A Continuation Principle for Fredholm maps I: Theory and Basics

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We prove an abstract and flexible continuation theorem for zeros of parametrized Fredholm maps between Banach spaces. It guarantees not only the existence of zeros to corresponding equations, but also that they form a continuum for parameters from a connected manifold. Our basic tools are transfer maps and a specific topological degree. The main result is tailor-made to solve boundary value problems over infinite time-intervals and for the (global) continuation of homoclinic solutions.

1 Introduction

This paper investigates the global solution structure of abstract equations

$$G(x,\lambda) = 0 \tag{O_λ}$$

depending on a parameter λ. Locally near a given reference solution (x^*,λ^*) this apparently is the setting of the classical implicit function theorem (e.g. [14, 27]), which found countless applications over the last century. Yet, it is of intrinsic interest to obtain information on the global structure of the solution branches. A first approach to this problem using Leray-Schauder degree theory is due to [25] (see also [14, pp. 321ff]), where applications to nonlinear elliptic partial differential equations are given, or [15]. A more recent contribution from [9] applies to Fredholm maps and allows to study boundary value problems on unbounded intervals (see [19, 20] or [23]). For real parameters λ, such global implicit function theorems roughly state that solution branches $\mathcal{C}$ to (O_λ) run from boundary to boundary of the domain of G, unless the set $\mathcal{C} - \{(x^*,\lambda^*)\}$ is connected (see Fig. 1). In case G is globally defined, then $\mathcal{C} - \{(x^*,\lambda^*)\}$ is either connected, or consists of two disjoint and unbounded branches (see Fig. 2).

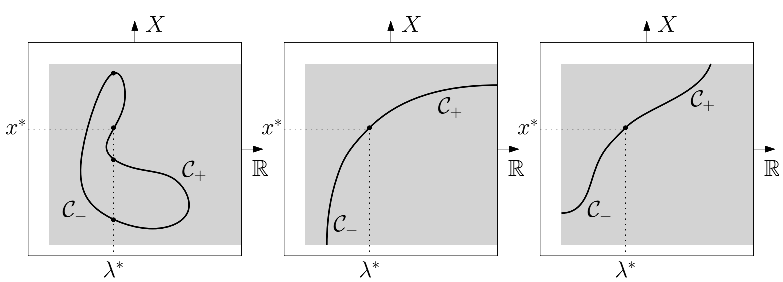

Fig. 1 Alternatives in global implicit function theorem, where the grey shaded area symbolizes the domain of G: The intersection $\mathcal{C}_- \cap \mathcal{C}_+$ is larger than just $\{(x^*,\lambda^*)\}$ (left) or, $\mathcal{C}_+$ is unbounded (center, here $\mathcal{C}_-$ touches the boundary of X) or $\mathcal{C}_-$ touches the boundary of Λ (right, while $\mathcal{C}_+$ touches the boundary of X)

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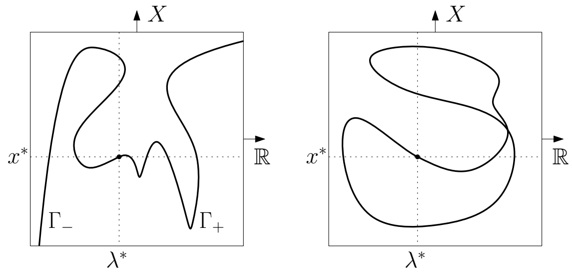

Fig. 2 Alternatives in global implicit function theorem for globally defined G: Two disjoint unbounded sets Γ_-, Γ_+ meet at (x^*, λ^*) (left) or, the difference $\mathcal{C} - \{(x^*, \lambda^*)\}$ is connected (right)

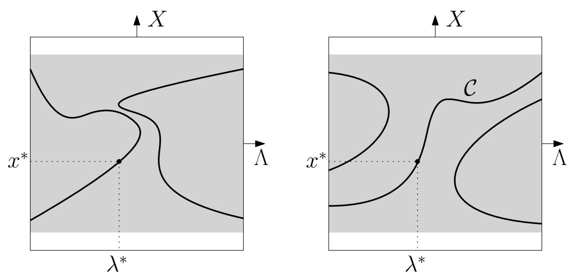

Fig. 3 Illustration of Thm. 1.1, where the grey shaded area symbolizes $O \times \Lambda$: For every parameter value λ there exists a solution of (O_λ) (left). Yet, only the solution branch $\mathcal{C}$ in the right figure covers the entire parameter space Λ and is guaranteed by Thm. 1.1.

The setting of *continuation principles* is slightly different. Rather than asking for the behavior of a global solution branch through a given solution (x^*, λ^*), one is interested in the question, whether a solution exists for every parameter λ, and if these solutions are part of a continuum? In equivalent terms, is there a solution branch covering the entire parameter space?

In order to describe our approach to this problem, let us be more specific. We deal with oriented Fredholm mappings $G\colon \overline{O} \times \Lambda \to Y$ between Banach spaces X, Y, where $O \subseteq X$ is nonempty, open and suppose

(A1) the parameter space Λ is a connected metrizable (Banach) manifold (or more general, a connected absolute neighborhood retract, cf. [8, p. 287, Cor. (5.4)]),

(A2) for any compact set $\Lambda_0 \subseteq \Lambda$ the restriction $G|_{\overline{O} \times \Lambda_0}$ is proper,

(A3) $G^{-1}(0) \cap ((\partial O) \times \Lambda) = \emptyset$ and $G(\cdot, \lambda)|_O$ is Fredholm of index 0 for all $\lambda \in \Lambda$,

(A4) there exists a $\lambda^* \in \Lambda$ such that $\deg(G(\cdot, \lambda^*), O, 0) \neq 0$,

where $\deg(G(\cdot, \lambda^*), O, 0)$ is an ambient topological degree, we are going to specify later. Under these assumptions, our main result reads as follows (cf. Fig. 3):

Theorem 1.1 (global continuation principle) *If* (A1–A4) *hold, then* $G^{-1}(0) \cap (O \times \Lambda)$ *has a connected component* $\mathcal{C}$ *which for every* $\lambda \in \Lambda$ *contains a solution* $x \in O$ *of* (O_λ).

The proof of Thm. 1.1 combines techniques from functional analysis (oriented Fredholm operators), algebraic topology (transfer homomorphisms) and differentiable manifolds, as well as topology (mapping degrees), and will be given in §3. As a prototypical result we would like to mention the approach in [1, Thm. 2.1]. Here, the structure of $G^{-1}(0)$ is studied for semilinear Fredholm maps $G(x, \lambda) = L_\lambda x + C_\lambda(x)$ for all $\lambda \in \Lambda$, where $L_\lambda\colon X \to Y$ is a linear Fredholm operator of index 0 and $C_\lambda\colon \mathcal{O} \to Y$ is completely continuous on an open, nonempty subset $\mathcal{O}$ of a vector bundle over Λ with fibers isomorphic to a Banach space X and Λ being a connected differentiable manifold without boundary.

One motivation for our work is an extension of [1] (or [15]). Rather than assuming a semilinear structure of G with completely continuous perturbation, we investigate (nonlinear) Fredholm operators ab initio. This requires to use different and novel techniques in the proof of Thm. 1.1:

First, following [7] and using cohomology theory, we extend the construction of transfer homomorphisms to finite-dimensional manifolds and prove several general properties. It should be noted that the results concerning the transfer homomorphisms are of independent interest and can be applied to further relevant problems not considered here. What is more, the study of this subject at hand is far from being complete. For example, the results obtained in §2 can be extended to functions defined on any topological bundle over a base Λ (observe that in the setting of §2, $\mathbb{M}^n \times \Lambda$ has a structure of a trivial topological bundle over Λ). The most general case requires additional techniques from cohomology theory, such as Thom classes, and is postponed to a future paper. More precisely, we essentially follow the Dold fixed point transfer [7]. However, the construction from [7] does not immediately apply, since parameterized mappings

$G\colon E\times\Lambda\to E$ between the same space E (being a Euclidean neighborhood retract) are considered, as opposed to our situation of maps $G\colon X\times\Lambda\to Y$ with possibly different Banach spaces X and Y. Whence, compared to [7, 1] the construction of the transfer homomorphism must be deeply modified. Our approach is based on advanced techniques from algebraic topology.

Second, since Fredholm mappings are not necessarily compact perturbations of the identity, the Leray-Schauder degree used in [1] does not apply. As an appropriate replacement serves a degree for (oriented) Fredholm maps. Among various corresponding constructions, the most popular ones are due to Benevieri and Furi [2], Fitzpatrick, Pejsachowicz and Rabier [10, 11, 21] or the topological degree obtained in Väth [27]. We found the latter one to have two advantages:

(1) The proof of Thm. 1.1 is extensively based on the reduction property (B.2) of a topological degree, shared by all the degrees mentioned above. However, what is interesting, this property was used directly in [27]. Thus, in order to obtain clear and transparent proofs, it appears natural to work with this degree.

(2) Our approach is based on transfer homomorphism methods involving homology and cohomology theory. This requires to relate the degree for Fredholm maps with the possibly lesser known (co)homological degree between finite-dimensional oriented manifolds constructed in [6, Chap. VIII]. Once again, the approach of [27] allows us to do this quickly.

Summarizing, although our arguments remain valid when using the Benevieri-Furi [2], as well as the Fitzpatrick-Pejsachowicz-Rabier degree [10, 21], we feel that the corresponding reasoning would be longer and more complicated.

Another motivation for our work are applications to boundary value problems on unbounded domains [19, 9, 20] or in the field of nonautonomous dynamical systems [22, 23, 24].

The paper is organized as follows: §2 introduces some preliminaries from the theory of differentiable manifolds, but essentially focusses on the technique of transfer homomorphisms. Here, the central Prop. 2.9 appears to be of independent interest. §3 is devoted to the proof of our main theoretical contribution, i.e. Thm. 1.1. Finally, our three appendices aim to help readers possibly unfamiliar with single perquisites of this paper, and intend to keep it largely self-contained. App. A introduces topological tools like direct sums of abelian groups (and their limits), the tautness of cohomology, cross products and Künneth's theorem for singular cohomology. The required degree theory is summarized in App. B. It presents essential properties of both the homological degree between oriented manifolds due to [6], as well as of the Benevieri-Furi degree [2].

Notation and preliminaries

In what follows, we use the notation $\mathbb{N}_0$ for $\mathbb{N}\cup\{0\}$. On a metric space X, $B_r(x)$ and $\bar{B}_r(x)$ are the open resp. closed r-balls centered in a point $x\in X$, the interior of a set $\Omega\subset X$ is Ω°, the closure is $\overline{\Omega}$ and $\partial\Omega$ the boundary.

For Banach spaces X,Y we denote the space of linear bounded operators from X to Y by $L(X,Y)$, $GL(X,Y)$ are the invertible elements and $\Phi_0(X,Y)$ the linear index 0 Fredholm operators. We briefly write $L(X):=L(X,X)$ (similarly for other spaces) and id_X for the identity map on X. Furthermore, $N(T):=T^{-1}(0)$ and $R(T):=TX$ are the *kernel* resp. the *range* of $T\in L(X,Y)$. Norms on finite-dimensional linear spaces are denoted by $|\cdot|$.

Being guided by [2], suppose that $C(T)$ denote the *correctors* of $T\in L(X,Y)$, that is, the set of $K\in L(X,Y)$ with $\dim R(K)<\infty$ and $T+K\in GL(X,Y)$. We call $K_1,K_2\in C(T)$ *equivalent*, if $\det((T+K_1)^{-1}(T+K_2)|_{X_0})>0$ holds, where X_0 is a finite-dimensional subspace of X containing the range $R((T+K_1)^{-1}(K_1-K_2))$. One can prove that $C(T)$ contains exactly two equivalence classes and $C(T)\neq\emptyset$ holds if and only if $T\in\Phi_0(X,Y)$. An *orientation* of $T\in\Phi_0(X,Y)$ is an equivalence class of correctors for T according to the above equivalence relation; by the opposite orientation of T we mean the complemented equivalence class in $C(T)$. An *oriented linear Fredholm operator* is a pair (T,σ) consisting of a $T\in\Phi_0(X,Y)$ and an orientation σ.

A closed subspace $Y_0\subseteq Y$ is *transversal* to $T\in\Phi_0(X,Y)$, if $R(T)+Y_0=Y$ and Y_0 is complemented in Y (or $T^{-1}(Y_0)$ is complemented in X). For an oriented Fredholm map (T,σ) and $Y_0\subseteq Y$ being transversal to T one has:

- If $X_0:=T^{-1}(Y_0)$, then the *inherited orientation* of $T_0:=T|_{X_0}$ is
$$\sigma_0:=\{K|_{X_0}\in L(X_0,Y_0)\mid K\in\sigma \text{ and } R(K)\subseteq Y_0\}\,, \tag{1.1}$$
$T|_{(X_0,Y_0)}\in\Phi_0(X_0,Y_0)$ and $\dim Y_0<\infty$ implies $\dim X_0=\dim Y_0$,
- Y_0 is transversal to T if and only if there are closed subspaces $Y_1\subseteq Y$ and $X_1\subseteq X$ satisfying $Y=Y_0\oplus Y_1$ and $X=X_0\oplus X_1$ such that $T|_{(X_1,Y_1)}\in GL(X_1,Y_1)$.

Given a nonempty, open set $O \subseteq X$ and a nonlinear C^1-map $F : O \to Y$, a point $x \in O$ (resp. $y \in Y$) is called *regular* (resp. *regular value*) of F, if the Fréchet derivative $DF(x) \in L(X,Y)$ is onto (resp. if each element of $F^{-1}(y)$ is regular). One speaks of a *Fredholm map* $F : \overline{O} \to Y$ with index 0, if $DF(x) \in \Phi_0(X,Y)$ holds for all $x \in O$. A submanifold $Y_0 \subseteq Y$ is called *transversal* to F on $M \subseteq X$, if for each $x \in M \cap F^{-1}(Y_0)$ the subspace $T_{F(x)}Y_0 \subseteq Y$ is transversal to $DF(x)$.

Typically dealing with mappings $G\colon O \times \Lambda \to Y$ depending on two variables, it is convenient to abbreviate $G_\lambda := G(\cdot,\lambda) : O \to Y$. Finally, a *generalized Fredholm homotopy* of index 0 is a continuous map $H\colon O \times [0,1] \to Y$ with continuous derivative $(x,t) \mapsto DH_t(x) \in \Phi_0(X,Y)$ for every $t \in [0,1]$.

2 Transfer homomorphism for oriented manifolds

We begin our investigations with preliminaries concerning orientations on manifolds and (co)homological fundamental classes. Throughout the paper we use the *Alexander-Spanier cohomology* $\check{H}^*$, singular cohomology H^* with rational coefficients (see [26]) and consider *singular homology* H_* with integer coefficients. The abbreviation ANR stands for *Absolute Neighborhood Retract*.

In the following, suppose $\mathbb{M}^n$ is an n-dimensional manifold (n-manifold), while $U \subseteq \mathbb{M}^n$ and $K \subseteq \mathbb{M}^n$ are generic open resp. compact and connected subsets $\neq \emptyset$.

Remark 2.1 For every $x \in \mathbb{M}^n$, the relative homology group $H_k(\mathbb{M}^n, \mathbb{M}^n - x)$ is an infinite cyclic group for $k = n$ and vanishes for $k \neq n$.

Given the disjoint union

$$\widetilde{\mathbb{M}^n} := \coprod_{x\in\mathbb{M}^n} H_n(\mathbb{M}^n, \mathbb{M}^n - x),$$

let $p\colon \widetilde{\mathbb{M}^n} \to \mathbb{M}^n$ be a mapping given by the condition $p^{-1}(x) = H_n(\mathbb{M}^n, \mathbb{M}^n - x)$ for any $x \in \mathbb{M}^n$. One can construct a topology on $\widetilde{\mathbb{M}^n}$ such that p becomes a covering map (see [5, pp. 391–393]). Let $q\colon \widetilde{\mathbb{M}^n} \to \mathbb{Z}$ be given by $q(k\alpha) = |k|$, where $\alpha \in H_n(\mathbb{M}^n, \mathbb{M}^n - x)$ is a generator and define

$$\Gamma K := \left\{ s\colon K \to \widetilde{\mathbb{M}^n} \mid p(s(x)) = x \text{ for all } x \in K \right\},$$

as well as a homomorphism

$$J_K\colon H_n(\mathbb{M}^n, \mathbb{M}^n - K) \to \Gamma K, \qquad J_K(\gamma)(x) = (i_{K,x})_*(\gamma) \text{ for all } x \in K,\ \gamma \in H_n(\mathbb{M}^n, \mathbb{M}^n - K),$$

where $(i_{K,x})_*\colon H_n(\mathbb{M}^n, \mathbb{M}^n - K) \to H_n(\mathbb{M}^n, \mathbb{M}^n - x)$ denotes the homomorphism induced by the inclusion.

An n-manifold $\mathbb{M}^n$ is said to be *orientable*, if there is a section $\mu \in \Gamma\mathbb{M}^n$ (called an *orientation* of $\mathbb{M}^n$) such that $q(\mu(x)) \equiv 1$ on $\mathbb{M}^n$ holds and the pair $(\mathbb{M}^n, \mu)$ is called *oriented n-manifold*. The restriction $\mu|_U$ is an orientation of the submanifold U (more precisely: for $x \in U$ one has $(\mu|_U)(x) = (j_{U,x})_*^{-1}(\mu(x))$, where $(j_{U,x})_*\colon H_n(U, U - x) \to H_n(\mathbb{M}^n, \mathbb{M}^n - x)$ is the excision isomorphism). For simplicity, the induced orientation $\mu|_U$ of U is again denoted by μ.

Moreover, J_K defines an isomorphism (see [5, pp. 395–397]) and there exists a unique class

$$\mu_K \in H_n(\mathbb{M}^n, \mathbb{M}^n - K) \text{ with } J_K(\mu_K) = \mu|_K,$$

where $\mu|_K\colon K \to \widetilde{\mathbb{M}^n}$ stands for the restriction of $\mu\colon \mathbb{M}^n \to \widetilde{\mathbb{M}^n}$ to K. This class μ_K is called *fundamental homology class* of K and characterized by the property that the inclusion homomorphism

$$(i_{K,x})_*\colon H_n(\mathbb{M}^n, \mathbb{M}^n - K) \to H_n(\mathbb{M}^n, \mathbb{M}^n - x) \text{ takes } \mu_K \text{ into the orientation } \mu(x) \text{ for all } x \in K.$$

Moreover, $(i_{K,x})_*$ is an isomorphism for all $x \in K$ (see [5, p. 397]). Thus, in this case, μ_K is a generator of the group $H_n(\mathbb{M}^n, \mathbb{M}^n - K)$ because of $H_n(\mathbb{M}^n, \mathbb{M}^n - x) = \mathbb{Z}$. If $K \subset U$, then one can consider the fundamental classes μ_K and μ_K^U w.r.t. $\mathbb{M}^n$ and U, respectively. The excision isomorphism $(j_{U,K})_*\colon H_n(U, U - K) \to H_n(\mathbb{M}^n, \mathbb{M}^n - K)$ takes μ_K^U into μ_K. Whence, one can skip the symbol U from the notation μ_K^U.

We will need the following result from [3, Prop. 6.6, Cor. 7.2]:

Theorem 2.2 (universal coefficient theorem) *If $K \subset U$ is a compact and connected subset, then*

$$I_K\colon H^n(U, U - K) \to \mathrm{Hom}(H_n(U, U - K), \mathbb{Q}), \qquad I_K([\varphi])([\alpha]) := \varphi(\alpha)$$

is an isomorphism.

For later convenience, the value of φ at α is denoted by $\langle\varphi, \alpha\rangle$. Based on Thm. 2.2 we introduce the *fundamental cohomology class* of a compact, connected set $K \subset U \subset (\mathbb{M}^n, \mu)$ as unique element $\mu^K \in H^n(U, U-K)$ so that $\langle\mu^K, \mu_K\rangle = 1$, where μ_K is the fundamental homology class of K (note $\mathrm{Hom}(H_n(U, U-K), \mathbb{Q}) \simeq \mathrm{Hom}(\mathbb{Z}, \mathbb{Q})$). The latter admits the following properties:

Lemma 2.3 *Let $U_1, U_2 \subseteq \mathbb{M}^n$ be open and assume K_1, K_2 are compact, connected with $K_2 \subset K_1 \subset U_1 \subset U_2$. If $h^*\colon H^n(U_2, U_2 - K_2) \to H^n(U_1, U_1 - K_1)$ denotes the homomorphism induced by the inclusion $h\colon (U_1, U_1 - K_1) \hookrightarrow (U_2, U_2 - K_2)$, then $h^*(\mu^{K_2}) = \mu^{K_1}$.*

P r o o f. Because of $h_*(\mu_{K_1}) = \mu_{K_2}$ (cf. [6, VIII, 2.8]) and

$$\langle\mu^{K_1}, \mu_{K_1}\rangle = 1 = \langle\mu^{K_2}, \mu_{K_2}\rangle = \langle\mu^{K_2}, h_*(\mu_{K_1})\rangle = \langle h^*(\mu^{K_2}), \mu_{K_1}\rangle$$

the uniqueness of the cohomological fundamental class yields $h^*(\mu^{K_2}) = \mu^{K_1}$. □

This property explains, analogously to the concept of a fundamental class in homology, why we can neglect the symbol U in the notation of $\mu^K \in H^n(U, U-K)$.

Corollary 2.4 *The homomorphism $i^*_{K,x}\colon H^n(\mathbb{M}^n, \mathbb{M}^n - x) \to H^n(\mathbb{M}^n, \mathbb{M}^n - K)$ is an isomorphism, which satisfies $i^*_{K,x}(\mu^x) = \mu^K$ for all $x \in K$.*

P r o o f. The first conclusion results from the following commutative diagram:

$$\begin{tikzcd} H^n(\mathbb{M}^n, \mathbb{M}^n - x) \arrow[r, "i^*_{K,x}"] \arrow[d, "I_x"', "\cong"] & H^n(\mathbb{M}^n, \mathbb{M}^n - K) \arrow[d, "\cong"', "I_K"] \\ \mathrm{Hom}(H_n(\mathbb{M}^n, \mathbb{M}^n - x), \mathbb{Q}) \arrow[r, "\mathrm{Hom}((i_{K,x})_*)", "\cong"'] & \mathrm{Hom}(H_n(\mathbb{M}^n, \mathbb{M}^n - K), \mathbb{Q}), \end{tikzcd}$$

where $\mathrm{Hom}((i_{K,x})_*)$ is given by $\mathrm{Hom}((i_{K,x})_*)(h) := h \circ (i_{K,x})_*$ for all $h \in \mathrm{Hom}(H_n(\mathbb{M}^n, \mathbb{M}^n - x), \mathbb{Q})$. The second conclusion is due to Lemma 2.3. □

If $\mathbb{M}^n$ is of class C^1, then orientability can be characterized as follows (see [3, pp. 347–348] or [13, pp. 267–268]): $\mathbb{M}^n$ is orientable if and only if there is an atlas $\mathcal{A}$ of $\mathbb{M}^n$ such that for (U, ϕ), $(V, \psi) \in \mathcal{A}$ the relation $D(\psi\phi^{-1}) > 0$ on $\phi(U \cap V) > 0$ holds (such an atlas $\mathcal{A}$ is called *orienting*). Suppose that $f\colon (\mathbb{M}^n, \mu) \to (\mathbb{N}^n, \widehat{\mu})$ is a homeomorphism between two oriented n-manifolds. We will say that *f preserves* (resp. *reverses*) the orientations of $\mathbb{M}^n$ and $\mathbb{N}^n$, provided that $f_*(\mu_x) = \widehat{\mu}_{f(x)}$ (resp. $f_*(\mu_x) = -\widehat{\mu}_{f(x)}$) for all $x \in \mathbb{M}^n$ holds. What is interesting [13, pp. 266–267, Thm. 3.4] is that, if $f\colon (\mathbb{R}^n, \mu) \to (\mathbb{R}^n, \mu)$ is a diffeomorphism, then $f_*\colon H_n(\mathbb{R}^n, \mathbb{R}^n - x) \to H_n(\mathbb{R}^n, \mathbb{R}^n - f(x))$ satisfies

$$f_*(\mu_x) = (\operatorname{sgn}\det Df(x)) \cdot \mu_{f(x)} \text{ for } x \in \mathbb{R}^n.$$

Suppose now that $(\mathbb{M}^n, \mu)$ is an oriented n-manifold and $A \subseteq X$, $B \subseteq Y$ are open.

Corollary 2.5 *If $K \subset \mathbb{M}^n$ is connected and compact, then*

$$\times\colon H^n(\mathbb{M}^n, \mathbb{M}^n - K) \otimes H^d(Y, B) \to H^{n+d}((\mathbb{M}^n, \mathbb{M}^n - K) \times (Y, B))$$

is an isomorphism for all $d \in \mathbb{N}_0$.

P r o o f. First, observe that Rem. 2.1, Cor. 2.4 and the universal coefficient Thm. 2.2 imply

$$H^p(\mathbb{M}^n, \mathbb{M}^n - K) = \begin{cases} 0 & \text{if} \quad p \neq n, \\ \mathbb{Q} & \text{if} \quad p = n, \end{cases}$$

hence, $\bigoplus_{p+q=n+d} H^p(\mathbb{M}^n, \mathbb{M}^n - K) \otimes H^q(Y, B) = H^n(\mathbb{M}^n, \mathbb{M}^n - K) \otimes H^d(Y, B)$ and from Künneth's Thm. A.5 we deduce the conclusion. □

We shall make use of the following suspension homomorphism:

Proposition 2.6 (suspension homomorphism) *If $K \subset \mathbb{M}^n$ is compact and connected with fundamental class μ^K, then the so-called* suspension homomorphism

$$\sigma^n \colon H^d(Y,B) \to H^{n+d}((\mathbb{M}^n, \mathbb{M}^n - K) \times (Y,B)), \qquad \sigma^n(\xi) := \mu^K \times \xi$$

is an isomorphism for all $d \in \mathbb{N}_0$.

Proof. Given $d \in \mathbb{N}_0$, observe that Cor. 2.5 implies that cross product

$$\times \colon H^n(\mathbb{M}^n, \mathbb{M}^n - K) \otimes H^d(Y,B) \to H^{n+d}((\mathbb{M}^n, \mathbb{M}^n - K) \times (Y,B))$$

is an isomorphism. The assertion follows, since $\mu^K \in H^n(\mathbb{M}^n, \mathbb{M}^n - K) = \mathbb{Q}$ is a generator. □

Corollary 2.7 *If $\mathrm{pr}_2 \colon (\mathbb{M}^n, \mathbb{M}^n - K) \times (Y,B) \to (Y,B)$ is the projection on the second component and*

$$\sigma^n \colon H^q(Y,B) \to H^{q+n}((\mathbb{M}^n, \mathbb{M}^n - K) \times (Y,B)),$$

then the suspension homomorphism satisfies

$$\sigma^n(\xi \smile \eta) = \sigma^n(\xi) \smile \mathrm{pr}_2^*(\eta) \text{ for all } \xi \in H^{q_1}(Y,B), \eta \in H^{q_2}(Y,B).$$

Proof. Let $q_1, q_2 \in \mathbb{N}_0$. If $\xi \in H^{q_1}(Y,B)$ and $\eta \in H^{q_2}(Y,B)$, then

$$\sigma^n(\xi \smile \eta) = \mu^K \times (\xi \smile \eta) = (\mu^K \smile 1_{\mathbb{M}^n}) \times (\xi \smile \eta) = (\mu^K \times \xi) \smile (1_{\mathbb{M}^n} \times \eta) = \sigma^n(\xi) \smile \mathrm{pr}_2^*(\eta)$$

yields the assertion. □

Next, we explain the relationship of the cohomological fundamental class with the suspension homomorphism from Prop. 2.6. For this purpose, we need the homomorphisms

$$\begin{aligned} \mathrm{DEG} \colon H^0(\{\lambda^*\}) &\to H^0(\{\lambda^*\}), & \mathrm{DEG}(1_{\lambda^*}) &= q \cdot 1_{\lambda^*}, \\ H \colon H^n(\mathbb{E}^n, \mathbb{E}^n - K) &\to H^n(\mathbb{M}^n, \mathbb{M}^n - x), & H(\widehat{\mu}^K) &= q \cdot \mu^x, \end{aligned}$$

where $1_{\lambda^*} \in H^0(\{\lambda^*\}) = \mathbb{Q}$, $\widehat{\mu}^K \in H^n(\mathbb{E}^n, \mathbb{E}^n - K)$ and $\mu^x \in H^n(\mathbb{M}^n, \mathbb{M}^n - x)$ are generators. Then

$$\begin{tikzcd}
H^0(\{\lambda^*\}) & & H^0(\{\lambda^*\}) \arrow[ll, "\mathrm{DEG}"'] \arrow[d, "\sigma^n", "\cong"'] \\
H^n((\mathbb{M}^n, \mathbb{M}^n - x) \times \{\lambda^*\}) \arrow[u, "(\sigma^n)^{-1}"', "\cong"] & & H^n((\mathbb{E}^n, \mathbb{E}^n - K) \times \{\lambda^*\}) \arrow[d, "(\mathrm{pr}_1^*)^{-1}", "\cong"'] \\
H^n(\mathbb{M}^n, \mathbb{M}^n - x) \arrow[u, "\mathrm{pr}_1^*"', "\cong"] & & H^n(\mathbb{E}^n, \mathbb{E}^n - K) \arrow[ll, "H"']
\end{tikzcd} \tag{2.1}$$

is commutative with the suspension isomorphisms (see also (A.4))

$$\sigma^n \colon H^0(\{\lambda^*\}) \to H^n((\mathbb{M}^n, \mathbb{M}^n - x) \times \{\lambda^*\}), \qquad \sigma^n \colon H^0(\{\lambda^*\}) \to H^n((\mathbb{E}^n, \mathbb{E}^n - K) \times \{\lambda^*\}).$$

On an oriented n-manifold $(\mathbb{M}^n, \mu)$ without boundary, an n-dimensional oriented normed space $(\mathbb{E}^n, \widehat{\mu})$ and an open, connected $O \subseteq \mathbb{M}^n$, we consider maps $g \colon O \times \Lambda \to \mathbb{E}^n$ satisfying:

(a1) The parameter space Λ is a connected ANR,

(a2) for every compact set $\Lambda_0 \subset \Lambda$, $g^{-1}(0) \cap (O \times \Lambda_0)$ is compact,

(a3) there exists $\lambda^* \in \Lambda$ with $\deg_0(g_{\lambda^*}, O, \mathbb{E}^n) \neq 0$, where $\deg_0(g_{\lambda^*}, O, \mathbb{E}^n)$ denotes the homological degree defined as in (B.1).

Fix a compact and connected subset $\Lambda_0 \subseteq \Lambda$. For a subset $\mathcal{U} \subseteq O \times \Lambda_0$ we define the λ-*fiber*

$$\mathcal{U}_\lambda := \{x \in O \mid (x, \lambda) \in \mathcal{U}\} \text{ for all } \lambda \in \Lambda_0$$

and abbreviate $\mathcal{S} := g^{-1}(0) \subset O \times \Lambda$, $\mathcal{S}|\mathcal{U} := g^{-1}(0) \cap \mathcal{U}$, $\mathcal{S}|\Lambda_0 := \mathcal{S} \cap (O \times \Lambda_0)$ for the set of zeros. Our construction of the transfer map is based on the subsequent topological result which follows from the proof of Lemma 3.22 contained in [16, p. 77]:

Lemma 2.8 *If $\mathbb{M}^n$ is connected and $C \subset \mathbb{M}^n \times \Lambda_0$ is compact, then there is a compact, connected set $M \subset \mathbb{M}^n$ with $C \subseteq M \times \Lambda_0 \subseteq \mathbb{M}^n \times \Lambda_0$.*

The above assumptions (a1-a3) imply further properties of g for all $\lambda \in \Lambda$:

- $\deg_0(g_\lambda, O, \mathbb{E}^n) \neq 0$ for all $\lambda \in \Lambda$ (the proof is that of Lemma 3.2 below),
- there exists a compact, connected $M \subset \mathbb{M}^n$ with $\mathcal{S}|\Lambda_0 \subset M \times \Lambda_0$ (by Lemma 2.8).

Let $\mathcal{U}$ be any open subset of $O \times \Lambda_0$ such that $\partial\mathcal{U} \cap \mathcal{S}|\Lambda_0 = \emptyset$ with $\deg_0(g_{\lambda^*}, \mathcal{U}_{\lambda^*}, \mathbb{E}^n) \neq 0$ for some $\lambda^* \in \Lambda_0$. This brings us into the position to consider the following diagram

$$\begin{tikzcd}
(\mathbb{E}^n, \mathbb{E}^n - 0) \times \mathcal{U} & (\mathcal{U}, \mathcal{U} - \mathcal{S}|\mathcal{U}) \arrow[l, "{(g,\mathrm{id})}"'] \arrow[r, "i_1"] & (\mathbb{M}^n \times \Lambda_0, \mathbb{M}^n \times \Lambda_0 - \mathcal{S}|\mathcal{U}) \\
(\mathbb{M}^n, \mathbb{M}^n - x) \times \Lambda_0 & (\mathbb{M}^n, \mathbb{M}^n - M) \times \Lambda_0 \arrow[l, "i_{x,M} \times \mathrm{id}_{\Lambda_0}"'] & (\mathbb{M}^n \times \Lambda_0, \mathbb{M}^n \times \Lambda_0 - M \times \Lambda_0) \arrow[l, "\mathrm{id}"'] \arrow[u, "i_2"]
\end{tikzcd} \tag{2.2}$$

for all $x \in M$, which requires some comments: First of all, the map

$$(g, \mathrm{id})\colon (\mathcal{U}, \mathcal{U} - \mathcal{S}|\mathcal{U}) \to (\mathbb{E}^n, \mathbb{E}^n - 0) \times \mathcal{U}, \qquad (g, \mathrm{id})(x) = (g(x), x)$$

is well-defined because $(\mathbb{E}^n, \mathbb{E}^n - 0) \times \mathcal{U} = (\mathbb{E}^n \times \mathcal{U}, (\mathbb{E}^n - 0) \times \mathcal{U})$,

$$(g, \mathrm{id})(\mathcal{U} - \mathcal{S}|\mathcal{U}) \subset (\mathbb{E}^n - 0) \times \mathcal{U}, \qquad (g, \mathrm{id})(\mathcal{U}) \subset \mathbb{E}^n \times \mathcal{U}.$$

Given $(\mathcal{X}; \mathcal{X}_1, \mathcal{X}_2) := (\mathbb{M}^n \times \Lambda_0, \mathcal{U}, \mathbb{M}^n \times \Lambda_0 - \mathcal{S}|\mathcal{U})$ one observes that

- $\mathcal{X}_1 \cap \mathcal{X}_2 = \mathcal{U} - \mathcal{S}|\mathcal{U}$, $\mathcal{X}_1 \cup \mathcal{X}_2 = \mathbb{M}^n \times \Lambda_0$,
- $\mathcal{U}$ is open in $\mathbb{M}^n \times \Lambda_0$, $\mathbb{M}^n \times \Lambda_0 - \mathcal{S}|\mathcal{U}$ is open in $\mathbb{M}^n \times \Lambda_0$ and hence $\mathcal{X}_1 \cup \mathcal{X}_2 = \mathcal{X}_1^\circ \cup \mathcal{X}_2^\circ$.

Thus, $(\mathcal{X}; \mathcal{X}_1, \mathcal{X}_2)$ is an excisive triad and therefore the inclusion $(\mathcal{X}_1, \mathcal{X}_1 \cap \mathcal{X}_2) \hookrightarrow (\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{X}_2)$ induces an isomorphism on cohomology groups in all dimensions (and the above inclusion coincides with the map i_1, cf. App. A.3). The Künneth Thm. A.5 implies that

$$\begin{tikzcd}
H^n(\mathbb{M}^n, \mathbb{M}^n - x) \otimes H^d(\Lambda_0) \arrow[r, "i^* \otimes \mathrm{id}^*"] \arrow[d, "\times"', "\cong"] & H^n(\mathbb{M}^n, \mathbb{M}^n - M) \otimes H^d(\Lambda_0) \arrow[d, "\cong"', "\times"] \\
H^{n+d}((\mathbb{M}^n, \mathbb{M}^n - x) \times \Lambda_0) \arrow[r, "i_3^*"] & H^{n+d}((\mathbb{M}^n, \mathbb{M}^n - M) \times \Lambda_0)
\end{tikzcd}$$

commutes. By applying functor cohomology to the diagram (2.2) and the suspension homomorphisms from Prop. 2.6 we arrive at the diagram

$$
\begin{tikzcd}
H^d(\mathcal{U}) \arrow[r, "\sigma^n", "\cong"'] \arrow[ddddd, dotted, "t_{\mathcal{U}}(g)"'] & H^{n+d}((\mathbb{E}^n,\mathbb{E}^n-0)\times\mathcal{U}) \arrow[d, "(g,\mathrm{id})^*"] \\
& H^{n+d}(\mathcal{U},\mathcal{U}-\mathcal{S}|\mathcal{U}) \arrow[d, "(i_1^*)^{-1}"] \\
& H^{n+d}(\mathbb{M}^n\times\Lambda_0,\mathbb{M}^n\times\Lambda_0-\mathcal{S}|\mathcal{U}) \arrow[d, "i_2^*"] \\
& H^{n+d}(\mathbb{M}^n\times\Lambda_0,\mathbb{M}^n\times\Lambda_0-M\times\Lambda_0) \arrow[d, "\mathrm{id}^*"] \\
& H^{n+d}((\mathbb{M}^n,\mathbb{M}^n-M)\times\Lambda_0) \arrow[d, "((i_{x,M}\times\mathrm{id}_{\Lambda_0})^*)^{-1}"] \\
H^d(\Lambda_0) & H^{n+d}((\mathbb{M}^n,\mathbb{M}^n-x)\times\Lambda_0) \arrow[l, "(\sigma^n)^{-1}"', "\cong"]
\end{tikzcd}
\tag{2.3}
$$

for all $d\in\mathbb{N}_0$, which in turn motivates

Definition 2.9 The composition in diagram (2.3) is denoted by $t_{\mathcal{U}}(g)\colon H^d(\mathcal{U})\to H^d(\Lambda_0)$ and called *transfer homomorphism* of g over $\mathcal{U}$.

In the remaining section, we derive those properties of the transfer homomorphism which are necessary in the proof of our main Thm. 1.1.

Proposition 2.10 (properties of the transfer homomorphism) *Given the diagram*

$$
\begin{tikzcd}
H^d(\Lambda_0) \arrow[r, "1_{\Lambda_0}\otimes\mathrm{id}", "injective"'] \arrow[d, "\pi^*"'] & H^0(\Lambda_0)\otimes H^d(\Lambda_0) \arrow[d, "\pi_0^*\otimes\mathrm{id}"] \\
H^d(\mathcal{U}) \arrow[d, "t_{\mathcal{U}}(g)"'] & H^0(\mathcal{U})\otimes H^d(\Lambda_0) \arrow[l, "\mathrm{id}_0^*\smile\pi^*"'] \arrow[d, "t_{\mathcal{U}}(g)\otimes\mathrm{id}"] \\
H^d(\Lambda_0) & H^0(\Lambda_0)\otimes H^d(\Lambda_0), \arrow[l, "\smile"', "\cong"]
\end{tikzcd}
\tag{2.4}
$$

where $\pi_0^*\colon H^0(\Lambda_0)\to H^0(\mathcal{U})$ is induced by $\pi\colon\mathcal{U}\to\Lambda_0$, the unit element of $H^0(\mathcal{U})$ is the constant map $1_{\Lambda_0}\colon H^0(\Lambda_0)\to H^0(\Lambda_0)$ and $\mathrm{id}_0^*\colon H^0(\mathcal{U})\to H^0(\mathcal{U})$ the identity, it holds:

(a) *The transfer homomorphism $t_{\mathcal{U}}(g)$ makes* (2.4) *commutative,*

(b) *if $t_{\mathcal{U}}(g)(1_{\mathcal{U}})\neq 0$, then $\pi^*\colon H^d(\Lambda_0)\to H^d(\mathcal{U})$ is injective.*

Proof. (a): Let $\mathrm{pr}_2\colon\mathbb{M}^n\times\Lambda_0\to\Lambda_0$ stand for the projection onto the second component. In the proof we use the cup and cross products introduced in Sects. A.3, A.4, whose properties are due to Prop. A.4. Let us proceed in five steps:

(I) We will prove that the diagram

$$
\begin{tikzcd}
H^d(\Lambda_0) \arrow[r, "1_{\Lambda_0}\otimes\mathrm{id}", "\cong"'] \arrow[d, "\pi^*"'] & H^0(\Lambda_0)\otimes H^d(\Lambda_0) \arrow[d, "\pi_0^*\otimes\mathrm{id}"] \\
H^d(\mathcal{U}) & H^0(\mathcal{U})\otimes H^d(\Lambda_0) \arrow[l, "\mathrm{id}_0^*\smile\pi^*"']
\end{tikzcd}
$$

is commutative. Actually, if $\xi \in H^d(\Lambda_0)$, then commutativity follows from

$$\begin{aligned}[(\mathrm{id}_0^* \smile \pi^*) \circ (\pi_0^* \otimes \mathrm{id}) \circ (1_{\Lambda_0} \otimes \mathrm{id})](\xi) &= [(\mathrm{id}_0^* \smile \pi^*) \circ (\pi_0^* \otimes \mathrm{id})](1_{\Lambda_0} \otimes \xi) \\ = [\mathrm{id}_0^* \smile \pi^*](\pi_0^*(1_{\Lambda_0}) \otimes \xi) &= [\mathrm{id}_0^* \smile \pi^*](1_{\mathcal{U}} \otimes \xi) = \mathrm{id}_0^*(1_{\mathcal{U}}) \smile \pi^*(\xi) \\ &= 1_{\mathcal{U}} \smile \pi^*(\xi) = \pi^*(\xi).\end{aligned}$$

(II) Now we show the commutativity of the diagram

$$\begin{tikzcd}
H^d(\mathcal{U}) \arrow[d, "\sigma^n"'] & & H^0(\mathcal{U}) \otimes H^d(\Lambda_0) \arrow[ll, "\mathrm{id}_0^* \smile \pi^*"'] \arrow[d, "\sigma^n \otimes \mathrm{id}"] \\
H^{n+d}((\mathbb{E}^n, \mathbb{E}^n - 0) \times \mathcal{U}) \arrow[d, "(g,\mathrm{id})^*"'] & & H^n((\mathbb{E}^n, \mathbb{E}^n - 0) \times \mathcal{U}) \otimes H^d(\Lambda_0) \arrow[d, "(g,\mathrm{id})^* \otimes \mathrm{id}"] \\
H^{n+d}(\mathcal{U}, \mathcal{U} - \mathcal{S}|\mathcal{U}) & & H^n(\mathcal{U}, \mathcal{U} - \mathcal{S}|\mathcal{U}) \otimes H^d(\Lambda_0). \arrow[ll, "\mathrm{id} \smile \pi^*"']
\end{tikzcd}$$

Indeed, if $\xi \in H^d(\Lambda_0)$ and $\eta \in H^0(\mathcal{U})$, then commutativity results from

$$\begin{aligned}&[(g,\mathrm{id})^* \circ \sigma^n \circ (\mathrm{id}_0^* \smile \pi^*)](\eta \otimes \xi) = [(g,\mathrm{id})^* \circ \sigma^n]\,(\eta \smile \pi^*(\xi)) \\ &= (g,\mathrm{id})^*(\sigma^n(\eta \smile \pi^*(\xi))) = (g,\mathrm{id})^*(\mu^0 \times [\eta \smile \pi^*(\xi)]) \\ &= [\Delta^*(g^* \times \mathrm{id}^*)]\left(\mu^0 \times [\eta \smile \pi^*(\xi)]\right) = \Delta^*\left(g^*(\mu^0) \times \mathrm{id}^*[\eta \smile \pi^*(\xi)]\right) \\ &= \Delta^*\left(g^*(\mu^0) \times [\eta \smile \pi^*(\xi)]\right) = g^*(\mu^0) \smile (\eta \smile \pi^*(\xi)) \\ &= (g^*(\mu^0) \smile \eta) \smile \pi^*(\xi) = [g^*(\mu^0) \smile \mathrm{id}^*(\eta)] \smile \pi^*(\xi) \\ &= \Delta^*[g^*(\mu^0) \times \mathrm{id}^*(\eta)] \smile \pi^*(\xi) = \Delta^*[(g \times \mathrm{id})^*](\mu^0 \times \eta) \smile \pi^*(\xi) \\ &= \Delta^*[(g \times \mathrm{id})^*](\sigma^n(\eta)) \smile \pi^*(\xi) = (g,\mathrm{id})^*(\sigma^n(\eta)) \smile \pi^*(\xi) \\ &= [(\mathrm{id} \smile \pi^*) \circ ((g,\mathrm{id})^* \otimes \mathrm{id}) \circ (\sigma^n \otimes \mathrm{id})](\eta \otimes \xi).\end{aligned}$$

(III) It is straight forward to show that the following diagram is commutes:

$$\begin{tikzcd}
H^{n+d}(\mathcal{U}, \mathcal{U} - \mathcal{S}|\mathcal{U}) & H^n(\mathcal{U}, \mathcal{U} - \mathcal{S}|\mathcal{U}) \otimes H^d(\Lambda_0) \arrow[l, "\mathrm{id} \smile \pi^*"'] \\
H^{n+d}(\mathbb{M}^n \times \Lambda_0, \mathbb{M}^n \times \Lambda_0 - \mathcal{S}|\mathcal{U}) \arrow[u, "i_1^*"] & H^n(\mathbb{M}^n \times \Lambda_0, \mathbb{M}^n \times \Lambda_0 - \mathcal{S}|\mathcal{U}) \otimes H^d(\Lambda_0). \arrow[l, "\mathrm{id} \smile \mathrm{pr}_2^*"'] \arrow[u, "i_1^* \otimes \mathrm{id}"']
\end{tikzcd}$$

(IV) Take the diagram

$$\begin{tikzcd}
H^{n+d}(\mathbb{M}^n \times \Lambda_0, \mathbb{M}^n \times \Lambda_0 - \mathcal{S}|\mathcal{U}) \arrow[d, "(i_1^*)^{-1}"'] & H^n(\mathbb{M}^n \times \Lambda_0, \mathbb{M}^n \times \Lambda_0 - \mathcal{S}|\mathcal{U}) \otimes H^d(\Lambda_0) \arrow[l, "\mathrm{id} \smile \mathrm{pr}_2^*"'] \arrow[d, "(i_1^*)^{-1} \otimes \mathrm{id}"] \\
H^{n+d}(\mathbb{M}^n \times \Lambda_0, \mathbb{M}^n \times \Lambda_0 - M \times \Lambda_0) \arrow[d, "i_2^*"'] & H^n(\mathbb{M}^n \times \Lambda_0, \mathbb{M}^n \times \Lambda_0 - M \times \Lambda_0) \otimes H^d(\Lambda_0) \arrow[l, "\mathrm{id} \smile \mathrm{pr}_2^*"'] \arrow[d, "i_2^* \otimes \mathrm{id}"] \\
H^{n+d}((\mathbb{M}^n, \mathbb{M}^n - M) \times \Lambda_0) \arrow[d, "((i_{x,M} \times \mathrm{id}_{\Lambda_0})^*)^{-1}"'] & H^n((\mathbb{M}^n, \mathbb{M}^n - M) \times \Lambda_0) \otimes H^d(\Lambda_0) \arrow[l, "\mathrm{id} \smile \mathrm{pr}_2^*"'] \arrow[d, "((i_{x,M} \times \mathrm{id}_{\Lambda_0})^*)^{-1} \otimes \mathrm{id}"] \\
H^{n+d}((\mathbb{M}^n, \mathbb{M}^n - x) \times \Lambda_0) & H^n((\mathbb{M}^n, \mathbb{M}^n - x) \times \Lambda_0) \otimes H^d(\Lambda_0), \arrow[l, "\mathrm{id} \smile \mathrm{pr}_2^*"']
\end{tikzcd}$$

whose associate homomorphisms are as in (2.3). Since $(\mathrm{id} \smile \mathrm{pr}_2^*)(\alpha \smile \beta) = \alpha \smile \mathrm{pr}_2^* \beta$ holds for any corresponding cohomology classes α and β, the commutativity follows.

(V) Finally, let us consider the diagram

$$\begin{tikzcd}
H^{n+d}((\mathbb{M}^n,\mathbb{M}^n-x)\times\Lambda_0) & H^n((\mathbb{M}^n,\mathbb{M}^n-x)\times\Lambda_0)\otimes H^d(\Lambda_0) \arrow[l, "\mathrm{id}\smile\mathrm{pr}_2^*"'] \\
H^d(\Lambda_0) \arrow[u, "\sigma^n"] & H^0(\Lambda_0)\otimes H^d(\Lambda_0). \arrow[l, "\smile"'] \arrow[u, "\sigma^n\otimes\mathrm{id}"']
\end{tikzcd}$$

It suffices to prove

$$\sigma^n(\xi\smile\eta)=\sigma^n(\xi)\smile\mathrm{pr}_2^*(\eta)$$

for all $\xi\in H^0(\Lambda_0)$, $\eta\in H^d(\Lambda_0)$, which follows from Cor. 2.7 for $K:=\{x\}$, $Y:=\Lambda_0$, $B:=\emptyset$, $q_1:=0$, $q_2:=d$. The conclusion results from the above steps and the construction of the transfer map (see Def. 2.9).

(b): Let $c_1:=t_{\mathcal{U}}(g)(1_{\mathcal{U}})$. As for the injectivity of $\pi^*\colon H^d(\Lambda_0)\to H^d(\mathcal{U})$, let us observe that $1_{\Lambda_0}\in H^0(\Lambda_0)=\mathbb{Q}$ is a generator,

$$t_{\mathcal{U}}\circ\pi_0^*\colon\mathbb{Q}\to\mathbb{Q},\qquad [t_{\mathcal{U}}\circ\pi_0^*](1_{\Lambda_0})=c_1\cdot 1_{\Lambda_0},\qquad c_1\neq 0.$$

Hence, the commutativity of (2.4) from (a) yields that $t_{\mathcal{U}}(g)\circ\pi^*\colon H^d(\Lambda_0)\to H^d(\Lambda_0)$ is a monomorphism and thus $\pi^*\colon H^d(\Lambda_0)\to H^d(\mathcal{U})$ is injective. This establishes Prop. 2.10. □

We conclude this section with a sufficient condition for a nontrivial transfer homomorphism:

Lemma 2.11 *If* $\deg(g_{\lambda^*},\mathcal{U}_{\lambda^*},0)\neq 0$ *holds for some parameter* $\lambda^*\in\Lambda_0$, *then the transfer homomorphism* $t_{\mathcal{U}}(g)\colon H^0(\mathcal{U})\to H^0(\Lambda_0)$ *has a non-zero value at* $1_{\mathcal{U}}$.

Proof. It suffices to show that

$$t_{\mathcal{U}}(g)(1_{\mathcal{U}})=\deg_0(g_{\lambda^*},\mathcal{U}_{\lambda^*},0)\cdot 1_{\Lambda_0}.$$

Thereto, consider

$$\begin{tikzcd}
(\mathbb{E}^n,\mathbb{E}^n-0)\times\mathcal{U} \arrow[r, "\mathrm{pr}"] & (\mathbb{E}^n,\mathbb{E}^n-0) \\
(\mathcal{U},\mathcal{U}-\mathcal{S}|\mathcal{U}) \arrow[u, "{(g,\mathrm{id})}"] \arrow[d, "i_1"'] & (\mathcal{U}_{\lambda^*},\mathcal{U}_{\lambda^*}-g_{\lambda^*}^{-1}(0)\cap\mathcal{U}_{\lambda^*}) \arrow[l, "j_{\lambda^*}"'] \arrow[u, "g_{\lambda^*}"'] \arrow[d, "j_1"] \\
(\mathbb{M}^n\times\Lambda_0,\mathbb{M}^n\times\Lambda_0-\mathcal{S}|\mathcal{U}) & (\mathbb{M}^n,\mathbb{M}^n-g_{\lambda^*}^{-1}(0)\cap\mathcal{U}_{\lambda^*}) \arrow[l, "j_{\lambda^*}"'] \\
(\mathbb{M}^n,\mathbb{M}^n-M)\times\Lambda_0 \arrow[r, "\mathrm{pr}"] \arrow[u, "i_2"] \arrow[d, "i_{x,M}\times\mathrm{id}_{\Lambda_0}"'] & (\mathbb{M}^n,\mathbb{M}^n-M) \arrow[u, "j_2"'] \arrow[d, "i_{x,M}"] \\
(\mathbb{M}^n,\mathbb{M}^n-x)\times\Lambda_0 \arrow[r, "\mathrm{pr}"] & (\mathbb{M}^n,\mathbb{M}^n-x),
\end{tikzcd}\tag{2.5}$$

where i_k,j_k and $i_{x,M}$ are the corresponding inclusions, pr denotes the projection, j_{λ^*} is given by $j_{\lambda^*}(x)=(x,\lambda^*)$ and $M\subset\mathbb{M}^n$ is compact, connected containing $\mathcal{S}|\mathcal{U}\subset M$. Thus, by applying the cohomology functor to (2.5) we obtain the commutative diagram

$$\begin{tikzcd}
H^n((\mathbb{E}^n,\mathbb{E}^n-0)\times\mathcal{U}) \arrow[d, "T_g"] & H^n(\mathbb{E}^n,\mathbb{E}^n-0) \arrow[l, "\mathrm{pr}^*"', "\cong"] \arrow[d, "H_g"] \\
H^n((\mathbb{M}^n,\mathbb{M}^n-x)\times\Lambda_0) & H^n(\mathbb{M}^n,\mathbb{M}^n-x) \arrow[l, "\mathrm{pr}^*"']
\end{tikzcd}\tag{2.6}$$

with

$$T_g = [(i_{x,M} \times \mathrm{id}_{\Lambda_0})^*]^{-1} \circ i_2^* \circ (i_1^*)^{-1} \circ (g, \mathrm{id})^* \text{ and } H_g = (i_{x,M}^*)^{-1} \circ j_2^* \circ (j_1^*)^{-1} \circ (g_{\lambda^*})^*.$$

Since μ^x is a generator of $H^n(\mathbb{M}^n, \mathbb{M}^n - x) \simeq \mathbb{Q}$, it follows that

$$H_g(\widehat{\mu}^{\,0}) = c_0 \cdot \mu^x, \tag{2.7}$$

where $c_0 \in \mathbb{Q}$. We will show that

$$c_0 = \deg(g_{\lambda^*}, \mathcal{U}_{\lambda^*}, 0).$$

Indeed, one has

$$\begin{aligned} c_0 &= c_0 \langle \mu^x, \mu_x \rangle = \langle c_0 \cdot \mu^x, \mu_x \rangle = \langle H_g(\widehat{\mu}^{\,0}), \mu_x \rangle = \langle [(i_{x,M}^*)^{-1} j_2^* (j_1^*)^{-1} (g_{\lambda^*})^*](\widehat{\mu}^{\,0}), \mu_x \rangle \\ &= \langle (g_{\lambda^*})^*(\widehat{\mu}^{\,0}), [(j_1)_*^{-1} (j_2)_* (i_{x,M})_*^{-1}](\mu_x) \rangle = \langle (g_{\lambda^*})^*(\widehat{\mu}^{\,0}), \mu_{g_{\lambda^*}^{-1}(0) \cap \mathcal{U}_{\lambda^*}} \rangle \\ &= \langle \widehat{\mu}^{\,0}, (g_{\lambda^*})_*(\mu_{g_{\lambda^*}^{-1}(0) \cap \mathcal{U}_{\lambda^*}}) \rangle = \langle \widehat{\mu}^{\,0}, \deg(g_{\lambda^*}, \mathcal{U}_{\lambda^*}, 0) \cdot \widehat{\mu}_0 \rangle \\ &= \deg(g_{\lambda^*}, \mathcal{U}_{\lambda^*}, 0) \cdot \langle \widehat{\mu}^{\,0}, \widehat{\mu}_0 \rangle = \deg(g_{\lambda^*}, \mathcal{U}_{\lambda^*}, 0), \end{aligned}$$

with the scalar products having properties discussed in [6, Chap. VII, pp. 187–189]

$$\langle \cdot, \cdot \rangle \colon H^n(\mathbb{M}^n, \mathbb{M}^n - x) \otimes H_n(\mathbb{M}^n, \mathbb{M}^n - x) \to \mathbb{Q} \quad \text{resp.} \quad \langle \cdot, \cdot \rangle \colon H^n(\mathbb{E}^n, \mathbb{E}^n - 0) \otimes H_n(\mathbb{E}^n, \mathbb{E}^n - 0) \to \mathbb{Q}.$$

Next, by applying the suspensions homomorphisms to (2.6) we obtain

$$\begin{tikzcd}
H^0(\mathcal{U}) \arrow[r, "i_{\lambda^*}^*"] \arrow[d, "\sigma^n", "\cong"'] & H^0(\{\lambda^*\}) \arrow[d, "\sigma^n", "\cong"'] \\
H^n((\mathbb{E}^n, \mathbb{E}^n - 0) \times \mathcal{U}) \arrow[r, "(\mathrm{id} \times i_{\lambda^*})^*"] \arrow[d, "\mathrm{id}"] & H^n((\mathbb{E}^n, \mathbb{E}^n - 0) \times \{\lambda^*\}) \arrow[d, "(\mathrm{pr}^*)^{-1}", "\cong"'] \\
H^n((\mathbb{E}^n, \mathbb{E}^n - 0) \times \mathcal{U}) \arrow[d, "T_g"] & H^n(\mathbb{E}^n, \mathbb{E}^n - 0) \arrow[l, "\mathrm{pr}^*"'] \arrow[d, "H_g"] \\
H^n((\mathbb{M}^n, \mathbb{M}^n - x) \times \Lambda_0) \arrow[d, "\mathrm{id}"] & H^n(\mathbb{M}^n, \mathbb{M}^n - x) \arrow[l, "\mathrm{pr}^*"'] \arrow[d, "\mathrm{pr}^*"] \\
H^n((\mathbb{M}^n, \mathbb{M}^n - x) \times \Lambda_0) \arrow[r, "(\mathrm{id} \times i_{\lambda^*})^*"] \arrow[d, "(\sigma^n)^{-1}", "\cong"'] & H^n((\mathbb{M}^n, \mathbb{M}^n - x) \times \{\lambda^*\}) \arrow[d, "(\sigma^n)^{-1}", "\cong"'] \\
H^0(\Lambda_0) \arrow[r, "i_{\lambda^*}^*", "\cong"'] & H^0(\{\lambda^*\})
\end{tikzcd}$$

and arrive at

$$\left(i_{\lambda^*}^* \circ (\sigma^n)^{-1} \circ T_g \circ \sigma^n\right)(1_{\mathcal{U}}) = \left((\sigma^n)^{-1} \circ \mathrm{pr}^* \circ H_g \circ (\mathrm{pr}^*)^{-1} \circ \sigma^n\right)(i_{\lambda^*}^*(1_{\mathcal{U}})).$$

Yet, in view of (2.3), one obtains

$$t_{\mathcal{U}|\Lambda_1}(g)(1_{\mathcal{U}}) = \left(i_{\lambda^*}^* \circ (\sigma^n)^{-1} \circ T_g \circ \sigma^n\right)(1_{\mathcal{U}}).$$

On the other hand, (2.1) and (2.7) imply that

$$((\sigma^n)^{-1} \circ \mathrm{pr}^* \circ H_g \circ (\mathrm{pr}^*)^{-1} \circ \sigma^n)(1_{\lambda^*}) = \mathrm{DEG}(1_{\lambda^*}) = \deg(g_{\lambda^*}, \mathcal{U}_{\lambda^*}, 0) \cdot 1_{\lambda^*}$$

Furthermore, basic properties of the unit cohomology classes (collected in App. A.3) imply both $i_{\lambda^*}^*(1_{\mathcal{U}}) = 1_{\lambda^*}$ and $i_{\lambda^*}^*(1_{\Lambda_0}) = 1_{\lambda^*}$. Finally, taking into account the above considerations, we conclude

$$t_{\mathcal{U}}(g)(1_{\mathcal{U}}) = \deg(g_{\lambda^*}, \mathcal{U}_{\lambda^*}, 0) \cdot 1_{\Lambda_0}.$$

This completes the proof. □

3 Proof of Thm. 1.1

Before proving Thm. 1.1 we illustrate its assumptions by means of the elementary

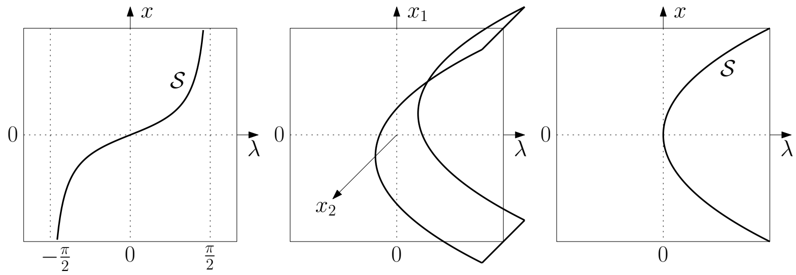

Fig. 4 Global branches not covering the parameter space $\Lambda = \mathbb{R}$ from Ex. 3.1(1) (left), (2) (center) and (3) (right)

Example 3.1 It is clear that Thm. 1.1 fails on a disconnected parameter space. Thus, we suppose $\Lambda = \mathbb{R}$ in the following, but throughout conclude that Λ is not covered by solutions:

(1) For $X = Y = \mathbb{R}$, $G(x, \lambda) := \arctan x - \lambda$ is Fredholm of index 0 for all λ, but we obtain an unbounded $G^{-1}([-\frac{\pi}{2}, \frac{\pi}{2}])$. Here, the properness assumption (A2) fails to hold (see Fig. 4 (left)).

(2) Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$ and consider $G(x, \lambda) := x_1^2 - \lambda$. Then the Fredholm index of $DG_\lambda(x)$ is equal to 1 for all $x \in \mathbb{R}^2$, i.e. (A3) is violated (cf. Fig. 4 (center)). For an example with the infinite-dimensional spaces $X = Y = c_0$ consisting of real sequences $(\phi_k)_{k\in\mathbb{N}}$ satisfying $\lim_{k\to\infty} \phi_k = 0$ (equipped with the sup-norm), consider the mapping $G\colon c_0 \times \Lambda \to c_0$, $G(\phi, \lambda) := (\phi_1^2 - \lambda, 0, \phi_2^2 - \frac{\lambda}{2}, 0, \phi_3^2 - \frac{\lambda}{3}, 0, \ldots)$, which is not Fredholm, due to the infinite-dimensional kernels of $DG_\lambda(x)$. Again, (A3) fails.

(3) If $X = Y = \mathbb{R}$, then $G(x, \lambda) := x^2 - \lambda$ is Fredholm of index 0, but $\deg(G_\lambda, O, 0) = 0$ vanishes for all λ and bounded, open $O \subseteq \mathbb{R}$. Hence, (A4) is not satisfied (see Fig. 4 (right)).

The proof of Thm. 1.1 needs several tools. We first abbreviate $\mathcal{S} := G^{-1}(0) \cap (O \times \Lambda)$ for the set of all solutions to (O_λ) in $O \times \Lambda$. The Benevieri-Furi degree of App. B.2 is denoted by $\deg(G_\lambda, O, 0)$.

Lemma 3.2 $\deg(G_\lambda, O, 0) \neq 0$ *for all* $\lambda \in \Lambda$.

Whence, property (bf1) from App. B.2 implies that (O_λ) has a solution for every $\lambda \in \Lambda$. It remains to show that all these solutions can be chosen from a continuum.

P r o o f. Fix $\lambda_1 \in \Lambda$. Since Λ is path-connected (connected ANRs are locally path-connected [18], thus as connected metric spaces, they are path-connected), it follows that there exists a continuous function $\sigma\colon [0, 1] \to \Lambda$ such that $\sigma(0) = \lambda^*$ and $\sigma(1) = \lambda_1$. Then the continuous mapping

$$H\colon O \times [0, 1] \to Y, \qquad H(x, t) := G(x, \sigma(t)) \text{ for all } x \in O, t \in [0, 1]$$

is an index 0 Fredholm homotopy. Since $H^{-1}(0) = G^{-1}(0) \cap (\overline{O} \times \sigma([0, 1]))$ is compact, we infer

$$0 \neq \deg(G_{\lambda^*}, O, 0) = \deg(H_0, O, 0) = \deg(H_1, O, 0) = \deg(G_{\lambda_1}, O, 0)$$

from the homotopy invariance property (bf2). □

Lemma 3.3 *The solution set $\mathcal{S}$ has a component $\mathcal{C}$ so that for every compact, connected subset $\Lambda_0 \subset \Lambda$ containing λ^*, every open neighborhood $\mathcal{U}$ of $\mathcal{C}|\Lambda_0$ in $O \times \Lambda_0$ satisfying $\mathcal{S}|\Lambda_0 \cap \partial\mathcal{U} = \emptyset$ contains a neighborhood $\mathcal{V}$ of $\mathcal{C}|\Lambda_0$ in $\mathcal{U}$ with $\mathcal{S}|\Lambda_0 \cap \partial\mathcal{V} = \emptyset$ and* $\deg(G_{\lambda^*}, \mathcal{V}_{\lambda^*}, 0) \neq 0$.

P r o o f. We argue by contradiction. That is, for any component $\mathcal{C}$ of $\mathcal{S}$ there exists a compact and connected subset $\Lambda_0 \subset \Lambda$ with $\lambda^* \in \Lambda_0$ and a neighborhood $\mathcal{U}_\mathcal{C}$ of $\mathcal{C}|\Lambda_0$ in $O \times \Lambda_0$ with $\mathcal{S}|\Lambda_0 \cap \partial\mathcal{U}_\mathcal{C} = \emptyset$ and such that any neighborhood $\mathcal{V}$ of $\mathcal{C}|\Lambda_0$ in $O \times \Lambda_0$ admits

$$\mathcal{V} \subset \mathcal{U}_\mathcal{C} \text{ and } \mathcal{S}_0 \cap \partial\mathcal{V} = \emptyset \quad \Longrightarrow \quad \deg(G_{\lambda^*}, \mathcal{V}_{\lambda^*}, 0) = 0. \tag{3.1}$$

The compactness of $G_{\lambda^*}^{-1}(0) \cap O$ implies that there exist finitely many components $\mathcal{C}_1, \ldots, \mathcal{C}_r$ of $\mathcal{S}$ such that the family of sets $U_i := \{x \in O \mid (x, \lambda^*) \in \mathcal{U}_{\mathcal{C}_i}\}$, $i = 1, \ldots, r$, is covering $G_{\lambda^*}^{-1}(0) \cap O$. If $C_{\lambda^*}^i := \{x \in O \mid (x, \lambda^*) \in \mathcal{C}_i\}$, then $C_{\lambda^*}^i \subset U_i$, $i = 1, \ldots, r$. Let us observe that (3.1) implies

$$\deg(G_{\lambda^*}, U_i, 0) = 0 \text{ for all } i = 1, \ldots, r.$$

Now we show $\deg(G_{\lambda^*}, U_1 \cup U_2, 0) = 0$ by distinguishing two cases:
Case 1: $\mathcal{U}_{\mathcal{C}_1} \cap \mathcal{U}_{\mathcal{C}_2} = \emptyset$: Then $U_1 \cap U_2 = \emptyset$ and hence

$$\deg(G_{\lambda^*}, U_1 \cup U_2, 0) = \deg(G_{\lambda^*}, U_1, 0) + \deg(G_{\lambda^*}, U_2, 0) = 0 + 0 = 0.$$

Case 2: $\mathcal{U}_{\mathcal{C}_1} \cap \mathcal{U}_{\mathcal{C}_2} \neq \emptyset$. Consider the following decomposition:

$$\mathcal{U}_{\mathcal{C}_1} \cup \mathcal{U}_{\mathcal{C}_2} = (\mathcal{U}_{\mathcal{C}_1} - \mathcal{U}_{\mathcal{C}_2}) \cup (\mathcal{U}_{\mathcal{C}_1} \cap \mathcal{U}_{\mathcal{C}_2}) \cup (\mathcal{U}_{\mathcal{C}_2} - \mathcal{U}_{\mathcal{C}_1}).$$

Since $\partial(\mathcal{U}_{\mathcal{C}_1} - \mathcal{U}_{\mathcal{C}_2}) \cup \partial(\mathcal{U}_{\mathcal{C}_2} - \mathcal{U}_{\mathcal{C}_1}) \cup \partial(\mathcal{U}_{\mathcal{C}_1} \cap \mathcal{U}_{\mathcal{C}_2}) \subseteq \partial\mathcal{U}_{\mathcal{C}_1} \cup \partial\mathcal{U}_{\mathcal{C}_2}$ and $\mathcal{S} \cap (\partial\mathcal{U}_{\mathcal{C}_1} \cup \partial\mathcal{U}_{\mathcal{C}_2}) = \emptyset$, it follows that $\mathcal{C}_1 \subset \mathcal{U}_{\mathcal{C}_1} - \mathcal{U}_{\mathcal{C}_2}$ or $\mathcal{C}_1 \subset \mathcal{U}_{\mathcal{C}_1} \cap \mathcal{U}_{\mathcal{C}_2}$. Observe that if $\mathcal{C}_1 \subset \mathcal{U}_{\mathcal{C}_1} - \mathcal{U}_{\mathcal{C}_2}$, then $\mathcal{C}_{\lambda^*}^1 \subset U_1 - U_2$. Thus the set $U_1 - U_2$ is a neighborhood of $\mathcal{C}_{\lambda^*}^1$ and consequently, by (3.1), we deduce that $\deg(G_{\lambda^*}, U_1 - U_2, 0) = 0$. Hence the additivity property (bf3) of the degree implies

$$0 = \deg(G_{\lambda^*}, U_1, 0) \overset{\text{(bf3)}}{=} \deg(G_{\lambda^*}, U_1 - U_2, 0) + \deg(G_{\lambda^*}, U_1 \cap U_2, 0)$$

and therefore $\deg(G_{\lambda^*}, U_1 \cap U_2, 0) = 0$. If $\mathcal{C}_1 \subset \mathcal{U}_{\mathcal{C}_1} \cap \mathcal{U}_{\mathcal{C}_2}$, then reasoning similarly as above we first conclude $\deg(G_{\lambda^*}, U_1 \cap U_2, 0) = 0$ and then that $\deg(G_{\lambda^*}, U_1 - U_2, 0) = 0$. What is more,

$$0 = \deg(G_{\lambda^*}, U_2, 0) \overset{\text{(bf3)}}{=} \deg(G_{\lambda^*}, U_2 - U_1, 0) + \deg(G_{\lambda^*}, U_1 \cap U_2, 0) \quad = \deg(G_{\lambda^*}, U_2 - U_1, 0) + 0,$$

which leads to $\deg(G_{\lambda^*}, U_2 - U_1, 0) = 0$. Consequently, we obtain

$$\deg(G_{\lambda^*}, U_1 \cup U_2, 0) \overset{\text{(bf3)}}{=} \deg(G_{\lambda^*}, U_1 - U_2, 0) + \deg(G_{\lambda^*}, U_1 \cap U_2, 0) + \deg(G_{\lambda^*}, U_2 - U_1, 0) = 0.$$

Note that the above considerations actually imply

$$\deg(G_{\lambda^*}, U_i \cup U_j, 0) = 0, \ i \neq j, \ 1 \leqslant i, j \leqslant r,$$

which is more general. Now assume by induction that for any sequence $1 \leqslant i_1 < i_2 < \ldots < i_s \leqslant r$ one has the relation

$$\deg(G_{\lambda^*}, \bigcup_{k=1}^{s} U_{i_k}, 0) = 0$$

and we will prove

$$\deg(G_{\lambda^*}, \bigcup_{k=1}^{s+1} U_{i_k}, 0) = 0.$$

Let $\bigcup_{k=1}^{s+1} U_{\mathcal{C}_{i_k}} = U_{\mathcal{C}_{i_1}} \cup U_{\widetilde{C}}$, where $U_{\widetilde{C}} := \bigcup_{k=2}^{s+1} U_{\mathcal{C}_{i_k}}$. From the induction step we know that

$$\deg(G_{\lambda^*}, \bigcup_{k=2}^{s+1} U_{i_k}, 0) = 0$$

holds and two cases arise: Either $U_{\mathcal{C}_{i_1}} \cap U_{\widetilde{C}} = \emptyset$ or $U_{\mathcal{C}_{i_1}} \cap U_{\widetilde{C}} \neq \emptyset$. Reasoning as in the case of the sets $\mathcal{U}_{\mathcal{C}_1}$ and $\mathcal{U}_{\mathcal{C}_2}$ we deduce that

$$\deg(G_{\lambda^*}, \bigcup_{k=1}^{s+1} U_{i_k}, 0) = 0.$$

Hence, taking $s := r$ and the excision property (bf3) of the topological degree yields

$$\deg(G_{\lambda^*}, O, 0) = \deg(G_{\lambda^*}, \bigcup_{i=1}^{r} U_i, 0) = 0,$$

which implies a contradiction, and Lemma 3.3 is shown. □

Lemma 3.4 (Whyburn's lemma, [4]) *Let P, Q be disjoint closed subsets of a compact metric space M. If there is no closed, connected subset of M that intersects both P and Q, then there exist disjoint closed subsets K_P and K_Q of M such that $P \subset K_P$, $Q \subset K_Q$ and $M = K_P \cup K_Q$.*

Lemma 3.5 *If $\mathcal{C} \subset O \times \Lambda$ is as in Lemma 3.3, then for any compact, connected set $\Lambda_0 \subset \Lambda$ containing λ^*, there exists a (descending) family $\{\mathcal{V}^\gamma\}_{\gamma\in\Gamma}$ of neighborhoods (in $O\times\Lambda_0$) of $\mathcal{C}|\Lambda_0$ with $\mathcal{S}|\Lambda_0\cap\partial\mathcal{V}^\gamma = \emptyset$, $\deg(G_{\lambda^*}, \mathcal{V}^\gamma_{\lambda^*}, 0) \neq 0$ and $\bigcap_{\gamma\in\Gamma} \mathcal{V}^\gamma = \mathcal{C}|\Lambda_0$.*

P r o o f. Fix a compact and connected subset $\Lambda_0 \subset \Lambda$ with $\lambda^* \in \Lambda_0$. We proceed in two steps and mimic the notation of Lemma 3.4.

(I) (special case). We assume $\mathcal{S} = \mathcal{C}$. Take any neighborhood $\mathcal{U}$ of $\mathcal{C}|\Lambda_0$ in $O \times \Lambda_0$. It is clear that $\partial\mathcal{U} \cap \mathcal{S}|\Lambda_0 = \emptyset$. Then from Lemma 3.3 it follows that there exists a neighborhood $\mathcal{V}$ of $\mathcal{C}|\Lambda_0$ in $O \times \Lambda_0$ contained in $\mathcal{U}$ such that

$$\deg(G_{\lambda^*}, \mathcal{V}_{\lambda^*}, 0) \neq 0 \text{ and } \partial\mathcal{V} \cap \mathcal{S}|\Lambda_0 = \emptyset.$$

For any $\varepsilon > 0$ we set $\mathcal{V}^\varepsilon := \mathcal{V} \cap O_\varepsilon(\mathcal{C}|\Lambda_0)$, where $O_\varepsilon(\mathcal{C}|\Lambda_0)$ is the ε-neighborhood of $\mathcal{C}|\Lambda_0$ in $O \times \Lambda_0$. Since $\mathcal{S}|\Lambda_0 = \mathcal{C}|\Lambda_0$ and $\partial\mathcal{V} \cap \mathcal{S}|\Lambda_0 = \emptyset$, we deduce that $\partial\mathcal{V}^\varepsilon \cap \mathcal{S}|\Lambda_0 = \emptyset$ (we take the boundary $\partial\mathcal{V}^\varepsilon$ in $O \times \Lambda_0$). Moreover, the excision property (bf3) of the degree implies

$$0 \neq \deg(G_{\lambda^*}, \mathcal{V}_{\lambda^*}, 0) = \deg(G_{\lambda^*}, \mathcal{V}^\varepsilon_{\lambda^*}, 0).$$

Then $\bigcap_{\varepsilon>0} \mathcal{V}^\varepsilon = C|\Lambda_0$ establishes the special case.

(II) (general case). Let us assume that $\mathcal{S}$ consists at least of two connected components. Let $\mathcal{S} = (\bigcup_{\alpha\in\Gamma} \mathcal{C}^\alpha) \cup \mathcal{C}$, where $\mathcal{C}^\alpha$ are the connected components of $\mathcal{S}$ and $\mathcal{C}$ is as above. Choose a finite subset $\Gamma_\alpha \subset \Gamma$ and define

$$M := \mathcal{S}|\Lambda_0, \qquad P_{\Gamma_\alpha} := \bigcup_{\beta\in\Gamma_\alpha} \mathcal{C}^\beta_0, \qquad Q_{\Gamma_\alpha} := \mathcal{C}|\Lambda_0.$$

Any $\mathcal{C}^\beta$ cannot be connected with $\mathcal{C}$ in the space $\mathcal{S}$, i.e., there is no closed, connected subset of $\mathcal{S}$ intersecting both $\mathcal{C}^\beta$ and $\mathcal{C}$. This follows from the fact that $\mathcal{C}^\beta$ and $\mathcal{C}$ are connected components of M. Thus, we can conclude that no connected subset of M intersects both P and Q. Whyburn's Lemma 3.4 shows that there exist two disjoint closed subsets $K_{P_{\Gamma_\alpha}}$ and $K_{Q_{\Gamma_\alpha}}$ of M fulfilling

$$P_{\Gamma_\alpha} \subset K_{P_{\Gamma_\alpha}}, \qquad Q_{\Gamma_\alpha} \subset K_{Q_{\Gamma_\alpha}}, \qquad M = K_{P_{\Gamma_\alpha}} \cup K_{Q_{\Gamma_\alpha}}.$$

Observe that $K_{P_{\Gamma_\alpha}}$ and $K_{Q_{\Gamma_\alpha}}$ are compact. Hence, there exists $\varepsilon_{\Gamma_\alpha} > 0$ with

$$O_\varepsilon\left(K_{P_{\Gamma_\alpha}}\right) \cap O_\varepsilon\left(K_{Q_{\Gamma_\alpha}}\right) = \emptyset \text{ for all } \varepsilon \leqslant \varepsilon_{\Gamma_\alpha} \tag{3.2}$$

and ε-neighborhoods taken w.r.t. $O \times \Lambda_0$. Hence, Lemma 3.3 implies for any $\mathcal{U}^{\Gamma_\alpha,\varepsilon} := O_\varepsilon(K_{Q_{\Gamma_\alpha}})$, $0 < \varepsilon \leqslant \varepsilon_{\Gamma_\alpha}$ there exists a neighborhood $\mathcal{V}^{\Gamma_\alpha,\varepsilon}$ of $\mathcal{C}|\Lambda_0$ contained in $\mathcal{U}^{\Gamma_\alpha,\varepsilon}$ satisfying

$$\deg(G_{\lambda^*}, \mathcal{V}^{\Gamma_\alpha,\varepsilon}_{\lambda^*}, 0) \neq 0, \qquad \partial\mathcal{V}^{\Gamma_\alpha,\varepsilon} \cap \mathcal{S}|\Lambda_0 = \emptyset$$

(notice that $\partial\mathcal{U}^{\Gamma_\alpha,\varepsilon} \cap \mathcal{S}|\Lambda_0 = \emptyset$ because of $\mathcal{S}|\Lambda_0 = K_{P_{\Gamma_\alpha}} \cup K_{Q_{\Gamma_\alpha}}$ and (3.2)), whence

$$\mathcal{C}|\Lambda_0 \subset \bigcap_{\varepsilon\leqslant\varepsilon_{\Gamma_\alpha}} \mathcal{V}^{\Gamma_\alpha,\varepsilon} \subset \bigcap_{\varepsilon\leqslant\varepsilon_{\Gamma_\alpha}} \mathcal{U}^{\Gamma_\alpha,\varepsilon} = K_{Q_{\Gamma_\alpha}},$$

$$\bigcap_{\Gamma_\alpha\subset\Gamma} K_{Q_{\Gamma_\alpha}} = \mathcal{C}|\Lambda_0 \text{ (since } \bigcup_{\Gamma_\alpha\subset\Gamma} K_{P_{\Gamma_\alpha}} = \mathcal{S}|\Lambda_0 - \mathcal{C}|\Lambda_0).$$

A family $\{\mathcal{V}^\gamma\}_{\gamma\in\Gamma}$ satisfies the required property with

$$\Gamma = \{\gamma = (\Gamma_\alpha, \varepsilon) \mid I_\alpha \subset \Gamma \text{ is finite}, \varepsilon \leqslant \varepsilon_{\Gamma_\alpha}\}.$$

This completes the proof. □

Note that Lemma 3.2, 3.3, 3.5 extend to parameterized maps between an oriented n-manifolds. The only difference in the proofs is that one employs topological degree for maps between oriented n-manifolds instead of the topological degree for Fredholm maps in Banach spaces.

Lemma 3.6 *The solution set $\mathcal{S}$ has a component $\mathcal{C}$ such that for each compact, connected ANR $\Lambda_0 \subset \Lambda$, there exists an injective homomorphism $\pi^*\colon \check{H}^*(\Lambda_0) \to \check{H}^*(C|\Lambda_0)$ induced by the restriction $\pi\colon \mathcal{C}|\Lambda_0 \to \Lambda_0$.*

Proof. Fix a compact, connected ANR $\Lambda_0 \neq \emptyset$, $\lambda^* \in \Lambda_0$ and let $\mathcal{C}$ be a component of $\mathcal{S}$ as in Lemma 3.3. Since $G|_{\overline{O}\times\Lambda_0}$ is proper, we infer that $\mathcal{C}|\Lambda_0$ is compact. Furthermore, Lemma 3.5 implies that there exists a family $\{\mathcal{V}^\gamma\}_{\gamma\in\Gamma}$ of neighborhoods of $\mathcal{C}|\Lambda_0$ in $O \times \Lambda_0$ such that

$$\mathcal{S}|\Lambda_0 \cap \partial\mathcal{V}^\gamma = \emptyset, \qquad \deg(G_{\lambda^*}, \mathcal{V}^\gamma_{\lambda^*}, 0) \neq 0, \qquad \bigcap_{\gamma\in\Gamma} \mathcal{V}^\gamma = \mathcal{C}|\Lambda_0.$$

Then the tautness property of the Alexander-Spanier cohomology (see (A.1)) implies that

$$\check{H}^*(\mathcal{C}|\Lambda_0) = \varinjlim\{H^q(\mathcal{V}^\gamma)\}. \tag{3.3}$$

Moreover, consider the family of homomorphisms $\{\pi^*_{\mathcal{V}^\gamma}\colon H^*(\Lambda_0) \to H^*(\mathcal{V}^\gamma) \mid \gamma \in \Gamma\}$, which forms a direct system of injective homomorphisms by Prop. 2.10 and Lemma 2.11. Hence, we obtain the direct limit

$$\varinjlim\{\pi^*_{\mathcal{V}^\gamma}\}\colon \check{H}^*(\Lambda_0) \to \check{H}^*(\mathcal{C}|\Lambda_0)$$

that by Rem. A.3 (see also (3.3)) coincides with the homomorphism $\pi^*\colon \check{H}^*(\Lambda_0) \to \check{H}^*(\mathcal{C}|\Lambda_0)$. Finally, since the direct limits preserve monomorphisms (see Cor. A.2), we obtain that $\varinjlim\{\pi^*_{\mathcal{V}^\gamma}\}$ is a monomorphism, which in turn implies that π^* is also injective. This completes the proof. □

Proof of Thm. 1.1. For $\mathcal{C} \subseteq \mathcal{S}$ as in Lemma 3.3, we prove $\mathcal{C}_\lambda \neq \emptyset$ for all $\lambda \in \Lambda$. It suffices to show that a homomorphism (induced by π) $\pi_0^*\colon \mathbb{Q} = \check{H}^0(\Lambda_0) \to \check{H}^0(\mathcal{C}|\Lambda_0)$ is nontrivial (notice that in the proof we take $\Lambda_0 \subset \Lambda$ to be $\Lambda_0 := \{\lambda\}$). For this purpose we consider the restriction $G_\lambda\colon O \to Y$ of $G\colon O \times \Lambda \to Y$ does not depend on the parameter $\lambda \in \Lambda$. Since G_λ is proper, $G_\lambda^{-1}(0)$ is compact and thus there exists a finite-dimensional subspace $Y_0 \subset Y$ and an open subset $\widetilde{O} \subseteq O$ of $G_\lambda^{-1}(0)$ in which G_λ is transversal to Y_0. Consequently, $M_\lambda := G_\lambda^{-1}(Y_0) \cap \widetilde{O}$ is a differentiable oriented manifold of dimension $\dim Y_0$. Using $G_\lambda^{-1}(0) \subset M_\lambda$ we deduce $\mathcal{C}|\Lambda_0 = \mathcal{C} \cap \pi^{-1}(\lambda) \subset M_\lambda$. Hence, it suffices to consider the map $G_\lambda|_{M_\lambda}\colon M_\lambda \to Y_0$. Using Lemma 3.5 it follows that there exists a family $\{\mathcal{V}^\gamma\}_{\gamma\in\Gamma}$ of neighborhoods of $\mathcal{C}|\Lambda_0$ (in $O \times \Lambda_0$) with

$$\mathcal{S}|\Lambda_0 \cap \partial\mathcal{V}^\gamma = \emptyset, \qquad \deg(G_\lambda, \mathcal{V}^\gamma_\lambda, 0) \neq 0, \qquad \bigcap_{\gamma\in\Gamma} \mathcal{V}^\gamma = \mathcal{C}|\Lambda_0.$$

One can assume w.l.o.g. that $\mathcal{V}^\gamma_\lambda \subset \widetilde{O}$ for all $\gamma \in \Gamma$ (because $\mathcal{V}^\gamma_\lambda$ is a descending family). Whence, in view of (B.2) one has $0 \neq \deg(G_\lambda, \mathcal{V}^\gamma_\lambda, 0) = \deg_B(G_\lambda|M_\lambda, \mathcal{V}^\gamma_\lambda \cap M_\lambda, 0)$, where the right-hand side denotes the C^1-Brouwer degree for manifolds (see [27, Chap. 9]). On the other hand, the uniqueness of the topological degree for maps defined on C^1-manifolds (see Rem. B.1) implies $\deg_B(G_\lambda|_{M_\lambda}, \mathcal{V}^\gamma_\lambda \cap M_\lambda, 0) = \deg_0(G_\lambda|_{M_\lambda}, \mathcal{V}^\gamma_\lambda \cap M_\lambda, Y_0)$, where the right-hand side denotes the homological degree. Therefore, Lemma 2.11 for manifolds implies that the homomorphism

$$(\pi|_{\mathcal{V}^\gamma_\lambda \cap M_\lambda})^*_0\colon H^0(\Lambda_0) \to H^0(\mathcal{V}^\gamma_\lambda \cap M_\lambda)$$

is injective. Finally, taking into account $\bigcap_{\gamma\in\Gamma}(\mathcal{V}^\gamma_\lambda \cap M_\lambda) = \mathcal{C}|\Lambda_0$ and Lemma 3.6 we deduce that the mapping $\pi_0^*\colon \check{H}^0(\Lambda_0) \to \check{H}^0(\mathcal{C}|\Lambda_0)$ is injective and hence nontrivial. Thus, $\check{H}^0(\mathcal{C}|\Lambda_0) \neq 0$, which in turn implies that $\mathcal{C}|\Lambda_0 \neq \emptyset$, concluding the present proof. □

Remark 3.7 Notice that Thm. 1.1 admits also a relative version as in [1, Thm. 2.1]. But the relative version in the case of all Fredholm maps requires some new techniques from the transfer homomorphism theory and is therefore left for future work.

Appendices

A Topological tools

A.1 Direct systems of abelian groups and their limits

A group is understood as additive abelian group throughout. Let $(\mathcal{D}, \leqslant)$ be a directed set and $\{G_\alpha\}_{\alpha\in\mathcal{D}}$ a family of groups. Given homomorphisms $g_{\alpha,\beta}\colon G_\alpha \to G_\beta$ with $\alpha \leqslant \beta$ in $\mathcal{D}$ such that $g_{\alpha,\alpha} = \mathrm{id}$ and $g_{\alpha,\gamma} = g_{\beta,\gamma} \circ g_{\alpha,\beta}$ for $\alpha \leqslant \beta \leqslant \gamma$, the family $\{G_\alpha\} = \{G_\alpha, g_{\alpha,\beta} \mid \alpha, \beta \in \mathcal{D}\}$ is called *direct system* of groups over $\mathcal{D}$.

The *direct limit group* $G^\infty = \varinjlim\{G_\alpha, g_{\alpha,\beta}\}$ is the quotient group of the direct sum of groups $G^\infty = \oplus_{\alpha\in\mathcal{D}} G_\alpha/\sim$ modulo the equivalence relation $\sim$ defined as follows: For $g_\alpha \in G_\alpha, g_\beta \in G_\beta$ set

$$g_\alpha \sim g_\beta \quad :\Longleftrightarrow \quad g_{\alpha,\gamma}(g_\alpha) = g_{\beta,\gamma}(g_\beta) \text{ for some } \gamma \in \mathcal{D} \text{ with } \alpha \leqslant \gamma \text{ and } \beta \leqslant \gamma.$$

In other words, $g_{\alpha,\beta}(g) \sim g$ for all $g \in G_\alpha$ and for all $\beta \in \mathcal{D}$ with $\alpha \leqslant \beta$.

The inclusions $G_\alpha \hookrightarrow \bigoplus_{\alpha\in\mathcal{D}} G_\alpha$ induce homomorphisms $i_\alpha\colon G_\alpha \to G^\infty$ with $i_\beta \circ g_{\alpha,\beta} = i_\alpha$, $\alpha \leqslant \beta$. We next explain how to use the algebraic notion of the limit of homomorphisms:

Proposition A.1 ([3, pp. 534–536]) *Let $\{G'_\alpha\}$, $\{G_\alpha\}$ and $\{G''_\alpha\}$ be direct system over the same directed set $\mathcal{D}$. If for each $\alpha \in \mathcal{D}$ there is an exact sequence $G'_\alpha \xrightarrow{f'_\alpha} G_\alpha \xrightarrow{f_\alpha} G''_\alpha$ such that*

$$\begin{tikzcd} G'_\alpha \arrow[r, "f'_\alpha"] \arrow[d, "g'_{\alpha,\beta}"'] & G_\alpha \arrow[r, "f_\alpha"] \arrow[d, "g_{\alpha,\beta}"] & G''_\alpha \arrow[d, "g''_{\alpha,\beta}"] \\ G'_\beta \arrow[r, "f'_\beta"] & G_\beta \arrow[r, "f_\beta"] & G''_\beta \end{tikzcd}$$

commutes, then there exist two unique homomorphisms

$$\varinjlim\{f_\alpha\}\colon (G^\infty, \{i_\alpha\}) \to (G''^\infty, \{i''_\alpha\}) \text{ and } \varinjlim\{f'_\alpha\}\colon (G'^\infty, \{i'_\alpha\}) \to (G^\infty, \{i_\alpha\})$$

such that $i''_\alpha \circ f_\alpha = (\varinjlim\{f_\alpha\}) \circ i_\alpha$, $i_\alpha \circ f_\alpha = (\varinjlim\{f'_\alpha\}) \circ i'_\alpha$ for all $\alpha \in \mathcal{D}$ and the induced sequence

$$\varinjlim\{G'_\alpha\} \xrightarrow{\varinjlim\{f'_\alpha\}} \varinjlim\{G_\alpha\} \xrightarrow{\varinjlim\{f_\alpha\}} \varinjlim\{G''_\alpha\} \text{ is exact.}$$

Corollary A.2 *If f_α is a monomorphism (or epimorphism) for every $\alpha \in \mathcal{D}$, then $\varinjlim\{f_\alpha\}$ is also a monomorphism (resp. epimorphism).*

A.2 Tautness of the cohomology

Let us assume that $A \subseteq X$ is a closed subset of a metric space X. By a *neighborhood* of A in X we shall understand an open subset U of X containing A in its interior. In this way we obtain the family $\{U_\alpha\}$ of neighborhoods of A in X. For inclusions of the form $i_{\alpha\beta}\colon U_\alpha \hookrightarrow U_\beta$ (if $U_\alpha \subset U_\beta$) and $j_\alpha\colon A \hookrightarrow U_\alpha$ we consider the following homomorphisms $i^*_{\alpha\beta}\colon \check{H}^q(U_\beta) \to \check{H}^q(U_\alpha)$ and $j^*_\alpha\colon \check{H}^q(U_\alpha) \to \check{H}^q(A)$, respectively. Then $\{\check{H}^q(U_\alpha), i^*_{\alpha\beta}\}$ is a direct system of abelian groups induced by the system $\{U_\alpha, i_{\alpha\beta}\}$. We say A is a *taut subspace* of X, if $\check{H}^q(A)$ is the direct limit of the direct system $\{\check{H}^q(U_\alpha), i^*_{\alpha\beta}\}$ for every $q \in \mathbb{N}_0$ (w.r.t. the homomorphisms j_α). This fact is denoted by the following symbol:

$$\varinjlim\{\check{H}^q(U_\alpha)\} = \check{H}^q(A) \text{ for all } q \in \mathbb{N}_0. \tag{A.1}$$

It is known that $X_0 \subset X$ is taut in X w.r.t. the Alexander-Spanier cohomology (see [16, p. 238]), provided X_0 is a compact subset of X or X_0 is a retract of some open subset of X. Since Alexander-Spanier cohomology and singular cohomology are functorially isomorphic ([16, p. 262] and [26, Chap. 6]) on the category of ANRs (see [8, Chap. IV]), it follows that tautness can be expressed using singular cohomology.

Remark A.3 Let $q \in \mathbb{N}_0$. If $X_0 \subset X$, $X'_0 \subset X'$ are compact subsets of ANRs X, Y, then

$$\varinjlim\{H^q(U_\alpha)\} = \check{H}^q(X_0)$$

and if $f\colon (X, X_0) \to (X', X'_0)$, then

$$\varinjlim\{f^*\colon H^q(U'_\alpha) \to H^q(U_\alpha)\} = f^*\colon \check{H}^q(X'_0) \to \check{H}^q(X_0)$$

holds.

A.3 Cup products

We continue with results on the cohomology functor needed throughout the text. First, we recall the notion of the cup product for singular cohomology H^* and for Alexander-Spanier cohomology $\check{H}^*$ (based on all cochains), respectively.

For subspaces $X_1, X_2 \subseteq X$ the triple $(X; X_1, X_2)$ is called a *triad*. It is denoted as *excisive* (for a given cohomology theory) if both the inclusions $(X_1, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_2)$ and $(X_2, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_1)$ induce isomorphisms of cohomology groups in all dimensions. A triad $(X; X_1, X_2)$ is excisive, if

$$X_1 \cup X_2 = X_1^\circ \cup X_2^\circ, \tag{A.2}$$

with the interior taken w.r.t. $X_1 \cup X_2$. This holds (see [16]) when X_1 and X_2 are open in X, or when $X_1 = \emptyset$ or $X_2 = \emptyset$. If a triad $(X; X_1, X_2)$ is excisive, then there is an internal product

$$\smile\colon H^p(X, X_1) \otimes H^q(X, X_2) \to H^{p+q}(X, X_1 \cup X_2). \tag{A.3}$$

The construction of the cup product (A.3) for Alexander-Spanier cohomology under the assumption that $(X; X_1, X_2)$ satisfies (A.2) can be found in [16]. A homomorphism $\varepsilon\colon C_0(X) \to \mathbb{Z}$ (resp. $\varepsilon\colon C^0(X) \to \mathbb{Z}$) sending singular 0-simplices σ (resp. any point $x \in X$) to $1 \in \mathbb{Z}$ may be considered as 0-cochain in singular cohomology theory (resp. in Alexander-Spanier cohomology). We write $1_X \in H^0(X)$ (resp. $1_X \in \check{H}^0(X)$) for its cohomology class and call it *unit cohomology class*.

Proposition A.4 (properties of the cup product, [12, 16]) *The cup product* (A.3) *satisfies:*

(a) *$\smile$ is natural, i.e., if $f\colon X \to Y$, then $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$,*

(b) *for any $\alpha \in H^*(X, A)$, $\alpha \smile 1_X = \alpha = 1_X \smile \alpha$,*

(c) *$\alpha \smile (\beta \smile \gamma) = (\alpha \smile \beta) \smile \gamma$.*

It is shown in [6, p. 288] that the cup product can also be defined for triads $(X; X_1, X_2)$ whenever $X_1, X_1 \subset X$ are locally compact subspaces of some Euclidean neighborhood retract (see [7, p. 81]). In particular, if X, X_1, X_2 are closed subspaces, [6, p. 288] provides a construction for the Čech cohomology theory which is in turn true for the Alexander-Spanier cohomology theory because both cohomology theories are functorially isomorphic on the category of paracompact spaces (see [26]).

A.4 Cross product and Künneth theorem for singular cohomology

Given subsets $A \subseteq X$, $B \subseteq Y$ such that the triad $(X \times Y; A \times Y, X \times B)$ is excisive, define

$$\times\colon H^p(X, A) \otimes H^q(Y, B) \to H^{p+q}((X, A) \times (Y, B)), \qquad \xi_1 \times \xi_2 := \mathrm{pr}_1^*(\xi_1) \smile \mathrm{pr}_2^*(\xi_2)$$

as external product, where pr_1 (resp. pr_2) is the projection of $X \times Y$ onto X (resp. Y).

The cohomological cross product admits the following properties for $p, p_i \in \mathbb{N}_0$:

- If $\mathrm{pr}_1\colon (X, A) \times Y \to (X, A)$, $\mathrm{pr}_2\colon X \times (Y, B) \to (Y, B)$ are corresponding projections, then
$$u \times 1_Y = \mathrm{pr}_1^*(u) \ \text{ for } \ u \in H^p(X, A) \ \text{ and } \ 1_X \times v = \mathrm{pr}_2^*(v) \ \text{ for } \ v \in H^p(Y, B), \tag{A.4}$$
- $(\xi_1 \times \xi_2) \times \xi_3 = \xi_1 \times (\xi_2 \times \xi_3)$ for $\xi_i \in H^{p_i}(X_i, A_i)$ with open $A_i \subset X_i$, $i = 1, 2, 3$,
- if $\Delta\colon (X, A_1 \cup A_2) \to (X, A_1) \times (X, A_2)$ is the diagonal map, then $\Delta^*(\xi \times \eta) = \xi \smile \eta$ holds for $\xi \in H^{p_1}(X, A_1)$ and $\eta \in H^{p_2}(X, A_2)$ with open $A_i \subset X$.

Let us state the following result for singular cohomology:

Theorem A.5 (Künneth theorem, [17, p. 197]) *Let $A \subset X$, $B \subset Y$ and $(X \times Y; A \times Y, X \times B)$ be an excisive triad. If $H_p(X, A)$ is finitely generated for all $p \in \mathbb{N}_0$, then the external cross product*

$$\times\colon \bigoplus_{p+q=n} H^p(X, A) \otimes H^q(Y, B) \to H^n((X, A) \times (Y, B))$$

defines an isomorphism.

Furthermore, the following diagram

$$\begin{tikzcd} H^p(X,A)\otimes H^q(Y,B) \arrow[r, "\times"] \arrow[d, "f^*\otimes g^*"'] & H^n((X,A)\times(Y,B)) \arrow[d, "(f\times g)^*"] \\ H^p(X',A')\otimes H^q(Y',B') \arrow[r, "\times"] & H^n((X',A')\times(Y',B')) \end{tikzcd}$$

is commutative, where $f\colon (X,A)\to(X',A')$ and $g\colon (Y,B)\to(Y',B')$.

B Topological degrees

We next discuss two topological degrees. The first one applies to mappings between differentiable n-manifolds $\mathbb{M}^n$ and coincides with the classical Brouwer degree in case $\mathbb{M}^n=\mathbb{R}^n$. Some properties playing a crucial role in the proof of Thm. 1.1 are shown. The second topological tool under the name of Benevieri-Furi degree is advantageous when dealing with differential equations. We concentrate on the main properties and provide facts helping us to calculate the Benevieri-Furi degree of some operators appearing in applications.

B.1 Homological degree between oriented manifolds

We are using singular homology. Let $f\colon U\to\mathbb{E}^n$, where U is an open subset of an oriented n-manifold $(\mathbb{M}^n,\mu)$, K is a compact and connected subset of an n-dimensional oriented normed space $(\mathbb{E}^n,\widehat{\mu})$, and $f^{-1}(K)$ is compact. Note that $f_*\colon H_n(U,U-f^{-1}(K))\to H_n(\mathbb{E}^n,\mathbb{E}^n-K)$ takes the fundamental class $\mu_{f^{-1}(K)}$ into the multiple of $\widehat{\mu}_K$.

Following [6, Chap. VIII] we introduce the *homological degree* $\deg_K(f,U,\mathbb{E}^n)$ of f over K as an integer satisfying the equality:

$$f_*(\mu_{f^{-1}(K)})=\deg_K(f,U,\mathbb{E}^n)\widehat{\mu}_K, \tag{B.1}$$

where $f_*\colon H_n(U,U-f^{-1}(K))\to H_n(\mathbb{E}^n,\mathbb{E}^n-K)$, and abbreviate $\deg_0:=\deg_{\{0\}}$.

The properties of the above defined degree important for us, read as follows:

(h0) (**normalization**) If f is a homeomorphism preserving (reversing) the orientations of $\mathbb{M}^n$ and $\mathbb{E}^n$, then it holds

$$\deg_0(f,\mathbb{M}^n,\mathbb{E}^n)=1 \text{ (resp. } \deg_0(f,\mathbb{M}^n,\mathbb{E}^n)=-1),$$

(h1) (**existence**) if $\deg_K(f,U,\mathbb{E}^n)\neq 0$, then $f^{-1}(K)\neq\emptyset$,

(h2) (**excision**) $\deg_K(f,U,\mathbb{E}^n)=\deg_K(f,\widetilde{U},\mathbb{E}^n)$, if $\widetilde{U}\subset U$ is open and $f^{-1}(K)\subset\widetilde{U}$,

(h3) (**homotopy invariance**) if $h\colon U\times[0,1]\to\mathbb{E}^n$ is a deformation and $K\subset\mathbb{E}^n$ is a compact, connected set such that $h^{-1}(K)$ is compact, then

$$\deg_K(h_0,U,\mathbb{E}^n)=\deg_K(h_1,U,\mathbb{E}^n),$$

(h4) (**additivity**) if U is a finite union of open sets U_i, $i=1,\dots,r$, such that $K_i':=f^{-1}(K)\cap U_i$ are mutually disjoint, then

$$\deg_K(f,U,\mathbb{E}^n)=\sum_{i=1}^r\deg_K(f|U_i,U_i,\mathbb{E}^n),$$

(h5) if $K'\subset f^{-1}(K)$ is compact, then $f_*\colon H_n(U,U-K')\to H_n(\mathbb{E}^n,\mathbb{E}^n-K)$ takes the fundamental class $\mu_{K'}$ into $\deg_K(f,U,\mathbb{E}^n)\widehat{\mu}_K$.

Remark B.1 Let $K=\{0\}$.

(1) If $\mathbb{M}^n=\mathbb{E}^n=\mathbb{R}^n$ and $f\colon U\to\mathbb{R}^n$, then $\deg_K(f,U,\mathbb{R}^n)$ reduces to the classical homological topological degree $\deg(f,U,0)$ (cf. [6, Chap. VIII]).

(2) If $(\mathbb{M}^n,\mu)$ and $(\mathbb{E}^n,\widehat{\mu})$ are of class C^1 and $f\colon (U,\mu)\to(\mathbb{E}^n,\widehat{\mu})$ is of class C^1, then [27, Thm. 9.56] implies that the homological degree for oriented n-manifolds defined by Dold coincides with the classical Brouwer degree (for manifolds of class C^1).

B.2 The Benevieri-Furi degree

Let $F\colon O \to Y$ denote an index 0 Fredholm map. For open $O \subseteq X$ the *Benevieri-Furi degree* $\deg(F, O, y) \in \mathbb{Z}$ (resp. $\deg(F, O, y) \in \mathbb{Z}_2$ in the nonoriented case) is constructed as follows:

Let $Y_0 \subset Y$ be a finite-dimensional submanifold transversal to F on an open neighborhood $O_0 \subset O$ of $F^{-1}(0)$, F oriented on O_0. The intersection $X_0 := O_0 \cap F^{-1}(Y_0)$ is either empty or a submanifold of the same dimension as Y_0 and of class C^1 (see [27, Thm. 8.55]). If $\dim Y_0 > 0$, then $F_0 := F|_{X_0} \in C^1(X_0, Y_0)$ satisfies the *reduction property*

$$\deg(F, O, 0) = \deg_B(F_0, X_0, 0), \tag{B.2}$$

where $\deg_B$ is the C^1-Brouwer degree. For $X_0 = \emptyset$ the right-hand side is set to be zero. In the oriented situation, the orientation of F_0 is defined as in (1.1). The sign of an oriented Fredholm operator (T, σ) is $\operatorname{sgn} T = 1$ if $0 \in \sigma$, $\operatorname{sgn} T = -1$ otherwise and $\operatorname{sgn} T = 0$ if $T \notin GL(X, Y)$.

Such a degree is uniquely determined and has the following properties (see [2, 27]):

(bf0) **(regular normalization)** $\deg(F, O, 0) = \sum_{x \in F^{-1}(0)} \operatorname{sgn} DF(x)$ is a finite sum, if 0 is a regular value of F,

(bf1) **(existence)** if $\deg(F, O, 0) \neq 0$, then $0 \in F(O)$,

(bf2) **(homotopy invariance)** if $H\colon O \times [0, 1] \to Y$ is a generalized (oriented) Fredholm homotopy of index 0 with $H^{-1}(0)$ being compact, then

$$\deg(H_0, O, 0) = \deg(H_1, O, 0),$$

(bf3) **(excision-additivity)** if $O_i \subset O$, $i \in I$, is a family of pairwise disjoint open subsets with $F^{-1}(0) \subset \bigcup_{i \in I} O_i$ such that $O_i \cap F^{-1}(0)$ is compact for all $i \in I$, then

$$\deg(F, O, 0) = \sum_{i \in I} \deg(F, O_i, 0),$$

where in the right-hand side has a finite number of nonzero summands,

(bf4) **(compatibility with the Brouwer degree)** if $\dim X = \dim Y < \infty$, then the Benevieri-Furi degree coincides with the C^1-Brouwer degree.

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