in [1, pp. 465ff, Chapt. 9] or [12]. In this context, fold, transcritical and pitchfork bifurcations are investigated under an invariant measure (in form of Lyapunov exponents). Furthermore, in [11] it is shown that an additively perturbed system with a pitchfork pattern has a one-point random attractor for all parameter values — a destruction of the unperturbed situation.

1.2. Notation

Throughout the paper, generic real Banach spaces are denoted by X, Y and equipped with norm $|\cdot|$; however, we consistently use the double bar notation $\|\cdot\|$ for norms on function or sequence spaces. The *interior* of a set $\Omega \subseteq X$ is denoted by Ω° and $B_\varepsilon(x)$ is the open ball with center x and radius $\varepsilon > 0$. For the *distance* of a point $x \in X$ to the set Ω we write $\operatorname{dist}_X(x, \Omega) := \inf_{y \in \Omega} |x - y|$. The space of bounded linear operators between X and Y is $L(X, Y)$, $L(X) := L(X, X)$ and for the corresponding toplinear isomorphisms we write $GL(X, Y)$. Given an operator $T \in L(X, Y)$, $R(T) := TX$ is the *range* and $N(T) := T^{-1}(0)$ the *kernel*. The *dual space* of X is X', $\langle x', x \rangle := x'(x)$ the duality product and $T' \in L(Y', X')$ is the *dual operator* to T. For a given subspace $X_0 \subseteq X$ the *annihilator* is defined as set of functionals $X_0^\perp := \{x' \in X' : \langle x', x_0 \rangle = 0 \text{ for all } x_0 \in X_0\}$.

The following terminology is tailor-made for time-dependent problems. A subset $\mathcal{A} \subseteq \mathbb{R} \times X$ is called *nonautonomous set* with *t-fiber* $\mathcal{A}(t) := \{x \in X : (t, x) \in \mathcal{A}\}$.

Finally, in the whole paper, $\Lambda, V \subseteq \mathbb{R}$ are nonempty open intervals, where Λ is interpreted as *parameter* and V as *perturbation space*.

2. Parabolic partial differential equations

First, we deal with nonautonomous semilinear parabolic PDEs in terms of abstract evolutionary differential equations in fractional power spaces. A prime example are Allen-Cahn equations $u_t = u_{xx} + g(t, u, \lambda, \varepsilon)$ under Dirichlet boundary conditions on open bounded domains $U \subseteq \mathbb{R}^d$, which can be formulated as evolutionary equation in $X = L^2(U)$ and the interpolation space $X^{1/2} = H_0^1(U)$ (see, for instance, [40, p. 269ff]). After reviewing the required Fredholm theory on linear equations and introducing a suitable spatial setting, the abstract Thm. A.1 yields the robustness of fold bifurcation patterns, whose dynamical interpretation will be given in Thm. 2.7.

Our approach relies on fractional power spaces (see our standard reference [16, pp. 24ff, Sect. 1.4] or [29, 40]) and linear parabolic equations. For this purpose, suppose that $A : D(A) \subseteq X \to X$ is a sectorial operator on X (cf. [16, pp. 16ff, Sect. 1.3]) and we can choose $a \in \mathbb{R}$ so that $A_a := A + a\,\mathrm{id}$ satisfies $\inf \Re\sigma(A_a) > 0$. We define fractional powers A_a^β, $\beta \in [0, 1)$, of A_a and *fractional power spaces*

$$X^\beta := D(A_a^\beta), \qquad\qquad |x|_\beta := \left|A_a^\beta x\right|.$$

Note that the graph norms $|\cdot|_\beta$ are equivalent for different choices of a and $(X^\beta, |\cdot|_\beta)$ become Banach spaces. They fulfill the continuous embedding $X^\beta \hookrightarrow X^\gamma$, X^β is a dense subspace of X^γ for $0 \le \gamma \le \beta$ (cf. [16, p. 29, Thm. 1.4.8]) and particularly

$$D(A) \hookrightarrow X^\beta \hookrightarrow X \quad \text{for all } 0 \le \beta < 1; \tag{2.1}$$

finally, the embedding $X^\beta \hookrightarrow X$ is compact, provided A has compact resolvent.

Given an interval $I \subseteq \mathbb{R}$ we suppose $B(\cdot) - A : I \to L(X^\beta, X)$ is locally Hölder continuous with exponent $\theta \in (0, 1)$. Under this assumption a linear parabolic equation

$$\boxed{\dot u + Au = B(t)u} \tag{L}$$

is well-posed: For every pair $(t_0, u_0) \in I \times X^\beta$ there exists a unique forward solution $T(\cdot, t_0)u_0 : [t_0, \infty) \cap I \to X^\beta$ of (L). The family $T(t, s)$, $s \le t$ of *transition operators* satisfies $T(t, s) \in L(X^\beta)$ and $T(t, t) = \mathrm{id}$, as well as the 2-parameter semigroup property $T(t, s)T(s, t_0) = T(t, t_0)$ for all triples $t_0 \le s \le t$ (cf. [16, pp. 191–192, Thm. 7.1.3]). Moreover, if A has a compact resolvent, then $T(t, s) \in L(X^\beta)$, $s < t$, becomes a compact operator (see [16, p. 196]).

The *adjoint* $T'(t, s)$ of a transition operator $T(t, s)$ is defined by virtue of

$$\langle x', T(t, s)x\rangle = \langle T'(s, t)x', x\rangle \quad \text{for all } x \in X,\ x' \in X' \text{ and } s \le t.$$

It is shown in [16, p. 205, Thm. 7.3.1] that $T'(s, t) \in L(X')$ is a backward 2-parameter semigroup on X, continuous in $s < t$, but only weak*-continuous in points $t = s$. Given $u_0' \in (X^\beta)'$, for Hölder exponents $\theta > \beta$ the mapping $T(t_0, \cdot)'u_0'$ with values in X' is continuously differentiable, satisfies $T'(t_0, t)u_0' \in D(A')$ for $t_0 < t$ and solves the adjoint equation

$$\dot u - A'u = -B(t)'u. \tag{2.2}$$

An *invariant projector* for (L) is a strongly continuous function $t \mapsto P_t \in L(X^\beta)$ with $P_t^2 = P_t$ and $P_t T(t,s) = T(t,s) P_s$ for all $s \le t$, $s, t \in I$. Having this terminology at our disposal, a linear parabolic equation (L) or the induced transition operator $T(t,s)$ is said to have an *exponential dichotomy* (ED for short) on I, if there exist reals $K \ge 1$, $\alpha > 0$ and an invariant projector P_t such that

- the restriction $T(t,s)|_{N(P_s)} : N(P_s) \to N(P_t)$, $s \le t$, is an isomorphism and we define $T(s,t)$ as the inverse,
- one has the exponential estimates

$$|T(t,s)P_s|_\beta \le K e^{-\alpha(t-s)}, \quad |T(s,t)[\mathrm{id} - P_t]|_\beta \le K e^{\alpha(s-t)} \quad \text{for all } s \le t$$

(cf. [16, p. 224, Def. 7.6.1]). Provided I is unbounded below and A has a compact resolvent, from [16, p. 226] we can conclude that the so-called *unstable bundle* of (L),

$$\mathcal{V}^- := \big\{(\tau, \xi) \in I \times X^\beta : \xi \in N(P_\tau)\big\}$$

is finite-dimensional, i.e. its fibers $\mathcal{V}^-(t) \subseteq X^\beta$, $t \in I$, have finite dimension. Criteria for (L) to possess an ED have been given in [16, p. 225] or [24]. In addition, an ED of $T(t,s)$ carries over to the adjoint equation (2.2) as follows:

Lemma 2.1 (Lin's lemma)**.** *If a transition operator $T(t,s)$ admits an ED with α, K and invariant projector P_t on I, then also the dual transition operator $T'(s,t)$ is exponentially dichotomic on I with $T'(t,s)P'_s = P'_t T'(t,s)$,*

$$|T'(t,s)Q'_s| \le K e^{-\alpha(s-t)}, \quad |T'(s,t)[\mathrm{id} - Q'_t]| \le K e^{\alpha(t-s)}$$

for all $t \le s$, with an invariant projector $Q'_t := \mathrm{id} - P'_t$ and

$$R(Q'_t) = N(P_t)^\perp, \qquad N(Q'_t) = R(P_t)^\perp. \tag{2.3}$$

Proof. See [27, p. 229]. □

We turn to semilinear parabolic equations and suppose throughout that their *state space* $\Omega \subseteq X^\beta$ is a nonempty open convex set. For this purpose it is reasonable to restrict to functions with values in $D(A)$ and convenient to define the sets

$$\Omega_A := \Omega \cap D(A)$$

equipped with the graph norm $|\cdot|_\beta$. For right hand sides $f : \mathbb{R} \times \Omega \times \Lambda \times V \to X$ we consider nonautonomous equations

$$\boxed{\dot{u} + Au = f(t, u, \lambda, \varepsilon),} \tag{$(D)^\varepsilon_\lambda$}$$

where $\lambda \in \Lambda$ serves as bifurcation parameter and $\varepsilon \in V$ as perturbation (or imperfection) parameter. For fixed pairs $(\lambda, \varepsilon) \in \Lambda \times V$ we equip the problem $(D)^\varepsilon_\lambda$ with an initial condition $u(t_0) = u_0$ for $t_0 \in \mathbb{R}$, $u_0 \in X$. Given instants $t_0 < \tau$, a continuous function $\phi : [t_0, \tau) \to \Omega$ is called *(classical) solution* to $(D)^\varepsilon_\lambda$, if

- $f(\cdot, \phi(\cdot), \lambda, \varepsilon) : [t_0, \tau) \to X$ is continuous,
- $\dot{\phi}(t)$ exists in X, $\phi(t) \in D(A)$ for all $t \in (t_0, \tau)$ and fulfills the *solution identity*

$$\dot{\phi}(t) + A\phi(t) \equiv f(t, \phi(t), \lambda, \varepsilon) \tag{2.4}$$

(cf. [30] in connection with existence criteria from [16, pp. 52ff, Sect. 3.3]). In particular, we are interested in bounded solutions of $(D)^{\varepsilon}_{\lambda}$, which are frequently smooth in time (see [15]). In this regard, an *entire* or *complete solution* of $(D)^{\varepsilon}_{\lambda}$ is a continuously differentiable function $\phi : \mathbb{R} \to X$ satisfying (2.4) on the entire axis $\mathbb{R}$. We speak of a *permanent solution*, if additionally $\inf_{t\in\mathbb{R}} \operatorname{dist}_{X^\beta}(\phi(t), \Omega) > 0$ holds.

Our subsequent assumptions hold for C^m-smooth right hand sides of $(D)^{\varepsilon}_{\lambda}$ with derivatives bounded on bounded sets uniformly in time.

Hypothesis. *Let $m \in \mathbb{N}$, $\beta \in [0, 1)$, $\lambda^* \in \Lambda$, $\varepsilon^* \in V$ be given. Suppose the nonlinearity $f : \mathbb{R} \times \Omega \times \Lambda \times V \to X$ is continuous and $f(t, \cdot)$ is a C^m-function, $t \in \mathbb{R}$, such that the following holds for $0 \le j \le m$:*

(H_0) *the derivatives $D^j_{(2,3,4)} f$ are Hölder continuous with exponent $\theta > \beta$ in the first argument and for all bounded subsets $B \subseteq \Omega$ one has*

$$\sup_{t\in\mathbb{R}} \sup_{u\in B} \left| D^j_{(2,3,4)} f(t, u, \lambda, \varepsilon) \right| < \infty \quad \textit{for all } \lambda \in \Lambda,\ \varepsilon \in V$$

(well-definedness) *and for all $(\lambda^*, \varepsilon^*) \in \Lambda \times V$ and $\rho > 0$ there is a $\delta > 0$ with*

$$|u - v|_\beta < \delta \quad \Rightarrow \quad \sup_{t\in\mathbb{R}} \left| D^j_{(2,3,4)} f(t, u, \lambda, \varepsilon) - D^j_{(2,3,4)} f(t, v, \lambda, \varepsilon) \right| < \rho$$

for all $u, v \in \Omega$ and $(\lambda, \varepsilon) \in B_\delta(\lambda^*, \varepsilon^*)$ (uniform continuity).

Next we establish an ambient functional analytical setting for semilinear parabolic problems $(D)^{\varepsilon}_{\lambda}$ as abstract equations in function spaces. Being interested in classical solutions it is reasonable to work with Hölder spaces. More precisely, given a closed subspace $Y \subseteq X$ and $\theta \in (0, 1)$, we define function spaces

$BC(Y)$ consisting of all bounded continuous functions $\phi : \mathbb{R} \to Y$ with norm

$$\|\phi\|_{0,Y} := \sup_{t\in\mathbb{R}} |\phi(t)|_Y\,,$$

$BC^\theta(Y)$ consisting of all bounded functions $\phi : \mathbb{R} \to Y$ satisfying a Hölder condition with exponent θ, equipped with norm

$$\|\phi\|_{\theta,Y} := \max\left\{ \|\phi\|_{0,Y}\,, [\phi]_{\theta,Y} \right\}, \qquad [\phi]_{\theta,Y} := \sup_{s<t} \tfrac{|\phi(t)-\phi(s)|_Y}{(t-s)^\theta},$$

$BC^{1+\theta}(Y)$ consisting of all continuous bounded functions $\phi : \mathbb{R} \to Y$, for which the derivative $\dot{\phi} : \mathbb{R} \to X$ is exists as continuous bounded function satisfying a Hölder condition with exponent θ, equipped with the norm

$$\|\phi\|_1 := \left\{ \|\phi\|_{0,Y}\,, \|\dot{\phi}\|_{\theta,X} \right\},$$

which are Banach spaces. We write $BC := BC(X)$ and proceed similarly for other functions spaces. The set of functions $\phi \in BC$ with values in Ω is denoted by $BC(\Omega)$ and a similar notation is used for the other function spaces. While convexity of Ω carries over to $BC^{1+\theta}(\Omega)$ and $BC^{\theta}(\Omega)$, these function sets are not necessarily open.

An operator formulation of the differential equation $(D)_\lambda^\varepsilon$ depends on appropriate substitution operators and their derivatives, formally defined by

$$F(\phi, \lambda, \varepsilon) := f(\cdot, \phi(\cdot), \lambda, \varepsilon), \quad F^{\upsilon}(\phi, \lambda, \varepsilon) := D_2^{\upsilon_1} D_3^{\upsilon_2} D_4^{\upsilon_3} f(\cdot, \phi(\cdot), \lambda, \varepsilon). \tag{2.5}$$

Here, $\upsilon = (\upsilon_1, \upsilon_2, \upsilon_3) \in \mathbb{N}_0^3$ is a *multiindex* of length $|\upsilon| := \upsilon_1 + \upsilon_2 + \upsilon_3 \leq m$.

Proposition 2.2. *Under (H_0) the operator $F : BC^{1+\theta}(\Omega) \times \Lambda \times V \to BC^{\theta}$ is well-defined and m-times continuously differentiable on $BC^{1+\theta}(\Omega)^{\circ} \times \Lambda \times V$ with*

$$D^{\upsilon} F(\phi, \lambda, \varepsilon) = F^{\upsilon}(\phi, \lambda, \varepsilon) \quad \textit{for all } \phi \in BC(\Omega)^{\circ},\ \lambda \in \Lambda,\ \varepsilon \in V.$$

Proof. For mappings $f(t, \cdot, \lambda, \varepsilon)$ with arguments u and values in the same Banach space, we have given a proof in [36, Prop. 3.4], provided the derivatives $D^j_{(2,3,4)} f$, $1 \leq j \leq m$, are continuous. Since also $D^j_{(2,3,4)} f(\cdot, \phi(\cdot), \lambda, \varepsilon)$ are Hölder continuous with exponent θ for $\phi \in BC^{1+\theta}(\Omega)$, the interested reader might check that these arguments also hold in the present situation $\Omega \subseteq X^{\beta} \subseteq X$. $\square$

Corollary 2.3. *Under (H_0) the operator $G : BC^{1+\theta}(\Omega_A) \times \Lambda \times V \to BC^{\theta}$,*

$$G(\phi, \lambda, \varepsilon) := \dot{\phi} + A\phi - F(\phi, \lambda, \varepsilon)$$

is well-defined and m-times continuously differentiable on $BC^{1+\theta}(\Omega_A)^{\circ} \times \Lambda \times V$.

Proof. First of all, with given functions $\phi \in BC^{1+\theta}(\Omega_A)$ one has

$$|A\phi(t)|_X \leq |A|_{L(X^{\beta}, X)} \|\phi\|_{0, X^{\beta}} \leq |A|_{L(X^{\beta}, X)} \|\phi\|_1 .$$

Moreover, thanks to $\|\dot{\phi}\|_{\theta, X} \leq \|\phi\|_1$, one sees that $\phi \mapsto \dot{\phi} + A\phi$ is bounded linear between $BC^{1+\theta}(X^{\beta})$ and BC^{θ}. Therefore, Prop. 2.2 yields the assertion. $\square$

We are looking for classical solutions of $(D)_\lambda^\varepsilon$, which can be characterized as zeros of the operator G from Cor. 2.3. Thus, the crucial tool for our analysis is:

Theorem 2.4. *Let $\lambda \in \Lambda$, $\varepsilon \in V$ and suppose (H_0) holds. If $\phi \in BC^{\theta}(\Omega)$ is an entire solution of $(D)_\lambda^\varepsilon$, then $\phi \in BC^{1+\theta}(\Omega_A)$ and*

$$G(\phi, \lambda, \varepsilon) = 0; \tag{2.6}$$

conversely, if $\phi \in BC^{\theta}(\Omega)$ has a derivative $\dot{\phi} : \mathbb{R} \to X$ and solves (2.6)*, then ϕ is an entire bounded solution of $(D)_\lambda^\varepsilon$ in $BC^{1+\theta}(\Omega_A)$.*

Proof. Let $\lambda \in \Lambda$, $\varepsilon \in V$ be given. By definition, an entire solution $\phi \in BC^\theta(\Omega)$ satisfies the estimate $\sup_{t\in\mathbb{R}} |\phi(t)|_{X^\beta} < \infty$ and $\phi : \mathbb{R} \to X$ is of class C^1. Thus, our assumption (H_0) guarantees the existence of a $C \geq 0$ such that

$$\begin{aligned} \left|\dot\phi(t)\right|_X &\stackrel{(2.4)}{\leq} |A\phi(t)|_X + |f(t,\phi(t),\lambda,\varepsilon)|_X \leq |A|_{L(X^\beta,X)} |\phi(t)|_{X^\beta} + C \\ &\leq |A|_{L(X^\beta,X)} \|\phi\|_{0,X^\beta} + C \quad \text{for all } t \in \mathbb{R}; \end{aligned}$$

since also $[\dot\phi]_{\theta,X} < \infty$ holds, we get $\phi \in BC^{1+\theta}(\Omega)$. Indeed, due to $\phi(t) \in D(A)$ for all $t \in \mathbb{R}$ it is $\phi \in BC^{1+\theta}(\Omega_A)$. Furthermore, (2.4) and (2.6) are obviously equivalent.

Conversely, if $\phi \in BC^\theta(\Omega_A)$ admits a derivative $\dot\phi : \mathbb{R} \to X$, then the relation (2.6) reads as identity $\dot\phi(t) \equiv -A\phi(t) + f(t,\phi(t),\varepsilon,\lambda)$, which implies $\dot\phi \in BC^\theta(X)$, $\phi \in BC^{1+\theta}(\Omega_A)$ and that ϕ is an entire solution to $(D)^\varepsilon_\lambda$. □

The characterization from Thm. 2.4 allows to rephrase evolutionary equations $(D)^\varepsilon_\lambda$ as abstract 2-parameter bifurcation problem $G(\phi,\lambda,\varepsilon) = 0$ in the sense of Appendix A, but also allows a dynamical interpretation in terms of solutions for $(D)^\varepsilon_\lambda$. It thus remains to establish an adequate Fredholm theory for the derivative D_1G, which is strongly connected to the above notion of an ED.

As preparatory remark, in various ways, EDs are an adequate nonautonomous hyperbolicity notion. In order to motivate this, we suppose the semilinear equation $(D)^{\varepsilon^*}_{\lambda^*}$ possesses an entire reference solution $\phi^* = \phi(\lambda^*,\varepsilon^*)$ in $BC^\theta(\Omega)$ for a fixed parameter pair $(\lambda^*,\varepsilon^*) \in \Lambda \times V$. We suppose that ϕ^* is *hyperbolic*, i.e. the *variational equation*

$$\boxed{\dot u + Au = D_2 f(t,\phi^*(t),\lambda^*,\varepsilon^*)u,} \qquad (V)^{\varepsilon^*}_{\lambda^*}$$

admits an ED on $\mathbb{R}$ — note that the transition operator $T(t,s)$ exists as above, since the mapping $\mathbb{R} \to L(X^\beta, X)$, $t \mapsto D_2 f(t,\phi^*(t),\lambda,\varepsilon) - A$ is Hölder continuous with exponent θ. Hence, for every inhomogeneous perturbation $\psi \in BC^\theta$ there exists a unique bounded solution of the linearly inhomogeneous equation

$$\dot u + Au = D_2 f(t,\phi^*(t),\lambda^*,\varepsilon^*)u + \psi(t)$$

(cf. [16, pp. 227–228, Thm. 7.6.3]). This property, in turn, means that the derivative $D_1G(\phi^*,\lambda^*,\varepsilon^*) \in L(BC^{1+\theta}(X^\beta), BC^\theta)$ to (2.6) is invertible. Accordingly, the implicit function theorem implies that the entire bounded solution ϕ^* persists under small variation of the parameters λ,ε (see [36, Thm. 3.8] for the related situation of FDEs).

Addressing the complementary case, for fixed imperfection parameters $\varepsilon \in V$ we say that a semilinear parabolic equation $(D)^\varepsilon_\lambda$ undergoes a *bifurcation* at $\lambda = \lambda^*$ along the entire solution ϕ^*, or ϕ^* *bifurcates* at λ^*, if there exists a convergent parameter sequence $(\lambda_n)_{n\in\mathbb{N}}$ in Λ with the limit λ^*, such that the semilinear equation $(D)^\varepsilon_{\lambda_n}$ has two distinct bounded entire solutions $\phi^1_\varepsilon(\lambda_n), \phi^2_\varepsilon(\lambda_n)$ satisfying

$$\lim_{n\to\infty} \phi^1_\varepsilon(\lambda_n) = \lim_{n\to\infty} \phi^2_\varepsilon(\lambda_n) = \phi^*;$$

this notion corresponds to the usual terminology from branching theory (cf., for example, [43, p. 358, Def. 8.1]). In order to provide sufficient bifurcation criteria, we deal with nonhyperbolic solutions ϕ^* fulfilling

Hypothesis. *For $\tau \in \mathbb{R}$ suppose $\phi^* \in BC^\theta(\Omega)$ is an entire solution of $(D)^{\varepsilon^*}_{\lambda^*}$ so that*

(H_1) *the variational equation $(V)^{\varepsilon^*}_{\lambda^*}$ admits an ED both on $[\tau, \infty)$ and $(-\infty, \tau]$ with respective projectors P_t^+, P_t^- and nonzero $\xi \in X^\beta$, $\xi' \in (X^\beta)'$ satisfying*

$$R(P_\tau^+) \cap N(P_\tau^-) = \operatorname{span}\{\xi\}, \quad (R(P_\tau^+) + N(P_\tau^-))^\perp = \operatorname{span}\{\xi'\}. \quad (2.7)$$

We apply the Fredholm theory established in [7, 44] for the weighted differential operator $L : BC^{1+\theta}(X^\beta) \to BC^\theta$ given by

$$(L\psi)(t) := \dot{\psi}(t) + A\psi(t) - D_2 f(t, \phi^*(t), \lambda^*, \varepsilon^*)\psi(t) \quad \text{for all } t \in \mathbb{R}$$

and obtain

Proposition 2.5. *If (H_0), (H_1) hold, then $L \in L(BC^{1+\theta}(X^\beta), BC^\theta)$ is an index 0 Fredholm operator with kernel $N(L) = \operatorname{span}\{T(\cdot, \tau)\xi\}$.*

Remark 2.1. (1) In the terminology of Appendix A this means that ϕ^* is a degenerate solution of $G(\phi, \lambda^*, \varepsilon^*) = 0$, while 0 is a simple eigenvalue of $L = D_1 G(\phi^*, \lambda^*, \varepsilon^*)$.

(2) A converse to Prop. 2.5 was shown by [23] in the following sense: If L is Fredholm, $N(L|_{BC^1((-\infty,\tau],X^\beta)})$ is finite dimensional or $N(L|_{BC^1([\tau,\infty),X^\beta)})$ is finite codimensional, then $(V)^{\varepsilon^*}_{\lambda^*}$ has EDs on $(-\infty, \tau]$ and $[\tau, \infty)$.

Proof. First, as in the proof of Cor. 2.3 we see that L is bounded. The remaining assertion follows from [7, Lemma 3.2] or [44, Thm. 1]. □

Corollary 2.6. *If (H_0), (H_1) hold, then the linear functional*

$$\mu : BC^\theta \to \mathbb{R}, \qquad \mu(\psi) := \int_{\mathbb{R}} \langle T'(\tau, s)\xi', \psi(s)\rangle \, ds$$

is continuous with $|\mu| \le \frac{2K}{\alpha} |\xi'|$ and $R(L) = N(\mu)$.

Proof. Referring to [7, Lemma 3.2] or [44, Thm. 1] we know that $R(L)$ consists of all functions $\phi \in BC^\theta(X^\beta)$ satisfying the condition $\int_{\mathbb{R}} \langle \psi'(s), \phi(s)\rangle \, ds = 0$ for all solutions $\psi' \in BC^{1+\theta}((X^\beta)')$ of the adjoint equation

$$\dot{u} - A'u = -D_2 f(t, \phi^*(t), \lambda^*, \varepsilon^*)'u \quad (2.8)$$

in X'. Due to Lemma 2.1 also (2.8) has EDs on $[\tau, \infty)$ and $(-\infty, \tau]$ with respective projectors $(\operatorname{id} - P_t^+)'$ and $(\operatorname{id} - P_t^-)'$. Thus, the bounded entire solutions of (2.8) are given by $\operatorname{span}\{T'(\cdot, \tau)\xi'\}$ and we have $R(L) = N(\mu)$. Using the explicit dichotomy estimates from Lemma 2.1 we obtain the claimed bound on the functional μ as follows

$$\begin{aligned}
|\mu(\psi)| &\le \int_{-\infty}^{\tau} \left|\langle T'(\tau, s)(P_\tau^-)'\xi', \psi(s)\rangle\right| \, ds + \int_{\tau}^{\infty} \left|\langle T'(\tau, s)(\operatorname{id} - P_\tau^+)'\xi', \psi(s)\rangle\right| \, ds \\
&\le \int_{-\infty}^{\tau} \left|T(\tau, s)P_s^-\right| |\xi'| \, |\psi(s)| \, ds + \int_{\tau}^{\infty} \left|T(\tau, s)(\operatorname{id} - P_s^+)\right| |\xi'| \, |\psi(s)| \, ds \\
&\le K |\xi'| \, \|\psi\|_{\theta, X^\beta} \left(\int_{-\infty}^{\tau} e^{\alpha(s-\tau)} \, ds + \int_{\tau}^{\infty} e^{\alpha(\tau - s)} \, ds \right)
\end{aligned}$$

for all $\psi \in BC^\theta(X^\beta)$ and this implies that Cor. 2.6 is established. □

These preparations eventually put us into the position to apply the abstract bifurcation and imperfection theorems from Appendix A. The first bifurcation result ensures that near a fold point (ϕ^*, λ^*) of $G(\cdot, \varepsilon^*)$ the perturbed solution portrait of $(D)^\varepsilon_\lambda$ essentially keeps the same shape close to $\varepsilon = \varepsilon^*$, namely a parabola-like curve. This means, the nonhyperbolic solutions to equation $(D)^\varepsilon_\lambda$ near $(\phi^*, \lambda^*, \varepsilon^*)$ stay on a smooth curve under variation of ε. Hence, the fold bifurcation scenario described in [34, Thms. 3.13 and 2.13] for ODEs and difference equations persists (cf. Fig. A.3):

Theorem 2.7 (fold bifurcation)**.** *Suppose that* (H_0)*,* (H_1) *are satisfied with* $m \geq 2$*. If*

$$g_{010} := -\int_{\mathbb{R}} \langle T'(\tau, s)\xi', D_3 f(s, \phi^*(s), \lambda^*, \varepsilon^*)\rangle \, ds \neq 0,$$

$$g_{200} := -\int_{\mathbb{R}} \langle T'(\tau, s)\xi', D_2^2 f(s, \phi^*(s), \lambda^*, \varepsilon^*) \, [T(s, \tau)\xi]^2\rangle \, ds \neq 0,$$

then the following holds true:

(a) *There exist open convex neighborhoods* $\Omega_0 \subseteq BC^{1+\theta}(\Omega)$ *of* ϕ^**,* $\Lambda_0 \subseteq \Lambda$ *of* λ^**,* $V_0 \subseteq V$ *of* ε^* *and* C^m*-functions* $\phi^\star : V_0 \to \Omega_0$*,* $\lambda^\star : V_0 \to \Lambda_0$ *with*

$$\phi^\star(\varepsilon^*) = \phi^*, \qquad \lambda^\star(\varepsilon^*) = \lambda^*, \qquad \dot{\lambda}^\star(\varepsilon^*) = -\lambda^* \frac{g_{001}}{g_{010}}$$

and each $\phi^\star(\varepsilon) : \mathbb{R} \to \Omega$*,* $\varepsilon \in V_0$*, is a fold bifurcating entire solution to the semilinear parabolic equation* $(D)^\varepsilon_{\lambda^\star(\varepsilon)}$*, where*

$$g_{001} := -\int_{\mathbb{R}} \langle T'(\tau, s)\xi', D_4 f(s, \phi^*(s), \lambda^*, \varepsilon^*)\rangle \, ds.$$

(b) *For every* $\varepsilon \in V_0$ *there exists an open neighborhood* $S_\varepsilon \subseteq \mathbb{R}$ *of* 0 *and* C^m*-functions* $\phi_\varepsilon : S_\varepsilon \to \Omega_0$*,* $\lambda_\varepsilon : S_\varepsilon \to \Lambda_0$ *with* $\phi_\varepsilon(0) = \phi^\star(\varepsilon)$*,* $\lambda_\varepsilon(0) = \lambda^\star(\varepsilon)$*,*

$$\dot{\lambda}_\varepsilon(0) = 0, \qquad \ddot{\lambda}_\varepsilon(0) \neq 0, \qquad \ddot{\lambda}_{\varepsilon^*}(0) = -\frac{g_{200}}{g_{010}}$$

and each $\phi_\varepsilon(s) : \mathbb{R} \to \Omega$*,* $s \neq 0$*, is a hyperbolic solution to equation* $(D)^\varepsilon_{\lambda_\varepsilon(s)}$ *in* $BC^{1+\theta}(\Omega)$*. Locally in the neighborhood* $\Omega_0 \times \Lambda_0$ *one has:*

(b_1) Subcritical case*: If* $g_{200}/g_{010} > 0$*, then* $(D)^\varepsilon_\lambda$ *has no entire solution in* $BC^{1+\theta}(\Omega)$ *for* $\lambda > \lambda^\star(\varepsilon)$*,* $\phi^\star(\varepsilon)$ *is the unique entire solution of* $(D)^\varepsilon_{\lambda^\star(\varepsilon)}$ *in* $BC^{1+\theta}(\Omega)$ *and* $(D)^\varepsilon_\lambda$ *has exactly two distinct entire bounded solutions for* $\lambda < \lambda^\star(\varepsilon)$*; they are in* $BC^{1+\theta}(\Omega)$*.*

(b_2) Supercritical case*: If* $g_{200}/g_{010} < 0$*, then* $(D)^\varepsilon_\lambda$ *has no entire solution in* $BC^{1+\theta}(\Omega)$ *for* $\lambda < \lambda^\star(\varepsilon)$*,* $\phi^\star(\varepsilon)$ *is the unique entire solution of* $(D)^\varepsilon_{\lambda^\star(\varepsilon)}$ *in* $BC^{1+\theta}(\Omega)$ *and* $(D)^\varepsilon_\lambda$ *has exactly two distinct entire bounded solutions for* $\lambda > \lambda^\star(\varepsilon)$*; they are in* $BC^{1+\theta}(\Omega)$*.*

Remark 2.2. In case of a bounded operator $A \in L(X)$ the semilinear parabolic equation $(D)^\varepsilon_\lambda$ reduces to an ordinary differential equation in the Banach space X. Our theory, and in particular the above Thm. 2.7 applies for $\beta = 0$, $X^\beta = X$ and with Hölder continuity assumptions replaced by solely continuity throughout.

Proof. Let us apply Thm. A.1 with $\mathcal{X} = BC^{1+\theta}(X^\beta)$, $\mathcal{Z} = BC^\theta$, $\Omega = BC^{1+\theta}(\Omega)^\circ$ and the C^m-mapping $G : \Omega^\circ \times \Lambda \times V \to \mathcal{Z}$ defined in Cor. 2.3. Above all, since ϕ^* is a permanent solution, we have the inclusion $\phi^* \in \Omega$. By assumption, from Thm. 2.4 we get $G(\phi^*, \lambda^*, \varepsilon^*) = 0$ and Prop. 2.2 with Cor. 2.3 ensure

$$D_1 G(\phi^*, \lambda^*, \varepsilon^*) = L.$$

Hence, Prop. 2.5 guarantees that the required Fredholm property (A.3) for the derivative L holds. Finally, the explicit partial derivatives of G obtained by Prop. 2.2 and Cor. 2.6 yield that the generic conditions (A.4) are fulfilled, where $x_1 = T(\cdot, \tau)\xi$. With this, the claim follows from Thm. A.1 and the interpretation given in Thm. 2.4. □

3. Carathéodory differential equations

This section is devoted to finite-dimensional nonautonomous differential equations, however now with only measurable time dependence — so-called Carathéodory differential equations (we briefly write CDE). Such problems typically occur in control theory dealing with ODEs $\dot{x} = g(x, u(t), \lambda, \varepsilon)$ subject to control functions $u \in L^\infty$ (see, e.g., [10]). Another motivation to study them comes from continuous random dynamical systems (see [1]), which are of the form

$$\dot{x} = g(\theta_t \xi, x, \lambda, \varepsilon). \tag{3.1}$$

Such differential equations are driven by a metric dynamical system $\theta_t : \Xi \to \Xi$, $t \in \mathbb{R}$, on a probability space $(\Xi, \mathcal{F}, \mathbb{P})$; in particular, this means the mapping $(t, \xi) \mapsto \theta_t \xi$ is measurable. Hence, under natural assumptions on the right hand side g, the random differential equation (3.1) gives rise to a CDE $\dot{x} = f_\xi(t, x, \lambda, \varepsilon) = g(\theta_t \xi, x, \lambda, \varepsilon)$ for almost every realization $\xi \in \Xi$.

Basic introductions to CDEs are given in [22, pp. 315ff] or [3]. We already considered such equations in [35] and heavily rely on the corresponding earlier preparations here. We employ Thm. A.2 in order to investigate a perturbed transcritical bifurcation pattern given in Thm. 3.4. In fact, we extend the previous situation from Sect. 2 by also considering homoclinic solutions, which have limit 0 in both time directions. A minimal example concludes our results.

For this purpose, we equip the space $\mathbb{R}^d$ with the Euclidean norm $|\cdot|$. In a natural way, the duality pairing on $\mathbb{R}^d$ becomes the dot product $\langle x, y \rangle = \sum_{j=1}^d x_j y_j$ and the dual operator T' to $T \in L(\mathbb{R}^d)$ is simply the transpose. Furthermore, measure theoretical terminology always refers to the Lebesgue measure and integral.

Given an interval $I \subseteq \mathbb{R}$, let us suppose that $A : I \to L(\mathbb{R}^d)$ is locally integrable and essentially bounded, i.e.,

$$\operatorname*{ess\,sup}_{t \in I} |A(t)| < \infty. \tag{3.2}$$

Under this assumption we consider a linear Carathéodory equation

$$\boxed{\dot{x} = A(t)x} \tag{CL}$$

with *transition operator* $\Phi(t,s)$, $t,s\in I$ (cf. [3, Def. 2.8]). As opposed to Sect. 2, now $\Phi(t,s)\in GL(\mathbb{R}^d)$ has invertible values. An *invariant projector* for (CL) is a function $t\mapsto P_t\in L(\mathbb{R}^d)$ with $P_t^2=P_t$ and $P_t\Phi(t,s)=\Phi(t,s)P_s$ for all $s,t\in I$.

A linear differential equation (CL) is said to have an *exponential dichotomy* (ED for short) on I, if there exist reals $K\geq 1$, $\alpha>0$ and an invariant projector such that

$$|\Phi(t,s)P_s|\leq Ke^{-\alpha(t-s)},\qquad |\Phi(s,t)[\mathrm{id}-P_t]|\leq Ke^{\alpha(s-t)}\quad \text{for all } s\leq t;$$

due to the invertibility of $\Phi(t,s)$ this definition is simpler than for parabolic PDEs.

Moving on to nonlinear differential equations, we suppose their *state space* $\Omega\subseteq\mathbb{R}^d$ is open and convex. For *Carathéodory functions* $f:\mathbb{R}\times\Omega\times\Lambda\times V\to\mathbb{R}^d$, i.e.

- for almost every $t\in\mathbb{R}$ the mapping $f(t,\cdot,\lambda,\varepsilon)$, $\lambda\in\Lambda$, $\varepsilon\in V$, is continuous,
- for every $(x,\lambda,\varepsilon)\in\Omega\times\Lambda\times V$ the mapping $f(\cdot,x,\lambda,\varepsilon)$ is measurable

(cf., [3, Def. 2.1]), a *Carathéodory differential equation* reads as

$$\boxed{\dot{x}=f(t,x,\lambda,\varepsilon),}\tag{$(C)_\lambda^\varepsilon$}$$

where $\lambda\in\Lambda$ is a bifurcation and $\varepsilon\in V$ a perturbation parameter. A *solution* of the CDE $(C)_\lambda^\varepsilon$ is an absolutely continuous function $\phi:I\to\Omega$ satisfying the solution identity $\dot{\phi}(t)=f(t,\phi(t),\lambda,\varepsilon)$ a.e. on an interval $I\subseteq\mathbb{R}$. Formally, *entire* (or *complete*) and *permanent solutions* of $(C)_\lambda^\varepsilon$ are defined as in Sect. 2. Beyond that, a *homoclinic solution* ϕ is entire and fulfills the limit relation $\lim_{t\to\pm\infty}\phi(t)=0$.

Lastly, the *general solution* of $(C)_\lambda^\varepsilon$ is the solution $\varphi_\lambda^\varepsilon(\cdot;t_0,\xi_0)$ satisfying the initial condition $x(t_0)=\xi_0$ for all pairs $(t_0,\xi_0)\in\mathbb{R}\times\Omega$. Note that backward solutions always exist and are unique.

Our assumptions on the right hand side f resemble the ones from Sect. 2, but are required to hold only a.e. in the time variable:

Hypothesis. *Let $m\in\mathbb{N}$, suppose $f:\mathbb{R}\times\Omega\times\Lambda\times V\to\mathbb{R}^d$ is a Carathéodory function and $f(t,\cdot)$ is a C^m-function a.e. in $t\in\mathbb{R}$ such that the following holds for $0\leq j\leq m$:*

(H_0) *For all bounded $B\subseteq\Omega$ one has*

$$\operatorname*{ess\,sup}_{t\in\mathbb{R}}\sup_{x\in B}\left|D^j_{(2,3,4)}f(t,x,\lambda,\varepsilon)\right|<\infty\quad \textit{for all } \lambda\in\Lambda,\ \varepsilon\in V$$

(well-definedness) *and for $(\lambda^*,\varepsilon^*)\in\Lambda\times V$ and $\rho>0$ there exists a $\delta>0$ with*

$$|x-y|<\delta\quad\Rightarrow\quad\operatorname*{ess\,sup}_{t\in\mathbb{R}}\left|D^j_{(2,3,4)}f(t,x,\lambda,\varepsilon)-D^j_{(2,3,4)}f(t,y,\lambda,\varepsilon)\right|<\rho$$

for all $x,y\in\Omega$ and $(\lambda,\varepsilon)\in B_\delta(\lambda^*,\varepsilon^*)$ (uniform continuity).

(H_1) *We have $0\in\Omega$ and $\lim_{t\to\pm\infty}f(t,0,\lambda,\varepsilon)=0$ for all $\lambda\in\Lambda$, $\varepsilon\in V$.*

Our subsequent goal is a suitable functional analytical formulation of CDEs $(C)_\lambda^\varepsilon$ as abstract equations in ambient function spaces. This will be covered by the spaces

$AC(\Omega)$ of (locally) absolutely continuous functions,

$L^{\infty}(\Omega)$ of essentially bounded functions and

$W^{1,\infty}(\Omega)$ of bounded functions $\phi : \mathbb{R} \to \Omega$ with essentially bounded weak derivative.

We often abbreviate $AC := AC(\mathbb{R}^d)$ and proceed accordingly with other functions spaces. The canonical norm on L^{∞} is $\|\phi\|_0 := \operatorname{ess\,sup}_{t\in\mathbb{R}} |\phi(t)|$, and we use the norm

$$\|\phi\|_1 := \max\left\{\|\phi\|_0, \left\|\dot{\phi}\right\|_0\right\}$$

on $W^{1,\infty}$. Both, L^{∞} and $W^{1,\infty}$ are Banach spaces with the closed subspaces

$$L_0^{\infty} := \left\{\phi \in L^{\infty} : \lim_{t\to\pm\infty} \phi(t) = 0\right\}, \quad W_0^{1,\infty} := \left\{\phi \in W^{1,\infty} : \phi, \dot{\phi} \in L_0^{\infty}\right\},$$

resp., and the homoclinic solutions to $(C)_{\lambda}^{\varepsilon}$ are contained in L_0^{∞}. An operator formulation of $(C)_{\lambda}^{\varepsilon}$ depends on appropriate substitution operators F defined as in (2.5).

Proposition 3.1. *Under (H_0) the operator $G : W^{1,\infty}(\Omega) \times \Lambda \times V \to L^{\infty}$,*

$$G(\phi, \lambda, \varepsilon) := \dot{\phi} - F(\phi, \lambda, \varepsilon)$$

is well-defined and m-times continuously differentiable on $W^{1,\infty}(\Omega)^{\circ} \times \Lambda \times V$. If (H_0) and (H_1) are satisfied, then the same holds for $G : W_0^{1,\infty}(\Omega) \times \Lambda \times V \to L_0^{\infty}$.

Proof. See [35, Cor. 2.2]. □

This yields our important counterpart to Thm. 2.4 in the present CDE setting:

Theorem 3.2. *For all parameters $\lambda \in \Lambda$, $\varepsilon \in V$ the following holds under (H_0):*

(a) If $\phi \in L^{\infty}(\Omega)$ has a derivative a.e. in $\mathbb{R}$ and is an entire solution of $(C)_{\lambda}^{\varepsilon}$, then $\phi \in W^{1,\infty}$ and

$$G(\phi, \lambda, \varepsilon) = 0; \tag{3.3}$$

conversely, if $\phi \in L^{\infty}(\Omega)$ has a derivative a.e. in $\mathbb{R}$ and solves (3.3)*, then ϕ is an entire bounded solution of $(C)_{\lambda}^{\varepsilon}$ in $W^{1,\infty}$.*

(b) Under (H_0)–(H_1), if $\phi \in L_0^{\infty}(\Omega)$ is an entire solution of $(C)_{\lambda}^{\varepsilon}$, then $\phi \in W_0^{1,\infty}$ and (3.3) *holds; conversely, if $\phi \in L_0^{\infty}(\Omega)$ has a derivative a.e. in $\mathbb{R}$ and solves* (3.3)*, then ϕ is an entire bounded solution of $(C)_{\lambda}^{\varepsilon}$ in $W_0^{1,\infty}$.*

Proof. We refer to [35, Thm. 2.3]. □

Given a bounded entire reference solution $\phi^* = \phi(\lambda^*, \varepsilon)$ to the CDE $(C)_{\lambda^*}^{\varepsilon^*}$, its hyperbolicity in terms of an ED on $\mathbb{R}$ for the variational equation

$$\boxed{\dot{x} = D_2 f(t, \phi^*(t), \lambda^*, \varepsilon^*)x} \tag{$(CV)_{\lambda^*}^{\varepsilon^*}$}$$

prevents possible bifurcations. Here, the definition of a bifurcating solution is literally the same as in Sect. 2. In order to derive sufficient bifurcation criteria, we assume the following kind of nonhyperbolicity:

Hypothesis. *Let $\tau \in \mathbb{R}$, $\lambda^* \in \Lambda$, $\varepsilon^* \in V$ be given and suppose that $(C)_{\lambda^*}^{\varepsilon^*}$ has an entire permanent solution $\phi^* \in L^\infty(\Omega)$ with*

(H_2) *the variational equation $(CV)_{\lambda^*}^{\varepsilon^*}$ admits an ED both on $[\tau, \infty)$ and $(-\infty, \tau]$ with respective projectors P_t^+, P_t^- and nonzero vectors $\xi, \xi' \in \mathbb{R}^d$ satisfying*

$$R(P_\tau^+) \cap N(P_\tau^-) = \operatorname{span}\{\xi\}, \qquad (R(P_\tau^+) + N(P_\tau^-))^\perp = \operatorname{span}\{\xi'\}.$$

Note that differing from the parabolic case, we now do not need to require Hölder continuity of the mapping $t \mapsto D_2 f(t, \phi^*(t), \lambda^*, \varepsilon^*)$ in order to show that the corresponding transition operator $\Phi(t,s)$ exists. The necessary Fredholm theory for $(CV)_{\lambda^*}^{\varepsilon^*}$ is essentially due to [31, 32], where the minor modifications to tackle measurable time dependence can be found in [35].

In order to treat bounded and homoclinic solutions to $(C)_\lambda^\varepsilon$ simultaneously, let $\mathcal{X}$ stand for either one of the function spaces $W^{1,\infty}$ or $W_0^{1,\infty}$, while $\mathcal{Z}$ denotes the respective space L^∞ or L_0^∞. Then the weighted differential operator

$$L : \mathcal{X} \to \mathcal{Z}, \quad (L\psi)(t) := \dot{\psi}(t) - D_2 f(t, \phi^*(t), \lambda^*, \varepsilon^*)\psi(t) \quad \text{for almost all } t \in \mathbb{R}$$

has the following properties:

Proposition 3.3. *If (H_0), (H_2) hold, then $L \in L(\mathcal{X}, \mathcal{Z})$ has the following properties:*

(a) *It is an index 0 Fredholm operator with $N(L) = \operatorname{span}\{\Phi(\cdot, \tau)\xi\}$,*
(b) *$R(L) = N(\mu)$ with a bounded linear functional*

$$\mu : \mathcal{Z} \to \mathbb{R}, \qquad \mu(\psi) := \int_{\mathbb{R}} \langle \Phi(\tau, s)'\xi', \psi(s)\rangle \, ds$$

satisfying $|\mu| \leq \frac{2K}{\alpha} |\xi'|$.

Proof. This is a special case of [35, Prop. 3.1 and Cor. 3.2]. □

The above tools are the crucial ingredients to apply the abstract results from Appendix A. Thus, in order to match our present set-up, we could reformulate Thm. 2.7 for CDEs with the space $\mathcal{X}$ being $W^{1,\infty}$ or $W_0^{1,\infty}$ and $\mathcal{Z}$ denoting L^∞ or L_0^∞, respectively. However, we instead study transcritical bifurcations and leave the fold bifurcation case of Thm. 2.7 to the interested reader.

We suppose that a solution branch for $(C)_\lambda^{\varepsilon^*}$ is known; more formally, this means

Hypothesis. *Suppose that*

(H_3) *$f(t, \phi^*(t), \lambda, \varepsilon^*) \equiv 0$ for all $\lambda \in \Lambda$ and almost all $t \in \mathbb{R}$.*

Remark 3.1. Based on abstract results of [28] we have demonstrated in [35] that the global assumption (H_3) can be replaced by a local condition on the partial derivatives of G. The resulting *crossing curve bifurcation*, formulated for CDEs in [35, Thm. 4.1], includes the transcritical and pitchfork patterns as special cases.

It should be clear that a given branch of solutions from (H_3) does not persist under variation of ε and precisely one obtains (cf. Fig. A.4)

Theorem 3.4 (imperfect transcritical bifurcation)**.** *Suppose that* (H_0), (H_2), (H_3) *are satisfied with* $m \geq 2$. *If* $\phi^* \in \mathcal{X} = W^{1,\infty}$ *and under the generic assumptions*

$$g_{110} := -\int_{\mathbb{R}} \langle \Phi(\tau,s)'\xi', D_2 D_3 f(s,\phi^*(s),\lambda^*,\varepsilon^*)\Phi(s,\tau)\xi\rangle \, ds \neq 0,$$
$$g_{001} := -\int_{\mathbb{R}} \langle \Phi(\tau,s)'\xi', D_4 f(s,\phi^*(s),\lambda^*,\varepsilon^*)\rangle \, ds \neq 0,$$
$$g_{200} := -\int_{\mathbb{R}} \langle \Phi(\tau,s)'\xi', D_2^2 f(s,\phi^*(s),\lambda^*,\varepsilon^*)[\Phi(s,\tau)\xi]^2\rangle \, ds \neq 0,$$

then the following holds true:

(a) *There exist open convex neighborhoods* $S \subseteq \mathbb{R}$ *of* 0, $\Omega_0 \subseteq \mathcal{X}(\Omega)$ *of* ϕ^*, $\Lambda_0 \subseteq \Lambda$ *of* λ^*, $V_0 \subseteq V$ *of* ε^* *and* C^m*-functions* $\phi^\star : S \to \Omega_0$, $\lambda^\star : S \to \Lambda_0$, $\varepsilon^\star : S \to V_0$ *with*

$$\phi^\star(0) = \phi^*, \qquad \dot{\phi}^\star(0) = -\frac{g_{110}}{g_{200}}\Phi(\cdot,\tau)\xi,$$
$$\lambda^\star(0) = \lambda^*, \qquad \dot{\lambda}^\star(0) = 1,$$
$$\varepsilon^\star(0) = \varepsilon^*, \qquad \dot{\varepsilon}^\star(0) = 0, \qquad \ddot{\varepsilon}^\star(0) = \frac{g_{110}^2}{g_{200}g_{001}} \neq 0$$

and each $\phi^\star(s)$, $s \in S$, *is a nonhyperbolic entire solution to* $(D)_{\lambda^\star(s)}^{\varepsilon^\star(s)}$ *in* $\mathcal{X}(\Omega)$.

(b) *Under the additional assumptions*

$$g_{001} > 0, \qquad g_{110} > 0, \qquad g_{200} > 0$$

one has locally in $\Omega_0 \times \Lambda_0$ *that for every* $\varepsilon \in V_0$ *there exist compact intervals* $S_\varepsilon = [-\delta_\varepsilon, \delta_\varepsilon]$, $\Lambda_\varepsilon = [\lambda^* - \rho_\varepsilon, \lambda^* + \rho_\varepsilon] \subset \Lambda_0$ *such that:*

(b_1) *If* $\varepsilon < \varepsilon^*$, *then the bounded entire solutions to* $(C)_\lambda^\varepsilon$ *in* $\Omega_0 \times \Lambda_0$ *consist of two disjoint branches* $\Gamma_\varepsilon^1 \dot{\cup} \Gamma_\varepsilon^2$ *with*

$$\Gamma_\varepsilon^i = \left\{(\phi_\varepsilon^i(\lambda),\lambda) \in \mathcal{X}(\Omega) \times \Lambda : \lambda \in \Lambda_\varepsilon\right\} \quad \text{for } i = 1,2;$$

here, $\phi_\varepsilon^i : \Lambda_\varepsilon \to \Omega_0$ *is a* C^{m-1}*-function and every* $\phi_\varepsilon^i(\lambda) : \mathbb{R} \to \Omega_0$, $\lambda \in \Lambda_0$, *is a hyperbolic solution of* $(C)_\lambda^\varepsilon$ *in* $\mathcal{X}(\Omega)$.

(b_2) *Besides the given branch (cf.* (H_3)*), the bounded entire solutions to* $(C)_\lambda^{\varepsilon^*}$ *in* $\Omega_0 \times \Lambda_0$ *consist of a branch*

$$\Gamma := \{(\phi_{\varepsilon^*}(s), \lambda_{\varepsilon^*}(s)) \in \mathcal{X}(\Omega) \times \Lambda : s \in S_{\varepsilon^*}\};$$

here, $\phi_{\varepsilon^*} : S_{\varepsilon^*} \to \Omega_0$, $\lambda_{\varepsilon^*} : S_{\varepsilon^*} \to \Lambda_0$ *are* C^{m-1}*-functions with*

$$\phi_{\varepsilon^*}(0) = \phi^*, \qquad \lambda_{\varepsilon^*}(0) = \lambda^*, \qquad \dot{\lambda}_{\varepsilon^*}(s) > 0 \quad \text{for all } s \in S_{\varepsilon^*}.$$

Every function $\phi_{\varepsilon^*}(s) : \mathbb{R} \to \Omega$, $s \neq 0$, *is a hyperbolic solution of* $(C)_{\lambda_{\varepsilon^*}(s)}^{\varepsilon^*}$ *in* $\mathcal{X}(\Omega)$ *and* ϕ^* *is a transcritical bifurcating solution to* $(C)_{\lambda^*}^{\varepsilon^*}$. *There exist exactly two entire bounded solutions to* $(C)_\lambda^{\varepsilon^*}$ *for* $\lambda \neq \lambda^*$ *and* ϕ^* *is the unique entire bounded solution to* $(C)_{\lambda^*}^{\varepsilon^*}$.

(b_3) *If $\varepsilon > \varepsilon^*$, then the bounded entire solutions to $(C)^\varepsilon_\lambda$ in $\Omega_0 \times \Lambda_0$ consist of two disjoint branches $\Gamma^+_\varepsilon \dot\cup \Gamma^-_\varepsilon$ with*

$$\Gamma^\pm_\varepsilon = \left\{ (\phi^\pm_\varepsilon(s), \lambda^\pm_\varepsilon(s)) \in \mathfrak{X}(\Omega) \times \Lambda : \, s \in S_\varepsilon \right\};$$

here, $\phi^\pm_\varepsilon : S_\varepsilon \to \Omega_0$, $\lambda^\pm_\varepsilon : \mathbb{R} \to \Lambda_0$ are C^{m-1}-functions with

$$\begin{aligned} \lambda^-_\varepsilon(\pm\delta_\varepsilon) &= \lambda^* - \rho_\varepsilon, & \lambda^+_\varepsilon(\pm\delta_\varepsilon) &= \lambda^* + \rho_\varepsilon, \\ \dot\lambda^-_\varepsilon(0) &= 0, & \dot\lambda^+_\varepsilon(0) &= 0, \\ \ddot\lambda^-_\varepsilon(0) &< 0, & \ddot\lambda^+_\varepsilon(0) &> 0. \end{aligned}$$

Every function $\phi^\pm_\varepsilon(s) : \mathbb{R} \to \Omega$, $s \neq 0$, is a hyperbolic solution of $(C)^\varepsilon_{\lambda^\pm_\varepsilon(s)}$ in $\mathfrak{X}(\Omega)$, $\phi^+_\varepsilon(0)$ is a supercritical fold bifurcating solution of $(C)^\varepsilon_{\lambda^+_\varepsilon(0)}$ and $\phi^-_\varepsilon(0)$ is a subcritical fold bifurcating solution of $(C)^\varepsilon_{\lambda^-_\varepsilon(0)}$. There exist exactly two entire bounded solutions to $(C)^\varepsilon_\lambda$ for $\lambda < \lambda^-_\varepsilon(0)$ or $\lambda > \lambda^+_\varepsilon(0)$, a unique bounded entire solution for $\lambda = \lambda^\pm_\varepsilon(0)$ and no bounded entire solution for $\lambda \in (\lambda^-_\varepsilon(0), \lambda^+_\varepsilon(0))$.

If (H_0)–(H_3) are satisfied, then the same holds with $\mathfrak{X} = W^{1,\infty}_0$.

Remark 3.2. Our Hypothesis (H_0) is clearly fulfilled for (piecewise) continuous right hand sides f. In this setting, Thm. 3.4 applies for (piecewise) continuous (in t) ODEs, with solutions in the spaces of bounded (piecewise) C^1-functions. Thus, Thm. 3.4 extends the unperturbed transcritical bifurcation described in [34, Cor. 3.15].

Proof. Suppose that $\mathfrak{X} = W^{1,\infty}$, $\mathfrak{Z} = L^\infty$ and $\Omega = \mathfrak{X}(\Omega)^\circ$. We make use of Thm. A.2 with the C^m-mapping $G : \Omega \times \Lambda \times V \to \mathfrak{Z}$ given in Prop. 3.1. From Thm. 3.2(a) above we see that $G(\phi^*, \lambda^*, \varepsilon^*) = 0$ holds and [35, Prop. 2.1] implies

$$\big(D_1 G(\phi^*, \lambda^*, \varepsilon^*)\psi\big)(t) = \dot\psi(t) - D_2 f(t, \phi^*(t), \lambda^*, \varepsilon^*)\psi(t) = (L\psi)(t) \quad \text{a.e. in } \mathbb{R}.$$

With this the assumption (A.3) follows from (H_2) and Prop. 3.3(a). Note that (H_3) ensures (A.6) and Prop. 3.3(b) yields the generic conditions (A.7), in connection with the explicit derivatives of G given in [35, Prop. 2.1] and the claim follows.

Having additionally (H_1) fulfilled, Thm. A.2 can be applied twice in the large space $W^{1,\infty}$ and in the subspace $W^{1,\infty}_0$, where the latter situation only applies under (H_1). Hence, the bifurcating solutions are unique in $W^{1,\infty}$ and exist in $W^{1,\infty}_0$. □

A minimal example under which Thm. 3.4 applies is

Example 3.1 (transcritical bifurcation). Let $\Omega = \mathbb{R}^2$, $\alpha > 0$, suppose that $\gamma, \delta \neq 0$ are parameters and that $b_0, c_0 \in L^\infty$ satisfy

$$\int_\mathbb{R} e^{-\alpha|s|} c_0(s)\, ds \neq 0;$$

for instance, this means that c_0 is not an odd function. We furthermore define piecewise constant functions $b, c : \mathbb{R} \to \mathbb{R}$,

$$b(t) := \begin{cases} \alpha, & t < 0, \\ -\alpha, & t \geq 0 \end{cases} \qquad c(t) := \begin{cases} -\alpha, & t < 0, \\ \alpha, & t \geq 0 \end{cases} \tag{3.4}$$

and consider the nonlinear CDE

$$\dot{x} = f(t, x, \lambda, \varepsilon) := \begin{pmatrix} b(t) & 0 \\ \gamma\lambda & c(t) \end{pmatrix} x + \delta \begin{pmatrix} 0 \\ x_1^2 \end{pmatrix} + \varepsilon \begin{pmatrix} b_0(t) \\ c_0(t) \end{pmatrix} \tag{3.5}$$

depending on a bifurcation parameter $\lambda \in \mathbb{R}$ and a perturbation parameter $\varepsilon \in \mathbb{R}$. It is clear that the spatially smooth right hand side $f : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ satisfies (H_0), (H_3) with $\Lambda = V = \mathbb{R}$, $\lambda^* = \varepsilon^* = 0$ and $\phi^* = 0$. From the derivative

$$D_2 f(t, 0, \lambda^*, \varepsilon^*) = \begin{pmatrix} b(t) & 0 \\ 0 & c(t) \end{pmatrix}$$

we see that the transition matrix of the associate variational equation $(CV)_{\lambda^*}^{\varepsilon^*}$ reads as

$$\Phi(t, s) := \begin{cases} \operatorname{diag}(e^{-\alpha(t-s)}, e^{\alpha(t-s)}), & t \geq s \geq 0, \\ \operatorname{diag}(e^{-\alpha(t+s)}, e^{\alpha(t+s)}), & t \geq 0 > s, \\ \operatorname{diag}(e^{\alpha(t-s)}, e^{-\alpha(t-s)}), & 0 > t \geq s; \end{cases}$$

we can moreover define $\Phi(t, s) := \Phi(s, t)^{-1}$ for times $t < s$. Thus, $(CV)_{\lambda^*}^{\varepsilon^*}$ admits an ED on the interval $[0, \infty)$ with projector $P_t^+ \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, as well as a dichotomy on $(-\infty, 0]$ with $P_t^- \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. This implies that the operator L has 1-dimensional kernel, index 0 and (H_2) is fulfilled, provided we choose vectors $\xi = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $\xi' = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. After these observations the bounded linear functional $\mu : L^\infty \to \mathbb{R}$ from Prop. 3.3(b) is

$$\mu(\psi) = \int_{\mathbb{R}} \langle \xi' \Phi(0, s)', \psi(s) \rangle \, ds = \int_{\mathbb{R}} e^{-\alpha|s|} \psi_2(s) \, ds \tag{3.6}$$

where $\psi = (\psi_1, \psi_2) \in L^\infty$. We compute the derivatives

$$D_2 D_3 f(t, 0, 0, 0)\zeta = \begin{pmatrix} 0 \\ \gamma\zeta_1 \end{pmatrix}, \qquad D_2^2 f(t, 0, 0, 0)\zeta^2 = 2\delta \begin{pmatrix} 0 \\ \zeta_1^2 \end{pmatrix},$$
$$D_4 f(t, 0, 0, 0) = \begin{pmatrix} b_0(t) \\ c_0(t) \end{pmatrix}$$

for all $t \in \mathbb{R}$, $\zeta \in \mathbb{R}^2$ and therefore the relation (3.6) ensures

$$g_{110} = -\frac{\gamma}{\alpha} \neq 0, \qquad g_{200} = -\frac{4\delta}{3\alpha} \neq 0, \qquad g_{001} = -\int_{\mathbb{R}} e^{-\alpha|s|} c_0(s) \, ds \neq 0.$$

Thus, Thm. 3.4 applies to the planar CDE (3.5) and in particular its assertion (b) follows under the assumptions $\gamma, \delta, g_{001} < 0$.

On the other hand, we can also quantitatively illustrate Thm. 3.4, since equation (3.5) is explicitly solvable. For the sake of a simple presentation we retreat to the case $b_0 = 0$ and a constant nonzero function c_0. The first component of the general solution $\varphi_\lambda^\varepsilon$ to (3.5) reads as $\varphi_\lambda^\varepsilon(t; 0, \eta)_1 = e^{-\alpha|t|}\eta_1$ for all $t \in \mathbb{R}$ and the variation of constants formula (cf. [3, Thm. 2.10]) yields the second component

$$\varphi_\lambda^\varepsilon(t; 0, \eta)_2 = e^{\alpha|t|}\eta_2 + \int_0^t e^{\alpha|t-s|} \left(\gamma\lambda e^{-\alpha|s|}\eta_1 + \delta e^{-2\alpha|s|}\eta_1^2 + \varepsilon c_0 \right) ds$$

for all $t \in \mathbb{R}$. Evaluating the integral, this implies the asymptotic representation

$$\varphi_\lambda^\varepsilon(t;0,\eta)_2 = \begin{cases} e^{\alpha t}\left(\eta_2 + \frac{\lambda\gamma}{2\alpha}\eta_1 + \frac{\delta}{3\alpha}\eta_1^2 + \frac{\varepsilon}{\alpha}c_0\right) + O(t) \text{ as } t \to \infty, \\ e^{-\alpha t}\left(\eta_2 - \frac{\lambda\gamma}{2\alpha}\eta_1 - \frac{\delta}{3\alpha}\eta_1^2 - \frac{\varepsilon}{\alpha}c_0\right) + O(t) \text{ as } t \to -\infty \end{cases}$$

from which we derive the 0-fibers

$$\begin{aligned} \mathcal{W}^+_{\lambda,\varepsilon}(0) &:= \left\{\eta \in \mathbb{R}^2 : \sup_{t\geq 0}|\varphi_\lambda^\varepsilon(t;0,\eta)| < \infty\right\} \\ &= \left\{\left(\eta_1, -\tfrac{\lambda\gamma}{2\alpha}\eta_1 - \tfrac{\delta}{3\alpha}\eta_1^2 - \tfrac{\varepsilon}{\alpha}c_0\right) \in \mathbb{R}^2 : \eta_1 \in \mathbb{R}\right\}, \\ \mathcal{W}^-_{\lambda,\varepsilon}(0) &:= \left\{\eta \in \mathbb{R}^2 : \sup_{t\leq 0}|\varphi_\lambda^\varepsilon(t;0,\eta)| < \infty\right\} \\ &= \left\{\left(\eta_1, \tfrac{\lambda\gamma}{2\alpha}\eta_1 + \tfrac{\delta}{3\alpha}\eta_1^2 + \tfrac{\varepsilon}{\alpha}c_0\right) \in \mathbb{R}^2 : \eta_1 \in \mathbb{R}\right\} \end{aligned}$$

of the stable resp. unstable integral manifolds of (3.5). Their intersection is given by

$$\mathcal{W}^+_{\lambda,\varepsilon}(0) \cap \mathcal{W}^-_{\lambda,\varepsilon}(0) = \begin{cases} \left\{\left(0, \eta^\pm_{\lambda,\varepsilon}\right)\right\}, & \lambda^2 \geq \frac{48\delta\varepsilon c_0}{9\gamma^2}, \\ \emptyset, & \text{else} \end{cases}$$

with $\eta^\pm_{\lambda,\varepsilon} := -\frac{3\gamma\lambda \pm \sqrt{9\gamma^2\lambda^2 - 48\delta\varepsilon c_0}}{4\delta}$ and for parameters $\gamma, \delta, c_0 < 0$ we explicitly get:

- (b_1) If $\varepsilon < 0$, then the initial conditions $x(0) = \left(0, \eta^\pm_{\lambda,\varepsilon}\right)$ lead to two distinct bounded entire solutions (see Fig. 2 (left)).
- (b_2) If $\varepsilon = 0$, then the initial conditions $x(0) = 0$ and $x(0) = \left(0, -\frac{3\gamma}{2\delta}\lambda\right)$ yield bounded entire solutions (see Fig. 2 (center)).
- (b_3) For $\varepsilon > 0$ we obtain $\lambda^\pm_\varepsilon(0) = \pm\frac{4}{3}\sqrt{3\delta\varepsilon c_0}\,|\gamma|$ and there exists no bounded entire solution for $\lambda \in (\lambda^-_\varepsilon(0), \lambda^+_\varepsilon(0))$. In case $\lambda < \lambda^-_\varepsilon(0)$ or $\lambda > \lambda^+_\varepsilon(0)$ the initial conditions $x(0) = \left(0, \eta^\pm_{\lambda,\varepsilon}\right)$ yields to two distinct bounded entire solutions. Finally, for $\lambda = \lambda^-_\varepsilon(0)$ the solution $\varphi^\varepsilon_\lambda\left(\cdot; 0, (0, -\frac{3\gamma}{4\delta}\lambda)\right)$ subcritically fold bifurcates, whereas for $\lambda = \lambda^+_\varepsilon(0)$ the solution $\varphi^\varepsilon_\lambda\left(\cdot; 0, (0, -\frac{3\gamma}{4\delta}\lambda)\right)$ supercritically fold bifurcates (see Fig. 2 (right)).

4. Difference equations

Finally, we turn towards discrete nonautonomous dynamical systems in form of difference equations. They necessitate the smallest technical preparations concerning well-posedness of forward initial value problems and well-definedness of operator equations. However, when it comes to numerical simulations, the essential understanding is due to the discrete situation. Dealing with difference equations in Hilbert spaces X, our results are also applicable to nonautonomous evolutionary differential equations, as long as their forward solutions generate a smooth 2-parameter semiflow

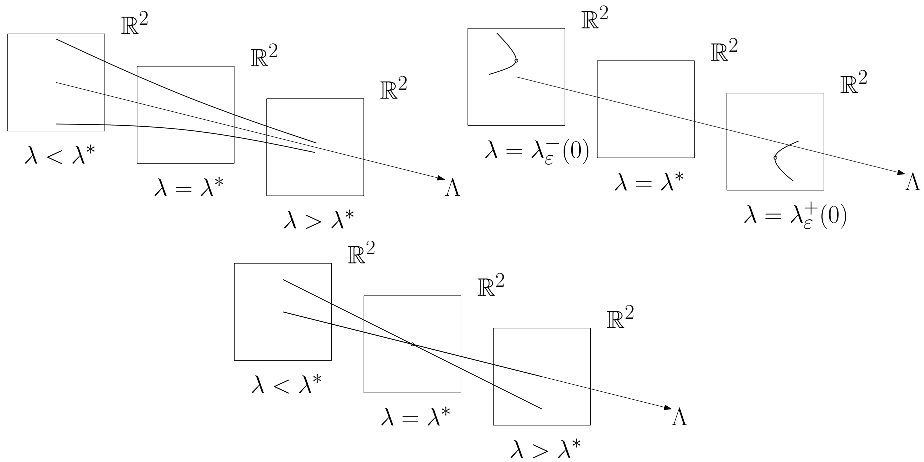

Figure 2: Initial values η for bounded entire solutions to the CDE (3.5):
Left ($\varepsilon < 0$): Two distinct bounded solutions for all λ
Center ($\varepsilon = 0$): Transcritical bifurcation at $\lambda^* = 0$
Right ($\varepsilon > 0$): Two fold bifurcations for $\lambda = \lambda_\varepsilon^\pm(0)$

$S_\lambda^\varepsilon(t,s)$, $s \le t$, on X. Indeed, rather than a differential, one alternatively investigates the nonautonomous difference equation $x_{k+1} = S_\lambda^\varepsilon(k+1,k)x_k$.

In this section, we elaborate on discrete equations in Hilbert spaces X. This restriction to inner product spaces is due to the necessity to define a natural generalized inverse (see below). We obtain and discuss a perturbed pitchfork bifurcation result in Thm. 4.4 using Thm. A.3 and apply it to a general cubic example. Compared to the previously considered differential equations, a slightly more delicate Fredholm theory is required in form of Cor. 4.3.

As usual, $\mathbb{Z}$ denotes the ring of integers, $\mathbb{N}$ are the positive integers and a *discrete interval* $\mathbb{I}$ is the intersection of a real interval with $\mathbb{Z}$; sometimes it is convenient to introduce the shifted interval $\mathbb{I}' := \{k \in \mathbb{I} : k+1 \in \mathbb{I}\}$. Given $\kappa \in \mathbb{Z}$ we define the unbounded discrete intervals $\mathbb{Z}_\kappa^+ := \{k \in \mathbb{Z} : \kappa \le k\}$ and $\mathbb{Z}_\kappa^- := \{k \in \mathbb{Z} : \kappa \ge k\}$. Due to our Hilbert space setting, the annihilator $X_0^\perp$ of a subspace X_0 is the orthogonal complement.

For linear operators $T \in L(X)$ with closed range $R(T)$ we define the *generalized inverse* $T^\dagger \in L(X)$ by linear extension based on the relation

$$T^\dagger x := \begin{cases} 0, & x \in R(T)^\perp, \\ T|_{N(T)^\perp}^{-1} x, & x \in R(T). \end{cases}$$

Following [8, p. 22, (c)], the generalized inverse fulfills $N(T^\dagger) = R(T)^\perp$ and moreover the decomposition $X = N(T) \oplus T^\dagger R(T)$. In case $\dim X < \infty$ one obtains the usual *Moore-Penrose inverse*.

With an operator sequence $A_k \in L(X)$, $k \in \mathbb{I}$, linear difference equations read as

$$\boxed{x_{k+1} = A_k x_k.} \tag{$L\Delta$}$$

As opposed to differential equations, where Hölder continuity or measurability assumptions were due, the *transition operator* $\Phi(k,l) \in L(X)$, $l \leq k$, $k,l \in \mathbb{I}$, for $(L\Delta)$ trivially exists in forward time and is given by the product

$$\Phi(k,l) := \begin{cases} \mathrm{id} & \text{for } k = l, \\ A_{k-1} \cdots A_l & \text{for } k > l; \end{cases}$$

if every A_k is invertible, we additionally set $\Phi(k,l) := A_k^{-1} \cdots A_{l-1}^{-1}$ for $k < l$. We say a sequence of projections $P_k \in L(X)$, $k \in \mathbb{I}$, is an *invariant projector*, provided

$$A_k P_k = P_{k+1} A_k \quad \text{for all } k \in \mathbb{I}' \tag{4.1}$$

and we speak of a *regular projector*, if the restriction $A_k : N(P_k) \to N(P_{k+1})$ is an isomorphism. This resembles the situation of Sect. 2 and hence, the restricted transition operator $\Phi(k,l)|_{N(P_l)} : N(P_l) \to N(P_k)$, $l \leq k$, is well-defined with a bounded inverse $\Phi(l,k)$; we can introduce *Green's function* as

$$\Gamma_P(k,l) := \begin{cases} \Phi(k,l)P_l & \text{for } k \geq l, \\ -\Phi(k,l)[\mathrm{id} - P_l] & \text{for } l > k. \end{cases} \tag{4.2}$$

A linear difference equation $(L\Delta)$ is said to have an *exponential dichotomy* (ED for short) on $\mathbb{I}$, if there exist reals $K \geq 1$, $\alpha \in (0,1)$ such that

$$|\Phi(k,l)P_l| \leq K\alpha^{k-l} \quad \text{for all } l \leq k, \quad |\Phi(k,l)[I - P_l]| \leq K\alpha^{l-k} \quad \text{for all } k \leq l$$

with some regular invariant projector P_k. Conditions yielding an ED on $\mathbb{Z}$ have been summarized in [36, Exs. 2.2–2.5] for various linear difference equations.

Now we turn to nonlinear difference equations. Suppose throughout that $\Omega \subseteq X$ is a nonempty open convex set. We consider functions $f_k : \Omega \times \Lambda \times V \to X$, $k \in \mathbb{Z}$, which are the right hand sides of nonautonomous difference equations

$$\boxed{x_{k+1} = f_k(x_k, \lambda, \varepsilon).} \tag{Δ}^\varepsilon_\lambda$$

For fixed parameter pairs $(\lambda, \varepsilon) \in \Lambda \times V$, an *entire* or *complete solution* of the difference equation $(\Delta)^\varepsilon_\lambda$ is a sequence $\phi = (\phi_k)_{k\in\mathbb{Z}}$ with $\phi_k \in \Omega$ satisfying the recursion $(\Delta)^\varepsilon_\lambda$ on the whole integer axis $\mathbb{Z}$. To emphasize the dependence on (λ, ε), we sometimes write $\phi(\lambda, \varepsilon)$. Provided $0 \in \Omega$, an entire solution satisfying the two-sided limit relation $\lim_{k\to\pm\infty} \phi_k = 0$ is called *homoclinic* to 0 and we speak of a *permanent solution*, if $\inf_{k\in\mathbb{Z}} \operatorname{dist}_{\mathbb{R}^d}(\phi_k, \Omega) > 0$.

The *general solution* $\varphi^\varepsilon_\lambda(\cdot; \kappa, \xi_0)$ fulfills the recursion $(\Delta)^\varepsilon_\lambda$ and the initial condition $x_\kappa = \xi_0$ for given initial pairs $(\kappa, \xi_0) \in \mathbb{Z} \times \Omega$. We do not impose invertibility of the mapping $f_k(\cdot; \lambda, \varepsilon)$ and thus backward solutions to $(\Delta)^\varepsilon_\lambda$ must not exist or be unique.

Hypothesis. *Let $m \in \mathbb{N}$ and suppose each $f_k : \Omega \times \Lambda \times V \to X$, $k \in \mathbb{Z}$, is a C^m-function such that the following holds for $0 \leq j \leq m$:*

(H_0) *For all bounded $B \subseteq \Omega$ one has*

$$\sup_{k\in\mathbb{Z}}\sup_{x\in B}\left|D^j f_k(x,\lambda,\varepsilon)\right| < \infty \quad \textit{for all } \lambda \in \Lambda,\ \varepsilon \in V$$

(well-definedness) *and for all $(\lambda^*, \varepsilon^*) \in \Lambda \times V$ and $\rho > 0$ there is a $\delta > 0$ with*

$$|x-y| < \delta \quad \Rightarrow \quad \sup_{k\in\mathbb{Z}}\left|D^j f_k(x,\lambda,\varepsilon) - D^j f_k(y,\lambda,\varepsilon)\right| < \rho \tag{4.3}$$

for all $x, y \in \Omega$ and $(\lambda, \varepsilon) \in B_\delta(\lambda^*, \varepsilon^*)$ (uniform continuity).

(H_1) *We have $0 \in \Omega$ and $\lim_{k\to\pm\infty} f_k(0,\lambda,\varepsilon) = 0$ for all $\lambda \in \Lambda$, $\varepsilon \in V$.*

As before, the subsequent step is a functional analytical formulation of difference equations $(\Delta)_\lambda^\varepsilon$ as abstract equations in sequence spaces. Thereto, the set of bounded sequences $\phi = (\phi_k)_{k\in\mathbb{Z}}$ with $\phi_k \in \Omega$ is denoted by $\ell^\infty(\Omega)$ and in case $0 \in \Omega$ we write $\ell_0(\Omega)$ for the space of sequences converging to 0 in both time directions. Convexity of Ω carries over to the spaces $\ell^\infty(\Omega), \ell_0(\Omega)$. We briefly write $\ell^\infty := \ell^\infty(X)$, $\ell_0 := \ell_0(X)$ or simply ℓ for one of these two spaces, which both are Banach spaces canonically equipped with norm

$$\|\phi\| := \sup_{k\in\mathbb{I}} |\phi_k|\,.$$

The essential operator formulation of $(\Delta)_\lambda^\varepsilon$ is simpler than for differential equations, since only the sequence spaces ℓ^∞ (resp. ℓ_0) are involved.

Theorem 4.1. *Let $(\lambda, \varepsilon) \in \Lambda \times V$ be fixed. A sequence ϕ in Ω is an entire solution of the difference equation $(\Delta)_\lambda^\varepsilon$, if and only if ϕ solves the nonlinear equation*

$$G(\phi, \lambda, \varepsilon) = 0$$

with a formally defined operator $G(\phi,\lambda,\varepsilon) = S\phi - F(\phi,\lambda,\varepsilon)$, where $(S\phi)_k := \phi_{k+1}$ and $(F(\phi,\lambda,\varepsilon))_k := f_k(\phi_k,\lambda,\varepsilon)$. Moreover, under (H_0) the mapping G fulfills:

(a) *$G : \ell^\infty(\Omega)\times\Lambda\times V \to \ell^\infty$ is well-defined and of class C^m on $\ell^\infty(\Omega)^\circ \times \Lambda \times V$,*
(b) *if (H_1) holds, then $G : \ell_0(\Omega)\times\Lambda\times V \to \ell_0$ is well-defined and of class C^m.*

Proof. See [36, Thm. 2.4 and Prop. 2.3]. □

Roughly, a bifurcation of an entire solution $\phi^* = \phi(\lambda^*, \varepsilon)$ to $(\Delta)_{\lambda^*}^\varepsilon$ is defined as above in terms of a change in the number of bounded or homoclinic solutions. As expected, bifurcation properties of ϕ^* necessarily depend on the *variational equation*

$$\boxed{x_{k+1} = D_1 f_k(\phi_k^*, \lambda^*, \varepsilon^*)x_k} \tag{$V\Delta$}_{\lambda^*}^{\varepsilon^*}$$

with transition operator Φ. We speak of a hyperbolic solution ϕ^*, if $(V\Delta)_{\lambda^*}^{\varepsilon^*}$ has an ED on $\mathbb{Z}$ and is thus robust under parameter variation (see [36, Thm. 2.11]). To observe bifurcations, we are interested in nonhyperbolic solutions ϕ^* of $(\Delta)_{\lambda^*}^{\varepsilon^*}$, i.e. in particular degenerate zeros for $G(\cdot, \lambda^*, \varepsilon^*)$ of the form:

Hypothesis. *Let $\kappa \in \mathbb{Z}$, $\lambda^* \in \Lambda$, $\varepsilon^* \in V$ be given, suppose that a difference equation $(\Delta)_{\lambda^*}^{\varepsilon^*}$ has an entire permanent solution $\phi^* \in \ell^\infty(\Omega)$ with*

(H_2) *the variational equation $(V\Delta)_{\lambda^*}^{\varepsilon^*}$ admits an ED both on $\mathbb{Z}_\kappa^+$ and $\mathbb{Z}_\kappa^-$ with respective projectors P_k^+, P_k^- and nonzero vectors $\xi, \xi' \in X$ satisfying*

$$R(P_\kappa^+) \cap N(P_\kappa^-) = \operatorname{span}\{\xi\}, \qquad (R(P_\kappa^+) + N(P_\kappa^-))^\perp = \operatorname{span}\{\xi'\}$$

and that $R(P_\kappa^+ + P_\kappa^- - \mathrm{id})$ is closed.

Remark 4.1. Referring to [8, p. 10, Thm. 1], the range of $P_\kappa^+ + P_\kappa^- - \mathrm{id}$ needs to be closed such that $P_\kappa^+ + P_\kappa^- - \mathrm{id} \in L(X)$ has a generalized inverse. This assumption holds under one of the following conditions:

- X or merely $R(P_\kappa^+ + P_\kappa^- - \mathrm{id})$ is finite-dimensional.
- Both kernels $N(P_\kappa^+)$ and $N(P_\kappa^+)$ are finite-dimensional — a situation frequently met in applications where the transition operator $\Phi(k, l)$, $l < k$, is compact (see [16, p. 226] for the continuous case, or Sect. 2). Indeed, $\dim N(P_\kappa^+) < \infty$ guarantees that P_κ^+ is Fredholm and $\dim N(P_\kappa^-) < \infty$ ensures that $\mathrm{id} - P_\kappa^-$ is finite-dimensional, hence compact. Therefore, the sum $P_\kappa^+ + P_\kappa^- - \mathrm{id}$ is Fredholm (see [43]) and has a generalized inverse (cf. [8, p. 12, Remarks (1)]).

We apply Fredholm theory of [4, 5] to the weighted difference operator

$$L : \ell \to \ell, \qquad (L\psi)_k := \psi_{k+1} - D_1 f_k(\phi_k^*, \lambda^*, \varepsilon^*)\psi_k \quad \text{for all } k \in \mathbb{Z}.$$

Proposition 4.2. *If (H_0), (H_2) hold, then $L \in L(\ell)$ has the following properties:*

(a) *It is an index 0 Fredholm operator with kernel $N(L) = \operatorname{span}\{\Phi(\cdot, \kappa)\xi\}$,*
(b) *$R(L) = N(\mu)$ with the linear bounded functional*

$$\mu : \ell \to \mathbb{R}, \qquad \mu(\psi) := \sum_{j \in \mathbb{Z}} \langle \Phi(\kappa, j+1)'\xi', \psi_j \rangle$$

satisfying $|\mu| \le K \frac{1+\alpha}{1-\alpha} |\xi'|$.

Remark 4.2. From a spectral theoretical perspective this means that 1 is a simple eigenvalue of the shift operator $T : \ell \to \ell$, $(T\psi)_k := D_1 f_{k-1}(\phi_{k-1}^*, \lambda^*, \varepsilon^*)\psi_{k-1}$. Also the dichotomy spectrum (see [2]) of $(V\Delta)_{\lambda^*}^{\varepsilon^*}$ contains 1.

Proof. This is a special case of [34, Lemma 2.9 and 2.12], which hold for difference equations in reflexive Banach spaces, hence in Hilbert spaces. $\square$

Corollary 4.3. *Let $\psi \in R(L)$. If $\mathfrak{X}_0 \subseteq \mathfrak{X}$ denotes a complement of $N(L)$, then the inverse of the restriction $L|_{\mathfrak{X}_0} : \mathfrak{X}_0 \to R(L)$ is given by $\overline{\psi} = L|_{\mathfrak{X}_0}^{-1}\psi$ with*

$$\overline{\psi_k} := \begin{cases} \Phi(k,\kappa)P_\kappa^+\xi_\kappa^* + \sum_{j=\kappa}^\infty \Gamma_{P^+}(k, j+1)\psi_j, & k \ge \kappa, \\ \Phi(k,\kappa)[\mathrm{id} - P_\kappa^-]\xi_\kappa^* + \sum_{j=-\infty}^{\kappa-1} \Gamma_{P^-}(k, j+1)\psi_j, & k \le \kappa, \end{cases} \tag{4.4}$$

$$\xi_\kappa^* := [P_\kappa^+ + P_\kappa^- - \mathrm{id}]^\dagger \left(\sum_{j=-\infty}^{\kappa-1} \Phi(\kappa, j+1)P_j^-\psi_j + \sum_{j=\kappa}^\infty \Phi(\kappa, j+1)[\mathrm{id} - P_j^+]\psi_j \right).$$

Proof. Let $\psi \in R(L) \subseteq \ell$ and $\xi_0 \in X$. Thanks to the dichotomy assumptions (H_2) on both semiaxes, we know that the bounded forward solutions $(\phi_k^+)_{k\in\mathbb{Z}_\kappa^+}$ to the linear inhomogeneous system

$$x_{k+1} = A_k x_k + \psi_k \tag{4.5}$$

are $\phi_k^+ = \Phi(k,\kappa)P_\kappa^+\xi_0 + \sum_{j=\kappa}^\infty \Gamma_{P^+}(k,j+1)\psi_j$, while the corresponding backward solutions $(\phi_k^-)_{k\in\mathbb{Z}_\kappa^-}$ are $\phi_k^- = \Phi(k,\kappa)[\mathrm{id} - P_\kappa^-]\xi_0 + \sum_{j=-\infty}^{\kappa-1}\Gamma_{P^-}(k,j+1)\psi_j$ (see [33, Lemma 2.7(ii)] or [17, p. 34, Satz 3.1.2(ii)] treating noninvertible equations in Banach spaces). Consequently, the initial values ξ_0 yielding bounded entire solutions to (4.5) can be deduced from the condition $\phi_\kappa^+ = \phi_\kappa^-$. This is equivalent to

$$\begin{aligned}(P_\kappa^+ + P_\kappa^- - \mathrm{id})\xi_0 \quad &= \quad \sum_{j=-\infty}^{\kappa-1}\Gamma_{P^-}(\kappa,j+1)\psi_j - \sum_{j=\kappa}^{\infty}\Gamma_{P^+}(\kappa,j+1)\psi_j \\ &\overset{(4.2)}{=} \quad \sum_{j=-\infty}^{\kappa-1}\Phi(\kappa,j+1)P_j^-\psi_j + \sum_{j=\kappa}^{\infty}\Phi(\kappa,j+1)(\mathrm{id} - P_j^+)\psi_j,\end{aligned}$$

which we solve for $\xi_0 \in X$. By Rem. 4.1 the generalized inverse to $P_\kappa^+ + P_\kappa^- - \mathrm{id}$ exists. Thus, the general solution $\xi_0 \in X$ to this linear equation is given by

$$\begin{aligned}\xi_0 =& [\mathrm{id} - (P_\kappa^+ + P_\kappa^- - \mathrm{id})^\dagger (P_\kappa^+ + P_\kappa^- - \mathrm{id})]\eta \\ &+ (P_\kappa^+ + P_\kappa^- - \mathrm{id})^\dagger\left(\sum_{j=-\infty}^{\kappa-1}\Phi(\kappa,j+1)P_j^-\psi_j + \sum_{j=\kappa}^{\infty}\Phi(\kappa,j+1)(\mathrm{id} - P_j^+)\psi_j\right)\end{aligned}$$

with any $\eta \in X$. Thanks to [8, p. 11] we have

$$R(\mathrm{id} - (P_\kappa^+ + P_\kappa^- - \mathrm{id})^\dagger(P_\kappa^+ + P_\kappa^- - \mathrm{id})) = N(P_\kappa^+ + P_\kappa^- - \mathrm{id})$$

and consequently $\xi_0 = \gamma\xi + \xi_\kappa^*$ with an arbitrary coefficient $\gamma \in \mathbb{R}$. This implies that the linear inhomogeneous equation (4.5) has a 1-parameter family of bounded solutions

$$\phi = \phi_\gamma + \overline{\psi} \quad \text{with } \phi_\gamma := \gamma\Phi(\cdot,\kappa)\xi \in N(L)$$

by Prop. 4.2(a). Due to the direct decomposition $\ell = N(L) \oplus \mathfrak{X}_0$ the unique solution in the complement $\mathfrak{X}_0$ is given by (4.4). □

In the following, we address a pitchfork bifurcation of entire solutions to $(\Delta)_\lambda^\varepsilon$ under perturbation. For this we again assume a given solution branch:

Hypothesis. *Suppose that for all* $\lambda \in \Lambda$ *one has*

(H_3) $f_k(\phi_k^*, \lambda, \varepsilon^*) \equiv 0$ *on* $\mathbb{Z}$.

With this, the pitchfork bifurcation pattern as described in [34, Cor. 3.15] unfolds as follows (cf. Fig. A.5):

Theorem 4.4 (imperfect pitchfork bifurcation)**.** *Suppose that* (H_0)*,* (H_2) *are satisfied with* $m \geq 2$*. In case* $\phi^* \in \ell = \ell^\infty$ *and under the assumptions*

$$\begin{aligned}
g_{110} &:= -\sum_{j\in\mathbb{Z}} \langle \Phi(\kappa, j+1)'\xi', D_1 D_2 f_j(\phi_j^*, \lambda^*, \varepsilon^*)\Phi(j,\kappa)\xi\rangle \neq 0,\\
g_{010} &:= -\sum_{j\in\mathbb{Z}} \langle \Phi(\kappa, j+1)'\xi', D_2 f_j(\phi_j^*, \lambda^*, \varepsilon^*)\rangle = 0,\\
g_{200} &:= -\sum_{j\in\mathbb{Z}} \langle \Phi(\kappa, j+1)'\xi', D_1^2 f_j(\phi_j^*, \lambda^*, \varepsilon^*)[\Phi(j,\kappa)\xi]^2\rangle = 0,\\
g_{001} &:= -\sum_{j\in\mathbb{Z}} \langle \Phi(\kappa, j+1)'\xi', D_3 f_j(\phi_j^*, \lambda^*, \varepsilon^*)\rangle \neq 0
\end{aligned}$$

and

$$g := g_{110} - \sum_{j\in\mathbb{Z}} \langle \Phi(\kappa, j+1)'\xi', D_1^2 f_j(\phi_j^*, \lambda^*, \varepsilon^*)\overline{D_2 f_j(\phi_j^*, \lambda^*, \varepsilon^*)}\Phi(j,\kappa)\xi\rangle \neq 0$$

with the notation $\overline{\psi}$ *from Cor. 4.3, the following holds true:*

(a) *There exist open convex neighborhoods* $S \subseteq \mathbb{R}$ *of* 0*,* $\Omega_0 \subseteq \ell(\Omega)$ *of* ϕ^**,* $\Lambda_0 \subseteq \Lambda$ *of* λ^**,* $V_0 \subseteq V$ *of* ε^* *and* C^m*-functions* $\phi^\star : S \to \Omega_0$*,* $\lambda^\star : S \to \Lambda_0$*,* $\varepsilon^\star : S \to V_0$ *with*

$$\begin{aligned}
&\phi^\star(0) = \phi^*, && \dot\phi^\star(0) = \Phi(\cdot,\kappa)\xi, &&\\
&\lambda^\star(0) = \lambda^*, && \dot\lambda^\star(0) = 0, &&\\
&\varepsilon^\star(0) = \varepsilon^*, && \dot\varepsilon^\star(0) = 0, && \ddot\varepsilon^\star(0) = 0
\end{aligned}$$

and every $\phi^\star(s)$*,* $s \in S$*, is a nonhyperbolic entire solution to* $(\Delta)_{\lambda^\star(s)}^{\varepsilon^\star(s)}$ *in* $\ell(\Omega)$*.*

(b) *Under the additional assumption* $m \geq 3$ *it is*

$$\dddot\varepsilon^\star(0) = 2\frac{h}{g_{001}}, \qquad \ddot\lambda^\star(0) = -\frac{h}{g}$$

with

$$\begin{aligned}
h :=& -\sum_{j\in\mathbb{Z}} \langle \Phi(\kappa, j+1)'\xi', D_1^3 f_j(\phi_j^*, \lambda^*, \varepsilon^*)[\Phi(j,\kappa)\xi]^3\rangle\\
&- 3\sum_{j\in\mathbb{Z}} \langle \Phi(\kappa, j+1)'\xi', D_1^2 f_j(\phi_j^*, \lambda^*, \varepsilon^*)\overline{D_1^2 f_j(\phi_j^*, \lambda^*, \varepsilon^*)[\Phi(j,\kappa)\xi]^2}\Phi(j,\kappa)\xi\rangle
\end{aligned}$$

and if additionally beyond (H_3) *also*

$$g_{110} > 0, \qquad g_{001} < 0, \qquad h < 0$$

hold, then $\dddot\varepsilon^\star(0) > 0$*,* $\ddot\lambda^\star(0) > 0$ *and locally in* $\Omega_0 \times \Lambda_0$ *for every* $\varepsilon \in V_0$ *there exist compact intervals* $S_\varepsilon = [-\delta_\varepsilon, \delta_\varepsilon]$*,* $[\lambda^* - \rho_\varepsilon, \lambda^* + \rho_\varepsilon] \subset \Lambda_0$ *such that:*

(b_1) *Besides the constant branch (cf. (H_3)), the bounded entire solutions to $(\Delta)_\lambda^{\varepsilon^*}$ in $\Omega_0 \times \Lambda_0$ consist of a branch*

$$\Gamma := \{(\phi_{\varepsilon^*}(s), \lambda_{\varepsilon^*}(s)) \in \ell(\Omega) \times \Lambda : s \in S_{\varepsilon^*}\};$$

here, $\phi_{\varepsilon^*} : S_{\varepsilon^*} \to \Omega_0$, $\lambda_{\varepsilon^*} : S_{\varepsilon^*} \to \Lambda_0$ are C^{m-1}-functions with

$$\lambda_{\varepsilon^*}(0) = \lambda^*, \quad \dot{\lambda}_{\varepsilon^*}(0) = 0, \quad \ddot{\lambda}_{\varepsilon^*}(0) > 0, \quad \lambda_{\varepsilon^*}(\pm\delta_{\varepsilon^*}) = \lambda^* + \rho_{\varepsilon^*}.$$

Every $\phi_{\varepsilon^*}(s) : \mathbb{Z} \to \Omega$, $s \neq 0$, is a hyperbolic solution of $(\Delta)_{\lambda_{\varepsilon^*}(s)}^{\varepsilon^*}$ in $\ell(\Omega)$ and ϕ^* is a supercritical pitchfork bifurcating solution of $(\Delta)_{\lambda^*}^{\varepsilon^*}$. There exists a unique entire bounded solution to $(\Delta)_\lambda^{\varepsilon^*}$ for $\lambda \leq \lambda^*$ and exactly three entire bounded solutions for $\lambda > \lambda^*$,

(b_2) *if $\varepsilon \neq \varepsilon^*$, then the bounded entire solutions to $(\Delta)_\lambda^\varepsilon$ in $\Omega_0 \times \Lambda_0$ consist of two disjoint branches $\Gamma_\varepsilon^+ \dot{\cup} \Gamma_\varepsilon^-$ with*

- *a first branch of the form*

$$\Gamma_\varepsilon^+ = \left\{(\phi_\varepsilon^+(s), \lambda_\varepsilon^+(s)) \in \ell(\Omega) \times \Lambda : s \in S_\varepsilon\right\};$$

here, $\phi_\varepsilon^+ : S_\varepsilon \to \Omega_0$, $\lambda_\varepsilon^+ : \mathbb{R} \to \Lambda_0$ are C^{m-1}-functions with

$$\lambda_\varepsilon^+(\pm\delta_\varepsilon) = \lambda_0 + \rho_\varepsilon, \qquad \dot{\lambda}_\varepsilon^+(0) = 0, \qquad \ddot{\lambda}_\varepsilon^+(0) > 0.$$

Every $\phi_\varepsilon^+(s)$, $s \neq 0$, is a hyperbolic solution of $(\Delta)_{\lambda_\varepsilon^+(s)}^\varepsilon$ and $\phi_\varepsilon^+(0)$ is a supercritical fold bifurcating solution to $(\Delta)_{\lambda_\varepsilon^+(0)}^\varepsilon$,

- *a second branch of the form*

$$\Gamma_\varepsilon^- = \left\{(\phi_\varepsilon^-(\lambda), \lambda) \in \ell(\Omega) \times \Lambda : \lambda \in \Lambda_\varepsilon\right\};$$

here, $\phi_\varepsilon^- : \Lambda_\varepsilon \to \Omega_0$ is a one-to-one C^{m-1}-functions. Every $\phi_\varepsilon^+(\lambda)$ is a hyperbolic solution of $(\Delta)_\lambda^\varepsilon$,

- *there is a unique entire bounded solution to $(\Delta)_\lambda^\varepsilon$ for $\lambda < \lambda_\varepsilon^+(0)$, exactly two bounded entire solution for $\lambda = \lambda_\varepsilon^+(0)$ and exactly three bounded entire solution for $\lambda > \lambda_\varepsilon^+(0)$.*

If (H_0)–(H_3) are satisfied, then the same holds with $\ell = \ell_0$.

Proof. We are going to apply Thm. A.3 with $\mathcal{X} = \mathcal{Z} = \ell^\infty$, $\Omega = \ell^\infty(\Omega)^\circ$ and the corresponding mapping G defined in Thm. 4.1. Since the entire solution ϕ^* is permanent, one has $\phi^* \in \Omega$ and Thm. 4.1 ensures (A.2). From Prop. 4.2(a) we deduce (A.3) with $x_1 = \Phi(\cdot, \kappa)\xi$ — the required partial derivatives are given in [36, Prop. 2.3]. The bounded functional μ from Prop. 4.2(b) guarantees the relations (A.10) and (A.11). Finally, the assumption (H_3) directly implies (A.6). We obtain the claim for $\ell = \ell^\infty$ from Thm. A.3 and the dynamical interpretation is given in Thm. 4.1.

Under also (H_1) the above arguments apply with $\ell = \ell_0$, too. Hence, uniqueness assertions hold in ℓ^∞, while existence of bifurcating solutions is given in ℓ_0. $\square$

Example 4.1 (planar cubic systems). Let $X = \mathbb{R}^2$ and suppose $\alpha \in (-1, 1) \setminus \{0\}$. We consider a nonautonomous difference equation $(\Delta)^{\varepsilon}_{\lambda}$ in $\Omega = \mathbb{R}^2$ with a smooth right hand side $f_k : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ of the form

$$x_{k+1} = f_k(x_k, \lambda, \varepsilon) := \begin{pmatrix} b_k & 0 \\ 0 & c_k \end{pmatrix} x_k + H_k(x_k, \lambda, \varepsilon). \tag{4.6}$$

The real sequences b_k, c_k, $k \in \mathbb{Z}$, are assumed to be piecewise constant

$$b_k := \begin{cases} \alpha^{-1}, & k < 0, \\ \alpha, & k \geq 0, \end{cases} \qquad c_k := \begin{cases} \alpha, & k < 0, \\ \alpha^{-1}, & k \geq 0 \end{cases} \tag{4.7}$$

and the nonlinearity $H_k : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ is of the form

$$\begin{aligned} H_k(x, \lambda, \varepsilon) :=& \lambda \begin{pmatrix} b_{10}(k)x_1 + b_{01}(k)x_2 \\ c_{10}(k)x_1 + c_{01}(k)x_2 \end{pmatrix} + \begin{pmatrix} b_{11}(k)x_1x_2 + b_{02}(k)x_2^2 \\ c_{11}(k)x_1x_2 + c_{02}(k)x_2^2 \end{pmatrix} \\ &+ \begin{pmatrix} b_{30}(k)x_1^3 + b_{21}x_1^2x_2 + b_{12}(k)x_1x_2^2 + b_{03}(k)x_2^3 \\ c_{30}(k)x_1^3 + c_{21}x_1^2x_2 + c_{12}(k)x_1x_2^2 + c_{03}(k)x_2^3 \end{pmatrix} + \varepsilon \begin{pmatrix} b_{00}(k) \\ c_{00}(k) \end{pmatrix} \end{aligned}$$

with bounded coefficient functions $b_{ij} : \mathbb{Z} \to \mathbb{R}$, $0 \leq i + j \leq 3$ satisfying

$$\sum_{j \in \mathbb{Z}} \alpha^{|j+1|} \alpha^{|j|} c_{10}(j) \neq 0, \qquad \sum_{j \in \mathbb{Z}} \alpha^{|j+1|} c_{00}(j) \neq 0. \tag{4.8}$$

Therefore, our hypotheses (H_0) and (H_3) are fulfilled for the trivial solution $\phi^* = 0$ and an imperfection parameter $\varepsilon^* = 0$. We are interested in the behavior of $(\Delta)^{\varepsilon}_{\lambda}$ resp. (4.6) close to the reference solution $\phi^* = 0$ near the bifurcation value $\lambda^* = 0$. The transition matrix of the associate variational equation $(V\Delta)^{\varepsilon^*}_{\lambda^*}$ reads as

$$\Phi(k, l) := \begin{cases} \operatorname{diag}(\alpha^{k-l}, \alpha^{l-k}), & k \geq l \geq 0, \\ \operatorname{diag}(\alpha^{k+l}, \alpha^{-k}\alpha^{-l}), & k \geq 0 > l, \\ \operatorname{diag}(\alpha^{l-k}, \alpha^{k-l}), & 0 > k \geq l; \end{cases}$$

we can extend $\Phi(k, l) := \Phi(l, k)^{-1}$ for $k < l$. From this, $(V\Delta)^{\varepsilon^*}_{\lambda^*}$ admits an ED on $\mathbb{Z}_0^+$ with projector $P_k^+ \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and an ED on $\mathbb{Z}_0^-$ with $P_k^- \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, the weighted shift operator L has 1-dimensional kernel, index 0. Thus, assumption (H_2) holds and we can choose $\xi = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $\xi' = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. After these observations the bounded linear functional $\mu : \ell^\infty \to \mathbb{R}$ from Prop. 4.2(b) is given by

$$\begin{aligned} \mu(\psi) &= \sum_{j \in \mathbb{Z}} \langle \xi' \Phi(0, j+1)', \psi_j \rangle \\ &= \sum_{j=-\infty}^{-2} \langle \xi' \Phi(0, j+1)', \psi_j \rangle + \langle \xi', \psi_{-1} \rangle + \sum_{j=0}^{\infty} \langle \xi' \Phi(0, j+1)', \psi_j \rangle \\ &= \sum_{j \in \mathbb{Z}} \alpha^{|j+1|} \psi_j^2 \end{aligned} \tag{4.9}$$

and we compute the necessary partial derivatives

$$
\begin{aligned}
&D_1D_2f_j(0,0,0)\Phi(j,0)\xi = \alpha^{|j|}\begin{pmatrix} b_{10}(j) \\ c_{10}(j) \end{pmatrix}, && D_2f_j(0,0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
&D_1^2f_j(0,0,0)[\Phi(j,0)\xi]^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, && D_3f_j(0,0,0) = \begin{pmatrix} b_{00}(j) \\ c_{00}(j) \end{pmatrix}, \\
&D_1^3f_j(0,0,0)[\Phi(j,0)\xi]^3 = 6\alpha^{3|j|}\begin{pmatrix} b_{30}(j) \\ c_{30}(j) \end{pmatrix}
\end{aligned}
$$

for all $j \in \mathbb{Z}$. Consequently, $\overline{D_2f_j(\phi_j^*, \lambda^*, \varepsilon^*)} \equiv 0$ on $\mathbb{Z}$, referring to (4.9) this yields

$$
\begin{aligned}
g_{110} &= -\sum_{j\in\mathbb{Z}} \alpha^{|j+1|}\alpha^{|j|}c_{10}(j), && g_{010} = 0, \\
g_{200} &= 0, && g_{001} = -\sum_{j\in\mathbb{Z}} \alpha^{|j+1|}c_{00}(j), \\
g = g_{110} &= -\sum_{j\in\mathbb{Z}} \alpha^{|j+1|}\alpha^{|j|}c_{10}(j), && h = -6\sum_{j\in\mathbb{Z}} \alpha^{|j+1|}\alpha^{3\alpha|j|}c_{30}(j)
\end{aligned}
$$

and due to our assumption (4.8) we know that Thm. 4.4 applies. In particular, for sequences $c_{00}, c_{10}, c_{30} : \mathbb{Z} \to \mathbb{R}$ fulfilling $g_{110} > 0$, $g_{001} < 0$ and $h < 0$ the trivial solution of $(\Delta)_\lambda^0$ exhibits a supercritical pitchfork bifurcation at $\lambda^* = 0$. This bifurcation perturbs according to assertion (b_2) from Thm. 4.4.

A. Perturbed analytical bifurcations

Our main assertions on the bifurcation of bounded entire solutions in Sects. 2–4 are deduced using abstract analytical bifurcation results from [41, 28]. In the following appendix, we formulate them using our previous notation of [34, 35].

Thereto, we assume that $\mathcal{X}, \mathcal{Z}$ are real Banach spaces and $\Omega \subseteq \mathcal{X}$, $\Lambda \subseteq \mathbb{R}$, $V \subseteq \mathbb{R}$ denote nonempty open neighborhoods of points $x_0 \in \mathcal{X}$, $\lambda_0 \in \mathbb{R}$, $\varepsilon_0 \in \mathbb{R}$ in the respective spaces. Given a C^m-mapping $G : \Omega \times \Lambda \times V \to \mathcal{Z}$, $m \geq 2$, we are interested in the set of solutions $x \in \Omega$ to an abstract 2-parameter problem

$$
G(x, \lambda, \varepsilon) = 0 \tag{A.1}
$$

near a given reference solution $(x_0, \lambda_0, \varepsilon_0)$, i.e.

$$
G(x_0, \lambda_0, \varepsilon_0) = 0. \tag{A.2}
$$

For the partial derivative $D_1G(x_0, \lambda_0, \varepsilon_0) \in L(\mathcal{X}, \mathcal{Z})$ we suppose

$$
\begin{aligned}
\dim N(D_1G(x_0, \lambda_0, \varepsilon_0)) &= \operatorname{codim} R(D_1G(x_0, \lambda_0, \varepsilon_0)) = 1, \\
N(D_1G(x_0, \lambda_0, \varepsilon_0)) &= \operatorname{span}\{x_1\}
\end{aligned} \tag{A.3}
$$

for some nonzero vector $x_1 \in \mathcal{X}$. A triple $(x_0, \lambda_0, \varepsilon_0) \in \Omega \times \Lambda \in V$ satisfying (A.2) and $N(D_1 G(x_0, \lambda_0, \varepsilon_0)) \neq \{0\}$ is called *degenerate solution* to (A.1). Finally, a *fold* or *turning point* (w.r.t. λ) of (A.1) is a solution satisfying (A.3) and

$$D_2 G(x_0, \lambda_0, \varepsilon_0) \notin R(D_1 G(x_0, \lambda_0, \varepsilon_0)).$$

Thanks to (A.3), the Fréchet derivative $D_1 G(x_0, \lambda_0, \varepsilon_0) \in L(\mathcal{X}, \mathcal{Z})$ is a Fredholm operator of index 0. Thus, the Hahn-Banach theorem yields the existence of a functional $\mu \in \mathcal{Z}'$ such that $N(\mu) = R(D_1 G(x_0, \lambda_0, \varepsilon_0))$.

For the sake of a brief notation we introduce the convenient abbreviations

$$G_{ijk} := D_1^i D_2^j D_3^k G(x_0, \lambda_0, \varepsilon_0), \qquad g_{ijk} := \mu(D_1^i D_2^j D_3^k G(x_0, \lambda_0, \varepsilon_0) x_1^i)$$

for all triples $(i, j, k) \in \mathbb{N}_0^3$ with $i + j + k \leq m$. Having this at hand, we can formulate the following persistence result for fold points:

Theorem A.1 (abstract fold bifurcation)**.** *Let $m \geq 2$. If* (A.2), (A.3) *and*

$$g_{010} \neq 0, \qquad g_{200} \neq 0 \tag{A.4}$$

are satisfied, then the following holds true (see Fig. A.3):

(a) *There exist open convex neighborhoods $\Omega_0 \subseteq \Omega$ of x_0, $\Lambda_0 \subseteq \Lambda$ of λ_0, $V_0 \subseteq V$ of ε_0 and C^m-functions $x^\star : V_0 \to \Omega_0$, $\lambda^\star : V_0 \to \Lambda_0$ with*

$$x^\star(\varepsilon_0) = x_0, \qquad \lambda^\star(\varepsilon_0) = \lambda_0, \qquad \dot{\lambda}^\star(\varepsilon_0) = -\lambda_0 \frac{g_{001}}{g_{010}}$$

and every triple $(x^\star(\varepsilon), \lambda^\star(\varepsilon), \varepsilon)$, $\varepsilon \in V_0$, is a fold point of (A.1).

(b) *For every $\varepsilon \in V_0$ there exists an open neighborhood $S_\varepsilon \subseteq \mathbb{R}$ of 0 such that $\{(x, \lambda) \in \Omega_0 \times \Lambda_0 : G(x, \lambda, \varepsilon) = 0\} = \Gamma_\varepsilon$ with the branch*

$$\Gamma_\varepsilon = \{(x_\varepsilon(s), \lambda_\varepsilon(s)) \in \Omega \times \Lambda : s \in S_\varepsilon\}$$

and C^m-functions $x_\varepsilon : S_\varepsilon \to \Omega_0$, $\lambda_\varepsilon : S_\varepsilon \to \Lambda_0$ satisfying $x_\varepsilon(0) = x^\star(\varepsilon)$, $\lambda_\varepsilon(0) = \lambda^\star(\varepsilon)$,

$$\dot{\lambda}_\varepsilon(0) = 0, \qquad \ddot{\lambda}_\varepsilon(0) \neq 0, \qquad \ddot{\lambda}_{\varepsilon_0}(0) = -\frac{g_{200}}{g_{010}},$$

where the triple $(x_\varepsilon(0), \lambda_\varepsilon(0), \varepsilon)$ is the unique degenerate solution to (A.1) *on Γ_ε and a fold point. For $\lambda \in \Lambda_0$ it holds*

(b_1) *if $g_{200}/g_{010} < 0$, then*

$$\#\{x \in \Omega_0 : G(x, \lambda, \varepsilon) = 0\} = \begin{cases} 0, & \lambda < \lambda^\star(\varepsilon), \\ 1, & \lambda = \lambda^\star(\varepsilon), \\ 2, & \lambda > \lambda^\star(\varepsilon), \end{cases}$$

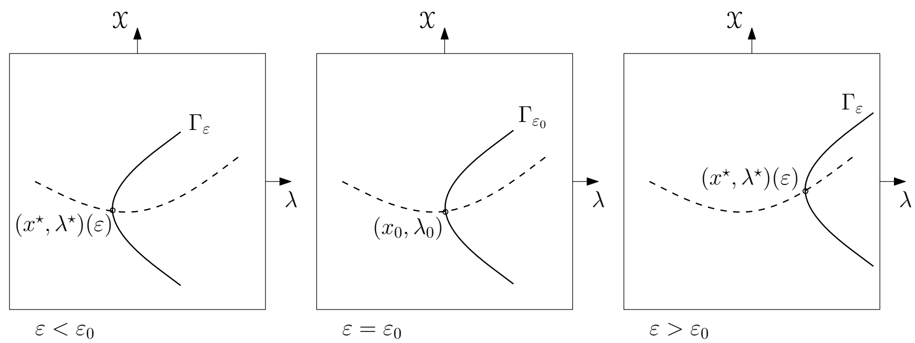

Figure A.3: Supercritical fold bifurcation from Thm. A.1, where the curve $(x^\star, \lambda^\star)(\varepsilon)$ of degenerate solutions is dashed.

(b_2) *if* $g_{200}/g_{010} > 0$*, then*

$$\#\{x \in \Omega_0 : G(x, \lambda, \varepsilon) = 0\} = \begin{cases} 0, & \lambda > \lambda^\star(\varepsilon), \\ 1, & \lambda = \lambda^\star(\varepsilon), \\ 2, & \lambda < \lambda^\star(\varepsilon). \end{cases}$$

Remark A.1. In the degenerate situation $g_{200} = 0$ a bifurcation of fold points occurs. For the behavior of the corresponding cusp point see [41, Thms. 2.2 and 2.3(2)].

Proof. See [41, Thms. 2.1 and 2.3(1)]. □

The Fredholm property (A.3) yields that the kernel $N(D_1G(x_0, \lambda_0, \varepsilon_0)) \subseteq \mathcal{X}$, as well as the range $R(D_1G(x_0, \lambda_0, \varepsilon_0)) \subseteq \mathcal{Z}$ split the respective spaces $\mathcal{X}$ and $\mathcal{Z}$, i.e. there exist two closed subspaces $\mathcal{X}_0 \subseteq \mathcal{X}$, $\mathcal{Z}_0 \subseteq \mathcal{Z}$ with

$$\mathcal{X} = \operatorname{span}\{x_1\} \oplus \mathcal{X}_0, \qquad \mathcal{Z} = \mathcal{Z}_0 \oplus R(D_1G(x_0, \lambda_0, \varepsilon_0)). \tag{A.5}$$

For our following result, we assume a constant solution branch, i.e.

$$G(x_0, \lambda, \varepsilon_0) \equiv 0 \quad \text{on } \Lambda. \tag{A.6}$$

Theorem A.2 (abstract transcritical bifurcation)**.** *Let* $m \geq 2$*. If* (A.2)*,* (A.3)*,* (A.6) *and the generic conditions*

$$g_{110} \neq 0, \qquad g_{001} \neq 0, \qquad g_{200} \neq 0 \tag{A.7}$$

are satisfied, then the following holds true (see Fig. A.4):

(a) *There exist open convex neighborhoods* $S \subseteq \mathbb{R}$ *of* 0*,* $\Omega_0 \subseteq \Omega$ *of* x_0*,* $\Lambda_0 \subseteq \Lambda$ *of* λ_0*,* $V_0 \subseteq V$ *of* ε_0 *and* C^m*-functions* $x^\star : S \to \Omega_0$*,* $\lambda^\star : S \to \Lambda_0$*,* $\varepsilon^\star : S \to V_0$ *with*

$$x^\star(0) = x_0, \qquad \dot{x}^\star(0) = -\frac{g_{110}}{g_{200}} x_1,$$

$$\lambda^\star(0) = \lambda_0, \qquad \dot\lambda^\star(0) = 1,$$

$$\varepsilon^\star(0) = \varepsilon_0, \qquad \dot\varepsilon^\star(0) = 0, \qquad \ddot\varepsilon^\star(0) = \frac{g_{110}^2}{g_{200}g_{001}} \neq 0$$

and every triple $(x^\star(s), \lambda^\star(s), \varepsilon^\star(s))$, $s \in S$, is a degenerate solution to (A.1).

(b) *Under the additional assumptions*

$$g_{001} > 0, \qquad g_{110} > 0, \qquad g_{200} > 0 \tag{A.8}$$

one has locally in $\Omega_0 \times \Lambda_0$ for every $\varepsilon \in V_0$ that there exist compact intervals $S_\varepsilon = [-\delta_\varepsilon, \delta_\varepsilon]$, $\Lambda_\varepsilon = [\lambda_0 - \rho_\varepsilon, \lambda_0 + \rho_\varepsilon] \subset \Lambda_0$ such that:

(b_1) *If $\varepsilon < \varepsilon_0$, then $\{(x, \lambda) \in \Omega_0 \times \Lambda_0 : G(x, \lambda, \varepsilon) = 0\} = \Gamma_\varepsilon^1 \dot\cup \Gamma_\varepsilon^2$ with the branches*

$$\Gamma_\varepsilon^i = \left\{(x_\varepsilon^i(\lambda), \lambda) \in \Omega \times \Lambda : \lambda \in \Lambda_\varepsilon\right\} \quad \textit{for } i = 1, 2$$

and C^{m-1}-functions $x_\varepsilon^i : \Lambda_0 \to \Omega_0$, where each $(x_\varepsilon^i(\lambda), \lambda, \varepsilon)$, $i = 1, 2$, is a nondegenerate solution to (A.1) *and*

$$\#\{x \in \Omega_0 : G(x, \lambda, \varepsilon) = 0\} = 2 \quad \textit{for all } \lambda \in \Lambda_\varepsilon,$$

(b_2) $\{(x, \lambda) \in \Omega_0 \times \Lambda_0 : G(x, \lambda, \varepsilon_0) = 0\} = \Gamma \cup \{(x_0, \lambda) : \lambda \in \Lambda_0\}$ *with*

$$\Gamma = \{(x_{\varepsilon_0}(s), \lambda_{\varepsilon_0}(s)) \in \Omega \times \Lambda : s \in S_{\varepsilon_0}\}$$

and C^{m-1}-functions $x_{\varepsilon_0} : S_{\varepsilon_0} \to \Omega_0$, $\lambda_{\varepsilon_0} : S_{\varepsilon_0} \to \Lambda_0$ satisfying $x_{\varepsilon_0}(0) = x_0$, $\lambda_{\varepsilon_0}(0) = \lambda_0$, $\dot\lambda_{\varepsilon_0}(s) > 0$ for all $s \in S_{\varepsilon_0}$; in particular, the triple $(x_0, \lambda_0, \varepsilon_0)$ is a transcritical bifurcation point of (A.1) *and*

$$\#\{x \in \Omega_0 : G(x, \lambda, \varepsilon) = 0\} = \begin{cases} 1, & \lambda = \lambda_0, \\ 2, & \lambda \in \Lambda_{\varepsilon_0} \setminus \{\lambda_0\}, \end{cases}$$

(b_3) *if $\varepsilon > \varepsilon_0$, then $\{(x, \lambda) \in \Omega_0 \times \Lambda_0 : G(x, \lambda, \varepsilon) = 0\} = \Gamma_\varepsilon^+ \dot\cup \Gamma_\varepsilon^-$ with the branches*

$$\Gamma_\varepsilon^\pm = \left\{(x_\varepsilon^\pm(s), \lambda_\varepsilon^\pm(s)) \in \Omega \times \Lambda : s \in S_\varepsilon\right\}$$

and C^{m-1}-functions $x_\varepsilon^\pm : S_\varepsilon \to \Omega_0$, $\lambda_\varepsilon^\pm : S_\varepsilon \to \Lambda_\varepsilon$ satisfying

$$\begin{aligned} \lambda_\varepsilon^-(\pm\delta_\varepsilon) &= \lambda_0 - \rho_\varepsilon, & \lambda_\varepsilon^+(\pm\delta_\varepsilon) &= \lambda_0 + \rho_\varepsilon, \\ \dot\lambda_\varepsilon^-(0) &= 0, & \dot\lambda_\varepsilon^+(0) &= 0, \\ \ddot\lambda_\varepsilon^-(0) &< 0, & \ddot\lambda_\varepsilon^+(0) &> 0, \end{aligned}$$

where the triple $(x_\varepsilon^\pm(0), \lambda_\varepsilon^\pm(0), \varepsilon)$ is the unique degenerate point of (A.1) *in $\Gamma_\varepsilon^\pm$, $(x_\varepsilon^-(0), \lambda_\varepsilon^-(0), \varepsilon)$ is a subcritical fold point, $(x_\varepsilon^+(0), \lambda_\varepsilon^+(0), \varepsilon)$ is a supercritical fold point and*

$$\#\{x \in \Omega_0 : G(x, \lambda, \varepsilon) = 0\} = \begin{cases} 2, & \lambda < \lambda_\varepsilon^-(0) \textit{ or } \lambda > \lambda_\varepsilon^+(0), \\ 0, & \lambda \in (\lambda_\varepsilon^-(0), \lambda_\varepsilon^+(0)), \\ 1, & \lambda \in \{\lambda_\varepsilon^-(0), \lambda_\varepsilon^+(0)\}. \end{cases}$$

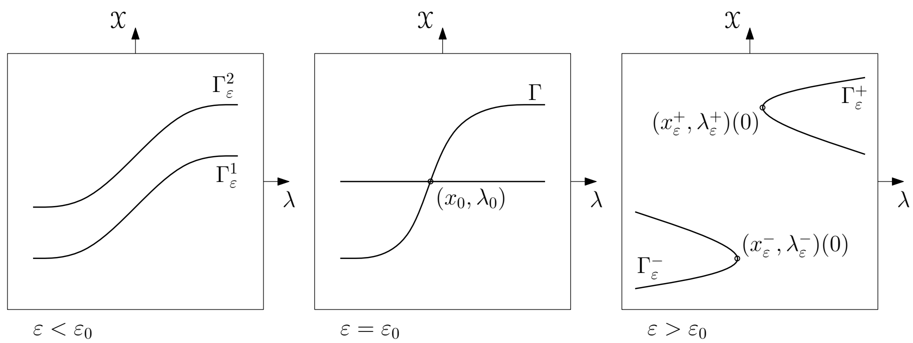

Figure A.4: Imperfect transcritical bifurcation from Thm. A.2

Remark A.2. (1) In the degenerate case $g_{200} = 0$ a pitchfork scenario formulated in the second part of [41, Thm. 2.4] occurs.

(2) One can also obtain the qualitative statement of Thm. A.2 without the global assumption (A.6) of a constant solution branch. For this, we refer to [41, Thm. 2.6] and also [28, Thm. 3.1].

Proof. See [41, Thms. 2.4 and 2.5] for part (a) and in particular [41, (4.28)] yields the formula for the derivative $\ddot{\varepsilon}^\star$. Moreover, the assertion (b) follows from [41, Thm. 2.5], since (A.8) implies the estimate $\ddot{\varepsilon}^\star(0) = \frac{g_{110}^2}{g_{200}g_{001}} > 0$. $\square$

We furthermore conclude that $D_1G(x_0, \lambda_0, \varepsilon_0)|_{\mathcal{X}_0} : \mathcal{X}_0 \to R(D_1G(x_0, \lambda_0, \varepsilon_0))$ is a toplinear isomorphism and under the assumption

$$y \in R(D_1G(x_0, \lambda_0, \varepsilon_0)) \tag{A.9}$$

there exists a unique solution $\overline{x} = -G_{100}^{-1}y \in X_0$ to the linear equation $y + G_{100}x = 0$.

Theorem A.3 (abstract pitchfork bifurcation)**.** *Let* $m \geq 2$*. If* (A.2), (A.3),

$$g_{110} \neq 0, \qquad g_{010} = 0, \qquad g_{200} = 0, \qquad g_{001} \neq 0 \tag{A.10}$$

and

$$g_{110} + \mu(D_1^2G(x_0, \lambda_0, \varepsilon_0)\overline{D_2G(x_0, \lambda_0, \varepsilon_0)}x_1) \neq 0 \tag{A.11}$$

are satisfied, then the following holds true (see Fig. A.5):

(a) *There exist open convex neighborhoods* $S \subseteq \mathbb{R}$ *of* 0, $\Omega_0 \subseteq \Omega$ *of* x_0, $\Lambda_0 \subseteq \Lambda$ *of* λ_0, $V_0 \subseteq V$ *of* ε_0 *and* C^m*-functions* $x^\star : S \to \Omega_0$, $\lambda^\star : S \to \Lambda_0$, $\varepsilon^\star : S \to V_0$ *with*

$$\begin{aligned} &x^\star(0) = x_0, && \dot{x}^\star(0) = x_1, && \\ &\lambda^\star(0) = \lambda_0, && \dot{\lambda}^\star(0) = 0, && \\ &\varepsilon^\star(0) = \varepsilon_0, && \dot{\varepsilon}^\star(0) = 0, && \ddot{\varepsilon}^\star(0) = 0 \end{aligned}$$

and every $(x^\star(s), \lambda^\star(s), \varepsilon^\star(s))$, $s \in S$, *is a degenerate solution to* (A.1).

(b) *Under the additional assumption* $m \geq 3$ *it is*

$$\dddot{\varepsilon}^\star(0) = 2\frac{g_{300} + 3\mu(G_{200}\overline{G_{200}}x_1)}{g_{001}},$$
$$\ddot{\lambda}^\star(0) = -\frac{g_{300} + 3\mu(G_{200}\overline{G_{200}}x_1)}{g_{110} + \mu(D_1^2 G(x_0, \lambda_0, \varepsilon_0)\overline{D_2 G(x_0, \lambda_0, \varepsilon_0)}x_1)}$$

and provided beyond (A.6) *also*

$$g_{110} > 0, \qquad g_{001} < 0, \qquad g_{300} + 3\mu(G_{200}\overline{G_{200}}x_1) < 0$$

hold, then $\dddot{\varepsilon}^\star(0) > 0$, $\ddot{\lambda}^\star(0) > 0$ *and locally in* $\Omega_0 \times \Lambda_0$ *for every* $\varepsilon \in V_0$ *there exist compact intervals* $S_\varepsilon = [-\delta_\varepsilon, \delta_\varepsilon]$, $[\lambda_0 - \rho_\varepsilon, \lambda_0 + \rho_\varepsilon] \subset \Lambda_0$ *such that:*

(b_1) $\{(x, \lambda) \in \Omega_0 \times \Lambda_0 : \, G(x, \lambda, \varepsilon_0) = 0\} = \Gamma \cup \{(x_0, \lambda) : \, \lambda \in \Lambda_0\}$ *with*

$$\Gamma = \{(x_{\varepsilon_0}(s), \lambda_{\varepsilon_0}(s)) \in \Omega \times \Lambda : \, s \in S_{\varepsilon_0}\}$$

and C^{m-1}*-functions* $x_{\varepsilon_0} : S_{\varepsilon_0} \to \Omega_0$, $\lambda_{\varepsilon_0} : S_{\varepsilon_0} \to \Lambda_0$ *with* $\lambda_{\varepsilon_0}(0) = \lambda_0$, $\dot{\lambda}_{\varepsilon_0}(0) = 0$, $\ddot{\lambda}_{\varepsilon_0}(0) > 0$ *and* $\lambda_{\varepsilon_0}(\pm\delta_{\varepsilon_0}) = \lambda_0 + \rho_{\varepsilon_0}$; *in particular, the triple* $(x_0, \lambda_0, \varepsilon_0)$ *is a supercritical pitchfork bifurcation point of* (A.1) *and*

$$\#\{x \in \Omega_0 : \, G(x, \lambda, \varepsilon) = 0\} = \begin{cases} 1, & \lambda \leq \lambda_0, \\ 3, & \lambda > \lambda_0, \end{cases}$$

(b_2) *for* $\varepsilon \neq \varepsilon_0$ *it is* $\{(x, \lambda) \in \Omega_0 \times \Lambda_0 : \, G(x, \lambda, \varepsilon) = 0\} = \Gamma_\varepsilon^+ \dot{\cup} \Gamma_\varepsilon^-$ *so that*

- *the first branch is of the form*

$$\Gamma_\varepsilon^+ = \left\{(x_\varepsilon^+(s), \lambda_\varepsilon^+(s)) \in \Omega \times \Lambda_\varepsilon : \, s \in S_\varepsilon\right\}$$

  with C^{m-1}*-functions* $x_\varepsilon^+ : S_\varepsilon \to \Omega_0$, $\lambda_\varepsilon^+ : S_\varepsilon \to \Lambda_0$ *satisfying* $\lambda_\varepsilon^+(\pm\delta_\varepsilon) = \lambda_0 + \rho_\varepsilon$, $\dot{\lambda}_\varepsilon^+(0) = 0$, $\ddot{\lambda}_\varepsilon^+(0) > 0$, *where* $(x_\varepsilon^+(0), \lambda_\varepsilon^+(0), \varepsilon)$ *is the unique degenerate solution on* Γ_ε^+ *and a supercritical fold point,*

- *the second branch is of the form*

$$\Gamma_\varepsilon^- = \left\{(x_\varepsilon^-(\lambda), \lambda) \in \Omega \times \Lambda : \, \lambda \in \Lambda_\varepsilon\right\}$$

  with a one-to-one C^{m-1}*-function* $x_\varepsilon^- : \Lambda_\varepsilon \to \Omega_0$, *where every triple* $(x_\varepsilon^-(\lambda), \lambda, \varepsilon)$ *is a nondegenerate solution to* (A.1)*,*

- $$\#\{x \in \Omega_0 : \, G(x, \lambda, \varepsilon) = 0\} = \begin{cases} 1, & \lambda < \lambda_\varepsilon^+(0), \\ 2, & \lambda = \lambda_\varepsilon^+(0), \\ 3, & \lambda > \lambda_\varepsilon^+(0). \end{cases}$$

Proof. See [28, Thm. 3.2] for (a) and [28, Thm. 4.1] for (b). □

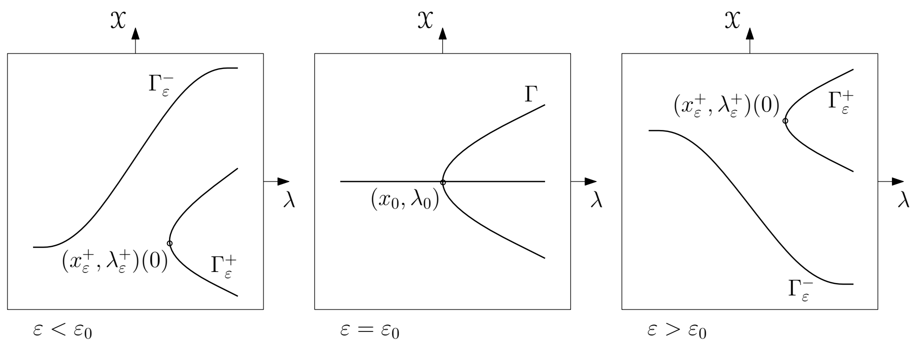

Figure A.5: Imperfect supercritical pitchfork bifurcation from Thm. A.3

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