Invariant Manifolds with Asymptotic Phase for Nonautonomous Difference Equations

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Abstract For autonomous difference equations with an invariant manifold conditions are known which guarantee that a solution approaching this manifold eventually behaves like a solution on this manifold. In this paper we extend the fundamental result in this context to difference equations which are nonautonomous and whose solutions are guaranteed only in forward time.

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1 Introduction

The concept of asymptotic phase originally occurred in connection with the approach of a solution of an autonomous ordinary differential equation to an orbitally asymptotically stable periodic solution. The well known Andronov-Witt-Theorem says that if all but one of the characteristic multipliers of a periodic solution $p(t)$ have modulus smaller than 1 then any nearby solution behaves asymptotically like a member of the family of periodic solutions $p(t+\varphi)$ where the phase shift φ is the parameter. For ordinary differential equations this result has been extended to manifolds of stationary or periodic solutions and to more general invariant manifolds in Aulbach [2, 3, 4] and in López-Fenner & Pinto [7], and for difference equations in Aulbach [4] and in López-Fenner & Pinto [8]. In the present paper we generalize the main result of [4] to the case of an nonautonomous equation whose right-hand side is allowed to be noninvertible and whose invariant manifold does not necessarily consist of stationary solutions. This result may also be considered as a discrete analog of the main result in [3].

The organization of this paper is as follows. In section 2 we introduce the notation underlying this paper and in section 3 we prove an auxiliary theorem on the reducibility of linear systems with a certain kind of exponential trichotomy. Section 4 contains another auxiliary result which describes a coordinate change by means of which the main result of this paper can be proved in section 5.

2 Preliminaries

We first fix the notation and introduce the basic concepts underlying this paper. $\mathbb{N}$ denotes the positive integers. A *discrete interval* I is defined to be the intersection of a real interval with the integers $\mathbb{Z} = \{0, \pm 1, \dots\}$. For any $\kappa \in \mathbb{Z}$ we use the abbreviations $\mathbb{Z}_\kappa^+ := [\kappa, \infty) \cap \mathbb{Z}$ and $\mathbb{Z}_\kappa^- := (-\infty, \kappa] \cap \mathbb{Z}$. The space of real $N \times N$-matrices is denoted by $\mathbb{R}^{N\times N}$ with the zero matrix 0_N, and $\mathcal{GL}_N(\mathbb{R})$ is the multiplicative group of invertible matrices in $\mathbb{R}^{N\times N}$ with the identity I_N. $\mathcal{N}(B) := B^{-1}(\{0\})$ denotes the nullspace of a matrix $B \in \mathbb{R}^{N\times N}$ and $\mathcal{R}(B) := B(\mathbb{R}^N)$ its range. For any $x \in \mathbb{R}^N$ the ball in $\mathbb{R}^N$ with center x and radius $\varepsilon > 0$ is denoted by $B_\varepsilon(x)$. Double bars $\|\cdot\|$ stand for an arbitrary norm on $\mathbb{R}^N$ and our matrix-norms are always induced by vector-norms. In particular, the norm $\|B\|_2 := \max_{\|x\|_2=1} \|Bx\|_2$ is induced by the Euclidean norm $\|x\|_2 := (\sum_{k=1}^N x_k^2)^{1/2}$. We write

$$x' = f(k, x) \tag{1}$$

for the difference equation $x(k+1) = f(k, x(k))$ with the right-hand side $f : I \times \mathbb{R}^N \to \mathbb{R}^N$ where I is a discrete interval. The expression $\lambda(k; \kappa, \xi)$ denotes the *general solution* of equation (1), i.e. $\lambda(\cdot\,; \kappa, \xi)$ solves equation (1) and satisfies the initial condition $\lambda(\kappa; \kappa, \xi) = \xi$ for $\kappa \in I$ and $\xi \in \mathbb{R}^N$. The general solution may be represented recursively as

$$\lambda(k; \kappa, \xi) := \begin{cases} \xi & \text{for } k = \kappa \\ f(k-1, \lambda(k-1; \kappa, \xi)) & \text{for } k > \kappa \end{cases}.$$

Given a matrix sequence $A : I \to \mathbb{R}^{N\times N}$ we define the *transition matrix* $\Phi(k, \kappa) \in \mathbb{R}^{N\times N}$ of the linear equation $x' = A(k)x$ as the mapping given by

$$\Phi(k, \kappa) := \begin{cases} I_N & \text{for } k = \kappa \\ A(k-1) \cdot \ldots \cdot A(\kappa) & \text{for } k > \kappa \end{cases}$$

and if $A(k)$ is invertible (in $\mathbb{R}^{N\times N}$) for $k \in \mathbb{Z}_\kappa^-$ then we set

$$\Phi(k, \kappa) := A(k)^{-1} \cdot \ldots \cdot A(\kappa - 1)^{-1} \quad \text{for } k < \kappa.$$

Finally, a point $\xi \in \mathbb{R}^N$ is called an *ω-limit point* of a mapping $\mu : \mathbb{Z}_\kappa^+ \to \mathbb{R}^N$ if there exists a sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}_\kappa^+$ with $\lim_{n\to\infty} k_n = \infty$ and $\lim_{n\to\infty} \mu(k_n) = \xi$.

3 Exponential Trichotomies and Reducibility

We consider a linear difference equation

$$x' = A(k)x \tag{2}$$

where the mapping $A : \mathbb{Z}_{\kappa_0}^+ \to \mathbb{R}^{N\times N}$, $\kappa_0 \in \mathbb{Z}$, is not assumed to have invertible values. Furthermore we consider two sequences of projections $P^-, P^+ : \mathbb{Z}_\kappa^+ \to \mathbb{R}^{N\times N}$, $\kappa \in \mathbb{Z}_{\kappa_0}^+$, with

$$P^-(k+1)A(k) \equiv A(k)P^-(k), \quad P^+(k+1)A(k) \equiv A(k)P^+(k) \quad \text{on } \mathbb{Z}_\kappa^+ \tag{3}$$

and we assume that the relation $P^-(k)P^+(k) \equiv P^+(k)P^-(k)$ holds on $\mathbb{Z}_\kappa^+$. Hence $I_N - P^-(k) - P^+(k)$ is a projection on $\mathbb{Z}_\kappa^+$ as well. The equation (2) is said to satisfy the *regularity condition* if the two mappings

$$\begin{aligned} A(k)|_{\mathcal{R}(P^+(k))} &: \mathcal{R}(P^+(k)) \to \mathcal{R}(P^+(k+1)), \\ A(k)|_{\mathcal{N}(P^+(k)+P^-(k))} &: \mathcal{N}(P^+(k)+P^-(k)) \to \mathcal{N}(P^+(k+1)+P^-(k+1)) \end{aligned}$$

are invertible for all $k \in \mathbb{Z}_\kappa^+$; they are well-defined because of the identities (3). If this is the case we can define the *extended transition matrix*

$$\Phi_{P^+}(k,l) = \begin{cases} \left[A(k)|_{\mathcal{R}(P^+(k))}\right]^{-1} \cdot \ldots \cdot \left[A(l-1)|_{\mathcal{R}(P^+(l-1))}\right]^{-1} & \text{for } k < l \\ I_{\mathcal{R}(P^+(l))} & \text{for } k = l \\ A(k-1)|_{\mathcal{R}(P^+(k-1))} \cdot \ldots \cdot A(l)|_{\mathcal{R}(P^+(l))} & \text{for } k > l \end{cases}$$

for $(k,l) \in (\mathbb{Z}_\kappa^+)^2$. The complementary expression $\Phi_{I_N-P^+-P^-}(k,l)$ is defined analogously. Finally, equation (2) is said to possess an *exponential trichotomy* if there exist real numbers $0 < \alpha < \beta$ and $K_1, K_2, K_3 \geq 1$ such that the following estimates hold true:

$$\begin{aligned} \left\|\Phi(k,l)P^-(l)\right\| &\leq K_1\alpha^{k-l} && \text{for } k \geq l \geq \kappa, \\ \left\|\Phi_{P^+}(k,l)P^+(l)\right\| &\leq K_2\beta^{k-l} && \text{for } l \geq k \geq \kappa, \\ \left\|\Phi_{I_N-P^--P^+}(k,l)\left[I_N - P^-(l) - P^+(l)\right]\right\| &\leq K_3 && \text{for } k,l \in \mathbb{Z}_\kappa^+. \end{aligned} \tag{4}$$

Remark 3.1 (1) If the coefficient matrices appearing in equation (2) are invertible, then the above notion of exponential trichotomy reduces to the corresponding notion used in López-Fenner & Pinto [8, Defintion 1.1]. For the differential equations case see Aulbach [3].

(2) If the coefficient matrices in equation (2) are independent of k, $A(k) \equiv A$, then this equation has an exponential trichotomy if all eigenvalues of A with modulus 1 are semi-simple.

Equation (2) is called *reducible* to an equation $x' = B(k)x$ with $B : \mathbb{Z}_\kappa^+ \to \mathbb{R}^{N\times N}$, if there exists a function $\Lambda : \mathbb{Z}_\kappa^+ \to \mathcal{GL}_N(\mathbb{R})$ with the following properties:

(i) Λ and $\Lambda(\cdot)^{-1}$ are bounded as functions from $\mathbb{Z}_\kappa^+$ to $\mathbb{R}^{N\times N}$,

(ii) the identity $\Lambda(k+1)B(k) \equiv A(k)\Lambda(k)$ holds on $\mathbb{Z}_\kappa^+$.

Later on we need the following reducibility result.

Theorem 3.2 *We suppose system (2) satisfies the following conditions:*

(i) It has an exponential trichotomy with constants α, β, K_1, K_2, K_3 and projections P^-, P^+ on $\mathbb{Z}_\kappa^+$, $\kappa \in \mathbb{Z}_{\kappa_0}^+$,

(ii) the ranks of the projections are constant on $\mathbb{Z}_\kappa^+$, $N^- :\equiv rk\, P^-(k)$, $N^+ :\equiv rk\, P^+(k)$.

Then the system (2) is reducible to a decoupled system

$$\begin{cases} u' = B^-(k)u \\ v' = B^+(k)v \\ w' = B^\star(k)w \end{cases} \tag{5}$$

with $B^- : \mathbb{Z}_\kappa^+ \to \mathbb{R}^{N^- \times N^-}$, $B^+ : \mathbb{Z}_\kappa^+ \to \mathcal{GL}_{N^+}(\mathbb{R})$ *and* $B^\star : \mathbb{Z}_\kappa^+ \to \mathcal{GL}_{N-N^--N^+}(\mathbb{R})$. *Moreover, the transition matrices* Ψ^-, Ψ^- *and* $\Psi^\star$ *of the subsystems* $u' = B^-(k)u$, $v' = B^+(k)v$ *and* $w' = B^\star(k)w$, *respectively, satisfy the estimates*

$$\begin{aligned} \left\|\Psi^-(k,l)\right\|_2 &\leq (2+K_1)^6(2+K_2)^2 K_1 \alpha^{k-l} && \text{for } k \geq l \geq \kappa, \\ \left\|\Psi^+(k,l)\right\|_2 &\leq (2+K_1)^6(2+K_2)^2 K_2 \beta^{k-l} && \text{for } l \geq k \geq \kappa, \qquad (6) \\ \left\|\Psi^\star(k,l)\right\|_2 &\leq (2+K_1)^6(2+K_2)^2 K_3 && \text{for } k,l \in \mathbb{Z}_\kappa^+. \qquad (7) \end{aligned}$$

Proof. (a) Because of the exponential trichotomy of system (2) we have

$$\left\|P^-(k)\right\|_2 \leq K_1, \quad \left\|P^+(k)\right\|_2 \leq K_2 \quad \text{for } k \in \mathbb{Z}_\kappa^+. \tag{8}$$

Using the methods in Gohberg, Kaashoek & Kos [6, Lemma 2.2] (for details see Pötzsche [9]) there exists a sequence $\Lambda : \mathbb{Z}_\kappa^+ \to \mathcal{GL}_N(\mathbb{R})$ such that on $\mathbb{Z}_\kappa^+$ we have

$$\begin{aligned} \Lambda(k)^{-1}P^-(k)\Lambda(k) &\equiv \begin{pmatrix} I_{N^-} & & \\ & 0_{N^+} & \\ & & 0_{N-N^--N^+} \end{pmatrix} =: D^-, \\ \Lambda(k)^{-1}P^+(k)\Lambda(k) &\equiv \begin{pmatrix} 0_{N^-} & & \\ & I_{N^+} & \\ & & 0_{N-N^--N^+} \end{pmatrix} =: D^+, \\ \Lambda(k)^{-1}\left[I_N - P^-(k) - P^+(k)\right]\Lambda(k) &\equiv \begin{pmatrix} 0_{N^-} & & \\ & 0_{N^+} & \\ & & I_{N-N^--N^+} \end{pmatrix} =: D^\star. \end{aligned}$$

and furthermore we get

$$\max\left\{\left\|\Lambda(k)\right\|_2, \left\|\Lambda(k)^{-1}\right\|_2\right\} \overset{(8)}{\leq} (2+K_1)^3(2+K_2) \quad \text{for } k \in \mathbb{Z}_\kappa^+. \tag{9}$$

Using Λ as a transformation, system (2) turns into the decoupled system (5) which moreover satisfies the regularity condition with respect to the constant projections D^+ and $D^\star$. This implies the invertibility of the matrices $B^+(k)$ and $B^\star(k)$ for all $k \in \mathbb{Z}_\kappa^+$.

(b) For the transition matrix Ψ^- we obtain

$$\begin{aligned} \left\|\Psi^-(k,l)\right\|_2 &= \left\|\Psi(k,l)D^-\right\|_2 = \left\|\Lambda(k)^{-1}\Phi(k,l)\Lambda(l)D^-\right\|_2 = \\ &= \left\|\Lambda(k)^{-1}\Phi(k,l)P^-(l)\Lambda(l)\right\|_2 \leq \\ &\overset{(9)}{\leq} (2+K_1)^6(2+K_2)^2\left\|\Phi(k,l)P^-(l)\right\|_2 \leq \\ &\overset{(4)}{\leq} K_1(2+K_1)^6(2+K_2)^2\alpha^{k-l} \quad \text{for } k \geq l \geq \kappa \end{aligned}$$

and using arguments as before one can see that Ψ^+ and Ψ^* satisfy the estimates (6) and (7). This completes the proof of Theorem 3.2. □

4 Transformation to Quasilinear Form

For the remainder of this paper we consider a difference equation

$$x' = f(k, x) \tag{10}$$

whose right-hand side $f : \mathbb{Z}^+_{\kappa_0} \times \mathbb{R}^N \to \mathbb{R}^N$, $\kappa_0 \in \mathbb{Z}$, has the property that $f(k, \cdot)$ is of class C^3 for any $k \in \mathbb{Z}^+_\kappa$, $\kappa \in \mathbb{Z}^+_{\kappa_0}$. We suppose that this system has an M-dimensional bounded invariant C^3-manifold $\mathcal{M} \subseteq \mathbb{R}^N$, this particularly means that for any initial point (κ, ξ) in $\mathbb{Z}^+_{\kappa_0} \times \mathcal{M}$ the corresponding solution $\lambda(k; \kappa, \xi)$ remains in $\mathcal{M}$ for all $k \in \mathbb{Z}^+_\kappa$. We furthermore suppose that any solution $\mu_0 : \mathbb{Z}^+_\kappa \to \mathbb{R}^N$ of (10) with initial value $\mu_0(\kappa) \in \mathcal{M}$ satisfies the following hypotheses:

(H1) The variational equation

$$y' = \frac{\partial f}{\partial x}(k, \mu_0(k))y$$

admits an exponential trichotomy with constants $0 < \alpha < 1 < \beta$, K_1, K_2, K_3 and projections P^-, P^+ whose ranks $N^- :\equiv \operatorname{rk} P^-(k)$ and $N^+ :\equiv \operatorname{rk} P^+(k)$ are constant on $\mathbb{Z}^+_\kappa$ and satisfy $N^- + N^+ = N - M$.

(H2) The limit

$$\lim_{y \to 0} \left[\frac{\partial f}{\partial x}(k, y + \mu_0(k)) - \frac{\partial f}{\partial x}(k, \mu_0(k)) \right] = 0_N$$

exists uniformly with respect to $k \in \mathbb{Z}^+_\kappa$.

(H3) There exists a neighborhood $V \subseteq \mathcal{M}$ of $\mu_0(\kappa)$ such that the derivatives

$$\left. \frac{\partial \lambda}{\partial \xi}(\cdot; \kappa, \cdot) \right|_{\mathbb{Z}^+_\kappa \times V} \quad \text{and} \quad \left. \frac{\partial^2 \lambda}{\partial \xi^2}(\cdot; \kappa, \cdot) \right|_{\mathbb{Z}^+_\kappa \times V}$$

are bounded.

The following theorem describes a change of coordinates which allows to transform system (10) into a particular "quasilinear" form which is suitable for further investigations in the next section.

Theorem 4.1 *For any solution $\mu_0 : \mathbb{Z}^+_\kappa \to \mathbb{R}^N$ of (10) with $\mu_0(\kappa) \in \mathcal{M}$ and satisfying the hypotheses (H1), (H2) and (H3) there exists a local transformation $\mathcal{T}_{\mu_0} : A_{\mu_0} \subseteq \mathbb{Z}^+_\kappa \times \mathbb{R}^N \to \mathbb{R}^N$ which transforms system (10) into a system of the form*

$$\begin{cases} \hat{u}' = B^-(k)\hat{u} + \hat{B}_1^-(k, \hat{u}, \hat{v}, \hat{w})\hat{u} + \hat{B}_2^-(k, \hat{u}, \hat{v}, \hat{w})\hat{v} \\ \hat{v}' = B^+(k)\hat{v} + \hat{B}_2^+(k, \hat{u}, \hat{v}, \hat{w})\hat{v} \\ \hat{w}' = \hat{w} + \hat{B}_1^\star(k, \hat{u}, \hat{v}, \hat{w})\hat{u} + \hat{B}_2^\star(k, \hat{u}, \hat{v}, \hat{w})\hat{v} \end{cases} \tag{11}$$

where $\hat{u} \in \mathbb{R}^{N^-}$, $\hat{v} \in \mathbb{R}^{N^+}$ and $\hat{w} \in \mathbb{R}^M$. Furthermore the following is true:

(a) The domain A_{μ_0} of the transformation $\mathcal{T}_{\mu_0}$ is a neighborhood of the "solution curve" $\{(k, \mu_0(k)) : k \in \mathbb{Z}_\kappa^+\}$ with the property that there exists some $\rho_1 > 0$ with

$$B_{\rho_1}(\mu_0(k)) \subseteq \left\{x \in \mathbb{R}^N : (k, x) \in A_{\mu_0}\right\} \qquad \text{for } k \in \mathbb{Z}_\kappa^+.$$

In addition, for any $k \in \mathbb{Z}_\kappa^+$ the mapping $\mathcal{T}_{\mu_0}(k, \cdot)$ is of class C^1 and satisfies the identity $\mathcal{T}_{\mu_0}(k, \mu_0(k)) \equiv 0$ on $\mathbb{Z}_\kappa^+$.

(b) The mappings B^- and B^+ are of type $B^- : \mathbb{Z}_\kappa^+ \to \mathbb{R}^{N^- \times N^-}$ and $B^+ : \mathbb{Z}_\kappa^+ \to \mathcal{GL}_{N^+}(\mathbb{R})$, respectively.

(c) The transition matrices Ψ^-, Ψ^+ of $\hat{u}' = B^-(k)\hat{u}$ and $\hat{v}' = B^+(k)\hat{v}$, respectively, satisfy the estimates

$$\begin{aligned} \left\|\Psi^-(k, l)\right\| &\leq \tilde{K}_1 \alpha^{k-l} && \text{for } k \geq l \geq \kappa, \\ \left\|\Psi^+(k, l)\right\| &\leq \tilde{K}_2 \beta^{k-l} && \text{for } l \geq k \geq \kappa \end{aligned}$$

with real constants $\tilde{K}_1, \tilde{K}_2 \geq 1$.

(d) The matrix-valued mappings $\hat{B}_1^-, \hat{B}_2^-, \hat{B}_2^+, \hat{B}_1^\star, \hat{B}_2^\star$ are continuous as functions of $(\hat{u}, \hat{v}, \hat{w})$ and they converge to the respective zero matrix uniformly with respect to $k \in \mathbb{Z}_\kappa^+$ as $(\hat{u}, \hat{v}, \hat{w}) \to (0, 0, 0)$.

(e) There exist real constants $c, C > 0$ with the following property: if $\mu, \bar{\mu} : \mathbb{Z}_\kappa^+ \to \mathbb{R}^N$ are any two solutions of equation (10) which satisfy $(k, \mu(k)), (k, \bar{\mu}(k)) \in A_{\mu_0}$ for all k in some subset $J \subseteq \mathbb{Z}_\kappa^+$ then the estimates

$$c \left\|\mu(k) - \bar{\mu}(k)\right\| \leq \left\|\mathcal{T}_{\mu_0}(k, \mu(k)) - \mathcal{T}_{\mu_0}(k, \bar{\mu}(k))\right\| \leq C \left\|\mu(k) - \bar{\mu}(k)\right\|$$

are valid for all $k \in J$.

Proof. We subdivide the proof in four steps.

(I) In order to decouple the linear part of system (10) we first use the transformation $y = x - \mu_0(k)$ to get from (10) the system

$$y' = \frac{\partial f}{\partial x}(k, \mu_0(k))y + r(k, y) \tag{12}$$

where the remainder term $r : \mathbb{Z}_\kappa^+ \times \mathbb{R}^N \to \mathbb{R}^N$ turns out to have two continuous partial derivatives with respect to $y \in \mathbb{R}^N$. Furthermore, we have

$$r(k, 0) \equiv 0 \quad \text{ on } \mathbb{Z}_\kappa^+ \tag{13}$$

as well as (cf. (H2))

$$\lim_{y \to 0} \frac{\partial r}{\partial y}(k, y) = 0 \tag{14}$$

uniformly with respect to $k \in \mathbb{Z}_\kappa^+$. Because of the assumption (H1) we may apply the Reducibility Theorem 3.2 to the linear part of system (12). This provides a transformation matrix $\Lambda : \mathbb{Z}_\kappa^+ \to \mathcal{GL}_N(\mathbb{R})$ which allows to decouple this system by means of the transformation $T_1 : \mathbb{Z}_\kappa^+ \times \mathbb{R}^N \to \mathbb{R}^N$ with $T_1(k, y) := \Lambda(k)^{-1} y$. In fact, the transformed system has the form

$$\begin{cases} u' = B^-(k)u + r^-(k, u, v, w) \\ v' = B^+(k)v + r^+(k, u, v, w) \\ w' = B^\star(k)w + r^\star(k, u, v, w) \end{cases} \tag{15}$$

where $B^- : \mathbb{Z}_\kappa^+ \to \mathbb{R}^{N^- \times N^-}$, $B^+ : \mathbb{Z}_\kappa^+ \to \mathcal{GL}_{N^+}(\mathbb{R})$ and $B^\star : \mathbb{Z}_\kappa^+ \to \mathcal{GL}_M(\mathbb{R})$. The phase space $\mathbb{R}^N$ is split in three parts according to $y = (u, v, w) \in \mathbb{R}^{N^-} \times \mathbb{R}^{N^+} \times \mathbb{R}^M$. Furthermore, the transition matrices Ψ^-, Ψ^+ and $\Psi^\star$ of the linear systems $u' = B^-(k)u$, $v' = B^+(k)v$ and $w' = B^\star(k)w$, respectively, obey the estimates

$$\begin{aligned} \left\|\Psi^-(k,l)\right\| &\le \tilde{K_1}\alpha^{k-l} \quad \text{for } k \ge l \ge \kappa, \\ \left\|\Psi^+(k,l)\right\| &\le \tilde{K_2}\beta^{k-l} \quad \text{for } l \ge k \ge \kappa, \\ \left\|\Psi^\star(k,l)\right\| &\le \tilde{K_3} \quad \text{for } k, l \in \mathbb{Z}_\kappa^+, \end{aligned} \tag{16}$$

where the constants $\tilde{K_1}, \tilde{K_2}, \tilde{K_3} \ge 1$ only depend on K_1, K_2, K_3 and the used norms (see Theorem 3.2(b)). The nonlinearities r^-, r^+ and $r^\star$ are twice continuously differentiable with respect to u, v and w. In addition, because of (13) we get

$$r^-(k, 0, 0, 0) \equiv 0, \quad r^+(k, 0, 0, 0) \equiv 0, \quad r^\star(k, 0, 0, 0) \equiv 0 \quad \text{on } \mathbb{Z}_\kappa^+$$

as well as (cf. (14))

$$\lim_{(u,v,w)\to(0,0,0)} \frac{\partial(r^-, r^+, r^\star)}{\partial(u, v, w)}(k, u, v, w) = 0 \tag{17}$$

uniformly with respect to $k \in \mathbb{Z}_\kappa^+$. It is worth noting here that both $\Lambda : \mathbb{Z}_\kappa^+ \to \mathcal{GL}_N(\mathbb{R})$ and $\Lambda(\cdot)^{-1}$ are bounded.

(II) We now determine a local coordinate change which makes the nonlinear terms of system (15) disappear on a set of the form $\mathbb{Z}_\kappa^+ \times \{0\} \times \{0\} \times B$ where $B \subseteq \mathbb{R}^M$ is an open neighborhood of 0. To this end let $X : B \to \mathcal{M}$ be a local C^3-coordinate system of the manifold $\mathcal{M}$ with $X(0) = \mu_0(\kappa)$ and $X(B) \subseteq V$. Then, for any $\eta \in B$ the function $\lambda(\cdot; \kappa, X(\eta))$ is a solution of (10) which because of the invariance of $\mathcal{M}$ remains in $\mathcal{M}$ for all $k \in \mathbb{Z}_\kappa^+$. Furthermore, $\lambda(\cdot; \kappa, X(\eta)) - \mu_0$ is a solution of system (12) and therefore the function

$$\upsilon(k; \eta) = \begin{pmatrix} \upsilon^-(k; \eta) \\ \upsilon^+(k; \eta) \\ \upsilon^\star(k; \eta) \end{pmatrix} := \Lambda(k)^{-1}(\lambda(k; \kappa, X(\eta)) - \mu_0(k)) \tag{18}$$

is a solution of (15) for any $\eta \in B$ which moreover vanishes identically for $\eta = 0$:

$$\upsilon(k; 0) \equiv 0 \quad \text{on } \mathbb{Z}_\kappa^+. \tag{19}$$

In addition the function $\upsilon(\cdot; \eta)$ is bounded for any fixed $\eta \in B$ since its values are in $\mathcal{M}$. Differentiating the corresponding solution identity with respect $\eta_i \in \mathbb{R}$ we get

$$\frac{\partial \upsilon}{\partial \eta_i}(k+1; \eta) \equiv \left[\begin{pmatrix} B^-(k) & & \\ & B^+(k) & \\ & & B^\star(k) \end{pmatrix} + \begin{pmatrix} \frac{\partial r^-}{\partial(u,v,w)}(k, \upsilon(k; \eta)) \\ \frac{\partial r^+}{\partial(u,v,w)}(k, \upsilon(k; \eta)) \\ \frac{\partial r^\star}{\partial(u,v,w)}(k, \upsilon(k; \eta)) \end{pmatrix} \right] \frac{\partial \upsilon}{\partial \eta_i}(k; \eta)$$

on $\mathbb{Z}_\kappa^+ \times B$ for $i = 1, \ldots, M$. According to (17) and (19) we get for $\eta = 0$

$$\frac{\partial \upsilon}{\partial \eta_i}(k+1;0) \equiv \begin{pmatrix} B^-(k) & & \\ & B^+(k) & \\ & & B^\star(k) \end{pmatrix} \frac{\partial \upsilon}{\partial \eta_i}(k;0) \quad \text{on } \mathbb{Z}_\kappa^+. \tag{20}$$

Thus the M functions $\frac{\partial \upsilon}{\partial \eta_1}(\cdot;0), \ldots, \frac{\partial \upsilon}{\partial \eta_M}(\cdot;0) : \mathbb{Z}_\kappa^+ \to \mathbb{R}^N$ are solutions of the linear system

$$\begin{cases} u' = B^-(k)u \\ v' = B^+(k)v \\ w' = B^\star(k)w \end{cases}. \tag{21}$$

Since $X : B \to X(B)$ is a diffeomorphism, the vectors $\frac{\partial X}{\partial \eta_1}(0), \ldots, \frac{\partial X}{\partial \eta_M}(0) \in \mathbb{R}^N$ are linearly independent, and because of the invertibility of the matrix $\Lambda(\kappa) \in \mathbb{R}^{N \times N}$ also the vectors

$$\frac{\partial \upsilon}{\partial \eta_i}(\kappa;0) \overset{(18)}{=} \Lambda(\kappa)^{-1} \frac{\partial X}{\partial \eta_i}(0) \quad \text{for } i = 1, \ldots, M$$

are linearly independent. Now we can choose the local coordinate system X von $\mathcal{M}$ such that the vectors $\frac{\partial \upsilon^\star}{\partial \eta_1}(\kappa;0), \ldots, \frac{\partial \upsilon^\star}{\partial \eta_M}(\kappa;0) \in \mathbb{R}^M$ are linearly independent and, since $B^\star(k) \in \mathbb{R}^{M \times M}$ is regular, we get the linear independence of the solutions $\frac{\partial \upsilon^\star}{\partial \eta_1}(\cdot;0), \ldots, \frac{\partial \upsilon^\star}{\partial \eta_M}(\cdot;0)$ of the M-dimensional linear system $w' = B^\star(k)w$. Altogether we thus have

$$\frac{\partial \upsilon^\star}{\partial \eta}(k;0) \in \mathcal{GL}_M(\mathbb{R}) \quad \text{for } k \in \mathbb{Z}_\kappa^+. \tag{22}$$

Furthermore we get the relation

$$\left\| \left[\frac{\partial \upsilon^\star}{\partial \eta}(k;0) \right]^{-1} \right\| = \left\| \left[\frac{\partial \upsilon^\star}{\partial \eta}(\kappa;0) \right]^{-1} \Psi^\star(\kappa,k) \right\| \le \tilde{K}_3 \left\| \left[\frac{\partial \upsilon^\star}{\partial \eta}(\kappa;0) \right]^{-1} \right\| \quad \text{for } k \in \mathbb{Z}_\kappa^+. \tag{23}$$

Finally, the function $\frac{\partial \upsilon}{\partial \eta}(\cdot;0)$ is bounded by assumption (H3) because we have

$$\frac{\partial \upsilon}{\partial \eta}(k;0) \overset{(18)}{=} \Lambda(k)^{-1} \frac{\partial \lambda}{\partial \xi}(k;\kappa,\mu_0(\kappa)) \frac{\partial X}{\partial \eta}(0) \quad \text{for } k \in \mathbb{Z}_\kappa^+. \tag{24}$$

Next we want to transform system (15) in such a way that the solutions corresponding to $\upsilon(\cdot;\eta)$, $\eta \in B$, get the form $(0,0,\eta)$. To this end we consider the mapping $S(k,u,v,w) := (u,v,0) + \upsilon(k;w)$ and notice that by Taylor's Theorem this mapping may be represented in the form

$$S(k,u,v,w) \overset{(19)}{=} \begin{pmatrix} u \\ v \\ 0 \end{pmatrix} + \frac{\partial \upsilon}{\partial \eta}(k;0)w + R_1(k,w)$$

where the remainder term $R_1 = (R_1^-, R_1^+, R_1^\star) : \mathbb{Z}_\kappa^+ \times B \to \mathbb{R}^N$ is twice continuously differentiable with respect to $w \in \mathbb{R}^M$ and satisfies $\lim_{w \to 0} \frac{R_1(k,w)}{\|w\|} = 0$. The mapping $\upsilon^\star : \mathbb{Z}_\kappa^+ \times B \to \mathbb{R}^M$ satisfies, because of (19), (22), (23) and

$$\frac{\partial^2 \upsilon}{\partial \eta^2}(k;\eta) \overset{(18)}{=} \Lambda(k)^{-1} \left[\frac{\partial^2 \lambda}{\partial \xi^2}(k;\kappa,X(\eta))DX(\eta) + \frac{\partial \lambda}{\partial \xi}(k;\kappa,X(\eta))D^2X(\eta) \right]$$

together with (H3) the assumptions of Lemma 6.1 (see Appendix). This provides a neighborhood $U^\star \subseteq B$ of 0, independent of k, where each $\upsilon_k^\star := \upsilon^\star(k;\cdot)$ is injective. Lemma 6.1 furthermore implies that $(\upsilon_k^\star)^{-1}$ is defined for all $k \in \mathbb{Z}_\kappa^+$ on a k-independent neighborhood $V^\star \subseteq \mathbb{R}^M$ of 0 with $V^\star \subseteq \upsilon^\star(k;U^\star)$. For the inverse of the coordinate change $S(k,\cdot)$ we get the representation

$$S(k,\cdot)^{-1}(u,v,w) = \begin{pmatrix} u - \upsilon^-(k;(\upsilon_k^\star)^{-1}(w)) \\ v - \upsilon^+(k;(\upsilon_k^\star)^{-1}(w)) \\ (\upsilon_k^\star)^{-1}(w) \end{pmatrix}$$

for all $(k,u,v,w) \in \mathbb{Z}_\kappa^+ \times \mathbb{R}^{N^-} \times \mathbb{R}^{N^+} \times V^\star$. Again, from Taylor's Theorem and the relation (19) we get

$$S(k,\cdot)^{-1}(u,v,w) = \begin{pmatrix} u - \frac{\partial \upsilon^-}{\partial \eta}(k;0)\left[\frac{\partial \upsilon^\star}{\partial \eta}(k;0)\right]^{-1} w \\ v - \frac{\partial \upsilon^+}{\partial \eta}(k;0)\left[\frac{\partial \upsilon^\star}{\partial \eta}(k;0)\right]^{-1} w \\ \left[\frac{\partial \upsilon^\star}{\partial \eta}(k;0)\right]^{-1} w \end{pmatrix} + R_2(k,w)$$

where the remainder term $R_2 = (R_2^-, R_2^+, R_2^\star) : \mathbb{Z}_\kappa^+ \times V^\star \to \mathbb{R}^N$ has two continuous partial derivatives with respect to $w \in \mathbb{R}^M$ and satisfies $\lim_{w\to 0} \frac{R_2(k,w)}{\|w\|} = 0$. Since the functions $\frac{\partial \upsilon}{\partial \eta_1}(\cdot;0), \dots, \frac{\partial \upsilon}{\partial \eta_M}(\cdot;0)$ are solutions of (21), in terms of the coordinates $(\bar{u},\bar{v},\bar{w}) := T_2(k,u,v,w) := S(k,\cdot)^{-1}(u,v,w)$ the transformed system has the simplified form

$$\begin{cases} \bar{u}' = B^-(k)\bar{u} + \bar{r}^-(k,\bar{u},\bar{v},\bar{w}) \\ \bar{v}' = B^+(k)\bar{v} + \bar{r}^+(k,\bar{u},\bar{v},\bar{w}) \\ \bar{w}' = \bar{w} + \bar{r}^\star(k,\bar{u},\bar{v},\bar{w}) \end{cases}. \tag{25}$$

Here, in view of (20) the nonlinearities $\bar{r}^-, \bar{r}^+$ and $\bar{r}^\star$ are defined as follows:

$$\begin{aligned}
\bar{r}^-(k,\bar{u},\bar{v},\bar{w}) &:= B^-(k)R_1^-(k,\bar{w}) + r^-(k,S(k,\bar{u},\bar{v},\bar{w})) - \\
&-\frac{\partial \upsilon^-}{\partial \eta}(k+1;0)\left[\frac{\partial \upsilon^\star}{\partial \eta}(k;0)\right]^{-1} R_1^\star(k,\bar{w}) - \\
&-\frac{\partial \upsilon^-}{\partial \eta}(k+1;0)\left[\frac{\partial \upsilon^\star}{\partial \eta}(k+1;0)\right]^{-1} r^\star(k,S(k,\bar{u},\bar{v},\bar{w})) + \\
&+R_2^-(k+1,\frac{\partial \upsilon^\star}{\partial \eta}(k+1;0)\bar{w} + B^\star(k)R_1^\star(k,\bar{w}) + r^\star(k,S(k,\bar{u},\bar{v},\bar{w}))), \\
\bar{r}^+(k,\bar{u},\bar{v},\bar{w}) &:= B^+(k)R_1^+(k,\bar{w}) + r^+(k,S(k,\bar{u},\bar{v},\bar{w})) - \\
&-\frac{\partial \upsilon^+}{\partial \eta}(k+1;0)\left[\frac{\partial \upsilon^\star}{\partial \eta}(k;0)\right]^{-1} R_1^\star(k,\bar{w}) - \\
&-\frac{\partial \upsilon^+}{\partial \eta}(k+1;0)\left[\frac{\partial \upsilon^\star}{\partial \eta}(k+1;0)\right]^{-1} r^\star(k,S(k,\bar{u},\bar{v},\bar{w})) + \\
&+R_2^+(k+1,\frac{\partial \upsilon^\star}{\partial \eta}(k+1;0)\bar{w} + B^\star(k)R_1^\star(k,\bar{w}) + r^\star(k,S(k,\bar{u},\bar{v},\bar{w})))
\end{aligned}$$

and

$$\begin{aligned}\bar r^\star(k,\bar u,\bar v,\bar w) &:= \left[\frac{\partial\upsilon^\star}{\partial\eta}(k;0)\right]^{-1} R_1^\star(k,\bar w)+\\ &+\left[\frac{\partial\upsilon^\star}{\partial\eta}(k+1;0)\right]^{-1} r^\star(k,S(k,\bar u,\bar v,\bar w))+\\ &+R_2^\star(k+1,\frac{\partial\upsilon^\star}{\partial\eta}(k+1;0)\bar w+B^\star(k)R_1^\star(k,\bar w)+r^\star(k,S(k,\bar u,\bar v,\bar w))).\end{aligned}$$

These functions have tree crucial properties: They have two continuous partial derivatives with respect to $(\bar u,\bar v,\bar w)$, together with the sequence $(\upsilon(k;\bar w))_{k\in\mathbb{Z}_\kappa^+}$ also the sequence $(S(k,\bar u,\bar v,\bar w))_{k\in\mathbb{Z}_\kappa^+}$ is bounded (for fixed $(\bar u,\bar v,\bar w)\in\mathbb{R}^{N^-}\times\mathbb{R}^{N^+}\times U^\star$), and from Lemma 6.1 and the relations (23) and (24) we get the boundedness of $(T_2(k,u,v,w))_{k\in\mathbb{Z}_\kappa^+}$ (for fixed $(u,v,w)\in\mathbb{R}^{N^-}\times\mathbb{R}^{N^+}\times V^\star$). Thus, for the nonlinear terms we get the relation

$$\lim_{(\bar u,\bar v,\bar w)\to(0,0,0)}\frac{\partial(\bar r^-,\bar r^+,\bar r^\star)}{\partial(\bar u,\bar v,\bar w)}(k,\bar u,\bar v,\bar w)=0$$

uniformly with respect to $k\in\mathbb{Z}_\kappa^+$. Since $\upsilon(\cdot;\eta)=S(\cdot,0,0,\eta)$ solves the system (15) we get for all $\eta\in U^\star$

$$\bar r^-(k,0,0,\eta)\equiv 0,\quad \bar r^+(k,0,0,\eta)\equiv 0,\quad \bar r^\star(k,0,0,\eta)\equiv 0\quad\text{ on }\mathbb{Z}_\kappa^+. \tag{26}$$

Hence, $(0,0,\eta)$, $\eta\in U^\star$, represents a family of stationary solutions of (25).

(III) In order to investigate system (25) we choose an open neighborhood $\tilde U\subseteq U$ of $0\in\mathbb{R}^N$ such that $(t\bar u,t\bar v,\bar w)\in U$ for any $(\bar u,\bar v,\bar w)\in\tilde U$ and all $t\in[0,1]$. By the Mean Value Theorem we then get for any $(\bar u,\bar v,\bar w)\in\tilde U$ and $k\in\mathbb{Z}_\kappa^+$ the relation

$$\begin{aligned}&\bar r^-(k,\bar u,\bar v,\bar w)=\\ =\;&\bar r^-(k,0,0,\bar w)+\int_0^1\frac{\partial\bar r^-}{\partial(\bar u,\bar v)}(k,t\bar u,t\bar v,\bar w)\,dt\begin{pmatrix}\bar u\\ \bar v\end{pmatrix}=\\ \overset{(26)}{=}\;&\left(\int_0^1\frac{\partial\bar r^-}{\partial\bar u}(k,t\bar u,t\bar v,\bar w)\,dt\right)\bar u+\left(\int_0^1\frac{\partial\bar r^-}{\partial\bar v}(k,t\bar u,t\bar v,\bar w)\,dt\right)\bar v.\end{aligned}$$

Analogous relations hold for the other nonlinear terms $\bar r^+$ and $\bar r^\star$. Using the abbreviations

$$\begin{aligned}B_1^-(k,\bar u,\bar v,\bar w)&:=\int_0^1\frac{\partial\bar r^-}{\partial\bar u}(k,t\bar u,t\bar v,\bar w)\,dt, & B_2^-(k,\bar u,\bar v,\bar w)&:=\int_0^1\frac{\partial\bar r^-}{\partial\bar v}(k,t\bar u,t\bar v,\bar w)\,dt,\\ B_1^+(k,\bar u,\bar v,\bar w)&:=\int_0^1\frac{\partial\bar r^+}{\partial\bar u}(k,t\bar u,t\bar v,\bar w)\,dt, & B_2^+(k,\bar u,\bar v,\bar w)&:=\int_0^1\frac{\partial\bar r^+}{\partial\bar v}(k,t\bar u,t\bar v,\bar w)\,dt,\\ B_1^\star(k,\bar u,\bar v,\bar w)&:=\int_0^1\frac{\partial\bar r^\star}{\partial\bar u}(k,t\bar u,t\bar v,\bar w)\,dt, & B_2^\star(k,\bar u,\bar v,\bar w)&:=\int_0^1\frac{\partial\bar r^\star}{\partial\bar v}(k,t\bar u,t\bar v,\bar w)\,dt,\end{aligned}$$

we get six matrix-valued functions which have continuous partial derivatives with respect to $(\bar u,\bar v,\bar w)$ and converge, by assumption (H2), to 0 uniformly with respect to $k\in\mathbb{Z}_\kappa^+$ as

$(\bar{u},\bar{v},\bar{w}) \to (0,0,0)$. The system (25) thus has the form

$$\begin{cases} \bar{u}' = B^-(k)\bar{u} + B_1^-(k,\bar{u},\bar{v},\bar{w})\bar{u} + B_2^-(k,\bar{u},\bar{v},\bar{w})\bar{v} \\ \bar{v}' = B^+(k)\bar{v} + B_1^+(k,\bar{u},\bar{v},\bar{w})\bar{u} + B_2^+(k,\bar{u},\bar{v},\bar{w})\bar{v} \\ \bar{w}' = \quad \bar{w} \quad + B_1^\star(k,\bar{u},\bar{v},\bar{w})\bar{u} + B_2^\star(k,\bar{u},\bar{v},\bar{w})\bar{v} \end{cases}. \tag{27}$$

In order to further decouple the system under consideration we now apply a theorem on the existence of local center-stable fiber bundles to systems (25) and (27). This result is a consequence of a local version of AULBACH, PÖTZSCHE & SIEGMUND [5, Theorem 4.11]. It provides a constant $\rho > 0$ and a function $s : \mathbb{Z}_\kappa^+ \times B_\rho(0) \subseteq \mathbb{Z}_\kappa^+ \times \mathbb{R}^{N^-} \times \mathbb{R}^M \to B_\rho(0) \subseteq \mathbb{R}^{N^+}$ which defines a local invariant fiber bundle S. The function s has the following properties:

(a) For all $k \in \mathbb{Z}_\kappa^+$ we have $s(k,0,0) = 0$ and $\frac{\partial s}{\partial(\bar{u},\bar{w})}(k,0,0) = 0$.

(b) For all points $(k,\bar{u},\bar{w}) \in \mathbb{Z}_\kappa^+ \times B_\rho(0)$ which have the property that $(B^-(k)\bar{u} + \bar{r}^-(k,\bar{u},s(k,\bar{u},\bar{w}),\bar{w}), \bar{w} + \bar{r}^\star(k,\bar{u},s(k,\bar{u},\bar{w}),\bar{w}))$ belongs to $B_\rho(0)$ we have

$$\begin{aligned} & s(k+1, B^-(k)\bar{u} + B_1^-(k,\bar{u},s(k,\bar{u},\bar{w}),\bar{w})\bar{u} + \\ & +B_2^-(k,\bar{u},s(k,\bar{u},\bar{w}),\bar{w})s(k,\bar{u},\bar{w}), \\ & \bar{w} + B_1^\star(k,\bar{u},s(k,\bar{u},\bar{w}),\bar{w})\bar{u} + \\ & +B_2^\star(k,\bar{u},s(k,\bar{u},\bar{w}),\bar{w})s(k,\bar{u},\bar{w})) = \\ = \; & B^+(k)s(k,\bar{u},\bar{w}) + B_1^+(k,\bar{u},s(k,\bar{u},\bar{w}),\bar{w})\bar{u} + \\ & +B_2^+(k,\bar{u},s(k,\bar{u},\bar{w}),\bar{w})s(k,\bar{u},\bar{w}). \end{aligned} \tag{28}$$

(c) For any $k \in \mathbb{Z}_\kappa^+$ the function $s(k,\cdot)$ is continuously differentiable.

Without loss of generality we may suppose that $B_\rho(0) \subseteq \tilde{U}$ in the following considerations. Since $(0,0,\eta) \in B_\rho(0)$ is a bounded (since stationary) solution of (25) we get

$$s(k,0,\eta) \equiv 0 \quad \text{on } \mathbb{Z}_\kappa^+ \tag{29}$$

for any $\eta \in B_\rho(0)$ which is sufficiently small. We now apply the local coordinate change

$$\begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{pmatrix} := T_3(k,\bar{u},\bar{v},\bar{w}) := \begin{pmatrix} \bar{u} \\ \bar{v} - s(k,\bar{u},\bar{w}) \\ \bar{w} \end{pmatrix} \tag{30}$$

to system (27). This yields the system

$$\begin{cases} \hat{u}' = B^-(k)\hat{u} \qquad\qquad\qquad\qquad\qquad\; + \hat{r}^-(k,\hat{u},\hat{v},\hat{w}) \\ \hat{v}' = B^+(k)\hat{v} + B_2^+(k,\hat{u},\hat{v},\hat{w})\hat{v} + \hat{r}^+(k,\hat{u},\hat{v},\hat{w}) \\ \hat{w}' = \quad \hat{w} \qquad\qquad\qquad\qquad\qquad\; + \hat{r}^\star(k,\hat{u},\hat{v},\hat{w}) \end{cases}$$

where we have used the abbreviations

$$\begin{aligned} \hat{r}^-(k,\hat{u},\hat{v},\hat{w}) \; := \; & B_1^-(k,\hat{u},\hat{v}+s(k,\hat{u},\hat{w}),\hat{w})\hat{u} + \\ & +B_2^-(k,\hat{u},\hat{v}+s(k,\hat{u},\hat{w}),\hat{w})(\hat{v}+s(k,\hat{u},\hat{w})), \end{aligned}$$

$$
\begin{aligned}
\hat r^\star(k,\hat u,\hat v,\hat w) &:= B_1^\star(k,\hat u,\hat v+s(k,\hat u,\hat w),\hat w)\hat u+\\
&\quad +B_2^\star(k,\hat u,\hat v+s(k,\hat u,\hat w),\hat w)(\hat v+s(k,\hat u,\hat w)),\\
\hat r^+(k,\hat u,\hat v,\hat w) &:= B^+(k)s(k,\hat u,\hat w)+B_1^+(k,\hat u,\hat v+s(k,\hat u,\hat w),\hat w)\hat u+\\
&\quad +B_2^+(k,\hat u,\hat v+s(k,\hat u,\hat w),\hat w)s(k,\hat u,\hat w)-\\
&\quad -s(k+1,\hat B^-(k)\hat u+B_1^-(k,\hat u,\hat v+s(k,\hat u,\hat w),\hat w)\hat u+\\
&\quad +B_2^-(k,\hat u,\hat v+s(k,\hat u,\hat w),\hat w)(\hat v+s(k,\hat u,\hat w)),\\
&\quad \hat w+B_1^\star(k,\hat u,\hat v+s(k,\hat u,\hat w),\hat w)\hat u+\\
&\quad +B_2^\star(k,\hat u,\hat v+s(k,\hat u,\hat w),\hat w)(\hat v+s(k,\hat u,\hat w))).
\end{aligned}
$$

Because of the identity (29) we get $\hat r^-(k,0,0,\eta)\equiv 0$, $\hat r^\star(k,0,0,\eta)\equiv 0$ on $\mathbb{Z}_\kappa^+$ for normwise sufficiently small $\eta\in B_\rho(0)$. As above we may write

$$
\begin{aligned}
\hat r^-(k,\hat u,\hat v,\hat w) &= \hat B_1^-(k,\hat u,\hat v,\hat w)\hat u+\hat B_2^-(k,\hat u,\hat v,\hat w)\hat v,\\
\hat r^\star(k,\hat u,\hat v,\hat w) &= \hat B_1^\star(k,\hat u,\hat v,\hat w)\hat u+\hat B_2^\star(k,\hat u,\hat v,\hat w)\hat v
\end{aligned}
$$

where

$$
\begin{aligned}
\hat B_1^-(k,\hat u,\hat v,\hat w) &:= \int_0^1 \frac{\partial \hat r^-}{\partial \hat u}(k,t\hat u,t\hat v,\hat w)\,dt, \quad \hat B_2^-(k,\hat u,\hat v,\hat w) := \int_0^1 \frac{\partial \hat r^-}{\partial \hat v}(k,t\hat u,t\hat v,\hat w)\,dt,\\
\hat B_1^\star(k,\hat u,\hat v,\hat w) &:= \int_0^1 \frac{\partial \hat r^\star}{\partial \hat u}(k,t\hat u,t\hat v,\hat w)\,dt, \quad \hat B_2^\star(k,\hat u,\hat v,\hat w) := \int_0^1 \frac{\partial \hat r^\star}{\partial \hat v}(k,t\hat u,t\hat v,\hat w)\,dt.
\end{aligned}
$$

Moreover we get

$$
\begin{aligned}
\hat r^+(k,\hat u,0,\hat w) &= B^+(k)s(k,\hat u,\hat w)+B_1^+(k,\hat u,s(k,\hat u,\hat w),\hat w)\hat u+\\
&\quad +B_2^+(k,\hat u,s(k,\hat u,\hat w),\hat w)s(k,\hat u,\hat w)-\\
&\quad -s(k+1,B^-(k)\hat u+B_1^-(k,\hat u,s(k,\hat u,\hat w),\hat w)\hat u+\\
&\quad +B_2^-(k,\hat u,s(k,\hat u,\hat w),\hat w)s(k,\hat u,\hat w),\\
&\quad \hat w+B_1^\star(k,\hat u,s(k,\hat u,\hat w),\hat w)\hat u+\\
&\quad +B_2^\star(k,\hat u,s(k,\hat u,\hat w),\hat w)s(k,\hat u,\hat w))=\\
&\overset{(28)}{=} 0 \quad \text{for } k\in\mathbb{Z}_\kappa^+
\end{aligned}
$$

and using the abbreviation

$$
\hat B_2^+(k,\hat u,\hat v,\hat w) := B_2^+(k,\hat u,\hat v,\hat w)+\int_0^1 \frac{\partial \hat r^+}{\partial \hat v}(k,\hat u,t\hat v,\hat w)\,dt
$$

we obtain the claimed form of the difference equation (11). Together with the function $s(k,\cdot)$ also the coordinate change $T_3(k,\cdot)$ is continuously differentiable for any $k\in\mathbb{Z}_\kappa^+$.

(IV) Defining the transformation $\mathcal{T}_{\mu_0}: A_{\mu_0}\to\mathbb{R}^N$ by the relation

$$
\mathcal{T}_{\mu_0}(k,x) := T_3\left(k,T_2(k,T_1(k,x-\mu_0(k)))\right)
$$

we get from the previous considerations the assertions of the theorem. Concerning statement (e) we note that the transformations T_1, S, T_3 and the inverses have bounded derivatives on their domain. □

5 The Main Result

The following theorem, the main result of this paper, may be considered as a discrete time version of the corresponding result on differential equations in AULBACH [3]. It turns out that, compared to the continuous time result, for the difference equations case we have to make two additional assumptions in order to take care of two well known deficiencies of discrete time solutions, the lack of backward existence and the disconnectedness.

Theorem 5.1 *We reconsider the difference equation*

$$x' = f(k, x) \tag{31}$$

dealt with in the previous section. In addition to the assumptions (H1), (H2) and (H3) we suppose that

(H4) $\mathcal{M}$ *is compact,*

(H5) $f(k,\cdot)|_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ *is surjective for any* $k \in \mathbb{Z}^+_{\kappa_0}$.

Then if $\mu : \mathbb{Z}^+_\kappa \to \mathbb{R}^N$ *is any solution of (31) with the properties*

(i) $\lim_{k\to\infty} [\mathcal{T}_{\mu_0}(k+1, \mu(k+1)) - \mathcal{T}_{\mu_0}(k, \mu(k))] = 0$ *for any solution* $\mu_0 : \mathbb{Z}^+_\kappa \to \mathbb{R}^N$ *of (31) with* $\mu_0(\kappa) \in \mathcal{M}$,

(ii) $\lim_{k\to\infty} \operatorname{dist}(\mu(k), \mathcal{M}) = 0$

then there exists a point $\xi \in \mathcal{M}$ *such that*

$$\lim_{k\to\infty} [\lambda(k; \kappa, \xi) - \mu(k)] = 0\,,$$

i.e. $\mathcal{M}$ *possesses an asymptotic phase.*

Proof. We proceed in three steps.

(I) Since assumption (H3) applies to all solutions of (31) starting on the manifold $\mathcal{M}$ and because $\mathcal{M}$ is compact, we have

$$\sup_{(k,\xi)\in\mathbb{Z}^+_\kappa\times\mathcal{M}} \left\| \frac{\partial\lambda}{\partial\xi}(k;\kappa,\xi) \right\| < \infty.$$

Thus there exists for any $\varepsilon > 0$ a $\delta = \delta(\varepsilon) > 0$ such that for all $\xi, \bar{\xi} \in \mathcal{M}$ the following implication is true:

$$\begin{array}{rll} & \left\|\lambda(k_0;\kappa,\xi) - \lambda(k_0;\kappa,\bar{\xi})\right\| < \delta & \text{for some } k_0 \in \mathbb{Z}^+_\kappa\,, \\ \Rightarrow & \left\|\lambda(k;\kappa,\xi) - \lambda(k;\kappa,\bar{\xi})\right\| < \varepsilon & \text{for } k \in \mathbb{Z}^+_{k_0}\,. \end{array} \tag{32}$$

(II) The compactness of $\mathcal{M}$ implies that, because of property (ii), the function μ has an ω-limit point $\eta \in \mathcal{M}$. Thus there exists a sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}^+_\kappa$ with

$$\eta = \lim_{n\to\infty} \mu(k_n). \tag{33}$$

Assumption (H5) then guarantees that the solutions of (31) on $\mathcal{M}$ have (not necessarily unique) backward continuations. Therefore there exists a sequence $(\eta_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ with

$$\eta = \lambda(k_n; \kappa, \eta_n) \quad \text{for } n \in \mathbb{N}\,. \tag{34}$$

Since $\mathcal{M}$ (and thus $(\eta_n)_{n\in\mathbb{N}}$) is bounded there exists a converging subsequence $(\eta_{n_m})_{m\in\mathbb{N}}$ whose limit $\xi := \lim_{m\to\infty} \eta_{n_m}$ belongs to the closed set $\mathcal{M}$. We therefore get the estimate

$$\begin{aligned} & \|\mu(k_{n_m}) - \lambda(k_{n_m}; \kappa, \xi)\| \le \\ \stackrel{(34)}{\le} \; & \|\mu(k_{n_m}) - \eta\| + \|\lambda(k_{n_m}; \kappa, \eta_{n_m}) - \lambda(k_{n_m}; \kappa, \xi)\| \quad \text{for } m \in \mathbb{N}, \end{aligned}$$

and using (32) and (33) we get

$$\lim_{m\to\infty} [\mu(k_{n_m}) - \lambda(k_{n_m}; \kappa, \xi)] = 0. \tag{35}$$

Consequently the solution $\lambda(\cdot; \kappa, \xi)$ lies in $\mathcal{M}$ and the function $\mu - \lambda(\cdot; \kappa, \xi)$ has 0 as ω-limit point. In order to simplify our notation from now on we write $(k_n)_{n\in\mathbb{N}}$ instead of $(k_{n_m})_{m\in\mathbb{N}}$.

(III) In order to show that the difference $\mu(k) - \lambda(k; \kappa, \xi)$ converges to 0 as $k \to \infty$ we notice that for the function $\nu(k) = (\nu^-, \nu^+, \nu^\star)(k) := \mathcal{T}_{\lambda(\cdot;\kappa,\xi)}(k, \mu(k))$ we have, because of Theorem 4.1(a),

$$\mathcal{T}_{\lambda(\cdot;\kappa,\xi)}(k, \lambda(k; \kappa, \xi)) \equiv 0 \quad \text{on } \mathbb{Z}_\kappa^+. \tag{36}$$

Because of (35) and the construction of $\mathcal{T}_{\lambda(\cdot;\kappa,\xi)}$ the point $0 \in \mathbb{R}^N$ is an ω-limit point of the function ν and it remains to be shown that $\nu(k)$ converges to 0 as $k \to \infty$. Assuming the contrary there exists a real number $\rho \in (0, \rho_1)$ ($\rho_1 > 0$ is defined in Theorem 4.1(a)) and because of assumption (i) there exists a sequence of nonempty $\mathbb{Z}$-intervals $J_n := [k_n, k_n^+]_\mathbb{Z}$, $n \in \mathbb{N}$, with $k_n, k_n^+ \in \mathbb{Z}_\kappa^+$, $k_n < k_n^+ < k_{n+1}$, such that

$$\lim_{n\to\infty} \nu(k_n) \;=\; 0, \tag{37}$$

$$\nu(k) \;\in\; B_\rho(0) \quad \text{for } k \in \bigcup_{n\in\mathbb{N}} J_n, \tag{38}$$

$$\nu(k_n^+) \;\in\; B_\rho(0) \setminus B_{\frac{\rho}{2}}(0) \quad \text{for } n \in \mathbb{N}. \tag{39}$$

On any discrete interval J_n the function ν is a solution of the linear homogeneous system

$$\begin{cases} u' = B^-(k)u + \hat{B}_1^-(k, \nu(k))u + \hat{B}_2^-(k, \nu(k))v \\ v' = B^+(k)v + \hat{B}_2^+(k, \nu(k))v \\ w' = w + \hat{B}_1^\star(k, \nu(k))u + \hat{B}_2^\star(k, \nu(k))v \end{cases} \tag{40}$$

where the transition matrices Ψ^- and Ψ^+ of $u' = B^-(k)u$ and $v' = B^+(k)v$, respectively, satisfy the estimates

$$\left\|\Psi^-(k,l)\right\| \le \tilde{K}_1 \alpha^{k-l} \quad \text{for } k \ge l \ge \kappa, \quad \left\|\Psi^+(k,l)\right\| \le \tilde{K}_2 \beta^{k-l} \quad \text{for } l \ge k \ge \kappa\,.$$

Without loss of generality we may suppose that $\rho > 0$ is so small that apart from the estimate

$$\rho < \min\left\{\rho_1, \frac{c}{2}\delta(\frac{\rho_1}{C})\right\} \tag{41}$$

(the positive constants c and C are those of Theorem 4.1(e)) the following estimates are true for all $k \in \bigcup_{n\in\mathbb{N}} J_n$:

$$\left\|\hat{B}_1^-(k,\nu(k))\right\| \le \min\left\{\frac{1-\alpha}{2\tilde{K}_1},\frac{\beta-1}{2\tilde{K}_2}\right\}, \qquad \left\|\hat{B}_2^-(k,\nu(k))\right\| \le \min\left\{\frac{1-\alpha}{2\tilde{K}_1},\frac{\beta-1}{2\tilde{K}_2}\right\},$$
$$\left\|\hat{B}_2^+(k,\nu(k))\right\| \le \min\left\{\frac{1-\alpha}{2\tilde{K}_1},\frac{\beta-1}{2\tilde{K}_2}\right\},$$
$$\left\|\hat{B}_1^\star(k,\nu(k))\right\| \le \min\left\{\frac{1-\alpha}{2\tilde{K}_1},\frac{\beta-1}{2\tilde{K}_2}\right\}, \qquad \left\|\hat{B}_2^\star(k,\nu(k))\right\| \le \min\left\{\frac{1-\alpha}{2\tilde{K}_1},\frac{\beta-1}{2\tilde{K}_2}\right\}.$$

Using Theorem 4.1(e) we get

$$\begin{aligned}\left\|\mu(k_n^+)-\lambda(k_n^+;\kappa,\xi)\right\| &\overset{(36)}{\le} \frac{1}{c}\left\|\mathcal{T}_{\lambda(\cdot;\kappa,\xi)}(k_n^+,\mu(k_n^+))\right\| < \\ &\overset{(38)}{<} \frac{\rho}{c} \overset{(41)}{\le} \frac{1}{2}\delta(\frac{\rho_1}{C}) \quad \text{for } n\in\mathbb{N},\end{aligned} \tag{42}$$

and since the sequence $(\mu(k_n^+))_{n\in\mathbb{N}}$ is bounded, because of the estimate (42), there exists an ω-limit point $\eta_0 := \lim_{m\to\infty}\mu(k_{n_m}^+) \in \mathcal{M}$ where $(k_{n_m}^+)_{m\in\mathbb{N}}$ is a subsequence of $(k_n^+)_{n\in\mathbb{N}}$.

As in the second step of this proof we get a point $\xi_0 \in \mathcal{M}$ such that

$$\lim_{l\to\infty}\left[\mu(k_{n_{m_l}}^+)-\lambda(k_{n_{m_l}}^+;\kappa,\xi_0)\right]=0 \tag{43}$$

where $(k_{n_{m_l}}^+)_{l\in\mathbb{N}}$ is a further subsequence of $(k_{n_m}^+)_{m\in\mathbb{N}}$. Using (42) this implies that for sufficiently large $l_0\in\mathbb{N}$ we get

$$\begin{aligned}&\left\|\lambda(k_{n_{m_l}}^+;\kappa,\xi)-\lambda(k_{n_{m_l}}^+;\kappa,\xi_0)\right\| \le \\ \le\ &\left\|\lambda(k_{n_{m_l}}^+;\kappa,\xi)-\mu(k_{n_{m_l}}^+)\right\| + \left\|\mu(k_{n_{m_l}}^+)-\lambda(k_{n_{m_l}}^+;\kappa,\xi_0)\right\| \le \\ \overset{(42)}{\le}\ &\delta(\frac{\rho_1}{C}) \quad \text{for } l\in\mathbb{Z}_{l_0}^+.\end{aligned}$$

Consequently, because of (32) we get from Theorem 4.1(e)

$$\left\|\mathcal{T}_{\lambda(\cdot;\kappa,\xi)}(k,\lambda(k;\kappa,\xi_0))\right\| \overset{(36)}{\le} C\left\|\lambda(k;\kappa,\xi_0)-\lambda(k;\kappa,\xi)\right\| \le \rho_1 \quad \text{for } k\in\mathbb{Z}_{n_{m_{l_0}}}^+.$$

Now we are in a position to apply Aulbach [4, Lemma 8.1] to the system (40) and its bounded solution

$$\nu_0(k) = (\nu_0^-,\nu_0^+,\nu_0^\star)(k) := \mathcal{T}_{\lambda(\cdot;\kappa,\xi)}(k,\lambda(k;\kappa,\xi_0)).$$

This provides a relation of the form

$$\lim_{k\to\infty}(\nu_0^-,\nu_0^+,\nu_0^\star)(k) = (0,0,w^*) \tag{44}$$

for some $w^*\in\mathbb{R}^M$. From (43) and Theorem 4.1(e) we conclude that the relation

$$\lim_{l\to\infty}\left[(\nu^-,\nu^+,\nu^\star)(k_{n_{m_l}}^+)-(\nu_0^-,\nu_0^+,\nu_0^\star)(k_{n_{m_l}}^+)\right]=(0,0,0)$$

holds true which, in turn, with (44) yields

$$\lim_{l\to\infty} (\nu^-, \nu^+, \nu^\star)(k^+_{n_{m_l}}) = (0, 0, w^*)\,. \tag{45}$$

Then using AULBACH [4, Lemma B.6] we see that there exist constants $C_1, C_2 > 0$ (which depend only on the growth rates α, β and $\tilde{K}_1, \tilde{K}_2$) with the property

$$\left\|\nu^\star(k^+_{n_{m_l}})\right\| \leq \left\|\nu^\star(k_{n_{m_l}})\right\| + C_1 \left\|\nu^-(k_{n_{m_l}})\right\| + C_2 \left\|\nu^+(k^+_{n_{m_l}})\right\|.$$

Because of (37) and (45) the sequence $(\nu^\star(k^+_{n_{m_l}}))_{l\in\mathbb{N}}$ and consequently also the sequence $(\nu(k^+_{n_{m_l}}))_{l\in\mathbb{N}}$ converges to 0 as $l \to \infty$. This, however, contradicts the relation (39). □

6 Appendix: Parameter-dependent Inverse Functions

For the reader's convenience we state here a qualitative Inverse Function Theorem which can be shown using ABRAHAM, MARSDEN & RATIU [1, Proposition 2.5.6].

Lemma 6.1 *Let Ω be an open neighborhood of the zero vector of some Banach space $\mathcal{X}$ and let $T : \mathcal{P} \times \Omega \to \mathcal{X}$ be a mapping such that $T(p, \cdot)$ is of class C^m ($m \geq 2$) for any p in some nonempty set $\mathcal{P}$. Furthermore assume the following:*

(i) $T(p, 0) \equiv 0$ on $\mathcal{P}$,

(ii) the partial derivatives $\frac{\partial T}{\partial x}(p, 0) : \mathcal{X} \to \mathcal{X}$ are invertible for $p \in \mathcal{P}$,

(iii) $M := \sup_{p\in\mathcal{P}} \|[\frac{\partial T}{\partial x}(p, 0)]^{-1}\| < \infty$,

(iv) $K := \sup_{(p,x)\in\mathcal{P}\times\overline{B_R(0)}} \|\frac{\partial^2 T}{\partial x^2}(p, x)\| < \infty$ for some $R > 0$ with $\overline{B_R(0)} \subseteq \Omega$.

Then, using the abbreviation $P := \min\left\{R, \frac{1}{2KM}\right\}$, there exists a uniquely determined mapping $S : \mathcal{P} \times \overline{B_{\frac{P}{2M}}(0)} \to \mathcal{X}$ with the following properties:

(a) S is bounded, more explicitly,

$$\|S(p, y)\| \leq P \quad \text{for } (p, y) \in \mathcal{P} \times \overline{B_{\frac{P}{2M}}(0)},$$

(b) $S(p, \cdot)$ is the inverse function of $T(p, \cdot)$, more explicitly,

$$T(p, S(p, y)) = y \quad \text{for } (p, y) \in \mathcal{P} \times \overline{B_{\frac{P}{2M}}(0)},$$

(c) $S(p, \cdot)|_{B_{\frac{P}{2M}}(0)}$ is of class C^m for each $p \in \mathcal{P}$.

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