LOCAL APPROXIMATION OF INVARIANT FIBER BUNDLES: AN ALGORITHMIC APPROACH

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Abstract. This paper contains an approach to compute Taylor approximations of invariant manifolds associated with arbitrary fixed reference solutions of nonautonomous difference equations. Our framework is sufficiently general to include, e.g., stable and unstable manifolds of periodic orbits, or classical center-stable/-unstable manifolds corresponding to equilibria. In addition, our focus is to give applicable and quantitative results.

Finally, in the appendix we present a short manual to the `Maple` program `IFB_Comp` to calculate Taylor approximations of invariant manifolds.

1. Preliminaries

1.1. **Introduction.** The role of invariant manifolds as a qualitative tool in the modern theory of autonomous dynamical systems cannot be overestimated (cf., e.g., [Shu87]). However, in general, it is difficult to determine invariant manifolds explicitly. Nevertheless, in many situations, it suffices to know only their Taylor approximation up to a certain order, like, e.g., in bifurcation theory or to apply Pliss's center manifold reduction.

Although this seems classical and well-established, even recently some papers on the Taylor approximation of invariant manifolds appeared (cf. [BK98, EvP04]). Beyond such systematic approaches, concrete computations can be found at many places, like, e.g., in the monograph [Kuz95, pp. 151–165, Section 5.4]. They all have in common that one has to solve a (possibly high-dimensional) linear algebraic equation to determine the desired Taylor coefficients, and, what is more important, they apply to the setting of autonomous equations only.

In this paper we present an algorithmic approach to obtain Taylor coefficients of invariant manifolds for nonautonomous difference equations, which is based on the theoretical results developed in [PR05]. The importance of a nonautonomous theory is due to the fact that, e.g., we are able to tackle more realistic problems with time-dependent parameters, or investigate the behavior near nonconstant solutions (cf. Subsection 3.1). Differing from the formal methods developed in [PR05], the present paper is focused on applicability of results: the propositions and theorems are quantitative to a large extend and necessary transformations of difference equations are given in a constructive way, such that they can be applied to given examples without further preparations.

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The appendix contains a brief description of our `Maple` program `IFB_Comp` to calculate Taylor approximations of invariant fiber bundles for nonautonomous difference equations.

1.2. **Notation.** The field of real numbers is denoted by $\mathbb{R}$, the complex numbers by $\mathbb{C}$, the integers by $\mathbb{Z}$, and for given $\kappa \in \mathbb{Z}$ we write $\mathbb{Z}_\kappa^+ := \{k \in \mathbb{Z} : \kappa \leq k\}$, $\mathbb{Z}_\kappa^- := \{k \in \mathbb{Z} : k \leq \kappa\}$, $\mathbb{N} := \mathbb{Z}_1^+$.

For arbitrary $N \in \mathbb{N}$, we consider the N-dimensional Euclidean space $\mathbb{R}^N$ with inner product $\langle x, y\rangle := \sum_{k=1}^N x_k y_k$ and induced norm $\|x\| := \sqrt{\langle x, x\rangle}$ for vectors $x, y \in \mathbb{R}^N$ with components x_k, y_k, respectively. Elements of $\mathbb{R}^N$ are always understood as columns throughout the paper. The *orthogonal complement* $V^\perp$ of a linear subspace $V \subseteq \mathbb{R}^N$ is given by $\{y \in \mathbb{R}^N : \langle x, y\rangle = 0 \text{ for all } x \in V\}$. The r-ball with center of x is denoted as $B_r^N(x) = \{y \in \mathbb{R}^N : \|x - y\| < r\}$; we abbreviate $B_r^N := B_r^N(0)$.

We write $\mathcal{L}(\mathbb{R}^N)$ for the set of real square matrices with N rows, $\mathcal{GL}(\mathbb{R}^N)$ for the subset of regular square matrices, $\mathbb{1}_N$ is the identity matrix and 0_N the zero matrix in $\mathcal{L}(\mathbb{R}^N)$. For $T \in \mathcal{L}(\mathbb{R}^N)$, the linear subspaces $\ker T := \{x \in \mathbb{R}^N : Tx = 0\}$ and $\operatorname{im} T := \{Tx \in \mathbb{R}^N : x \in \mathbb{R}^N\}$ denote the *kernel* and *range* of T, respectively; the determinant of T is denoted by $\det T$. Finally, the *spectrum* of T is given by the set $\sigma(T) := \{\lambda \in \mathbb{C} : \det(\lambda \mathbb{1}_N - T) = 0\}$.

It is important to point out that, for a vector- or matrix-valued sequence x, we use the convenient notation $x'(k) = x(k+1)$. The *k-fiber* of a set $S \subseteq \mathbb{Z} \times \mathbb{R}^N$ is given by $S(k) := \{x \in \mathbb{R}^N : (k, x) \in S\}$.

1.3. **Linear difference equations.** With a matrix sequence $A : \mathbb{Z} \to \mathcal{L}(\mathbb{R}^N)$ we define the *transition matrix* $\Phi(k, \kappa) \in \mathcal{L}(\mathbb{R}^N)$ of the linear difference equation

$$x' = A(k)x \tag{1.1}$$

in $\mathbb{R}^N$ as the mapping

$$\Phi(k, \kappa) := \begin{cases} \mathbb{1}_N & \text{for } k = \kappa \\ A(k-1)\cdots A(\kappa) & \text{for } k > \kappa \end{cases},$$

and if $A(k)$ is invertible for $k < \kappa$, then $\Phi(k, \kappa) := A(k)^{-1} \cdots A(\kappa - 1)^{-1}$.

A projection-valued mapping $P_+ : \mathbb{Z} \to \mathcal{L}(\mathbb{R}^N)$ is called an *invariant projector* of (1.1) if

$$P_+'(k)A(k) = A(k)P_+(k) \quad \text{for } k \in \mathbb{Z} \tag{1.2}$$

holds, and an invariant projector P_+ is denoted as *regular* if

$$A(k)|_{\ker P_+(k)} : \ker P_+(k) \to \ker P_+'(k) \text{ is bijective for all } k \in \mathbb{Z}. \tag{1.3}$$

Then the restriction $\bar{\Phi}(k, \kappa) := \Phi(k, \kappa)|_{\ker P_+(\kappa)} : \ker P_+(\kappa) \to \ker P_+(k)$, $\kappa \leq k$, is a well-defined isomorphism, and we write $\bar{\Phi}(\kappa, k)$ for its inverse. Let $\mathbb{I}$ denote either $\mathbb{Z}$ or $\mathbb{Z}_\kappa^+$. Then the linear difference equation (1.1) is said to possess an *exponential dichotomy* on $\mathbb{I}$ (ED for short) with *rates* $0 < \alpha_+ < \alpha_-$ if there exists a regular invariant projector $P_+ : \mathbb{Z} \to \mathcal{L}(\mathbb{R}^N)$ such that the dichotomy estimates

$$\sup_{l,k\in\mathbb{I},l\le k} \|\Phi(k,l)P_+(l)\| \, \alpha_+^{l-k} < \infty, \qquad \sup_{l,k\in\mathbb{I},l\le k} \left\|\bar{\Phi}(l,k)P_-(k)\right\| \alpha_-^{k-l} < \infty \tag{1.4}$$

are satisfied, where $P_-(k) := \mathbb{1}_N - P_+(k)$ denotes the *complementary projector*. In the following, the symbol $P_\pm$ simultaneously stands for P_+ or P_-, respectively, and we proceed accordingly with our further notation. Hence, the set

$$\mathcal{V}^\pm := \left\{(k,x) \in \mathbb{Z}\times\mathbb{R}^N : x \in \operatorname{im} P_\pm(k)\right\} \tag{1.5}$$

is invariant w.r.t. (1.1), i.e., its fibers satisfy $\Phi(k,\kappa)\mathcal{V}^\pm(\kappa) \subseteq \mathcal{V}^\pm(k)$ for $\kappa \le k$.

1.4. **Statement of the problem.** After these preparation we can present our primary objectives. Thereto, let $\tilde{U} \subseteq \mathbb{R}^N$ be a nonempty open convex set and $f : \tilde{U}\times\mathbb{Z} \to \mathbb{R}^N$ be a mapping. We consider the nonautonomous difference equation

$$x' = f(x,k), \tag{1.6}$$

whose maximal forward solution satisfying the initial condition $x(k_0) = x_0$ is denoted by $\varphi(\cdot, k_0, x_0)$ for $k_0 \in \mathbb{Z}$ and $x_0 \in \tilde{U}$.

Let us assume there exists a fixed reference solution $\nu : \mathbb{Z} \to \tilde{U}$ of (1.6) with $B_r^N(\nu(k)) \subseteq \tilde{U}$ for $k \in \mathbb{Z}$ and some $r > 0$. Typical examples of such reference solutions are equilibria, periodic or homo-/heteroclinic solutions, but we do not restrict ourself to such a situation here. Rather it is our goal to describe the domain of exponential attraction for ν and to provide local approximations of it.

Thereto, we say a solution μ of (1.6) is *exponentially decaying to* ν *on* $\mathbb{Z}_\kappa^\pm$ if μ exists on $\mathbb{Z}_\kappa^\pm$ for some $\kappa \in \mathbb{Z}$ and satisfies

$$\sup_{k\in\mathbb{Z}_\kappa^\pm} \|\mu(k) - \nu(k)\| \, \alpha_\pm^k < \infty$$

for some $\alpha_+ < 1 < \alpha_-$. Our global set-up will be as follows:

Hypothesis. *Let $f : \tilde{U}\times\mathbb{Z} \to \mathbb{R}^N$ be a mapping such that the partial derivatives $D_1 f, \dots, D_1^m f$ w.r.t. the first variable exist and are continuous for some $m \ge 2$. Moreover, we assume the following:*

- (H_1) *The* variational equation

$$x' = D_1 f(\nu(k), k)x \tag{1.7}$$

possesses an ED on $\mathbb{Z}$ with rates α_+, α_- and invariant projector P_+.

- (H_2) *For each $n \in \{2, \dots, m\}$ there exist reals $K_n \ge 0$ and points $x_n \in \tilde{U}$ such that $\|D_1^n f(x_n, k)\| \le K_n$ and for each bounded set $\Omega \subseteq \tilde{U}$ there exists a real $K \ge 0$ with $\|D_1^m f(x,k)\| \le K$ for all $x \in \Omega$ and $k \in \mathbb{Z}$.*

It is a consequence of these assumptions that the nonlinear difference equation (1.6) possesses two locally invariant sets $\mathcal{S}_\nu^+$ and $\mathcal{S}_\nu^-$, which are graphs of functions $s_\nu^\pm(\cdot, k)$ over the affine subspaces $\nu(k) + \mathcal{V}^\pm(k)$. To see this, apply [PR05, Theorem 3.2] to (3.1). More precisely, there exist $\rho > 0$ and mappings

$$s_\nu^\pm : \left\{(x,k) \in \mathbb{R}^N\times\mathbb{Z} : x \in (\nu(k) + \mathcal{V}^\pm(k)) \cap B_\rho^N(\nu(k))\right\} \to \mathbb{R}^N$$

satisfying $s_\nu^\pm(\nu(k),k) \equiv 0$ on $\mathbb{Z}$, $\lim_{x\to 0} D_1 s_\nu(x+\nu(k),k) = 0$ uniformly in $k \in \mathbb{Z}$,

$$s_\nu^\pm(\nu(k)+P_\pm(k)x,k) \in \mathcal{V}^\mp(k) \quad \text{for } k \in \mathbb{Z},\, x \in B_\rho^N,$$

such that the graphs $\mathcal{S}_\nu^\pm := \left\{(k,\xi+s_\nu^\pm(\xi,k)) : \xi \in (\nu(k)+\mathcal{V}^\pm(k)) \cap B_\rho^N(\nu(k))\right\}$ are *locally invariant fiber bundles* (IFBs for short) of (1.6). This means,

$$(k_0,x_0) \in \mathcal{S}_\nu^\pm \quad \Rightarrow \quad (k,\varphi(k;k_0,x_0)) \in \mathcal{S}_\nu^\pm \tag{1.8}$$

holds for $k \geq k_0$ as long as $\varphi(k;k_0,x_0)$ remains in the domain of definition for $s_\nu^\pm(\cdot,k)$. Moreover, we have

$$\mathcal{S}_\nu^+ \cap \mathcal{S}_\nu^- = \left\{(k,\nu(k)) \in \mathbb{Z}\times\mathbb{R}^N : k \in \mathbb{Z}\right\}.$$

In this context, $\mathcal{S}_\nu^+$ and $\mathcal{S}_\nu^-$ are denoted as *pseudo-stable* and *pseudo-unstable* fiber bundle of ν, respectively.

To illuminate this rather general framework, we close the section with a dynamic description of the sets $\mathcal{S}_\nu^+$ and $\mathcal{S}_\nu^-$ (cf. [Pöt98, p. 87, Satz 2.4.8]).

Remark 1.1.
(1) Under the assumption that the variational equation (1.7) possesses an ED with $\alpha_+ < \alpha_- = 1$ we have:

- if a solution μ of (1.6) is exponentially decaying to ν on $\mathbb{Z}_\kappa^+$, then there exists a $\kappa^* \in \mathbb{Z}_\kappa^+$ with $(k,\mu(k)) \in \mathcal{S}_\nu^+$ for all $k \in \mathbb{Z}_{\kappa^*}^+$,
- on the other hand, there exists a $\rho_1 \in (0,\rho)$ such that every solution μ with $\mu(\kappa) \in \mathcal{S}_\nu^+(\kappa) \cap B_{\rho_1}^N(\nu(\kappa))$ decays exponentially to ν on $\mathbb{Z}_\kappa^+$,
- if a solution μ exists on $\mathbb{Z}_\kappa^-$ and satisfies $\mu(k) \in B_\rho^N(\nu(k))$ for all $k \in \mathbb{Z}_\kappa^-$, then $((k,\mu(k)) \in \mathcal{S}_\nu^-$ for all $k \in \mathbb{Z}_\kappa^-$,

and $\mathcal{S}_\nu^+$ is denoted the *stable* and $\mathcal{S}_\nu^-$ the *center-unstable fiber bundle* of ν.
(2) Under the assumption that (1.7) has an ED with $1 = \alpha_+ < \alpha_-$ we have:

- if a solution μ of (1.6) is exponentially decaying to ν on $\mathbb{Z}_\kappa^-$, then there exists a $\kappa^* \in \mathbb{Z}_\kappa^-$ with $(k,\mu(k)) \in \mathcal{S}_\nu^-$ for all $k \in \mathbb{Z}_{\kappa^*}^-$,
- on the other hand, there exists a $\rho_1 \in (0,\rho)$ such that every solution μ with $\mu(\kappa) \in \mathcal{S}_\nu^-(\kappa) \cap B_{\rho_1}^N(\nu(\kappa))$ decays exponentially to ν on $\mathbb{Z}_\kappa^-$,
- if a solution μ exists on $\mathbb{Z}_\kappa^+$ and satisfies $\mu(k) \in B_\rho^N(\nu(k))$ for all $k \in \mathbb{Z}_\kappa^+$, then $((k,\mu(k)) \in \mathcal{S}_\nu^+$ for all $k \in \mathbb{Z}_\kappa^+$,

and $\mathcal{S}_\nu^+$ is denoted the *center-stable* and $\mathcal{S}_\nu^-$ the *unstable fiber bundle* of ν.

This terminology corresponds to the autonomous situation of invariant manifolds considered, e.g., in [Shu87]. It is the aim of this paper to obtain local approximations of these sets in form of Taylor expansions.

2. Sufficient Criteria for an Exponential Dichotomy

Even though an ED is a generic property in the class of linear systems with bounded coefficient sequences (cf. [AM96]), it is difficult to verify an assumption like (H_1) for a given nonautonomous equation. This section, however, contains sufficient criteria for exponential dichotomies in certain special cases.

We need some preparations from linear algebra (cf. [HS74, pp. 109–133]). Let $T \in \mathcal{L}(\mathbb{R}^N)$ and $0 < \alpha_+ < \alpha_-$. We say that T possesses an (α_+, α_-)-*spectral decomposition* if the sets

$$\sigma^+ := \{\lambda \in \sigma(T) : |\lambda| \le \alpha_+\}, \qquad \sigma^- := \{\lambda \in \sigma(T) : \alpha_- \le |\lambda|\}$$

are nonempty with $\sigma(T) = \sigma^+ \cup \sigma^-$, i.e., $\sigma(T)$ can be separated by an annulus with center 0 and radii $\alpha_+ < \alpha_-$. Having this at hand, we define

$$V_T^\pm := \bigoplus_{\substack{\lambda\in\sigma^\pm,\\ \Im\lambda=0}} \ker\left(T - \lambda \mathbb{1}_N\right)^N \oplus \bigoplus_{\substack{\lambda\in\sigma^\pm,\\ \Im\lambda>0}} \ker\left(T^2 - 2\Re\lambda T + |\lambda|^2\, \mathbb{1}_N\right)^N$$

and $n_\pm := \dim V_T^\pm$. Let $\{x_1^\pm, \dots, x_{n_\pm}^\pm\}$ be a basis of $V_T^\pm$. Using the regular matrix $C := (x_1^+, \dots, x_{n_+}^+, x_1^-, \dots, x_{n_-}^-) \in \mathcal{L}(\mathbb{R}^N)$ we introduce the projections

$$Q_T^+ := C \begin{pmatrix} \mathbb{1}_{n_+} & \\ & 0_{n_-} \end{pmatrix} C^{-1}, \qquad Q_T^- := C \begin{pmatrix} 0_{n_+} & \\ & \mathbb{1}_{n_-} \end{pmatrix} C^{-1},$$

which are complementary and fulfill $\ker Q_T^\pm = V_T^\mp$, $\operatorname{im} Q_T^\pm = V_T^\pm$.

2.1. **Autonomous equations.** In this subsection we assume that the mapping A in (1.1) does not depend on $k \in \mathbb{Z}$, i.e., we consider an autonomous linear difference equation of the form

$$x' = Ax. \tag{2.1}$$

Here, an eigenvalue λ of $A \in \mathcal{L}(\mathbb{R}^N)$ is said to be *semisimple* if its algebraic and geometric multiplicities coincide.

Proposition 2.1. *Let $0 < \alpha_+ < \alpha_-$ be reals, assume the coefficient matrix A possesses an (α_+, α_-)-spectral decomposition and the eigenvalues of A with modulus α_+ and α_- are semisimple. Then* (1.1) *possesses an ED on $\mathbb{Z}$ with rates α_+, α_- and constant invariant projector Q_A^+.*

Proof. See [Pöt98, p. 25, Satz 1.4.11] and [Kal92, p. 105, Satz 2.1.3.2]. □

2.2. **Periodic equations.** Let $\omega \in \mathbb{N}$. The difference equation (1.1) is said to be *ω-periodic* if $A(k) = A(k + \omega)$ for all $k \in \mathbb{Z}$. A 1-periodic equation (1.1) corresponds to the autonomous case (2.1).

Before stating the subsequent proposition, we note that for all $k \in \mathbb{Z}$ the matrix $M_\omega(k) := \Phi(k+\omega, k)$ has the same eigenvalues as the so-called *monodromy matrix* $M_\omega(0)$ (cf., e.g., [Zha99, p. 51, Theorem 2.8]). They are denoted as *Floquet multipliers* of (1.1). A Floquet theory for periodic difference equations can be found in [Aga92, Section 2.9, pp. 68–71].

Proposition 2.2. *Let $0 < \alpha_+ < \alpha_-$ be reals, assume the monodromy matrix $M_\omega(0)$ possesses an $(\alpha_+^\omega, \alpha_-^\omega)$-spectral decomposition and the Floquet multipliers with modulus α_+^ω and α_-^ω are semisimple. Then* (1.1) *possesses an ED on $\mathbb{Z}$ with α_+, α_- and an ω-periodic invariant projector $P_+(k) := Q_{M_\omega(k)}^+$ for all $k \in \mathbb{Z}$.*

Proof. We first prove that the projector defined by $P_+(k) := Q^+_{M_\omega(k)}$ for $k \in \mathbb{Z}$ is invariant, i.e., satisfies (1.2). Thereto, choose $k \in \mathbb{Z}$ and $x \in \mathbb{R}^N$. We decompose $x = x_1 + x_2$ with $x_1 \in \ker P_+(k)$ and $x_2 \in \operatorname{im} P_+(k)$; thus $x_1 \in V^-_{M_\omega(k)}$. Since the asserted equation is linear we assume w.l.o.g. that there exists a $\lambda \in \sigma^+$ such that $x_1 \in \ker \left((M_\omega(k)^2 - 2\Re\lambda M_\omega(k) + |\lambda|^2 \mathbb{1}_N\right)^N$. The periodicity of A yields

$$\begin{aligned}
&\left(M_\omega(k)^2 - 2\Re\lambda M_\omega(k) + |\lambda|^2 \mathbb{1}_N\right)^N x_1 = 0\\
\Rightarrow\quad & A(k+\omega)\left(M_\omega(k)^2 - 2\Re\lambda M_\omega(k) + |\lambda|^2 \mathbb{1}_N\right)^N x_1 = 0\\
\Rightarrow\quad & \left(M'_\omega(k)^2 - 2\Re\lambda M'_\omega(k) + |\lambda|^2 \mathbb{1}_N\right)^N A(k)x_1 = 0,
\end{aligned}$$

hence $A(k)x_1 \in V^-_{M'_\omega(k)} = \ker P'_+(k)$ and we have $P'_+(k)A(k)x_1 = A(k)P_+(k)x_1$. Analogously, $P'_+(k)A(k)x_2 = A(k)P_+(k)x_2$ follows. This shows the invariance of P_+. Using Proposition 2.1 we have

$$\|\Phi(k\omega, \kappa\omega)P_+(\kappa\omega)\| \le \tilde{K}_+\alpha_+^{k\omega-\kappa\omega}, \qquad \|\Phi(\kappa\omega, k\omega)P_-(k\omega)\| \le \tilde{K}_-\alpha_-^{\kappa\omega-k\omega}$$

for $k \ge \kappa$ and certain $\tilde{K}_\pm \ge 1$. We define

$$\hat{K}_+ := \max_{i=1}^{k}\max_{j=0}^{i-1} \|\Phi(i,j)P_+(j)\|\alpha_+^{j-i}, \quad \hat{K}_- := \max_{i=1}^{k}\max_{j=0}^{i-1} \|\Phi(j,i)P_-(i)\|\alpha_-^{i-j},$$

and it follows directly that (1.1) possess an ED on $\mathbb{Z}$ with rates α_+, α_-, bounds $\tilde{K}_+\hat{K}_+, \tilde{K}_-\hat{K}_-$ (cf. (1.4)) and invariant projector P_+. □

2.3. **Further criteria.** The next result is motivated by [Cop78, p. 70, Lemma 1].

Proposition 2.3. *Let $\kappa \in \mathbb{Z}$, $B : \mathbb{Z} \to \mathcal{L}(\mathbb{R}^N)$ and P_+ denote a regular invariant projector for* (1.1)*. Moreover, assume* (1.1) *possesses an ED on $\mathbb{Z}^+_\kappa$ with α_-, α_+, P_+ and that there exists a sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}$, $\lim_{n\to\infty} k_n = \infty$ such that*

$$\lim_{n\to\infty}\sup_{k\in J} \|A(k+k_n) - B(k)\| = 0 \quad \textit{for } J \subset \mathbb{Z} \textit{ finite.}$$

Then the linear difference equation $x' = B(k)x$ possesses an ED on $\mathbb{Z}$ with α_+, α_- and the invariant projector $Q_+(k) := \lim_{n\to\infty} P_+(k+k_n)$.

Proof. For all $n \in \mathbb{N}$, the translated equation $x' = A(k+k_n)x$ possesses the transition matrix $\Phi_n(k,\kappa) = \Phi(k+k_n, \kappa+k_n)$. Furthermore, it satisfies (1.3) with the invariant projector $P^n_+(k) := P_+(k+k_n)$, and due to the dichotomy assumptions there exist constants $K_+, K_- \ge 1$ with

$$\left\|\Phi_n(k,l)P^n_+(l)\right\| \le K_+\alpha_+^{k-l}, \qquad \left\|\bar{\Phi}_n(l,k)P^n_-(k)\right\| \le K_-\alpha_-^{l-k} \tag{2.2}$$

for $k \ge l \ge \kappa - k_n$. Since $\left\|P^n_+(k)\right\| \le K_+$ for all $k \ge \kappa - k_n$, passing over to a subsequence of $(k_n)_{n\in\mathbb{N}}$ yields the existence of $Q_+(k) := \lim_{n\to\infty} P^n_+(k)$ for all $k \in \mathbb{Z}$. On the other hand, $\Psi(k,l) := \lim_{n\to\infty} \Phi_n(k,l)$ is the transition matrix of $x' = B(k)x$ and taking the limit $n \to \infty$ in (2.2) leads to

$$\|\Psi(k,l)Q_+(l)\| \le K_+\alpha_+^{k-l}, \quad \left\|\bar{\Psi}(l,k)Q_-(k)\right\| \le K_-\alpha_-^{l-k} \quad \text{for } k \ge l > -\infty.$$

Since invariant projectors for ED on $\mathbb{Z}$ are uniquely determined (cf. [Kal94, p. 12]), one sees that Q_+ does not depend on the chosen subsequence. □

Beyond the above, there exist certain other conditions leading to an ED of equation (1.1) or (1.7). They can be subdivided into three classes using the key words: *Slowly varying coefficients* (cf. [Pöt04b, Corollary 3.6]); *Diagonal dominance* (cf. [Cop78, p. 55–56, Proposition 3] and [Pal77] for ODEs) and *Lyapunov functions* (cf. [Cop78, p. 61, Proposition 2] for ODEs).

3. Transformation of Difference Equations

In this section we describe how a nonautonomous difference equation (1.6) can be brought into a (decoupled) form, such that it is comparatively simple to calculate its IFBs, instead of working with the original system. Precisely, one has to proceed in two steps:

3.1. **Equation of perturbed motion.** Under the transformation $\mathcal{T}_k^1 : x \mapsto x - \nu(k)$ the difference equation (1.6) becomes

$$x' = D_1 f(\nu(k), k)x + f_\nu(x, k) \tag{3.1}$$

with $f_\nu(x, k) := f(x + \nu(k), k) - f(\nu(k), k) - D_1 f(\nu(k), k)x$ defined on $B_\rho^N \times \mathbb{Z}$. Note that (3.1) possesses the trivial solution.

3.2. **Lyapunov transformation.** Now it is our aim to decouple (1.7) without destroying the dynamical features of (3.1). We make use of a Lyapunov transformation (cf. [Pöt98, p. 166, Lemma A.6.1]). Thereto, let $P_+ : \mathbb{Z} \to \mathcal{L}(\mathbb{R}^N)$ be the invariant projector from (H_1) associated with the ED of (1.7). Then, due to the regularity condition (1.3), the fibers $\mathcal{V}^\pm(k)$, $k \in \mathbb{Z}$, possess constant dimensions $n_\pm$ with $n_+ + n_- = N$. For each $k \in \mathbb{Z}$, let $\{x_1(k), \dots, x_{n_+}(k)\}$ be an orthonormal basis of $\operatorname{im} P_+(k)$ and $\{y_1(k), \dots, y_{n_-}(k)\}$ be an orthonormal basis of $(\operatorname{im} P_+(k))^\perp$. Such orthonormal basis can be obtained using a Gram-Schmidt procedure (cf. [Hig96, pp. 376ff]) in lower dimensions.

Setting $C(k) := (x_1(k), \dots, x_{n_+}(k), y_1(k), \dots, y_{n_-}(k)) \in \mathcal{GL}(\mathbb{R}^N)$ yields

$$C(k)^{-1} P_+(k) C(k) = \begin{pmatrix} \mathbb{1}_{n_-} & R(k) \\ & 0_{n_+} \end{pmatrix},$$

and the mapping $\Lambda(k) := C(k) \begin{pmatrix} \mathbb{1}_{n_-} & -R(k) \\ & \mathbb{1}_{n_+} \end{pmatrix}$ is indeed a Lyapunov transformation, since we have

$$\|\Lambda(k)\| \le 2 + \|P_+(k)\|, \qquad \left\|\Lambda(k)^{-1}\right\| \le 1 + \|P_+(k)\| \quad \text{for } k \in \mathbb{Z}$$

(see [Pöt98, p. 28, Definition 1.5.1] for details); note that $P_+ : \mathbb{Z} \to \mathcal{L}(\mathbb{R}^N)$ is bounded due to (1.4). Thus, applying the transformation $\mathcal{T}_k^2 : x \mapsto \Lambda(k)x$ to (3.1) yields a nonautonomous difference equation of the form

$$\begin{aligned} x_+' &= A_+(k)x_+ + F_+(x_+, x_-, k) \\ x_-' &= A_-(k)x_- + F_-(x_+, x_-, k) \end{aligned} \tag{3.2}$$

(see [Pöt98, pp. 29–30, Lemma 1.5.4]) with $A_+ : \mathbb{Z} \to \mathcal{L}(\mathbb{R}^{n_+})$, $A_- : \mathbb{Z} \to \mathcal{GL}(\mathbb{R}^{n_-})$ given by

$$\begin{pmatrix} A_+(k) & \\ & A_-(k) \end{pmatrix} := \Lambda'(k)^{-1} D_1 f(\nu(k), k) \Lambda(k) \quad \text{for } k \in \mathbb{Z}$$

and maps $F_+ : B^{n_+}_{\rho_+} \times B^{n_-}_{\rho_-} \times \mathbb{Z} \to \mathbb{R}^{n_+}$, $F_- : B^{n_+}_{\rho_+} \times B^{n_-}_{\rho_-} \times \mathbb{Z} \to \mathbb{R}^{n_-}$ being m-times continuously differentiable w.r.t. (x_+, x_-) for some $\rho_+, \rho_- > 0$, and defined by

$$\begin{pmatrix} F_+(x_+, x_-, k) \\ F_-(x_+, x_-, k) \end{pmatrix} := \Lambda'(k) f_\nu \left(\Lambda(k)^{-1} \begin{pmatrix} x_+ \\ x_- \end{pmatrix}, k \right).$$

Then the assumptions (H_1) and (H_2) guarantee:

(i) The transition matrices Φ_+ and Φ_- of $x'_+ = A_+(k)x_+$ and $x'_- = A_-(k)x_-$, respectively, satisfy for all $k, l \in \mathbb{Z}$ the estimates

$$\|\Phi_+(k,l)\| \le K_+ \alpha_+^{k-l}, \qquad \|\Phi_-(l,k)\| \le K_- \alpha_-^{l-k} \quad \text{for } k \ge l. \tag{3.3}$$

(ii) We have $(F_+, F_-)(0,0,k) \equiv (0,0)$ on $\mathbb{Z}$ and the partial derivatives satisfy

$$\lim_{(x_+,x_-)\to(0,0)} D_{(1,2)}(F_+, F_-)(x_+, x_-, k) = 0 \quad \text{uniformly in } k \in \mathbb{Z}.$$

Remark 3.1. Let P_+ be ω-periodic. Then it is obvious from the above construction that the mapping Λ inherits the periodicity of P_+ if one chooses the basis of $\operatorname{im} P_+(k)$ and of $(\operatorname{im} P_+(k))^\perp$ accordingly.

4. Invariant Fiber Bundles

In this section we state an existence result for IFBs of the difference equation (3.2) and describe a method to compute Taylor approximations of them.

Proposition 4.1. *Assume (H_1) and (H_2) hold. Then there exist neighborhoods $U_+ \subseteq \mathbb{R}^{n_+}$, $U_- \subseteq \mathbb{R}^{n_-}$ of zero such that:*

(a) *There exists a continuous mapping $s^+ : U_+ \times \mathbb{Z} \to \mathbb{R}^{n_-}$ satisfying:*

(a_1) *Under the* gap condition

$$\alpha_+^m < \alpha_-, \tag{4.1}$$

s^+ is m-times continuously differentiable in the first argument, with $\lim_{\xi\to 0} D_1 s^+(\xi,k) = 0$ uniformly in $k \in \mathbb{Z}$,

(a_2) *the* invariance equation

$$s^+(A_+(\kappa)\xi + F_+(\kappa,\xi,s^+(\xi,\kappa)),\kappa+1) = A_-(\kappa)s^+(\xi,\kappa) + F_-(\xi,s^+(\xi,\kappa),\kappa)$$

holds for $(\xi,\kappa) \in U_+ \times \mathbb{Z}$ with $A_+(\kappa)\xi + F_+(\kappa,\xi,s^+(\xi,\kappa) \in U_+$,

(a_3) *s^+ is ω-periodic in the second argument if (3.1) is ω-periodic,*

(a_4) *its graph $S^+ := \{(\kappa,\xi,s^+(\xi,\kappa)) : \kappa \in \mathbb{Z}, \xi \in U_+\}$ is a pseudo-stable fiber bundle of (3.2) corresponding to its zero solution.*

(b) *There exists a continuous mapping $s^- : U_- \times \mathbb{Z} \to \mathbb{R}^{n_+}$ satisfying:*

(b_1) *Under the* gap condition

$$\alpha_+ < \alpha_-^m, \tag{4.2}$$

s^- is m-times continuously differentiable in the first argument, with $\lim_{\xi\to 0} D_1 s^-(\xi,k) = 0$ uniformly in $k \in \mathbb{Z}$,

(b_2) *the* invariance equation

$$s^-(A_-(\kappa)\xi + F_-(\kappa,\xi,s^-(\xi,\kappa)),\kappa+1) = A_+(\kappa)s^-(\xi,\kappa) + F_+(\xi,s^-(\xi,\kappa),\kappa)$$

holds for $(\xi,\kappa) \in U_- \times \mathbb{Z}$ with $A_-(\kappa)\xi + F_-(\kappa,\xi,s^-(\xi,\kappa) \in U_-$,

(b_3) *s^- is ω-periodic in the second argument if* (3.1) *is ω-periodic,*
(b_4) *its graph $S^- := \{(\kappa, s^-(\xi,\kappa),\xi) : \kappa \in \mathbb{Z}, \xi \in U_-\}$ is a pseudo-unstable fiber bundle of* (3.2) *corresponding to its zero solution.*

Proof. Using a standard cut-off technique one modifies (3.2) appropriately and applies [PS04, Theorem 4.1]. The periodicity assertion follows from [Aul98, Corollary 4.2] (see also [Pöt04a, Theorem 2.4]). □

Proposition 4.2. *Assume* (H_1) *and* (H_2) *hold. Then the IFBs $\mathcal{S}^\pm$ of* (3.2) *(cf. Proposition 4.1) and $\mathcal{S}_\nu^\pm$ of* (1.6) *are related by*

$$\mathcal{S}_\nu^\pm(k) = \nu(k) + \Lambda(k)^{-1}\mathcal{S}^\pm(k) \quad \text{for } k \in \mathbb{Z}.$$

Proof. This is obvious from the transformations $\mathcal{T}_k^1, \mathcal{T}_k^2$ applied to (1.6) to obtain (3.2) given in Section 3. □

Our final goal is to obtain Taylor approximations of the IFB $\mathcal{S}_\nu^\pm$ for (1.6). It is sufficient to concentrate on the IFB $\mathcal{S}^\pm$ for (3.2), since $\mathcal{S}_\nu^\pm$ and $\mathcal{S}^\pm$ are related by Proposition 4.2.

To deduce such a result, we present a formal approach using Fréchet derivatives (cf. [Lan93, Chapter XIII]) leading to a compact convenient notation. Although partial derivatives have the advantage that our formulas could be implemented instantly on a computer, the resulting expressions turn out to be immense — in particular for higher order derivatives. Yet, some further notation is needed:

Let $k, N, M \in \mathbb{N}$. For an k-tuple of the same vector $x \in \mathbb{R}^N$ we write $x^{(k)} := (x, \ldots, x)$. The linear space of symmetric k-linear mappings from $(\mathbb{R}^N)^k$ to $\mathbb{R}^M$ is denoted by $\mathcal{L}_k(\mathbb{R}^N, \mathbb{R}^M)$. With $T \in \mathcal{L}(\mathbb{R}^N)$ and $X \in \mathcal{L}_k(\mathbb{R}^N, \mathbb{R}^M)$ we abbreviate

$$Xx_1 \cdots x_k := X(x_1, \ldots, x_k), \qquad X_T x_1 \cdots x_k := X(Tx_1, \ldots, Tx_k).$$

Moreover, with given $j, l \in \mathbb{N}$, we write

$$P_j^<(l) := \left\{ (M_1, \ldots, M_j) \;\middle|\; \begin{array}{l} M_i \subseteq \{1, \ldots, l\} \text{ and } M_i \neq \emptyset \text{ for } i \in \{1, \ldots, j\}, \\ M_1 \cup \ldots \cup M_j = \{1, \ldots, l\}, \\ M_i \cap M_k = \emptyset \text{ for } i \neq k, i, k \in \{1, \ldots, j\}, \\ \max M_i < \max M_{i+1} \text{ for } i \in \{1, \ldots, j-1\} \end{array} \right\}$$

for the set of *ordered partitions* of $\{1, \ldots, l\}$ with length j and $\#M$ for the cardinality of a finite set $M \subset \mathbb{N}$. For a set $M = \{m_1, \ldots, m_k\} \subseteq \{1, \ldots, l\}$ we write $Xx_M := Xx_{m_1} \cdots x_{m_k}$ for $k \leq l$ and vectors $x_1, \ldots, x_l \in \mathbb{R}^N$.

We are interested in local approximations for the mapping $s^\pm$ from Proposition 4.1. The latter one guarantees under the gap conditions (4.1), (4.2) that $s^\pm(\cdot, k) : U_\pm \to \mathbb{R}^{n_\mp}$, $k \in \mathbb{Z}$, is m-times continuously differentiable and Taylor's theorem (cf. [Lan93, p. 350]) implies the representation

$$s^\pm(x,k) = \sum_{n=2}^{m} \frac{1}{n!} s_n^\pm(k) x^{(n)} + R^\pm(x,k)$$

with coefficient functions $s_n^\pm : \mathbb{Z} \to \mathcal{L}_n(\mathbb{R}^{n_\pm}, \mathbb{R}^{n_\mp})$ given by $s_n^\pm(k) := D_1^n s^\pm(0,k)$ and a remainder $R^\pm$ satisfying $\lim_{x\to 0} \frac{R^\pm(x,k)}{\|x\|^m} = 0$.

- It is convenient to introduce the function $S^{\pm}: U_{\pm}\times\mathbb{Z} \to \mathbb{R}^N$,
$$S^+(x,k) := \begin{pmatrix} x \\ s^+(x,k) \end{pmatrix}, \qquad S^-(x,k) := \begin{pmatrix} s^-(x,k) \\ x \end{pmatrix}$$
and its partial derivatives $S_n^{\pm}(k) := D_1^n S^{\pm}(0,k)$.
- We also introduce the function $g^{\pm}(x,k) := A_{\pm}(k)x + F_{\pm}(S^{\pm}(x,k),k)$ with partial derivatives
$$g_1^{\pm}(k)x_1 = A_{\pm}(k)x_1,$$
$$g_n^{\pm}(k)x_1\cdots x_n = \sum_{l=2}^{n} \sum_{(M_1,\dots,M_l)\in P_l^<(n)} D_1^l F_{\pm}(0,0,k) S^{\pm}_{\#M_1}(k)x_{M_1}\cdots S^{\pm}_{\#M_l}(k)x_{M_l}$$
for $n \in \{2,\dots,m\}$.

Now it is a consequence of [PR05] that the sequence $s_n^{\pm}: \mathbb{Z} \to \mathcal{L}_n(\mathbb{R}^{n_\pm}, \mathbb{R}^{n_\mp})$ is the unique bounded solution of the so-called *homological equation*

$$X'_{A_{\pm}(k)} = A_{\mp}(k)X + H_n^{\pm}(k) \tag{4.3}$$

with the inhomogeneity $H_2^{\pm}(k) := D_1^2 F_{\mp}(0,0,k)$ and

$$\begin{aligned} H_n^{\pm}(k)x_1\cdots x_n &:= D_1^n F_{\mp}(0,0,k)x_1\cdots x_n \\ &\quad + \sum_{l=2}^{n-1} \sum_{(M_1,\dots,M_l)\in P_l^<(n)} \big(D_1^l F_{\mp}(0,0,k) S^{\pm}_{\#M_1}(k)x_{M_1}\cdots S^{\pm}_{\#M_l}(k)x_{M_l} \\ &\quad - s_l^{\pm}(k+1) g^{\pm}_{\#M_1}(k)x_{M_1}\cdots g^{\pm}_{\#M_l}(k)x_{M_l}\big) \quad \text{for } x_1,\dots,x_n \in \mathbb{R}^{n_\pm}. \end{aligned}$$

This yields the following

Theorem 4.3. *Assume* (H_1), (H_2) *and*
$$\sup_{k\in\mathbb{Z}} \|D_1^n f(\nu(k),k)\| < \infty \quad \text{for } n \in \{2,\dots,m\}$$
hold. Then one has:

(a) *Under the gap condition* (4.1) *the mapping* $s^+ : U_+\times\mathbb{Z} \to \mathbb{R}^{n_-}$ *from Proposition 4.1(a) possesses the derivatives*

$$D_1^n s^+(0,k) = -\sum_{j=k}^{\infty} \Phi_-(k,j+1) H_n^+(j)_{\Phi_+(j+1,k)} \quad \text{for } n \in \{2,\dots,m\}, \tag{4.4}$$

(b) *under the gap condition* (4.2) *the mapping* $s^- : U_-\times\mathbb{Z} \to \mathbb{R}^{n_+}$ *from Proposition 4.1(b) possesses the derivatives*

$$D_1^n s^-(0,k) = \sum_{j=-\infty}^{k-1} \Phi_+(k,j+1) H_n^-(j)_{\Phi_-(j+1,k)} \quad \text{for } n \in \{2,\dots,m\}. \tag{4.5}$$

Proof. See [PR05, Theorem 4.2]. □

Remark 4.4.
(1) To avoid a repetitive computation of the infinite series in (4.4) and (4.5), we recommend to calculate $s_n^{\pm}(\kappa)$ for some fixed time $\kappa \in \mathbb{Z}$ and then use the homological equation (4.3) to determine subsequent values $s_n^{\pm}(k)$ recursively for $k > \kappa$.
(2) In case the difference equation (3.1) is ω-periodic for some $\omega \in \mathbb{N}$, the Taylor coefficients $s_n^{\pm}(k)$ inherit this periodicity for $n \in \{2, \dots, m\}$. Consequently, due to the variation of constants formula applied to the homological equation (4.3) and using $s_n^{\pm}(k+\omega) = s_n^{\pm}(k)$, one gets the relations

$$s_n^{+}(k) = \Phi_-(k, \omega+k) s_n^{+}(k)_{\Phi_+(\omega+k,k)} - \sum_{i=k}^{\omega+k-1} \Phi_-(k, i+1) H_n^{+}(i)_{\Phi_+(i,k)},$$

$$s_n^{-}(k) = \Phi_+(\omega+k, k) s_n^{-}(k)_{\Phi_-(k,\omega+k)} + \sum_{i=k}^{\omega+k-1} \Phi_+(\omega+k, i+1) H_n^{-}(i)_{\Phi_-(i,\omega+k)}.$$

For $0 \le k < \omega$ they yield algebraic equations to determine $s_n^{\pm}(0), \dots, s_n^{\pm}(\omega-1)$. In addition, these formulas are generalizations of the multilinear Sylvester equations obtained in the autonomous case (see, e.g., [BK98, Theorem 2.4]).

While the infinite series (4.4), (4.5) to determine the partial derivatives in Proposition 4.1 provide an analytical solution of our problem, they seem to be of restricted practical use due to the limit process involved. However, it is possible to obtain an a priori error estimate:

Corollary 4.1 (error estimates)**.** *Choose a real $\gamma_n^{\pm} > \sup_{k\in\mathbb{Z}} \|H_n^{\pm}(k)\|$, let $\varepsilon > 0$ be arbitrary and $k, K \in \mathbb{Z}$. Then, for finite approximations to the series* (4.4) *and* (4.5)*, the following holds:*

(a) *In case $K - k > \log_{\frac{\alpha_-}{\alpha_+^n}} \left(\frac{K_+^n K_- \gamma_n^+}{\varepsilon(\alpha_- - \alpha_+^n)} \right)$ one has*

$$\left\| - \sum_{j=k}^{K} \Phi_-(k, j+1) H_n^{+}(j)_{\Phi_+(j,k)} - D_1^n s^{+}(0, k) \right\| < \varepsilon,$$

(b) *in case $k - K > \log_{\frac{\alpha_-^n}{\alpha_+}} \left(\frac{K_+ K_-^n \gamma_n^-}{\varepsilon(\alpha_-^n - \alpha_+)} \right)$ one has*

$$\left\| \sum_{j=K}^{k-1} \Phi_+(k, j+1) H_n^{-}(j)_{\Phi_-(j,k)} - D_1^n s^{-}(0, k) \right\| < \varepsilon,$$

with

$$K_+ := \sup_{l \le k} \|\Phi(k,l) P_+(l)\| \, \alpha_+^{l-k}, \qquad K_- := \sup_{l \le k} \left\|\bar{\Phi}(l,k) P_-(k)\right\| \alpha_-^{k-l}.$$

Proof. See [PR05, Corollary 4.1]. □

5. Examples

This section contains two examples how to apply the results above. While the first example is more on a demonstration level, the second one deals with a periodic problem. Precisely, we calculate a 4th order Taylor approximation for the stable and unstable manifolds corresponding to a hyperbolic 2-periodic orbit of the Henon map.

Example 5.1. Consider the following nonautonomous difference equation describing a Flour beetle population (cf., e.g., [CD95])

$$\begin{aligned} x_1' &= b x_3 e^{-c_1(k)x_3 - c_2(k)x_1} \\ x_2' &= (1-\mu_1)x_1 \\ x_3' &= x_2 e^{-c_3(k)x_3} + (1-\mu_2)x_3 \end{aligned} \tag{5.1}$$

with parameters $b > 0$, $\mu_1, \mu_2 \in (0,1)$ and bounded sequences $c_1, c_2, c_3 : \mathbb{Z} \to (0, \infty)$. The linearization in $(0,0,0)$ has a real eigenvalue $\rho \in (1-\mu_2, \infty)$ and a complex-conjugated pair $\lambda_{1/2}$ satisfying $\left|\lambda_{1/2}\right| < \rho$. Hence, we have a 2-dimensional pseudo-stable and a 1-dimensional pseudo-unstable fiber bundle.

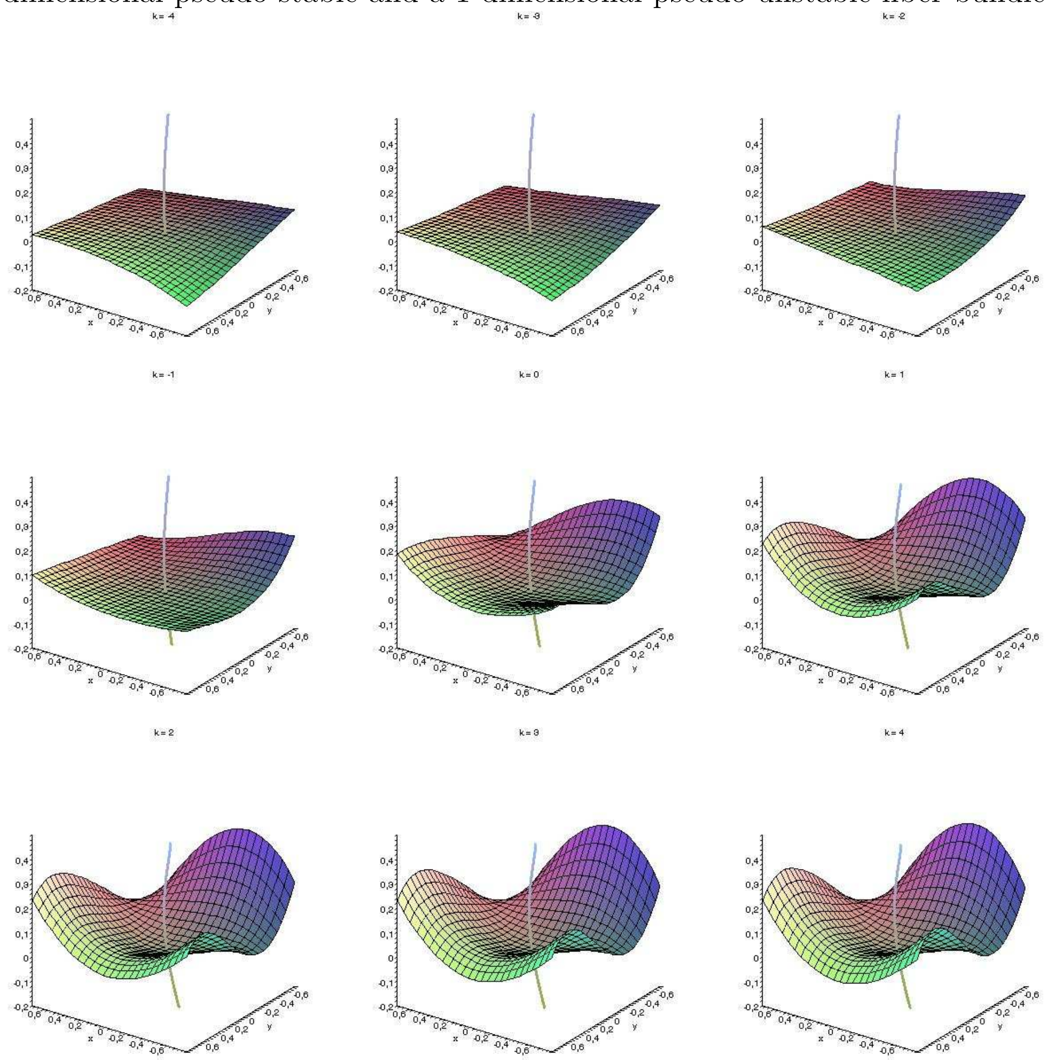

FIGURE 1. Stable and center-unstable fiber bundle corresponding to the zero solution for the flour beetle model (5.1), $k \in \{-4, \dots, 4\}$

For our numerical calculations we fix the parameters $b := 0.65$, $\mu_1 := 0.11$, $\mu_2 := 0.58$ and set $c_1(k) := 0.92 + 0.45\arctan(k)$, $c_2(k) := 0.9 + 0.13\arctan(k)$, $c_3(k) := 0.18 + 0.06\arctan(k)$. This yields the eigenvalues $-0.26 \pm 0.67i$, 1. If we apply the transformation $\mathcal{T}_k^2$ to (5.1), then the corresponding IFBs of the transformed system can be found in Figure 1.

Example 5.2. Consider the Henon map

$$\begin{aligned} x_1' &= 1 + x_2 - ax_1^2 \\ x_2' &= bx_1 \end{aligned} \tag{5.2}$$

with parameters $a := \frac{7}{5}$, $b := \frac{3}{10}$. We are going to study its behavior close to the 2-periodic solution $\nu(k) = (\frac{1}{4} + (-1)^k\frac{\sqrt{413}}{28}, \frac{3}{40} - 3(-1)^k\frac{\sqrt{413}}{280})$. The corresponding equation of perturbed motion is given by

$$x' = \begin{pmatrix} -\frac{7}{10} - (-1)^k\frac{\sqrt{413}}{10} & 1 \\ \frac{3}{10} & 0 \end{pmatrix} x + \begin{pmatrix} -\frac{7}{5}x_1^2 \\ 0 \end{pmatrix} \tag{5.3}$$

with a 2-periodic linear part. Then its monodromy matrix reads as $\Phi(2,0) = \begin{pmatrix} -\frac{167}{50} & -\frac{7}{10} + \frac{\sqrt{413}}{10} \\ -\frac{21}{100} - \frac{3\sqrt{413}}{100} & \frac{3}{10} \end{pmatrix}$ and the Floquet multipliers turn out to be $-\frac{38}{25} \pm \frac{\sqrt{5551}}{50}$. Due to Proposition 2.2 we obtain an ED on $\mathbb{Z}$ for the linear part of equation (5.3) with $\alpha_+ = \frac{1}{5}\sqrt{\frac{\sqrt{5551}}{2} - 38}$, $\alpha_- = \frac{1}{5}\sqrt{38 + \frac{\sqrt{5551}}{2}}$ and invariant projector

$$P_+(k) = \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{5551}}{122} & \frac{5\sqrt{5551}\left(7-(-1)^k\sqrt{413}\right)}{11102} \\ \frac{3\sqrt{5551}(7+(-1)^k\sqrt{413})}{22204} & \frac{1}{2} - \frac{\sqrt{5551}}{122} \end{pmatrix}.$$

This leads to the Lyapunov transformation

$$\Lambda(k) = \frac{1}{\mu(k)}\begin{pmatrix} \lambda_{11}(k) & 1 \\ \lambda_{21}(k) & \lambda_{22}(k) \end{pmatrix}$$

with $\mu(k) := \frac{1}{5}\sqrt{\frac{952}{13} - \frac{33\sqrt{5551}}{52} + (-1)^k\sqrt{59}(\frac{19\sqrt{7}}{13} - \frac{7\sqrt{793}}{52})}$ and

$$\begin{aligned} \lambda_{11}(k) &= \tfrac{82176625\sqrt{5551}}{413557573106} - \tfrac{575236375}{13559264692} + (-1)^k(\tfrac{5177127375\sqrt{46787}}{5376248450378} - \tfrac{82176625\sqrt{413}}{13559264692}), \\ \lambda_{21}(k) &= \tfrac{29501408375\sqrt{7}\sqrt{793}}{21504993801512} + \tfrac{13230436625}{352540881992} + (-1)^k(\tfrac{1725709125\sqrt{46787}}{21504993801512} - \tfrac{246529875\sqrt{413}}{352540881992}) \\ \lambda_{22}(k) &= \tfrac{7}{20} - \tfrac{\sqrt{5551}}{260} + (-1)^k(\tfrac{\sqrt{413}}{20} - \tfrac{\sqrt{46787}}{260}) \end{aligned}$$

To avoid such extensive expressions we switch to a floating point notation from now on, which is sufficient for our numerical purposes. Then the transformed equation (5.3) is given by

$$\begin{aligned} y_1' = &- (0.0593 - (-1)^k 0.1828)y_1 + (0.1256 - (-1)^k 0.1612)y_1^2 \\ &- (0.3835 - (-1)^k 0.2334)y_1y_2 + (0.0867 - (-1)^k 0.2449)y_2^2 \\ y_2' = &- (0.5950 + (-1)^k 1.8341)y_2 - (0.8491 - (-1)^k 0.5169)y_1^2 \\ &+ (0.7684 - (-1)^k 2.1688)y_1y_2 - (1.5273 - (-1)^k 0.0515)y_2^2 \end{aligned}$$

and the invariant fiber bundles read as

$$s(x,k) = s_0(x) + (-1)^k s_1(x), \qquad r(x,k) = r_0(x) - (-1)^k r_1(x)$$

with

$$\begin{aligned} s_0(x) &= 0.4760x^2 - 1.2467x^3 + 3.7690x^4 - 12.5193x^5 + O(x^5), \\ s_1(x) &= 0.6196x^2 - 1.3355x^3 + 3.8381x^4 - 12.5799x^5 + O(x^5), \\ r_0(y) &= 0.0031y^2 + 0.0184y^3 + 0.0106y^4 + 0.0031y^5 + O(y^5), \\ r_1(y) &= 0.0088y^2 + 0.0623y^3 + 0.0231y^4 + 0.0088y^5 + O(y^5). \end{aligned}$$

The following figure visualizes these invariant fiber bundles.

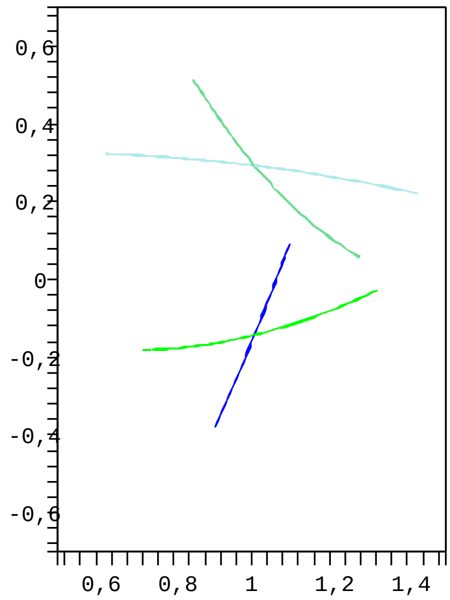

Figure 2. Locally stable and unstable fiber bundle corresponding to the 2-periodic orbit $\{\nu(0), \nu(1)\}$ the Henon map (5.2)

Appendix: A Manual to IFB_Comp

To approximate the infinite sums (4.4) and (4.5) in Theorem 4.3 we have written the `Maple` program `IFB_Comp`, which can be downloaded from the URL

`http://www.math.uni-augsburg.de/ana/dyn_sys/visual_e.hmtl`

In this appendix we present a few remarks on the usage of `IFB_Comp` assuming the reader is familiar with the computer algebra system `Maple`. One essentially has to proceed in two steps:

(1) Input the system data (dimensions, linear and nonlinear part) as explained in the program.

(2) Then execute the procedures `Main` and `Output`.
- `Main = Main(`k^-`,`k^+`,`p`,`o`,`b`)` is the procedure to compute the Taylor approximation of order o for the k-fibers of the pseudo-stable or pseudo-unstable bundle for $k = k^-, \dots, k^+$. The argument p (standard value: 10) describes how many terms of the infinite sums in (4.4) and (4.5) are computed. To compute the pseudo-stable bundle choose $b := 0$, for the pseudo-unstable bundle set $b := 1$.
- `Output=Output(`b`, `k`, `x^-`, `x^+`, `p`)` is the procedure to plot the k-fiber of the corresponding bundle. As in the procedure `Main`, b stands for type of the bundle. x^- and x^+ determine the area of output which is given by $[x^-, x^+]^N$. The argument p (standard value: 1000) describes the accuracy of the output.

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