TAYLOR APPROXIMATION OF INVARIANT FIBER BUNDLES FOR NONAUTONOMOUS DIFFERENCE EQUATIONS

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ABSTRACT. Invariant fiber bundles generalize invariant manifolds to nonautonomous difference equations. In this paper we develop a method to calculate their Taylor approximation, which is of crucial importance, e.g., for an application of the reduction principle in a nonautonomous setting.

1. INTRODUCTION

Many processes in physics, biology or other sciences are modeled by nonlinear *autonomous* difference equations (maps). To understand their often complicated dynamical behavior it is a well-established and powerful tool to use the concept of invariant manifolds. By doing so, e.g., in dynamical bifurcation theory, it is often possible to reduce the dimension of the equations considerably, since bounded bifurcating objects like stationary (periodic, homoclinic, ...) solutions or invariant curves typically lay on an invariant manifold, the center manifold. Another more classical application comes from stability theory in critical cases: If the linearization of a difference equation in an equilibrium possesses eigenvalues inside and on the unit circle of the complex plane, then stability properties depend on the equation reduced to its center-unstable manifold. In any case, to carry out this reduction, one needs to know the invariant manifolds quantitatively, or at least an approximation of them.

The paper at hand provides an important necessary tool to apply such a reduction in a *nonautonomous* framework, which can be motivated from an applied, as well as from a purely mathematical perspective: Concerning the applications, many models become more realistic if their intrinsic parameters are assumed to be time-dependent. On the other inner-mathematical side, the investigation of the dynamical behavior close to a non-constant solution canonically leads to a nonautonomous problem in form of the equation of perturbed motion (see Remark 3.1(1)). Having, for instance, a nonautonomous bifurcation theory available (cf. [Joh89, JY94, JM03, LRS02]), it is our hope that the introduced procedures can be helpful to simplify and reduce problems.

More precisely, we present a formal approach to compute higher order local approximations of invariant fiber bundles near steady states for nonautonomous difference equations. The invariant fiber bundles under consideration canonically generalize invariant manifolds to explicitly time-depended right-hand sides and include general pseudo-stable/-unstable manifolds, like, e.g., the classical stable/unstable, the above mentioned center-stable/-unstable, as well as strongly stable/unstable manifolds. The desired Taylor coefficients are determined by bounded solutions of a linear difference equation in the space of multilinear mappings. Furthermore, we provide an explicit expression for these solutions in terms of so-called Lyapunov-Perron sums (cf. Theorem 4.2) and indicate how to compute them numerically.

For autonomous difference equations, such approximations via Taylor expansions have been studied, e.g., in the monograph [Kuz95, pp. 151–165, Section 5.4] or the papers [Has80, Sim89, BK98, EvP04]. There the situation is simpler, since Taylor coefficients of invariant manifolds

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are (uniquely) determined by algebraic equations, i.e., so-called multilinear Sylvester equations. Hence, our overall approach is not completely new, since it canonically generalizes results from [BK98, EvP04] or explicit analytical computations (see, e.g., the projection method in [Kuz95]). Nevertheless, beyond the importance of nonautonomous techniques, we think it is useful and interesting to show that the algebraic problems from the well-established autonomous theory become problems related to perturbation theory of difference equations in a nonautonomous setting.

The outline of the paper is as follows. First, in Section 2, we establish our basic terminology and a crucial result on the existence of bounded solutions for linear difference equations in spaces of multilinear mappings. Section 3 sets up the necessary theoretical background on invariant fiber bundles; in particular it addresses the question of their uniqueness. In Section 4 we derive a linear difference equation for the Taylor coefficients of the invariant fiber bundles and solve it analytically. We demonstrate our results in Section 5 by some numerical and analytical examples. The latter ones make use of the reduction principle for nonautonomous difference equations, i.e., the fact that stability properties are completely determined by the behavior on the center-unstable fiber bundle (cf. [Pöt04]).

We close this introduction by pointing out different approaches to the numerical computation of invariant manifolds for difference equations: [Omb95] is based on the Lyapunov-Perron method, while [FK94] uses the graph transform method and the results of [HOV95] are based on invariant foliations. [DH97] use subdivision techniques to obtain global approximations and [ARS04a, ARS04b] generalize the corresponding results to nonautonomous difference equations. These approaches are based on the approximation of pullback attractors as considered, e.g., in [Kl00, CKS01].

Finally, for related results and further references in the continuous case of ordinary differential equations we refer to [PR04]. There, the methods are partially parallel to the present paper; yet, some differences need to be pointed out:

- In case of nonautonomous ODEs, the invariant fiber bundles are typically denoted as integral manifolds. Then the invariance equation for such manifolds is a first order partial differential equation and not a functional equation as in our discrete setting (see (3.6)). Hence, one needs different tools to analyze it, yielding another homological equation (see (4.4)).
- Moreover, the integral manifolds of ODEs need to satisfy certain continuity assumptions for their partial derivatives, which — due to the trivial topology on the integers — are redundant for difference equations.
- On the other hand, in contrast to the continuous case, we do not assume invertibility. We only require a regularity condition for the linearization (see (2.6)), which is crucial also for the existence of the invariant fiber bundles.

2. Preliminaries

Above all, let us introduce our notation. A *discrete interval* is the intersection of a (real) interval with the integers $\mathbb{Z}$, $\mathbb{Z}_\kappa^+ := \{k \in \mathbb{Z} : \kappa \le k\}$ for some $\kappa \in \mathbb{Z}$, and $\mathbb{N}$ are the positive integers. We write $\mathbb{R}$ for the real and $\mathbb{C}$ for the complex field; $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

The Banach spaces $\mathcal{X}, \mathcal{Y}$ are real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$) throughout this paper and their norm is denoted by $\|\cdot\|$. We write $I_{\mathcal{X}}$ for the identity map on $\mathcal{X}$ and $\mathcal{X}^*$ for the dual space of $\mathcal{X}$. With $n \in \mathbb{N}$, $\mathcal{L}_n(\mathcal{X};\mathcal{Y})$ is the Banach space of symmetric n-linear continuous operators from $\mathcal{X}^n$ to $\mathcal{Y}$, $\mathcal{L}_n(\mathcal{X}) := \mathcal{L}_n(\mathcal{X};\mathcal{X})$, $\mathcal{L}(\mathcal{X};\mathcal{Y}) := \mathcal{L}_1(\mathcal{X};\mathcal{Y})$ is the space of continuous homomorphisms from $\mathcal{X}$ to $\mathcal{Y}$ and $\mathcal{L}(\mathcal{X}) := \mathcal{L}(\mathcal{X};\mathcal{X})$ is the space of continuous endomorphisms on $\mathcal{X}$. For a mapping $X \in \mathcal{L}_n(\mathcal{X};\mathcal{Y})$ we abbreviate $Xx_1 \cdots x_n := X(x_1, \dots, x_n)$. With a closed subspace $\mathcal{X}_1 \subseteq \mathcal{X}$ and $T \in \mathcal{L}(\mathcal{X}_1;\mathcal{X})$, we define $X_T \in \mathcal{L}_n(\mathcal{X}_1;\mathcal{Y})$ by

$$X_T x_1 \cdots x_n := X(Tx_1, \dots, Tx_n) \quad \text{for } x_1, \dots, x_n \in \mathcal{X}_1$$

and obtain (cf. [Lan93, p. 68]) the norm estimate

$$\|X_T\| \leq \|T\|^n \|X\| \quad \text{for } n \in \mathbb{N}\,. \tag{2.1}$$

The range of T is denoted by $\mathcal{R}(T) := T(\mathcal{X}_1)$.

For the rest of the paper, $\mathbb{I}$ denotes a discrete interval unbounded above. For an n-tuple of the same vector $x \in \mathcal{X}$ we use the abbreviation $x^n := (x, \ldots, x) \in \mathcal{X}^n$. In a normed space, $B_\rho(x_0)$ is the ball with center x_0 and radius $\rho > 0$. Let $\tilde{U} \subseteq \mathcal{X}$ be nonempty and open. We say a mapping $F : \tilde{U}\times\mathbb{I}\to\mathcal{Y}$ is *uniformly bounded* if it maps bounded subsets of $\tilde{U}$ into bounded sets (uniformly w.r.t. $\mathbb{I}$); that is, if for any bounded $\Omega \subseteq \tilde{U}$ there exists an $M \geq 0$ such that $\|F(x,t)\| \leq M$ for all $x \in \Omega$, $t \in \mathbb{I}$. We write $D\bar{F}$ for the Fréchet derivative of a differentiable mapping $\bar{F} : \tilde{U} \to \mathcal{Y}$, and in case $F : \tilde{U}\times\mathbb{I}\to\mathcal{Y}$ depends differentiably on the first variable, then its partial derivative is denoted by D_1F. Higher order derivatives $D^n\bar{F}$ or D_1^nF are defined inductively.

We use the notation

$$x(k+1) = f(x(k), k) \tag{2.2}$$

to denote ordinary difference equations (OΔEs) with a *right-hand side* $f : \tilde{U}\times\mathbb{I}\to\mathcal{X}$. A sequence $\nu : I\to\mathcal{X}$, $I \subseteq \mathbb{I}$ is a discrete interval, is said to solve (2.2) on $I \subseteq \mathbb{I}$ if $\nu(k+1) = f(\nu(k), k)$ as long as ν exists, i.e., as long as $k+1 \in I$ and $\nu(k) \in \tilde{U}$ holds for $k \in I$. Let λ denote the *general solution* of equation (2.2), i.e., $\lambda(\cdot; k_0, x_0)$ solves (2.2) and satisfies the initial condition $\lambda(k_0; k_0, x_0) = x_0$ for $k_0 \in \mathbb{I}$, $x_0 \in \tilde{U}$. In forward time, $\lambda(\cdot; k_0, x_0)$ can be defined recursively

$$\lambda(k; k_0, x_0) := \begin{cases} x_0 & \text{for } k = k_0 \\ f(\lambda(k-1; k_0, x_0), k-1) & \text{for } k > k_0 \end{cases}, \tag{2.3}$$

as long as $\lambda(k-1; k_0, x_0) \in \tilde{U}$, while solutions of (2.2) need not to exist or need not to be unique in backward time without further assumptions.

Given an operator sequence $A : \mathbb{I}\to\mathcal{L}(\mathcal{X})$ we define the *transition operator* $\Phi(k,\kappa) \in \mathcal{L}(\mathcal{X})$ of the linear OΔE

$$x(k+1) = A(k)x(k) \tag{2.4}$$

in $\mathcal{X}$ as the mapping

$$\Phi(k,\kappa) := \begin{cases} I_{\mathcal{X}} & \text{for } k = \kappa \\ A(k-1)\cdots A(\kappa) & \text{for } k > \kappa \end{cases}.$$

A projection-valued sequence $P_- : \mathbb{I}\to\mathcal{L}(\mathcal{X})$ is said to be an *invariant projector* if

$$P_-(k+1)A(k) = A(k)P_-(k) \quad \text{for } k \in \mathbb{I} \tag{2.5}$$

holds. The complementary projector $P_+ : \mathbb{I}\to\mathcal{L}(\mathcal{X})$, defined by $P_+(k) := I_{\mathcal{X}} - P_-(k)$ for all $k \in \mathbb{I}$, is also an invariant projector. In case

$$\bar{A}(k) := A(k)|_{\mathcal{R}(P_-(k))} : \mathcal{R}(P_-(k))\to\mathcal{R}(P_-(k+1)) \text{ is invertible for all } k \in \mathbb{I}\,, \tag{2.6}$$

we say that P_- is *regular*. Then the so-called *extended transition operator*

$$\Phi_-(k,\kappa) := \begin{cases} \bar{A}(k-1)\cdots\bar{A}(\kappa) & \text{for } \kappa < k \\ I_{\mathcal{R}(P_-(\kappa))} & \text{for } k = \kappa \\ \bar{A}(k)^{-1}\cdots\bar{A}(\kappa-1)^{-1} & \text{for } k < \kappa \end{cases}$$

is well-defined. Particularly in Section 4 we are interested in linear OΔEs in $\mathcal{L}_n(\mathcal{X})$ of the form

$$X(k+1)_{A(k)P_+(k)} = A(k)X(k)_{P_+(k)} \quad \text{and} \quad X(k+1)_{A(k)P_-(k)} = A(k)X(k)_{P_-(k)}\,, \tag{2.7}$$

where P_- is a regular invariant projector. It is worth mentioning that these equations are not OΔEs of the form (2.2) since the projectors $P_\pm(k)$ are noninvertible in general. (Henceforth the symbol $P_\pm$ simultaneously stands for P_+ or P_-, respectively. We proceed similarly with our further notation.) It is easy to see that, given $\kappa \in \mathbb{I}$ and initial state $\Xi \in \mathcal{L}_n\big(\mathcal{X}; \mathcal{R}(P_-(\kappa))\big)$ with $\Xi_{P_+(\kappa)} = \Xi$,

$$\Lambda_+(k,\kappa)\Xi := \Phi_-(k,\kappa)\Xi_{\Phi(\kappa,k)P_+(k)} \quad \text{for } k \le \kappa \tag{2.8}$$

defines the uniquely determined backward solution $\Lambda_+(\cdot,\kappa)\Xi$ of the first equation (in (2.7)) satisfying $(\Lambda_+(k,\kappa)\Xi)_{P_+(k)} = \Lambda_+(k,\kappa)\Xi$ for all $k \le \kappa$. In the same way, given initial time $\kappa \in \mathbb{I}$ and initial state $\Xi \in \mathcal{L}_n\big(\mathcal{X}; \mathcal{R}(P_+(\kappa))\big)$ with $\Xi_{P_-(\kappa)} = \Xi$,

$$\Lambda_-(k,\kappa)\Xi := \Phi(k,\kappa)\Xi_{\Phi_-(\kappa,k)P_-(k)} \quad \text{for } k \ge \kappa \tag{2.9}$$

defines the uniquely determined forward solution $\Lambda_-(\cdot,\kappa)\Xi$ of the second equation (in (2.7)) which fulfills $(\Lambda_-(k,\kappa)\Xi)_{P_-(k)} = \Lambda_-(k,\kappa)\Xi$ for all $k \ge \kappa$.

In order to construct invariant fiber bundles (see Section 3) of nonautonomous OΔEs we need an appropriate hyperbolicity notion for their linear part.

Hypothesis. *Assume the mapping $A : \mathbb{I} \to \mathcal{L}(\mathcal{X})$ satisfies:*

(H_1) *Hypothesis on linear part: The linear difference equation* (2.4) *possesses an* exponential dichotomy*, i.e., there exists a regular invariant projector $P_- : \mathbb{I} \to \mathcal{L}(\mathcal{X})$ such that for all $k, \kappa \in \mathbb{I}$ the estimates*

$$\|\Phi(k,\kappa)P_+(\kappa)\| \le K_+\alpha^{k-\kappa}, \qquad \|\Phi_-(\kappa,k)P_-(k)\| \le K_-\beta^{\kappa-k} \quad \textit{for } \kappa \le k, \tag{2.10}$$

hold with real constants $K_+, K_- \ge 1$, $0 < \alpha < \beta$.

Remark 2.1. (H_1) means that the dichotomy spectrum (cf. [AS01]) of (2.4) is disjoint from (α, β). In the autonomous case, i.e., if $A_0 := A(k)$ does not depend on $k \in \mathbb{I}$, it is sufficient to assume that the spectrum $\sigma(A_0) \subseteq \mathbb{C}$ of the operator $A_0 \in \mathcal{L}(\mathcal{X})$ can be separated into a "pseudo-stable" spectral part $\sigma_+ \subseteq B_\alpha(0)$, $0 < \alpha$, and a disjoint "pseudo-unstable" spectral part σ_- outside a circle with center 0 and radius $\beta > \alpha$ in the complex plane. Then $P_\pm$ are constant (in $k \in \mathbb{I}$) and given by the spectral projectors related to $\sigma_\pm$, respectively (cf. [Kat80]).

Our first result deals with perturbations of such linear systems (2.7) in multilinear spaces $\mathcal{L}_n(\mathcal{X})$. For this, we need the notion of quasiboundedness. With reals $\gamma > 0$ and a fixed integer $\kappa \in \mathbb{I}$, we say a sequence $\nu : \mathbb{I} \to \mathcal{X}$ is γ-*quasibounded* if $\|\nu\|_{\kappa,\gamma} := \sup_{k \in \mathbb{I}} \|\nu(k)\| \gamma^{\kappa-k} < \infty$ holds. Obviously 1-quasiboundedness coincides with the classical notion of boundedness.

Lemma 2.2 (quasibounded solutions)**.** *Suppose (H_1) holds, let $n \in \mathbb{N}$, $\kappa \in \mathbb{I}$, $\gamma > 0$ and assume $H^\pm : \mathbb{I} \to \mathcal{L}_n(\mathcal{X})$ is γ-quasibounded with $H^\pm(k) \in \mathcal{L}_n\big(\mathcal{X}; \mathcal{R}(P_\mp(k+1))\big)$ for $k \in \mathbb{I}$. Then for the OΔE*

$$X(k+1)_{A(k)P_\pm(k)} = A(k)X(k)_{P_\pm(k)} + H^\pm(k)_{P_\pm(k)} \tag{2.11}$$

in $\mathcal{L}_n(\mathcal{X})$ the following holds:

(a) *In case $\gamma < \frac{\beta}{\alpha^n}$, there exists a unique γ-quasibounded solution $\Gamma_+ : \mathbb{I} \to \mathcal{L}_n(\mathcal{X})$ of* (2.11) *with*

$$\Gamma_+(k) = \Gamma_+(k)_{P_+(k)} \in \mathcal{L}_n\big(\mathcal{X}; \mathcal{R}(P_-(k))\big) \quad \textit{for } k \in \mathbb{I}, \tag{2.12}$$

given by

$$\Gamma_+(k) := -\sum_{j=k}^{\infty} \Phi_-(k, j+1)H^+(j)_{\Phi(j,k)P_+(k)} \tag{2.13}$$

and satisfying the estimate $\|\Gamma_+\|_{\kappa,\gamma} \le \frac{K_-K_+^n}{\beta-\gamma\alpha^n} \|H^+\|_{\kappa,\gamma}$,

(b) *in case $\mathbb{I} = \mathbb{Z}$ and $\gamma > \frac{\alpha}{\beta^n}$, there exists a unique γ-quasibounded solution $\Gamma_- : \mathbb{I} \to \mathcal{L}_n(\mathcal{X})$ of* (2.11) *with*

$$\Gamma_-(k) = \Gamma_-(k)_{P_-(k)} \in \mathcal{L}_n\big(\mathcal{X}; \mathcal{R}(P_+(k))\big) \quad \textit{for } k \in \mathbb{I},$$

given by

$$\Gamma_-(k) := \sum_{j=-\infty}^{k-1} \Phi(k, j+1) H^-(j)_{\Phi_-(j,k)P_-(k)}$$

and satisfying the estimate $\|\Gamma_-\|_{\kappa,\gamma} \leq \frac{K_+K_-^n}{\gamma\beta^n-\alpha} \|H^-\|_{\kappa,\gamma}$.

Proof. (a) We subdivide the proof into two steps:

(I) We first consider $H^+(k) \equiv 0$ on $\mathbb{I}$. Then equation (2.11) coincides with (2.7). Let $\Gamma_+ : \mathbb{I} \to \mathcal{L}_n(\mathcal{X})$ be a γ-quasibounded solution of (2.11) satisfying (2.12). Then taking the limit $k \to \infty$ in the inequality

$$\begin{aligned} \|\Gamma_+(\kappa)\| &\overset{(2.8)}{=} \left\|\Phi_-(\kappa, k)\Gamma_+(k)_{\Phi(k,\kappa)P_+(\kappa)}\right\| \\ &\overset{(2.12)}{\leq} \|\Phi_-(\kappa, k)P_-(k)\| \left\|\Gamma_+(k)_{\Phi(k,\kappa)P_+(\kappa)}\right\| \\ &\overset{(2.1)}{\leq} \|\Phi_-(\kappa, k)P_-(k)\| \|\Gamma_+(k)\| \|\Phi(k, \kappa)P_+(\kappa)\|^n \\ &\overset{(2.10)}{\leq} K_-K_+^n \left(\frac{\gamma\alpha^n}{\beta}\right)^{k-\kappa} \|\Gamma_+\|_{\kappa,\gamma} \quad \text{for } k \geq \kappa \end{aligned}$$

yields $\Gamma_+(\kappa) = 0$. Since $\kappa \in \mathbb{I}$ was arbitrary, the zero solution of (2.11) is the only γ-quasibounded solution satisfying (2.12).

(II) We now omit the restriction on H^+ and note that the sequence Γ_+ from (2.13) is well-defined, since the estimate

$$\begin{aligned} \|\Gamma_+(k)\| &\overset{(2.13)}{\leq} \sum_{j=k}^{\infty} \left\|\Phi_-(k, j+1)P_-(j+1)H^+(j)_{\Phi(j,k)P_+(k)}\right\| \\ &\overset{(2.1)}{\leq} \sum_{j=k}^{\infty} \|\Phi_-(k, j+1)P_-(j+1)\| \left\|H^+(j)\right\| \|\Phi(j, k)P_+(k)\|^n \\ &\overset{(2.10)}{\leq} \frac{K_-K_+^n\gamma^{k-\kappa}}{\beta} \sum_{j=k}^{\infty} \left(\frac{\gamma\alpha^n}{\beta}\right)^{j-k} \left\|H^+\right\|_{\kappa,\gamma} \\ &= \frac{K_-K_+^n}{\beta-\gamma\alpha^n} \left\|H^+\right\|_{\kappa,\gamma} \gamma^{k-\kappa} \quad \text{for } k \in \mathbb{I} \end{aligned}$$

holds, which in turn yields the claimed estimate for $\|\Gamma_+\|_{\kappa,\gamma}$. Moreover, it is easy to see from (2.5) that Γ_+ satisfies (2.12). Γ_+ is a solution of (2.11) since

$$\begin{aligned} \Gamma_+(k+1)_{A(k)P_+(k)} &\overset{(2.13)}{\equiv} -\sum_{j=k+1}^{\infty} \Phi_-(k+1, j+1)H^+(j)_{\Phi(j,k+1)P_+(k+1)A(k)P_+(k)} \\ &\overset{(2.5)}{\equiv} -\sum_{j=k}^{\infty} \bar{A}(k)\Phi_-(k, j+1)H^+(j)_{\Phi(j,k)P_+(k)} + H^+(k)_{P_+(k)} \\ &\overset{(2.13)}{\equiv} A(k)\Gamma_+(k)_{P_+(k)} + H^+(k)_{P_+(k)} \quad \text{on } \mathbb{I}. \end{aligned}$$

Finally, the uniqueness statement results from step (I), because the difference of any two γ-quasibounded solutions of (2.11) is a γ-quasibounded solution of (2.7) and therefore identically vanishing.

(b) This can be shown similarly. □

3. Invariant Fiber Bundles

In this section we introduce and summarize some fundamental facts concerning invariant fiber bundles of OΔEs. For the autonomous and center manifold situation, [Car81, pp. 33–36, Section 2.8] or [Shu87, Chapter 5, pp. 33–70] are good references. We, nevertheless, consider nonautonomous OΔEs of the form

$$x(k+1) = A(k)x(k) + F(x(k), k) \tag{3.1$_F$}$$

with a mapping $F : U_0 \times \mathbb{I} \to \mathcal{X}$, where $U_0 \subseteq \mathcal{X}$ is an open convex neighborhood of $0 \in \mathcal{X}$.

Hypothesis. *Let $m \in \mathbb{N}$ and assume the mapping $F : U_0 \times \mathbb{I} \to \mathcal{X}$ satisfies:*

(H_2) *Hypothesis on nonlinearity: F is m-times continuously Fréchet differentiable in the first argument, $F(0, k) \equiv 0$ on $\mathbb{I}$, for the partial derivative of F we have the limit relation*

$$\lim_{x \to 0} D_1 F(x, k) = 0 \quad \textit{uniformly in } k \in \mathbb{I} \tag{3.2}$$

and $D_1^m F$ is uniformly bounded.

Remark 3.1. (1) In applications one typically obtains $(3.1)_F$ from (2.2) as *equation of perturbed motion*. Thereto, let $\nu : \mathbb{I} \to \tilde{U}$ be a fixed reference solution of (2.2) satisfying $B_r(\nu(k)) \subseteq \tilde{U}$ for all $k \in \mathbb{I}$ with some $r > 0$. To investigate the local behavior of (2.2) close to ν, one considers the OΔE $(3.1)_F$ with

$$A(k) := D_1 f(\nu(k), k), \qquad F(x, k) := f(x + \nu(k), k) - f(\nu(k), k) - D_1 f(\nu(k), k)x$$

and $U_0 = B_r(0)$, under the assumptions that $f(\cdot, k)$ is m-times continuously differentiable with a uniformly bounded partial derivative $D_1^m f$ and that

$$\lim_{x \to 0} [D_1 f(x + \nu(k), k) - D_1 f(\nu(k), k)] = 0 \quad \text{uniformly in } k \in \mathbb{I},$$

holds to guarantee the limit relation (3.2).

(2) It is a consequence of the mean value inequality (cf. [Lan93, p. 342, Corollary 4.3]) that under (H_2) also F and its partial derivatives $D_1^n F$ are uniformly bounded for $n \in \{1, \ldots, m-1\}$.

Our next aim is to introduce a nonautonomous counterpart of an invariant manifold for $(3.1)_F$. To that end, let $P_\pm : \mathbb{I} \to \mathcal{L}(\mathcal{X})$ be the invariant projector of (2.4) introduced in (H_1), λ denotes the general solution to $(3.1)_F$ and $U \subseteq U_0$ is an open convex neighborhood of 0. Assume $s^\pm : U \times \mathbb{I} \to \mathcal{X}$ is a mapping, continuously Fréchet differentiable in the first argument and satisfying

$$s^\pm(0, k) \equiv 0 \quad \text{on } \mathbb{I}, \qquad \lim_{x \to 0} D_1 s^\pm(x, k) = 0 \quad \text{uniformly in } k \in \mathbb{I}, \tag{3.3}$$

$$s^\pm(x, k) = s^\pm(P_\pm(k)x, k) \in \mathcal{R}(P_\mp(k)) \quad \text{for } k \in \mathbb{I},\ x \in U. \tag{3.4}$$

Then the graph

$$\mathcal{S}^\pm := \left\{(\kappa, \xi + s^\pm(\xi, \kappa)) \in \mathbb{I} \times \mathcal{X} : \xi \in \mathcal{R}(P_\pm(\kappa)) \cap U\right\}$$

is called a *locally invariant fiber bundle* (IFB for short) of the nonlinear OΔE $(3.1)_F$ if the implication

$$(k_0, x_0) \in \mathcal{S}^\pm \quad \Rightarrow \quad (k, \lambda(k; k_0, x_0)) \in \mathcal{S}^\pm \tag{3.5}$$

holds for all $k \geq k_0$ as long as $\lambda(k; k_0, x_0) \in U$. The *$k$-fiber* of $\mathcal{S}^\pm$ is given by the set

$$\mathcal{S}^\pm(k) := \left\{x \in \mathcal{X} : (k, x) \in \mathcal{S}^\pm\right\}.$$

One speaks of a *C^m-fiber bundle* of $(3.1)_F$ if the partial derivatives $D_1^n s^\pm$ exist and are continuous for $n \in \{1, \ldots, m\}$. In case $U_0 = \mathcal{X}$ we say $\mathcal{S}^\pm$ is a *globally IFB* of $(3.1)_F$ if the implication (3.5) holds for all $k \geq k_0$. Geometrically, the conditions (3.3) imply that the IFB $\mathcal{S}^\pm$ contains the zero solution of $(3.1)_F$, and $\mathcal{S}^\pm$ is fiber-wise tangent to the vector bundle $\{(\kappa, \xi) \in \mathbb{I} \times \mathcal{X} : \xi \in \mathcal{R}(P_\pm(\kappa))\}$, while (3.4) yields that each fiber $\mathcal{S}^\pm(k)$ is a graph over $\mathcal{R}(P_\pm(k)) \cap U$.

Locally IFBs satisfy the following nonlinear functional equation, named as *invariance equation*

$$\begin{aligned} &A(k)s^\pm(\xi, k) + P_\mp(k+1)F(\xi + s^\pm(\xi, k), k) \\ &\qquad = s^\pm(A(k)\xi + P_\pm(k+1)F(\xi + s^\pm(\xi, k), k), k+1) \end{aligned} \tag{3.6}$$

for all $k \in \mathbb{I}$, $\xi \in \mathcal{R}(P_\pm(k)) \cap U$ such that $A(k)\xi + P_\pm(k+1)F(\xi + s^\pm(\xi, k), k) \in U$. Moreover, by passing over to a sufficiently small neighborhood U of 0 it is easy to see (cf. (3.3)–(3.4)) that

$$\mathcal{S}^+ \cap \mathcal{S}^- = \mathbb{I} \times \{0\}$$

holds, i.e., $\mathcal{S}^+$ and $\mathcal{S}^-$ intersect only along the trivial solution of $(3.1)_F$.

$\mathcal{S}^+$ and $\mathcal{S}^-$ are denoted as *pseudo-stable* and *pseudo-unstable* fiber bundle of $(3.1)_F$, respectively. To be more specific, $\mathcal{S}^+$ is called *center-stable* fiber bundle in case $\beta > 1$, *stable* fiber bundle in the hyperbolic situation $\alpha < 1 < \beta$ and *strongly stable* fiber bundle in case $\beta < 1$. Under the additional assumption $\mathbb{I} = \mathbb{Z}$, $\mathcal{S}^-$ is called *center-unstable* fiber bundle in case $\alpha < 1$, *unstable* fiber bundle in the hyperbolic situation $\alpha < 1 < \beta$ and *strongly unstable* fiber bundle in case $1 < \alpha$. In the light of Remark 2.1 this terminology corresponds to the autonomous situation of invariant manifolds considered, e.g., in [Shu87].

Concerning the existence of locally IFBs, due to our general Banach space setting we have to impose the assumption that $\mathcal{X}$ is a C^m-Banach space; that is, the norm on $\mathcal{X}$ is of class C^m away from 0. A characterization of such spaces, as well as examples, are contained in [KM97, pp. 127–152, Section 13]. Then, on $\mathcal{X}$, there exists a *C^m-cut-off function* $\chi : \mathcal{X} \to [0, 1]$ with the properties

$$\chi(x) \equiv 1 \quad \text{on } x \in B_1(0), \qquad \chi(x) \equiv 0 \quad \text{on } x \in \mathcal{X} \setminus B_2(0) \tag{3.7}$$

(cf. [AMR88, p. 473, Lemma 4.2.13]). It is possible to choose $r > 0$ so that $B_{2r}(0) \subseteq U_0$, and define the mapping $F_r : \mathcal{X} \times \mathbb{I} \to \mathcal{X}$,

$$F_r(x, k) := \begin{cases} \chi\left(\frac{x}{r}\right) F(x, k) & \text{for } x \in B_{2r}(0) \\ 0 & \text{else} \end{cases} . \tag{3.8}$$

Hence, F_r satisfies $F_r(x, k) = F(x, k)$ for $x \in B_r(0)$, $k \in \mathbb{I}$, due to (H_2) it has globally bounded partial derivatives $D_1^n F_r$ for $n \in \{1, \dots, m\}$ (uniformly in $k \in \mathbb{I}$), and by [Pöt98, p. 73, Lemma 2.3.2] the following limit relation holds true (cf. (3.2)):

$$\lim_{r \searrow 0} |F_r|_1 = 0 \qquad \text{with } |F_r|_1 := \sup_{(x,k) \in \mathcal{X} \times \mathbb{I}} \|D_1 F_r(x, k)\| . \tag{3.9}$$

Consequently, one can choose $r > 0$ so small that

$$|F_r|_1 < \begin{cases} \dfrac{\min\left\{\frac{\beta-\alpha}{2}, \alpha\left(\sqrt[m]{\frac{\alpha+\beta}{\alpha+\alpha^m}} - 1\right)\right\}}{2(K_+ + K_-)} & \text{if } \alpha^m < \beta \\ \dfrac{\min\left\{\frac{\beta-\alpha}{2}, \beta\left(1 - \sqrt[m]{\frac{\alpha+\beta}{\beta+\beta^m}}\right)\right\}}{2(K_+ + K_-)} & \text{if } \alpha < \beta^m \end{cases} \tag{3.10}$$

holds and we arrive at

Theorem 3.2 (existence of locally IFBs)**.** *Suppose (H_1)–(H_2) hold and that $\mathcal{X}$ is a C^m-Banach space. Then there exist reals $\rho_0 > 0$, $\gamma_0, \dots, \gamma_m \geq 0$ such that the following holds with $U = B_{\rho_0}(0)$:*

(a) *Under the gap condition*

$$\alpha^m < \beta \tag{3.11}$$

the $O\Delta E$ $(3.1)_F$ possesses a local pseudo-stable C^m-IFB $\mathcal{S}^+$,

(b) *for $\mathbb{I} = \mathbb{Z}$ and under the gap condition*

$$\alpha < \beta^m \tag{3.12}$$

the $O\Delta E$ $(3.1)_F$ possesses a local pseudo-unstable C^m-IFB $\mathcal{S}^-$,

(c) *the corresponding mapping $s^\pm : U \times \mathbb{I} \to \mathcal{X}$ satisfies*

$$\left\|D_1^n s^\pm(x, k)\right\| \leq \gamma_n \quad \text{for } x \in U,\ k \in \mathbb{I},\ n \in \{0, \dots, m\}, \tag{3.13}$$

(d) *if the mappings A and F are periodic in k with period $\theta \in \mathbb{N}$, then*

$$s^{\pm}(x, k+\theta) = s^{\pm}(x, k) \quad \text{for } x \in \mathcal{X},\ k \in \mathbb{I}\,,$$

and if the OΔE $(3.1)_F$ is autonomous, then the mapping $s^{\pm}$ is independent of $k \in \mathbb{I}$, i.e., the set $\{\xi + s^{\pm}(\xi) \in \mathcal{X} : \xi \in \mathcal{R}(P_{\pm}) \cap U\}$ is a locally invariant manifold of $(3.1)_F$.

Proof. One shows the existence of the mapping $s^{\pm} : U\times\mathbb{I}\to\mathcal{X}$ by applying a general theorem on IFBs (cf. [PS04, Theorem 3.5]) to the modified OΔE $(3.1)_{F_r}$, where $r > 0$ is chosen so small that (3.10) holds. The smoothness assertion follows from [PS04, Theorem 4.1], and the fact that $s^{\pm}$ satisfies the limit relation in (3.3) can be seen as in [Pöt98, p. 64, Korollar 2.2.15]. After all, the assertion (d) follows from [Aul98, Corollary 4.2]. □

It is well-known that, even under Hypotheses (H_1)–(H_2), e.g., center-unstable fiber bundles are not unique in general (cf. [Pöt04, Example 2.3]). However, they can be obtained as restrictions of uniquely determined globally IFBs of appropriately modified OΔEs, and calculated using Taylor expansions. We will show this under

Hypothesis. *Let $\mathcal{X}$ be a C^m-Banach space and assume:*

(H_3) *$A : \mathbb{I}\to\mathcal{L}(\mathcal{X})$ is bounded.*

Proposition 3.3 (globally IFBs). *Suppose (H_1)–(H_3) hold and let $\mathcal{S}^{\pm}$ denote a C^m-IFB of $(3.1)_F$, where the corresponding mapping $s^{\pm} : U\times\mathbb{I}\to\mathcal{X}$ possesses a uniformly bounded derivative $D_1^m s^{\pm}$. In case $\mathcal{S}^{+}$ is considered, assume* (3.11) *holds, and in case of $\mathcal{S}^{-}$, assume $\mathbb{I} = \mathbb{R}$ and* (3.12)*. Then there exists a $\rho > 0$ and mappings $\bar{F}_{\rho} : \mathcal{X}\times\mathbb{I}\to\mathcal{X}$, $s^{\pm}_{\rho} : \mathcal{X}\times\mathbb{I}\to\mathcal{X}$ such that the following holds:*

(a) *The graph*

$$\mathcal{S}^{\pm}_{\rho} := \left\{(\kappa, \xi + s^{\pm}_{\rho}(\xi, \kappa)) \in \mathbb{I}\times\mathcal{X} : \xi \in \mathcal{R}(P_{\pm}(\kappa))\right\} \tag{3.14}$$

is a globally C^m-IFB of $(3.1)_{\bar{F}_{\rho}}$,

(b) *$\bar{F}_{\rho}(x, k) = F(x, k)$ for all $x \in B_{\rho}(0)$, $k \in \mathbb{I}$,*

(c) *$s^{\pm}_{\rho}(x, k) = s^{\pm}(x, k)$ for all $x \in B_{\rho}(0)$, $k \in \mathbb{I}$, and $\mathcal{S}^{\pm}_{\rho} \cap (\mathbb{I}\times B_{\rho}(0)) = \mathcal{S}^{\pm} \cap (\mathbb{I}\times B_{\rho}(0))$.*

Proof. First of all, let $\Omega \subseteq U_0$ be a neighborhood of 0 in $\mathcal{X}$, we fix a C^m-cut-off function $\chi : \mathcal{X}\to[0, 1]$ satisfying (3.7) as introduced above. Choose a real number $r > 0$ so small that $B_{2r}(0) \subseteq \Omega$ and $B_{3r}(0) \subseteq U_0$. Now the proof is subdivided into two parts.

(I) We start by proving a special case and suppose that

$$\mathcal{V}^{\pm}(\Omega) := \{(\kappa, \xi) \in \mathbb{I}\times\mathcal{X} : \xi \in \mathcal{R}(P_{\pm}(\kappa)) \cap \Omega\}$$

is a locally IFB of $(3.1)_F$; that is, we can represent $\mathcal{V}^{\pm}(\Omega)$ as graph of the mapping $s^{\pm} : \Omega\times\mathbb{I}\to\mathcal{X}$, $s^{\pm}(x, k) \equiv 0$. Then the invariance equation (3.6) for $(3.1)_F$ boils down to

$$P_{\mp}(k+1)F(\xi, k) = 0 \quad \text{for } k \in \mathbb{I},\ \xi \in \mathcal{R}(P_{\pm}(k)) \cap \Omega\,.$$

We define F_r as in (3.8) and obtain

$$P_{\mp}(k+1)F_r(\xi, k) = 0 \quad \text{for } k \in \mathbb{I},\ \xi \in \mathcal{R}(P_{\pm}(k))\,, \tag{3.15}$$

since $P_{\mp}(k+1)F(\xi, k) = 0$ for $\xi \in \mathcal{R}(P_{\pm}(k)) \cap B_{2r}(0)$ and $\chi\left(\frac{x}{r}\right) = 0$ for $\|x\| \geq 2r$ (cf. (3.7)). To verify the (forward) invariance of $\mathcal{V}^{\pm}(\mathcal{X})$ under the modified OΔE $(3.1)_{F_r}$, we pick an arbitrary $k \in \mathbb{I}$ and get for the general solution $\bar{\lambda}$ of $(3.1)_{F_r}$,

$$\begin{aligned} P_{\mp}(k+1)\bar{\lambda}(k+1; k, \xi) &\overset{(2.3)}{=} P_{\mp}(k+1)\left(A(k)\xi + F_r(\xi, k)\right) \\ &\overset{(3.15)}{=} P_{\mp}(k+1)A(k)P_{\pm}(k)\xi \overset{(2.5)}{=} 0 \quad \text{for } \xi \in \mathcal{R}(P_{\pm}(k))\,, \end{aligned}$$

i.e., $\bar{\lambda}(k+1; k, \xi) \in \mathcal{R}(P_{\pm}(k+1))$. Choosing $r > 0$ so small that (3.10) holds, we can apply a general result on the existence of IFBs (cf. [PS04, Theorem 3.5, 4.1]), which yields a unique globally IFB $\mathcal{S}^{\pm}_r$ for $(3.1)_{F_r}$, representable as graph over $\mathcal{V}^{\pm}(\mathcal{X})$. Hence, $\mathcal{S}^{\pm}_r = \mathcal{V}^{\pm}(\mathcal{X})$ and, moreover, the assertions of Proposition 3.3 are evidently satisfied with $s_r^\pm(x,k) \equiv 0$, $\rho = r$ and $\bar{F}_\rho = F_r$.

(II) Now consider the general situation, when $\mathcal{S}^\pm$ is a locally IFB for $(3.1)_F$ given by a mapping $s^\pm : U \times \mathbb{I} \to \mathcal{X}$. We first define the C^m-mapping $s_r^\pm(\cdot,k) : \mathcal{X} \to \mathcal{X}$, $k \in \mathbb{I}$, by

$$s_r^\pm(x,k) := \begin{cases} \chi\left(\frac{x}{r}\right) s^\pm(x,k) & \text{for } x \in B_{2r}(0) \\ 0 & \text{else} \end{cases}. \tag{3.16}$$

Then the partial derivatives $D_1^n s_r^\pm(\cdot,k)$, $n \in \{1,\dots,m\}$, are globally bounded (uniformly in $k \in \mathbb{I}$) and from (3.3) we obtain the limit relation (cf. [Pöt98, p. 73, Lemma 2.3.2])

$$\lim_{r \searrow 0} \left|s_r^\pm\right|_1 = 0\,. \tag{3.17}$$

Particularly it is possible to choose $r > 0$ so small that

$$\left\|D_1 s_r^\pm(x,k)\right\| < \tfrac{1}{2} \quad \text{for } x \in \mathcal{X},\ k \in \mathbb{I} \tag{3.18}$$

holds. Next we define a C^m-diffeomorphism $\Psi_k : \mathcal{X} \to \mathcal{X}$ by

$$\Psi_k(x) := x - s_r^\pm(x,k)\,;$$

the inverse $\Psi_k^{-1} : \mathcal{X} \to \mathcal{X}$ is given by $\Psi_k^{-1}(x) = x + s_r^\pm(x,k)$. Under the change of variables $x \mapsto \Psi_k(x)$ the OΔE $(3.1)_F$ takes the form $(3.1)_G$ with $G : B_{2r}(0) \times \mathbb{I} \to \mathcal{X}$ of class C^m in the first variable and given by

$$\begin{aligned} G(x,k) := {} & A(k) s_r^\pm(x,k) + F(x + s_r^\pm(x,k),k) \\ & - s_r^\pm\big(A(k)x + F(x + s_r^\pm(x,k),k), k+1\big)\,. \end{aligned} \tag{3.19}$$

Note that $G(\cdot,k)$ is defined on $B_{2r}(0)$, since we have from the mean value inequality

$$\left\|x + s_r^\pm(x,k)\right\| \overset{(3.3)}{\leq} \|x\| + \left\|s_r^\pm(x,k) - s_r^\pm(0,k)\right\| \overset{(3.18)}{\leq} \tfrac{3}{2}\|x\| < 3r \quad \text{for } x \in B_{2r}(0),\ k \in \mathbb{I}$$

and therefore the inclusion $x + s_r^\pm(x,k) \in U_0$. Due to (3.3) we have $G(0,k) \equiv 0$ on $\mathbb{I}$; (H_3) and (3.2)–(3.3) leads to $\lim_{x \to 0} D_1 G(x,k) = 0$ uniformly in $k \in \mathbb{I}$. Also the invariance equation (3.6) implies that

$$P_\mp(k+1) G(\xi,k) \overset{(2.5)}{=} 0 \quad \text{for } k \in \mathbb{I},\ \xi \in \mathcal{R}(P_\pm(k)) \cap B_{2r}(0)\,. \tag{3.20}$$

Consequently, $\mathcal{V}^\pm(B_{2r}(0))$ is a locally IFB of $(3.1)_G$, and the results from step (I) imply that $\mathcal{V}^\pm(\mathcal{X})$ is the unique globally IFB of $(3.1)_{G_r}$ with $G_r : \mathcal{X} \times \mathbb{I} \to \mathcal{X}$ given by

$$G_r(x,k) := \begin{cases} \chi\left(\frac{x}{r}\right) G(x,k) & \text{for } x \in B_{2r}(0) \\ 0 & \text{else} \end{cases}.$$

Furthermore, $|G_r|_1 := \sup_{(x,k) \in \mathcal{X} \times \mathbb{I}} \|D_1 G_r(x,k)\|$ can be made smaller than any given positive number. Now, if we apply the inverse transformation $x \mapsto \Psi_k^{-1}(x)$ to $(3.1)_{G_r}$, one gets an OΔE of the form $(3.1)_{\bar{F}_r}$ with $\bar{F}_r : \mathcal{X} \times \mathbb{I} \to \mathcal{X}$,

$$\begin{aligned} \bar{F}_r(x,k) := {} & -A(k) s_r^\pm(x,k) + G_r(x - s_r^\pm(x,k),k) \\ & + s_r^\pm\big(A(k)x + G_r(x - s_r^\pm(x,k),k)), k+1\big)\,. \end{aligned} \tag{3.21}$$

Since $G_r(\cdot,k)$, $s_r^\pm(\cdot,k)$ are of class C^m with globally bounded derivatives, we obtain from (H_3) that also $\bar{F}_r(\cdot,k)$ is of class C^m with globally bounded partial derivatives. The product rule (cf. [Lan93, p. 336]) yields the estimate

$$\left\|D_1 \bar{F}_r(x,k)\right\| \leq 2 \sup_{k \in \mathbb{I}} \|A(k)\| \left|s_r^\pm\right|_1 + |G_r|_1 \left(1 + \left|s_r^\pm\right|_1\right)^2$$

for all $x \in \mathcal{X}$, $k \in \mathbb{I}$, and consequently, for sufficiently small $r > 0$, it is possible to fulfill (3.10).

Finally, choose a real $\rho \in (0,r)$ so small that the inclusion $B_\rho(0) \subseteq \Psi_k^{-1}(B_r(0))$ holds for all $k \in \mathbb{I}$, which is possible due to [Pöt98, p. 160, Lemma A.5.1]. Substituting (3.19) into (3.21) gives us the identity $\bar{F}_\rho(x,k) = F(x,k)$ for $x \in B_\rho(0)$, $k \in \mathbb{I}$. From (3.16) it is obvious that $s_\rho^\pm(x,k) = s(x,k)$ if $x \in B_\rho(0)$. Hence, $\mathcal{S}_\rho^\pm \cap (\mathbb{I} \times B_\rho(0)) = \mathcal{S}^\pm \cap (\mathbb{I} \times B_\rho(0))$. Since $\mathcal{V}^\pm(\mathcal{X})$ is the unique globally IFB of $(3.1)_{G_\rho}$ and $\Psi_k^{-1}(\mathcal{V}^\pm(\mathcal{X})(k)) = \mathcal{S}_\rho^\pm(k)$, the set $\mathcal{S}_\rho^\pm$ is invariant under $(3.1)_{\bar{F}_\rho}$. But we have (3.10), so by [PS04, Theorem 3.5, 4.1], $\mathcal{S}_\rho^\pm$ is the unique globally IFB of $(3.1)_{F_r}$. □

Our next theorem states that all IFBs $\mathcal{S}^\pm$ of $(3.1)_F$ possess the same Taylor series up to order m. Moreover, it enables us to calculate IFBs using approximate solutions of the invariance equation (3.6).

Theorem 3.4 (Taylor expansion)**.** *Suppose (H_1)–(H_3) hold and let $\mathcal{S}^\pm$ denote a C^m-IFB of $(3.1)_F$, where the corresponding mapping $s^\pm : U\times\mathbb{I}\to\mathcal{X}$ possesses a uniformly bounded derivative $D_1^m s^\pm$. In case $\mathcal{S}^+$ is considered, assume* (3.11) *holds, and in case of $\mathcal{S}^-$, assume $\mathbb{I} = \mathbb{R}$ and* (3.12)*. If a mapping $\sigma : \mathcal{X}\times\mathbb{I}\to\mathcal{X}$ is m-times continuously differentiable in the first variable and satisfies*

(i) *$\sigma(0, k) \equiv 0$ on $\mathbb{I}$, $\lim_{x\to 0} D_1\sigma(x, k) = 0$ uniformly in $k \in \mathbb{I}$, $D_1^m\sigma$ is uniformly bounded and $\sigma(x, k) = \sigma(P_\pm(k)x, k) \in \mathcal{R}(P_\mp(k))$ for all $k \in \mathbb{I}$, $x \in \mathcal{X}$,*
(ii) *with a real $r > 0$ so small that $x + \sigma(x, k) \in U_0$ holds for all $k \in \mathbb{I}$, $x \in B_r(0)$, the mapping $\mathcal{M}_k\sigma : B_r(0)\to\mathcal{X}$,*

$$\begin{aligned}(\mathcal{M}_k\sigma)(x) := {} & A(k)\sigma(x, k) + P_\mp(k + 1)F(x + \sigma(x, k), k)\\ & - \sigma(A(k)P_\pm(k)x + P_\pm(k + 1)F(x + \sigma(x, k), k), k + 1)\end{aligned}$$

satisfies

$$D^n(\mathcal{M}_k\sigma)(0) = 0 \quad \text{for } n \in \{1, \dots, m\}\,,\ k \in \mathbb{I}\,, \tag{3.22}$$

then we have $D_1^n\sigma(0, k) = D_1^n s^\pm(0, k)$ for all $k \in \mathbb{I}$, $n \in \{1, \dots, m\}$.

Remark 3.5. The assumption (i) of Theorem 3.4 is satisfied by polynomials of the form

$$\sigma(x, k) = \sum_{n=2}^{m} \sigma_n(k)_{P_\pm(k)}x^n$$

with bounded coefficient sequences $\sigma_n : \mathbb{I}\to\mathcal{L}_n(\mathcal{X})$ satisfying $\sigma_n(k) \in \mathcal{L}_n\big(\mathcal{X}; \mathcal{R}(P_\mp(k))\big)$ for all $n \in \{2, \dots, m\}$, $k \in \mathbb{I}$.

Proof. Define a C^m-diffeomorphism $\Psi_k : \mathcal{X}\to\mathcal{X}$, $k \in \mathbb{I}$, by $\Psi_k(x) := x - \sigma(x, k)$. Then the change of variables $x \mapsto \Psi_k(x)$ transforms the OΔE $(3.1)_F$ into $(3.1)_G$ with

$$\begin{aligned}G(x, k) := {} & A(k)\sigma(x, k) + F(x + \sigma(x, k), k)\\ & - \sigma(A(k)P_\pm(k)x + P_\pm(k + 1)F(x + \sigma(x, k), k), k + 1)\,.\end{aligned}$$

From our assumption (i) we have $G(0, k) \equiv 0$ on $\mathbb{I}$, and a consequence of (3.2) together with (H_3) is $\lim_{x\to 0} D_1G(x, k) = 0$ uniformly in $k \in \mathbb{I}$. Moreover, it follows from (2.5) that $P_\mp(k + 1)G(x, k)_{P_\pm(k)} = (\mathcal{M}_k\sigma)(x)$ and (3.22) yields the identity

$$P_\mp(k + 1)D_1^n G(0, k) \equiv 0 \quad \text{on } \mathbb{I} \tag{3.23}$$

for $n \in \{1, \dots, m\}$. Also the graph $\{(\kappa, \xi + (s^\pm - \sigma)(\xi, \kappa)) \in \mathbb{I}\times\mathcal{X} : \xi \in \mathcal{R}(P_\pm(\kappa))\}$ is a locally IFB for $(3.1)_G$. An application of Proposition 3.3 to $(3.1)_G$ then guarantees the existence of a $\rho > 0$ and a mapping $s_\rho^\pm : \mathcal{X}\times\mathbb{I}\to\mathcal{X}$ with $s_\rho^\pm(x, k) \equiv (s^\pm - \sigma)(x, k)$ on $B_\rho(0)\times\mathbb{I}$. The construction of the mapping $s_\rho^\pm$ in [PS04, Theorem 4.1] in connection with (3.23) implies

$$D_1^n(s^\pm - \sigma)(0, k) \equiv D_1^n s_\rho^\pm(0, k) \equiv 0 \quad \text{on } \mathbb{I}$$

for $n \in \{2, \dots, m\}$. This proves the assertion. □

4. Taylor Expansion of Invariant Fiber Bundles

In the situation that the OΔE $(3.1)_F$ possesses a C^m-IFB $\mathcal{S}^\pm$ with $m \geq 2$, it is natural to approximate the corresponding mapping $s^\pm$ by Taylor expansion. In this section we derive the equations that the corresponding Taylor coefficients need to satisfy, and prove that they are uniquely solvable if certain gap conditions on the linear part of $(3.1)_F$ are satisfied.

Above all, we quote a version of the higher order chain rule for Fréchet derivatives in order to express higher derivatives of the invariance equation (3.6). With given $j, l \in \mathbb{N}$ we write

$$P_j^<(l) := \left\{ (N_1, \ldots, N_j) \;\middle|\; \begin{array}{l} N_i \subseteq \{1, \ldots, l\} \text{ and } N_i \neq \emptyset \text{ for } i \in \{1, \ldots, j\}, \\ N_1 \cup \ldots \cup N_j = \{1, \ldots, l\}, \\ N_i \cap N_k = \emptyset \text{ for } i \neq k,\ i, k \in \{1, \ldots, j\}, \\ \max N_i < \max N_{i+1} \text{ for } i \in \{1, \ldots, j-1\} \end{array} \right\}$$

for the set of *ordered partitions* of $\{1, \ldots, l\}$ with length j and $\# N$ for the cardinality of a finite set $N \subset \mathbb{N}$. In case $N = \{n_1, \ldots, n_k\} \subseteq \{1, \ldots, l\}$ for $k \in \mathbb{N}$, $k \leq l$, we abbreviate $D^k g(x) x_N := D^k g(x) x_{n_1} \cdots x_{n_k}$ for vectors $x, x_1, \ldots, x_l \in \mathcal{X}$, where $g : \mathcal{X} \to \mathcal{X}$ is assumed to be l-times continuously differentiable.

Lemma 4.1 (chain rule)**.** *Given $m \in \mathbb{N}$, open sets $U, V \subseteq \mathcal{X}$ and mappings $g : U \to \mathcal{X}$, $f : V \to \mathcal{X}$ of class C^m with $g(U) \subseteq V$. Then the composition $f \circ g : U \to \mathcal{X}$ is m-times continuously differentiable and for $l \in \{1, \ldots, m\}$, $x \in U$ the derivatives are given by*

$$D^l (f \circ g)(x) x_1 \cdots x_l = \sum_{j=1}^{l} \sum_{(N_1, \ldots, N_j) \in P_j^<(l)} D^j f(g(x)) D^{\# N_1} g(x) x_{N_1} \cdots D^{\# N_j} g(x) x_{N_j}$$

for any $x_1, \ldots, x_l \in \mathcal{X}$.

Proof. See [Ryb91, Theorem 2]. $\square$

We are interested in local approximations of a mapping $s^\pm : U \times \mathbb{I} \to \mathcal{X}$ defining a C^m-IFB of $(3.1)_F$. Taylor's Theorem (cf. [Lan93, p. 350]) together with (3.3) implies the representation

$$s^\pm(x, k) = \sum_{n=2}^{m} \frac{1}{n!} s_n^\pm(k) x^n + R_m^\pm(x, k) \tag{4.1}$$

with coefficient functions $s_n^\pm : \mathbb{I} \to \mathcal{L}_n(\mathcal{X})$ given by $s_n^\pm(k) := D_1^n s^\pm(0, k)$ and a remainder $R_m^\pm$ satisfying $\lim_{x \to 0} \frac{R_m^\pm(x,k)}{\|x\|^m} = 0$. Theorem 3.4 guarantees that $s_n^\pm(k)$ is uniquely determined by the mapping from Theorem 3.2. Due to (3.13) the sequences $s_n^\pm$ are bounded, i.e., one has $\|s_n^\pm(k)\| \leq \gamma_n$ for $k \in \mathbb{I}$, $n \in \{2, \ldots, m\}$ with reals $\gamma_2, \ldots, \gamma_m \geq 0$. We need notational preparations:

- It is convenient to introduce $S^\pm : U \times \mathbb{I} \to \mathcal{X}$, $S^\pm(x, k) := P_\pm(k) x + s^\pm(x, k)$, satisfying

$$D_1 S^\pm(0, k) \overset{(3.3)}{=} P_\pm(k), \qquad D_1^n S^\pm(0, k) = D_1^n s^\pm(0, k) \quad \text{for } k \in \mathbb{I} \tag{4.2}$$

  and $n \in \{2, \ldots, m\}$. Hence, for the derivatives $S_n^\pm(k) := D_1^n S^\pm(0, k)$ we have the estimates

$$\left\| S_1^\pm(k) \right\| \overset{(2.10)}{\leq} K_\pm, \qquad \left\| S_n^\pm(k) \right\| \overset{(3.13)}{\leq} \gamma_n \quad \text{for } n \in \{2, \ldots, m\}. \tag{4.3}$$

- We abbreviate $g^\pm(x, k) := P_\pm(k+1) \left[A(k) x + F(P_\pm(k) x + s^\pm(x, k), k) \right]$ and the chain rule from Lemma 4.1 yields that the corresponding partial derivatives $g_n^\pm(k) := D_1^n g^\pm(0, k)$ are given by (cf. (3.2)–(3.3))

$$g_1^\pm(k) x_1 \overset{(2.5)}{=} A(k) P_\pm(k) x_1,$$

$$g_n^\pm(k) x_1 \cdots x_n = \sum_{l=2}^{n} \sum_{(N_1, \ldots, N_l) \in P_l^<(n)} P_\pm(k+1) D_1^l F(0, k) S_{\# N_1}^\pm(k)_{P_\pm(k)} x_{N_1} \cdots S_{\# N_l}^\pm(k)_{P_\pm(k)} x_{N_l}$$

for all $x_1,\dots,x_n\in\mathcal{X}$ and $n\in\{2,\dots,m\}$. Moreover, the uniform boundedness of D_1^lF (cf. (H_2)) and the estimates (2.1), (2.10), (4.3) imply that $g_n^\pm:\mathbb{I}\to\mathcal{L}_n(\mathcal{X})$ are bounded sequences for $n\in\{2,\dots,m\}$.

Note that the mappings $S^\pm$ and $g^\pm$ satisfy (cf. (3.4))

$$S^\pm(x,k)=S^\pm(P_\pm(k)x,k),\qquad g^\pm(x,k)=g^\pm(P_\pm(k)x,k)\quad\text{for } x\in B_r(0),\ k\in\mathbb{I}\,,$$

where $r>0$ is chosen so small that $S^\pm(x,k)\in U_0$, $g^\pm(x,k)\in U$ for $x\in B_r(0)$, $k\in\mathbb{I}$. Directly from the invariance equation (3.6) and (3.4) we get

$$\begin{aligned}&A(k)s^\pm(x,k)+P_\mp(k+1)F(P_\pm(k)x+s^\pm(x,k),k)\\&\qquad=s^\pm(A(k)P_\pm(k)x+P_\pm(k+1)F(P_\pm(k)x+s^\pm(x,k),k),k+1)\end{aligned}$$

and using the notation introduced above, this reads as

$$A(k)s^\pm(x,k)+P_\mp(k+1)F(S^\pm(x,k),k)=s^\pm(g^\pm(x,k),k+1)$$

for all $k\in\mathbb{I}$, $x\in B_r(0)$. If we differentiate this identity using Lemma 4.1 and set $x=0$, one gets

$$\begin{aligned}&s_n^\pm(k+1)_{A(k)P_\pm(k)}x_1\cdots x_n\\&\qquad+\sum_{l=2}^{n-1}\sum_{(N_1,\dots,N_l)\in P_l^<(n)}s_l^\pm(k+1)g_{\#N_1}^\pm(k)_{P_\pm(k)}x_{N_1}\cdots g_{\#N_l}^\pm(k)_{P_\pm(k)}x_{N_l}\\&=A(k)s_n^\pm(k)_{P_\pm(k)}x_1\cdots x_n+P_\mp(k+1)\big[D_1^nF(0,k)_{P_\pm(k)}x_1\cdots x_n\\&\qquad+\sum_{l=2}^{n-1}\sum_{(N_1,\dots,N_l)\in P_l^<(n)}D_1^lF(0,k)S_{\#N_1}^\pm(k)_{P_\pm(k)}x_{N_1}\cdots S_{\#N_l}^\pm(k)_{P_\pm(k)}x_{N_l}\big]\end{aligned}$$

for $n\in\{2,\dots,m\}$ and $x_1,\dots,x_n\in\mathcal{X}$. Therefore, $s_n^\pm:\mathbb{I}\to\mathcal{L}_n(\mathcal{X})$ is a solution of the linear OΔE

$$X(k+1)_{A(k)P_\pm(k)}=A(k)X(k)_{P_\pm(k)}+H_n^\pm(k)_{P_\pm(k)}\,,\tag{4.4}$$

denoted as *homological equation* for $\mathcal{S}^\pm$ with inhomogeneities $H_n^\pm:\mathbb{I}\to\mathcal{L}_n(\mathcal{X})$ defined by

$$\begin{aligned}H_n^\pm(k)x_1\cdots x_n:=&P_\mp(k+1)\Big[D_1^nF(0,k)_{P^\pm(k)}x_1\cdots x_n\\&+\sum_{l=2}^{n-1}\sum_{(N_1,\dots,N_l)\in P_l^<(n)}\big(D_1^lF(0,k)S_{\#N_1}^\pm(k)_{P_\pm(k)}x_{N_1}\cdots S_{\#N_l}^\pm(k)_{P_\pm(k)}x_{N_l}\\&\qquad-s_l^\pm(k+1)g_{\#N_1}^\pm(k)_{P_\pm(k)}x_{N_1}\cdots g_{\#N_l}^\pm(k)_{P_\pm(k)}x_{N_l}\big)\Big]\,.\end{aligned}\tag{4.5}$$

Obviously, one has $H_2^\pm(k)=P_\mp(k+1)D_1^2F(0,k)_{P_\pm(k)}$ and for $n\in\{3,\dots,m\}$ the values $H_n^\pm(k)$ only depend on $s_2^\pm,\dots,s_{n-1}^\pm$. This leads to the following

Theorem 4.2. *Suppose (H_1)–(H_3) are satisfied and consider a mapping $s^\pm:U\times\mathbb{I}\to\mathcal{X}$ from Theorem 3.2. Then the following holds:*

(a) *The coefficients $s_n^+:\mathbb{I}\to\mathcal{L}_n(\mathcal{X})$, $n\in\{2,\dots,m\}$, in the Taylor expansion* (4.1) *of the mapping $s^+:U\times\mathbb{I}\to\mathcal{X}$ can be determined recursively from the* Lyapunov-Perron sums

$$s_n^+(k)=-\sum_{j=k}^{\infty}\Phi_-(k,j+1)H_n^+(j)_{\Phi(j,k)P_+(k)}\quad\textit{for } n\in\{2,\dots,m\}\ ,\tag{4.6}$$

(b) *in case* $\mathbb{I} = \mathbb{Z}$ *the coefficients* $s_n^- : \mathbb{I} \to \mathcal{L}_n(\mathcal{X})$, $n \in \{2, \dots, m\}$, *in the Taylor expansion* (4.1) *of the mapping* $s^- : U \times \mathbb{I} \to \mathcal{X}$ *can be determined recursively from the* Lyapunov-Perron sums

$$s_n^-(k) = \sum_{j=-\infty}^{k-1} \Phi(k, j+1) H_n^-(j)_{\Phi_-(j,k)P_-(k)} \quad \text{for } n \in \{2, \dots, m\} \,. \tag{4.7}$$

Remark 4.3. For an autonomous OΔE $(3.1)_F$, the sequences (4.6), (4.7) are constant and exactly the stationary solutions of the homological equation (4.4). Then (4.4) reduces to the algebraic problem discussed in [BK98].

Proof. In the explanations preceding Theorem 4.2 we have seen that the sequence $s_n^\pm : \mathbb{I} \to \mathcal{L}_n(\mathcal{X})$ is a bounded solution of the homological equation (4.4). Moreover, it follows recursively from (H_2), (4.3), (2.10) and (4.5) that each inhomogeneity $H_n^\pm$ is bounded, i.e., 1-quasibounded. Consequently, due to the gap conditions (3.11) and (3.12), it yields from Lemma 2.2 that $s_n^\pm$ has the claimed appearance. $\square$

While the infinite series (4.6), (4.7) to determine the Taylor coefficients in Theorem 4.2 provide an analytical solution of our problem, they seem to be of little practical use due to the limit process involved and due to their abstract formulation using multilinear mappings. In the remaining section we try to overcome this deficit:

Corollary 4.1 (error estimates)**.** *Choose a real* $\gamma_n^\pm > \sup_{k \in \mathbb{I}} \|H_n^\pm(k)\|$ *and let* $\varepsilon > 0$ *be arbitrary. Then, for finite approximations to the series* (4.6) *and* (4.7), *the following holds:*

(a) *One has* $\left\| -\sum_{j=k}^{K} \Phi_-(k, j+1) H_n^+(j)_{\Phi(j,k)P_+(k)} - s_n^+(k) \right\| < \varepsilon$ *for all* $k, K \in \mathbb{I}$ *satisfying*

$$K - k > \log_{\frac{\beta}{\alpha^n}} \left(\frac{K_+^n K_- \gamma_n^+}{\varepsilon(\beta - \alpha^n)} \right),$$

(b) *one has* $\left\| \sum_{j=K}^{k-1} \Phi(k, j+1) H_n^-(j)_{\Phi_-(j,k)P_-(k)} - s_n^-(k) \right\| < \varepsilon$ *for all* $k, K \in \mathbb{I}$ *satisfying*

$$k - K > \log_{\frac{\beta^n}{\alpha}} \left(\frac{K_+ K_-^n \gamma_n^-}{\varepsilon(\beta^n - \alpha)} \right).$$

Proof. The assertions yield by easy estimates for the remainder of the geometric series. $\square$

Finally, we present notions from multilinear and tensor algebra (cf., e.g., [Gre78]) to obtain explicit matrix representations of the sequences $s_n^\pm$. Thereto, let $\mathcal{Y}$ and $\mathcal{Z}$ be Banach spaces over $\mathbb{F}$. Given $y_1^*, \dots, y_n^* \in \mathcal{Y}^*$ and $z \in \mathcal{Z}$, we define the element $(y_1^* \vee \cdots \vee y_n^*) \otimes z \in \mathcal{L}_n(\mathcal{Y}; \mathcal{Z})$ by

$$((y_1^* \vee \cdots \vee y_n^*) \otimes z) u_1 \cdots u_n := z \cdot \prod_{\nu=1}^{n} y_\nu^* u_\nu \quad \text{for } u_1, \dots, u_n \in \mathcal{Y} \,.$$

It is worth mentioning that not every element in $\mathcal{L}_n(\mathcal{Y}; \mathcal{Z})$ possesses such a representation.

Let $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Z}_1$ and $\mathcal{Z}_2$ be finite-dimensional Banach spaces over $\mathbb{F}$ with $N := \dim \mathcal{Y}_1 = \dim \mathcal{Y}_2$ and $M := \dim \mathcal{Z}_1 = \dim \mathcal{Z}_2$. We choose $S \in \mathcal{L}(\mathcal{Y}_2; \mathcal{Y}_1)$ and $T \in \mathcal{L}(\mathcal{Z}_1; \mathcal{Z}_2)$ arbitrarily and denote by $S^* \in \mathcal{L}(\mathcal{Y}_1^*; \mathcal{Y}_2^*)$ the dual linear mapping of S, i.e., $(S^* y^*) u := y^*(Su)$ for all $y^* \in \mathcal{Y}_1^*$ and $u \in \mathcal{Y}_2$. Then $(S^* \vee \cdots \vee S^*) \otimes T \in \mathcal{L}(\mathcal{L}_n(\mathcal{Y}_1; \mathcal{Z}_1); \mathcal{L}_n(\mathcal{Y}_2; \mathcal{Z}_2))$ is defined by

$$\begin{aligned} &\big(((S^* \vee \cdots \vee S^*) \otimes T)((y_1^* \vee \cdots \vee y_n^*) \otimes z)\big) u_1 \cdots u_n \\ &\qquad := ((S^* y_1^* \vee \cdots \vee S^* y_n^*) \otimes Tz) u_1 \cdots u_n = Tz \cdot \prod_{\nu=1}^{n} (S^* y_\nu^*) u_\nu \\ &\qquad = Tz \cdot \prod_{\nu=1}^{n} y_\nu^*(S u_\nu) = T((y_1^* \vee \cdots \vee y_n^*) \otimes z)(S u_1, \dots, S u_n) \end{aligned}$$

for any $(y_1^* \vee \cdots \vee y_n^*) \otimes z \in \mathcal{L}_n(\mathcal{Y}_1; \mathcal{Z}_1)$ and $u_1, \ldots, u_n \in \mathcal{Y}_2$. This implies

$$((S^* \vee \cdots \vee S^*) \otimes T)(X) = TX_S \quad \text{for } X \in \mathcal{L}_n(\mathcal{Y}_1; \mathcal{Z}_1)\,. \tag{4.8}$$

Let $\{y_1^i, \ldots, y_N^i\}$ and $\{z_1^i, \ldots, z_M^i\}$ be ordered bases of $\mathcal{Y}_i$ and $\mathcal{Z}_i$, respectively ($i \in \{1, 2\}$). We denote by $\{y_1^{i*}, \ldots, y_N^{i*}\}$ the corresponding dual basis of $\mathcal{Y}_i^*$. Then the set

$$\left\{(y_{r_1}^{i*} \vee \cdots \vee y_{r_n}^{i*}) \otimes z_r^i : 1 \leq r_1 \leq \ldots \leq r_n \leq N,\, r \in \{1, \ldots, M\}\right\} \tag{4.9}$$

is a basis of $\mathcal{L}_n(\mathcal{Y}_i; \mathcal{Z}_i)$. This basis contains $K := Md(N, n)$ elements, where $d(j, l) := \binom{j+l-1}{l}$ for $j, l \in \mathbb{N}$. We order this basis lexicographically with priority to the first components of $(r_1, \ldots, r_n, r)$. Let $\hat{S} \in \mathbb{F}^{N \times N}$ and $\hat{T} \in \mathbb{F}^{M \times M}$ be the matrix representations of S and T, respectively. Then $\hat{S}^T$ is the matrix representation for S^*, and it is possible to show that the matrix representation for the product $(S^* \vee \cdots \vee S^*) \otimes T \in \mathcal{L}(\mathcal{L}_n(\mathcal{Y}_1; \mathcal{Z}_1); \mathcal{L}_n(\mathcal{Y}_2; \mathcal{Z}_2))$ is given by

$$(\hat{S}^T \vee \cdots \vee \hat{S}^T) \otimes \hat{T} \in \mathbb{F}^{K \times K}\,,$$

where $\otimes$ is the Kronecker product, and we define $Q_m := \bigvee_{j=1}^m Q \in \mathbb{F}^{d(N,m) \times d(N,m)}$ for a matrix $Q \in \mathbb{F}^{N \times N}$ recursively by

$$Q_1 := Q\,, \qquad Q_{m+1} := \begin{pmatrix} q_{11} Q_m^{(N,N)} & q_{12} Q_m^{(N,N-1)} & \cdots & q_{1N} Q_m^{(N,1)} \\ q_{21} Q_m^{(N-1,N)} & q_{22} Q_m^{(N-1,N-1)} & \cdots & q_{2N} Q_m^{(N-1,1)} \\ \vdots & \vdots & & \vdots \\ q_{N1} Q_m^{(1,N)} & q_{N2} Q_m^{(1,N-1)} & \cdots & q_{NN} Q_m^{(1,1)} \end{pmatrix}$$

with the matrices $Q_m^{(j,l)}$ consisting of the last $d(j, m)$ rows and last $d(l, m)$ columns of the matrix Q_m for $l, j \in \{1, \ldots, N\}$.

We now define $\mathcal{Y}_k := \mathcal{R}(P_\pm(k))$ and $\mathcal{Z}_k := \mathcal{R}(P_\mp(k))$ for all $k \in \mathbb{I}$ and consider the linear mappings $\Phi(k, \kappa) : \mathcal{R}(P_+(\kappa)) \to \mathcal{R}(P_+(k))$ and $\Phi_-(k, \kappa) : \mathcal{R}(P_-(\kappa)) \to \mathcal{R}(P_-(k))$ and the multilinear mappings $s_n^\pm(k) \in \mathcal{L}_n(\mathcal{Y}_k; \mathcal{Z}_k)$ and $H_n^\pm(k) \in \mathcal{L}_n(\mathcal{Y}_k; \mathcal{Z}_{k+1})$. Due to (4.8), the formulas of Theorem 4.2 can then be written as

$$s_n^+(k) = -\sum_{j=k}^{\infty} ((\Phi(j, k)^* \vee \cdots \vee \Phi(j, k)^*) \otimes \Phi_-(k, j+1))(H_n^+(j))\,,$$

$$s_n^-(k) = \sum_{j=-\infty}^{k-1} ((\Phi_-(j, k)^* \vee \cdots \vee \Phi_-(j, k)^*) \otimes \Phi(k, j+1))(H_n^-(j))\,.$$

For $k \in \mathbb{I}$ let $\{y_1^k, \ldots, y_N^k\}$ and $\{z_1^k, \ldots, z_M^k\}$ be ordered bases of $\mathcal{Y}_k$ and $\mathcal{Z}_k$, respectively, and let (4.9) be an ordered basis of $\mathcal{L}_n(\mathcal{Y}_k; \mathcal{Z}_k)$. Finally, let $\mathrm{vec}_k : \mathcal{L}_n(\mathcal{Y}_k; \mathcal{Z}_k) \to \mathbb{F}^{K \times K}$ be the isomorphism which assigns to each multilinear form its coordinate vector with respect to the basis (4.9). If we write $\hat{\Phi}(k, \kappa)$ and $\hat{\Phi}_-(k, \kappa)$ for the matrix representations of the mappings $\Phi(k, \kappa)$ and $\Phi_-(k, \kappa)$, respectively, then the corresponding matrix equations for the formulas of Theorem 4.2 are given by

$$\mathrm{vec}_k(s_n^+(k)) = -\sum_{j=k}^{\infty} ((\hat{\Phi}(j, k)^T \vee \cdots \vee \hat{\Phi}(j, k)^T) \otimes \hat{\Phi}_-(k, j+1))\, \mathrm{vec}_j(H_n^+(j))\,,$$

$$\mathrm{vec}_k(s_n^-(k)) = \sum_{j=-\infty}^{k-1} ((\hat{\Phi}_-(j, k)^T \vee \cdots \vee \hat{\Phi}_-(j, k)^T) \otimes \hat{\Phi}(k, j+1))\, \mathrm{vec}_j(H_n^-(j))\,.$$

5. Examples

In this section we present several examples — basically from mathematical biology — to illustrate our results. We, however, disclaim a biological interpretation of the variables involved and refer to the corresponding references.

Example 5.1 (discrete epidemic model, cf. [CCL77])**.** Let $(\alpha_k)_{k\in\mathbb{I}}$, $(\beta_k)_{k\in\mathbb{I}}$ denote bounded real sequences and $\gamma > 0$. Consider the 1-dimensional second-order nonautonomous OΔE

$$y(k+2) = (1-\alpha_k y(k+1) - \beta_k y(k))\left(1-e^{-\gamma y(k+1)}\right), \tag{5.1}$$

which is equivalent to the 2-dimensional first-order system (2.2) with

$$f(x_1,x_2,k) := \begin{pmatrix} x_2 \\ (1-\alpha_k x_2 - \beta_k x_1)\left(1-e^{-\gamma x_2}\right) \end{pmatrix}.$$

The linear transformation $x \mapsto Tx$ with $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & \gamma \end{smallmatrix}\right)$, $T^{-1} = \left(\begin{smallmatrix} 1 & -\frac{1}{\gamma} \\ 0 & \frac{1}{\gamma} \end{smallmatrix}\right)$ applied to (2.2) yields the OΔE

$$x(k+1) = \begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix} x(k) + F(x(k),k), \tag{5.2}$$

where we have abbreviated

$$F(x_1,x_2,k) := \left[\gamma x_2 - \frac{1-\alpha_k\gamma x_2 - \beta_k(x_1+x_2)}{\gamma}\left(1-e^{-\gamma^2 x_2}\right)\right]\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Evidently, (5.2) satisfies (H_1)–(H_3) with an arbitrary $m \in \mathbb{N}$, where the dichotomy data is given by $\alpha \in (0,\gamma)$, $\beta = \gamma$, $K_\pm = 1$ and $P_+ = \left(\begin{smallmatrix} 1 & \\ & 0 \end{smallmatrix}\right)$; consequently, Theorem 3.2 applies. Concerning the pseudo-stable fiber bundle $\mathcal{S}^+$ of (5.2) it is easy to see that $\mathcal{S}^+ = \mathbb{I}\times\mathbb{R}\times\{0\}$ and $s^+(x_1,k) \equiv 0$ holds. On the other hand, in case $\mathbb{I} = \mathbb{Z}$, formula (4.7) from Theorem 4.2 implies that the coefficients s_n^- of the pseudo-unstable fiber bundle $\mathcal{S}^-$ of (5.2) can be computed explicitly; the first three are given by

$$\begin{aligned} s_2^-(k) =& \tfrac{1}{\gamma}\left(\gamma^2 + 2\alpha_{k-1}\gamma + 2\beta_{k-1}\right), \\ s_3^-(k) =& \tfrac{3\beta_{k-1}}{\gamma^2}s_2^-(k-1) + \tfrac{3\gamma^3+6\alpha_{k-1}\gamma^2+6\gamma\beta_{k-1}}{\gamma^2}s_2^-(k) - 3\alpha_{k-1}\gamma - 3\beta_{k-1} - \gamma^2, \\ s_4^-(k) =& \tfrac{12\beta_{k-1}}{\gamma^2}s_2^-(k-1)s_2^-(k) - \tfrac{6\beta_{k-1}}{\gamma}s_2^-(k-1) \\ & - \tfrac{24\gamma^3\beta_{k-1}+12\gamma\beta_{k-1}^2+7\gamma^5+24\gamma^4\alpha_{k-1}+12\gamma^3{\alpha_{k-1}}^2+24\gamma^2\alpha_{k-1}\beta_{k-1}}{\gamma^3}s_2^-(k) \\ & + \tfrac{4\beta_{k-1}}{\gamma^3}s_3^-(k-1) + \tfrac{12\beta_{k-1}\gamma^2+6\gamma^4+12\alpha_{k-1}\gamma^3}{\gamma^3}s_3^-(k) + \gamma^3 + 4\alpha_{k-1}\gamma^2 + 4\beta_{k-1}\gamma. \end{aligned} \tag{5.3}$$

The stability properties of the zero solution of (5.2) (or (5.1)) depend on the parameter γ. We have asymptotic stability in case $\gamma \in (0,1)$ (cf. [Aga92, p. 256, Corollary 5.6.3]), instability for $\gamma \in (1,\infty)$ (cf. [Aga92, p. 256, Theorem 5.6.4]), and the critical situation $\gamma = 1$ will be considered in Example 5.5.

Example 5.2 (flour beetle model, cf. [KC96])**.** Let $a \in (0,1)$, $b > 0$ be reals and $(\beta_k)_{k\in\mathbb{I}}$, $(\delta_k)_{k\in\mathbb{I}}$ denote bounded sequences in $[0,\infty)$. We consider the 1-dimensional third-order nonautonomous OΔE

$$y(k+3) = ay(k+2) + by(k)e^{-\beta_k y(k+2) - \delta_k y(k)}, \tag{5.4}$$

which is equivalent to the 3-dimensional first-order system (2.2) with

$$f(x_1,x_2,x_3,k) := \begin{pmatrix} x_2 \\ x_3 \\ ax_3 + bx_1 e^{-\beta_k x_3 - \delta_k x_1} \end{pmatrix}.$$

The time-varying coefficients β_k, δ_k describe the only significant source of pupal mortality in (5.4), the adult cannibalism (cf. [KC96]). For the sake of our analysis we retreat to the situation $a = \frac{b^2-\gamma^6}{b\gamma^2}$, where $\gamma > 0$ is a real number. This implies that $D_1 f(0,k)$ possesses a pair of complex-conjugated eigenvalues with modulus γ. To guarantee $a \in (0,1)$ we additionally assume

$$\gamma \in \left(\sqrt{\omega - \tfrac{b}{3\omega}}, \sqrt[3]{b}\right) \qquad \text{with } \omega := \sqrt[3]{\frac{b^2}{2} + \frac{\sqrt{3b^3(4+27b)}}{18}}.$$

The linear transformation $x \mapsto Tx$ with

$$T := \begin{pmatrix} \frac{\gamma^6-2b^2}{2b^2\gamma^2} & \frac{\gamma\sqrt{4b^2-\gamma^6}}{2b^2} & 1 \\ -\frac{\gamma^2}{2b} & -\frac{\sqrt{4b^2-\gamma^6}}{2b\gamma} & \frac{b}{\gamma^2} \\ 1 & 0 & \frac{b^2}{\gamma^4} \end{pmatrix}$$

applied to (2.2) yields the OΔE

$$x(k+1) = \begin{pmatrix} \sigma & \rho & 0 \\ -\rho & \sigma & 0 \\ 0 & 0 & \frac{b}{\gamma^2} \end{pmatrix} x(k) + F(x(k), k)\,, \tag{5.5}$$

with $\sigma := -\frac{\gamma^4}{2b}$, $\rho := \frac{\gamma\sqrt{4b^2-\gamma^6}}{2b}$ and we have abbreviated

$$F(x,k) := T^{-1} f(Tx,k) - \begin{pmatrix} \sigma & \rho & 0 \\ -\rho & \sigma & 0 \\ 0 & 0 & \frac{b}{\gamma^2} \end{pmatrix} x\,.$$

It is easy to see that (5.5) satisfies (H_1)–(H_3), where the dichotomy data is given by $\alpha = \gamma$, $\beta = \frac{b}{\gamma^2}$, $K_\pm = 1$ and $P_+ := \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$. Again, Theorem 3.2 applies, and to obtain a quadratic approximation to the pseudo-stable fiber bundle $\mathcal{S}^+$ we make the ansatz

$$s^+(x_1,x_2,k) = s_{20}(k)x_1^2 + s_{11}(k)x_1x_2 + s_{02}(k)x_2^2 + O(3)\,;$$

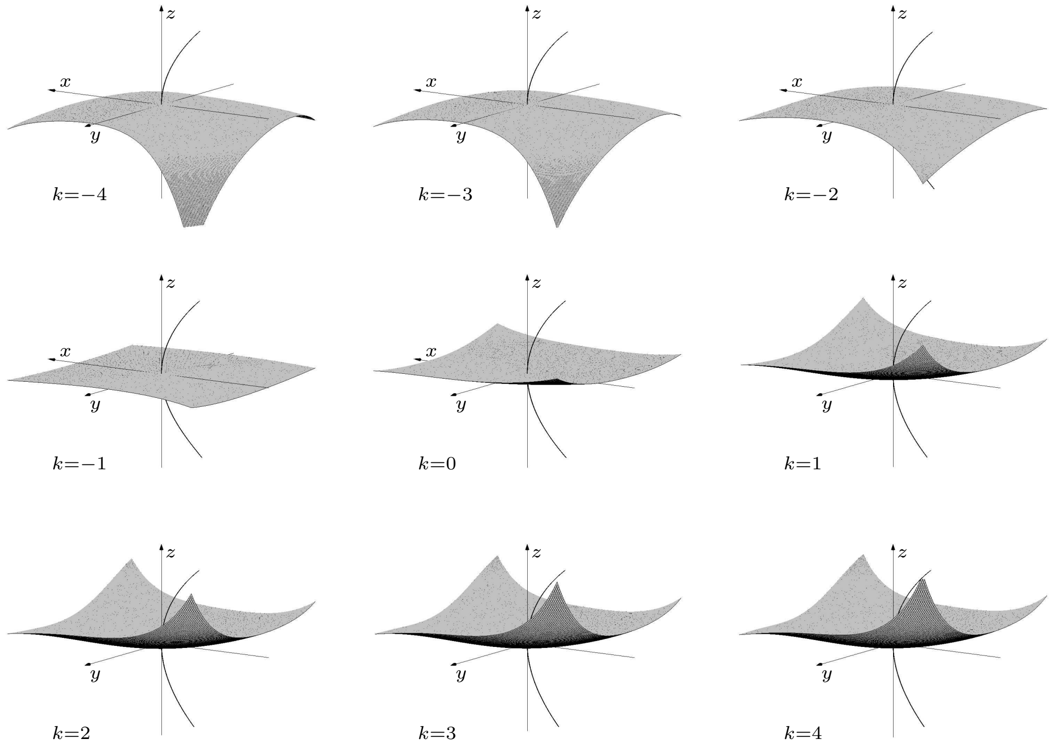
then the homological equation (4.4) for $s_2^+ : \mathbb{I} \to \mathcal{L}_2(\mathbb{R}^2;\mathbb{R}) \cong \mathbb{R}^3$ reduces to the linear system

$$\begin{aligned}\begin{pmatrix} s_{20}(k+1) \\ s_{11}(k+1) \\ s_{02}(k+1) \end{pmatrix} &= \begin{pmatrix} \frac{\gamma^2}{4b} & \frac{\sqrt{4b^2-\gamma^6}}{4b\gamma} & \frac{\gamma^6-4b^2}{4b\gamma^4} \\ \frac{\sqrt{4b^2-\gamma^6}}{2b\gamma} & \frac{\gamma^6-2b^2}{2b\gamma^4} & \frac{\sqrt{4b^2-\gamma^6}}{2b\gamma} \\ \frac{\gamma^6-4b^2}{4b\gamma^4} & \frac{\sqrt{4b^2-\gamma^6}}{4b\gamma} & \frac{\gamma^2}{4b} \end{pmatrix} \begin{pmatrix} s_{20}(k) \\ s_{11}(k) \\ s_{02}(k) \end{pmatrix} \\ &\quad + \begin{pmatrix} \frac{\gamma^4(3b^2-\gamma^6)(b^2\gamma^2\beta_k - 3b^2\delta_k + \gamma^6\delta_k)}{4b^5(2\gamma^6+b^2)} \\ \frac{\gamma(\gamma^6-4b^2)(2\gamma^2b^4\beta_k - 3b^4\delta_k - \gamma^8\beta_k b^2 + 4b^2\gamma^6\delta_k - \gamma^{12}\delta_k)}{2\sqrt{(\gamma^6-4b^2)}(2\gamma^6+b^2)b^5} \\ \frac{(\gamma^6-4b^2)(b^2-\gamma^6)(-\delta_k b^2 + b^2\gamma^2\beta_k + \gamma^6\delta_k)}{4\gamma^2b^5(2\gamma^6+b^2)} \end{pmatrix}.\end{aligned}$$

On the other side, for $\mathbb{I} = \mathbb{Z}$ the quadratic coefficient $s_2^- : \mathbb{I} \to \mathcal{L}_2(\mathbb{R};\mathbb{R}^2) \cong \mathbb{R}^2$ of the pseudo-unstable fiber bundle $\mathcal{S}^-$ is a solution of the linear OΔE

$$s_2^-(k+1) = \frac{\gamma^5}{2b^3} \begin{pmatrix} -\gamma^3 & \sqrt{4b^2-\gamma^6} \\ \sqrt{4b^2-\gamma^6} & -\gamma^3 \end{pmatrix} s_2^-(k) - \begin{pmatrix} \frac{4\gamma^6(\beta_k b^2 + \gamma^4\delta_k)}{b(2\gamma^6+b^2)} \\ \frac{4\gamma^3(\beta_k b^4 + \delta_k\gamma^4 b^2 - \gamma^6\beta_k b^2 - \gamma^{10}\delta_k)}{b\sqrt{4b^2-\gamma^6}(2\gamma^6+b^2)} \end{pmatrix}.$$

In order to approximate the IFBs $\mathcal{S}^+$ and $\mathcal{S}^-$ of (5.5) numerically using Theorem 4.2, we fix the parameters $b := \frac{11}{10}$, $\gamma := \frac{9}{10}$ (leading to $a = \frac{678559}{891000}$) and consider cannibalism rates $\beta_k := 1 - \frac{1}{\pi}\arctan k$, $\delta_k := 1 + \frac{1}{\pi}\arctan k$. Hence, cannibalism becomes stationary as $k \to \pm\infty$. Then the dichotomy rates for (5.5) are given by $\alpha = \frac{9}{10}$, $\beta = \frac{110}{81}$. We have computed an approximation of the stable and unstable fiber bundle of (5.5) up to order 6. The following figure visualizes corresponding fibers $\mathcal{S}^-(k), \mathcal{S}^+(k)$ for $k \in \{-4,\dots,4\}$ in the cube $[-0.6, 0.6]^3$.



The URL `http://www.math.uni-augsburg.de/ana/dyn_sys/visual_e.hmtl` contains an animation of these fiber bundles for $k \in \{-20, \dots, 20\}$, as well as a MAPLE program to calculate them.

Using the following theorem, one is able to maintain stability properties of solutions for $(3.1)_F$ from the corresponding properties of the zero solution of a finite dimensional OΔE — provided the center-unstable fiber bundles are known. For the definition of the corresponding stability notions we refer to, e.g., [Aga92, p. 240, Definition 5.4.1].

Theorem 5.3 (reduction principle)**.** *Suppose* (H_1)–(H_2) *are satisfied with* $\alpha < 1$, $\mathbb{I} = \mathbb{Z}$ *and a constant invariant projector* P_+ *with finite-dimensional kernel. If the zero solution of the* reduced equation

$$x(k+1) = A(k)x(k) + P_- F(x(k) + s^-(x(k), k), k) \tag{5.6}$$

in $\mathcal{R}(P_-)$ *is stable (asymptotically stable, unstable, respectively), then also the zero solution of* $(3.1)_F$ *is stable (asymptotically stable, unstable, respectively).*

Proof. See [Pöt04, Theorem 3.5]. □

Consider the OΔE (2.2) with a scalar right-hand side $f : \tilde{U} \times \mathbb{I} \to \mathbb{R}$ such that $f(0, k) \equiv 0$ on $\mathbb{I}$, where $\tilde{U} \subseteq \mathbb{R}$ is an open neighborhood of 0. In addition to the notions of stability mentioned in Theorem 5.3, we define the following notions of semi-stability: the zero solution of (2.2) is called *left-stable* (*right-stable*) if the corresponding stability definitions hold in left-sided (right-sided) neighborhoods of 0. The next proposition provides sufficient conditions concerning the right-hand side of (2.2) for the stability behavior of the zero solution.

Proposition 5.4 (stability of 1-dimensional nonautonomous OΔEs)**.** *Assume that* f *is* 2*-times continuously differentiable w.r.t. the first variable. Then the following holds:*

(a) *If there exists a* $\kappa \in \mathbb{I}$ *such that* $D_1 f(0,k) = 1$ *for all* $k \in \mathbb{Z}_\kappa^+$ *and either*

$$\liminf_{x \to 0, k \to \infty} \frac{D_1 f(x,k) - 1}{x} > 0 \quad \textit{or} \quad \limsup_{x \to 0, k \to \infty} \frac{D_1 f(x,k) - 1}{x} < 0$$

holds, then the zero solution of (2.2) *is unstable. If, in addition, in the first case, we have*

$$\limsup_{x \to 0, k \to \infty} \frac{D_1 f(x,k) - 1}{x} < \infty\,,$$

then the zero solution of (2.2) *is asymptotically left-stable. If, in the second case, also*

$$\liminf_{x \to 0, k \to \infty} \frac{D_1 f(x,k) - 1}{x} > -\infty$$

holds, then the zero solution of (2.2) *is asymptotically right-stable,*

(b) *if there exists a* $\kappa \in \mathbb{I}$ *such that* $D_1 f(0,k) = 1$, $D_1^2 f(0,k) = 0$ *for all* $k \in \mathbb{Z}_\kappa^+$, *and*

$$\liminf_{x \to 0, k \to \infty} \frac{D_1^2 f(x,k)}{x} > 0\,,$$

then the zero solution of (2.2) *is unstable,*

(c) *if there exists a* $\kappa \in \mathbb{I}$ *such that* $D_1 f(0,k) = 1$, $D_1^2 f(0,k) = 0$ *for all* $k \in \mathbb{Z}_\kappa^+$ *and*

$$\limsup_{x \to 0, k \to \infty} \frac{D_1^2 f(x,k)}{x} < 0 \quad \textit{and} \quad \liminf_{x \to 0, k \to \infty} \frac{D_1^2 f(x,k)}{x} > -\infty\,,$$

then the zero solution of (2.2) *is asymptotically stable.*

Proof. (a) The relation $\liminf_{x \to 0, k \to \infty} \frac{D_1 f(x,k) - 1}{x} > 0$ implies that there exists a $\kappa_1 \in \mathbb{Z}_\kappa^+$ and two constants $\delta, \eta > 0$ such that

$$\frac{D_1 f(x,k) - 1}{x} \geq \eta \quad \text{for } k \in \mathbb{Z}_{\kappa_1}^+ \text{ and } x \in (-\delta, \delta) \setminus \{0\}\,.$$

Thus, the mean value theorem (cf. [Lan93, p.341, Theorem 4.2]) implies

$$f(x,k) = \int_0^1 D_1 f(tx,k) x \, dt \geq x + \frac{1}{2}\eta x^2 \quad \text{for } k \in \mathbb{Z}_{\kappa_1}^+ \text{ and } x \in (-\delta, \delta) \setminus \{0\}\,,$$

which is obviously sufficient for the instability of the zero solution of (2.2). We now prove that this solution is left-attractive under the additional assumption $\limsup_{x \to 0, k \to \infty} \frac{D_1 f(x,k) - 1}{x} < \infty$. This relation implies the existence of a $\kappa_2 \in \mathbb{Z}_{\kappa_1}^+$ and two constants $0 < \delta_1 < \delta$, $\gamma > 0$ such that

$$D_1 f(x,k) \geq 1 + x\gamma \quad \text{for } k \in \mathbb{Z}_{\kappa_2}^+ \text{ and } x \in (-\delta_1, 0)\,.$$

Hence,

$$f(x,k) = \int_0^1 D_1 f(tx,k) x \, dt \leq x + \frac{1}{2}\gamma x^2 \quad \text{for } k \in \mathbb{Z}_{\kappa_2}^+ \text{ and } x \in (-\delta_1, 0)\,.$$

Therefore, there exists a $0 < \delta_2 < \delta_1$ such that

$$x + \frac{1}{2}\eta x^2 \leq f(x,k) \leq 0 \quad \text{for } k \in \mathbb{Z}_{\kappa_2}^+ \text{ and } x \in (-\delta_2, 0)\,.$$

This means that the zero solution is left-attractive. The second assertion can be proved analogously.

(b) Due to the hypotheses there exists a $\kappa_1 \in \mathbb{Z}_\kappa^+$ and two constants $\delta > 0$ and $\eta > 0$ such that

$$\frac{D_1^2 f(x,k)}{x} \geq \eta \quad \text{for } k \in \mathbb{Z}_{\kappa_1}^+ \text{ and } x \in (-\delta, \delta) \setminus \{0\}\,.$$

Thus, Taylor's Theorem implies for all $k \in \mathbb{Z}_{\kappa_1}^+$

$$f(k,x) = D_1 f(0,k) x + \int_0^1 \frac{1-t}{2} D_1^2 f(tx,k) x^2 \, dt \geq x + \frac{1}{6}\eta x^3 \quad \text{for } x \in (0, \delta)\,.$$

This relation is obviously sufficient for instability of the zero solution of (2.2).

(c) This assertion can be shown using arguments from (a) and (b). □

Example 5.5. Let $\mathbb{I} = \mathbb{Z}$. We consider the OΔE (5.1) from Example 5.1 for the critical parameter value $\gamma = 1$. The reduced equation (5.6) corresponding to (5.2) is given by

(5.7)
$$x_2(k+1) = x_2(k) - (1 + 2\alpha_k + 2\beta_k)x_2(k)^2 + (1 - 3\beta_k s_2^-(k) + 3\alpha_k + 3\beta_k)x_2(k)^3 + O(x_2(k)^4)\,.$$

Hence, due to the reduction principle from Theorem 5.3, the stability of the zero solution depends on the behavior of the sequence $(1+2\alpha_k+2\beta_k)_{k\in\mathbb{Z}}$. Proposition 5.4 yields that the zero solution of (5.7) is asymptotically left-stable if $\limsup_{k\to\infty}(\alpha_k+\beta_k) < -\frac{1}{2}$, and asymptotically right-stable if $\liminf_{k\to\infty}(\alpha_k+\beta_k) > -\frac{1}{2}$. In any case, the zero solutions of (5.2) and (5.6) are unstable in the above situation. In the degenerate case $1 + 2\alpha_k + 2\beta_k \equiv 0$ on $\mathbb{Z}_\kappa^+$, one has to take the center-unstable fiber bundle $\mathcal{S}^-$ of (5.2) into account. Keeping in mind (5.3), one has

$$x_2(k+1) = x_2(k) + \left[1 - 3\beta_k\left(-2\alpha_{k-1} + 2\beta_{k-1}\right) + 3\alpha_k\right] x_2(k)^3 + O(x_2(k)^4)\,.$$

We define the sequence $\gamma_k := [-\beta_k(-2\alpha_{k-1} + 2\beta_{k-1}) + \alpha_k]$ for $k \in \mathbb{Z}$. Then the zero solutions of (5.1), (5.7), (5.2) and (5.6) are unstable if $\liminf_{k\to\infty}\gamma_k > -\frac{1}{3}$, and asymptotically stable if $\limsup_{k\to\infty}\gamma_k < -\frac{1}{3}$ holds.

The following delay-difference equation is a generalization of a model discussed, e.g., in [KL92].

Example 5.6 (Pielou's discrete logistic model)**.** Let $N > 1$ be an integer and let $\gamma : \mathbb{Z}\to(0,\infty)$, $\delta_1,\dots,\delta_N : \mathbb{Z}\to\mathbb{R}$ be bounded sequences. Moreover, we abbreviate $\delta(k) := \sum_{i=1}^N \delta_i(k)$. Consider the delay difference equation

$$y(k+1) = \frac{y(k)\left(1-\gamma(k)y(k)^2\right)}{1+\sum_{i=1}^N \delta_i(k)y(k-i+1)}\,, \tag{5.8}$$

which possesses the equilibrium 0. Setting $x_{N-i+1}(k) := y(k-i+1)$, $i \in \{1,\dots,N\}$, leads to the equivalent first order OΔE (2.2) in $\mathbb{R}^N$ with the right-hand side

$$f(x,k) := \begin{pmatrix} x_2 \\ \vdots \\ x_N \\ \frac{x_N\left(1-\gamma(k)x_N^2\right)}{1+\sum_{i=1}^N \delta_{N-i+1}(k)x_i(k)} \end{pmatrix}.$$

Its linearization in $0 \in \mathbb{R}^N$ is given by

$$D_1 f(0,k) = \begin{pmatrix} 0 & 1 & 0 & & & & \\ & 0 & 1 & 0 & & & \\ & & 0 & 1 & 0 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 0 & 1 & 0 \\ & & & & & 0 & 1 \\ & & & & & & 1 \end{pmatrix},$$

does not depend on $k \in \mathbb{Z}$, and obviously has the eigenvalues 0 (multiplicity $N-1$), 1 (multiplicity 1). Consequently, it is possible to apply Theorem 3.2 and Theorem 5.3. Thereto, it is advantageous to transform $D_1 f(0,k)$ into Jordan canonical form, which can be done using the matrices

$$T := \begin{pmatrix} 1 & & & 1 \\ & \ddots & & \vdots \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix}, \qquad T^{-1} = \begin{pmatrix} 1 & & & -1 \\ & \ddots & & \vdots \\ & & \ddots & -1 \\ & & & 1 \end{pmatrix}.$$

If we apply this transformation $x \mapsto Tx$ to (2.2), then the reduced equation (5.6) reads as

$$x_N(k+1) = \frac{x_N(k)\left(1 - \gamma(k)x_N(k)^2\right)}{1 + \sum_{i=1}^{N-1} \delta_{N-i+1}(k)(x_N(k) + s_i^-(x_N(k), k)) + \delta_1(k)x_N(k)}$$

where $s^- = (s_1^-, \ldots, s_{N-1}^-) : U \times \mathbb{Z} \to \mathbb{R}^{N-1}$, $U \subseteq \mathbb{R}$ an open neighborhood of 0, parameterizes the the center-unstable fiber bundle $\mathcal{S}^-$ of the transformed equation. Using the Taylor expansions

$$s_i^-(x_N, k) = \sum_{n=2}^{m} \frac{s_{i,n}^-(k)}{n!} x_N^n + R_{i,m}(x, k) \quad \text{for } i \in \{1, \ldots, N-1\} ,$$

this, in turn, leads to the representation

(5.9)

$$x_N(k+1) = x_N(k) - 2\delta(k)x_N(k)^2 - 3\left(2\gamma(k) - 2\delta(k)^2 + \sum_{i=1}^{N-1} \delta_{N-i+1}(k)s_{i,2}(k)\right) x_N(k)^3 + O(x_N(k)^4) ,$$

where the sequence $(s_{1,2}, \ldots, s_{N-1,2}) : \mathbb{Z} \to \mathbb{R}^{N-1}$ is the unique bounded solution of the linear OΔE

$$x(k+1) = \begin{pmatrix} 0 & 1 & 0 & & & & \\ & 0 & 1 & 0 & & & \\ & & 0 & 1 & 0 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 0 & 1 & 0 \\ & & & & & 0 & 1 \\ & & & & & & 0 \end{pmatrix} x(k) + 2\delta(k) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \tag{5.10}$$

in $\mathbb{R}^{N-1}$. Now the factor $\delta(k)$ determines the stability of the zero solution to (5.9). We have asymptotical left-stability if $\limsup_{k\to\infty} \delta(k) < 0$, and asymptotical right-stability if $\liminf_{k\to\infty} \delta(k) > 0$. In the degenerate case $\delta(k) \equiv 0$ on $\mathbb{Z}_\kappa^+$ the reduced equation (5.9) simplifies to

$$x_N(k+1) = x_N(k) - 3\left(2\gamma(k) + \sum_{i=1}^{N-1} \delta_{N-i+1}(k)s_{i,2}(k)\right) x_N(k)^3 + O(x_N(k)^4)$$

and from (5.10) one obtains $s_{i,2}(k) \equiv 0$ on $\mathbb{Z}_\kappa^+$ for $i \in \{1, \ldots, N-1\}$, i.e.,

$$x_N(k+1) = x_N(k) - 6\gamma(k)x_N(k)^3 + O(x_N(k)^4) .$$

Hence, the zero solutions of (5.9) and (5.8) are unstable if $\limsup_{k\to\infty} \gamma(k) < 0$, and asymptotically stable if $\liminf_{k\to\infty} \gamma(k) > 0$ holds.

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