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Stability of Center Fiber Bundles for Nonautonomous Difference Equations

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Abstract. Center fiber bundles are the generalization of center manifolds to nonautonomous difference equations. Following closely an idea of Palmer, we deduce the essential stability properties of such fiber bundles, namely their asymptotic phase and a reduction principle.

1 Introduction and preliminaries

With the aid of elementary tools like the variation of constants formula and Gronwall's inequality, Palmer [11] proved a useful lemma in the theory of center manifolds for finite dimensional autonomous ordinary differential equations. It means that as long as a solution remains in the vicinity of an equilibrium with given center manifold, it must be close to some solution on the mentioned manifold. This lemma is helpful, since such stability properties as asymptotic phase and Pliss's reduction principle are derived simply and directly. In this paper we show that the same result also holds in the situation of nonautonomous difference equations in arbitrary Banach spaces, with the restriction that we need a finite dimensional center manifold to deduce its asymptotic phase and the reduction principle, since a compactness argument is involved. The importance of a nonautonomous theory is due to the fact that the investigation of nonconstant solutions canonically leads to time dependent problems in form of the equation of perturbed motion. To our best knowledge, related results under global assumptions on the nonlinearities can be found solely in Wanner [14] and Janglajew [8], whereas our situation is more realistic and — concerning the spectrum of the linear part — more general (cf. Hypothesis 2.1). We only refer to Aulbach [2], Carr [7] and Palmer [11] for further references on ordinary differential equations.

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Now we introduce our notation. $\mathbb{Z}$ stands for the integers and $\mathbb{N}$ is the set of positive integers. A *discrete interval* is the intersection of a real interval with the integers. For $\kappa, K \in \mathbb{Z}$ we write $\mathbb{Z}_\kappa^+ := \{k \in \mathbb{Z} : \kappa \leq k\}$ and $[\kappa, K]_\mathbb{Z} := [\kappa, K] \cap \mathbb{Z}$. The Banach spaces X, Y are real or complex throughout this paper and their norm is denoted by $\|\cdot\|$. $L(X)$ is the Banach space of continuous endomorphisms on X, I_X the identity map on X and $GL(X)$ the multiplicative group of bijective mappings in $L(X)$. On the Cartesian product $X \times Y$ we always use the norm

$$\|(x, y)\| := \max\{\|x\|, \|y\|\}. \tag{1.1}$$

In a normed space, $B_\epsilon(x)$ is the ball with center x and radius $\epsilon > 0$. In case a mapping $F : \mathbb{Z}\times Y \to X$ depends differentiably on the second variable, then its partial derivative is denoted by D_2F.

We write

$$x' = f(k, x) \tag{1.2}$$

to denote the difference equation $x(k + 1) = f(k, x(k))$, with the *right-hand side* $f : \mathbb{Z}\times U \to X$, where U is a subset of the Banach space X. A sequence $\nu : I \to X$ is said to solve (1.2) on a discrete interval I, if $\nu(k + 1) = f(k, \nu(k))$ as long as ν exists, i.e., as long as $\nu(k) \in U$ holds for $k \in I$. Let φ denote the *general solution* of equation (1.2), i.e., $\varphi(\cdot; \kappa, \xi)$ solves (1.2) and satisfies the initial condition $\varphi(\kappa; \kappa, \xi) = \xi$ for $\kappa \in I$, $\xi \in U$. In forward time, $\varphi(\cdot; \kappa, \xi)$ can be defined recursively

$$\varphi(k; \kappa, \xi) := \begin{cases} \xi & \text{for } k = \kappa \\ f(k - 1, \varphi(k - 1; \kappa, \xi)) & \text{for } k > \kappa \end{cases},$$

as long as $\varphi(k - 1; \kappa, \xi) \in U$. Given an operator sequence $A : \mathbb{Z} \to L(X)$ we define the *evolution operator* $\Phi(k, \kappa) \in L(X)$ of the linear difference equation

$$x' = A(k)x$$

as the mapping

$$\Phi(k, \kappa) := \begin{cases} I_X & \text{for } k = \kappa \\ A(k - 1) \cdots A(\kappa) & \text{for } k > \kappa \end{cases}$$

and if $A(k)$ is invertible (in $L(X)$) for $k < \kappa$, then

$$\Phi(k, \kappa) := A(k)^{-1} \cdots A(\kappa - 1)^{-1} \quad \text{for } k < \kappa.$$

2 Existence of center fiber bundles

In this section we repeat and summarize some basic facts about center fiber bundles. For the autonomous situation, Carr [7, pp. 33–36, Section 2.8] is still a good reference. However, our nonautonomous setting is as follows:

Hypothesis 2.1 Let $U \subseteq X \times Y$ be an open neighborhood of $(0, 0)$, $m \in \mathbb{N}$ and consider a system of nonautonomous difference equations

$$\begin{cases} x' = A(k)x + F(k, x, y) \\ y' = B(k)y + G(k, x, y) \end{cases}, \tag{2.1}$$

where X, Y are arbitrary Banach spaces, $A : \mathbb{Z} \to L(X)$, $B : \mathbb{Z} \to GL(Y)$ and the mappings $F : \mathbb{Z}\times U \to X$, $G : \mathbb{Z}\times U \to Y$ are m-times continuously differentiable with respect to (x, y). Moreover, we assume:

(i) Hypothesis on linear part: The evolution operators Φ and Ψ of the systems $x' = A(k)x$ and $y' = B(k)y$, respectively, satisfy for all $k, l \in \mathbb{Z}$ the estimates

$$\|\Phi(k,l)\| \leq K_1 \alpha^{k-l} \quad \text{for } k \geq l, \qquad \|\Psi(k,l)\| \leq K_2 \beta^{k-l} \quad \text{for } l \geq k, \tag{2.2}$$

with real constants $K_1, K_2 \geq 1$ and α, β with $0 < \alpha < 1$, $\alpha < \beta$.

(ii) Hypothesis on nonlinearities: We have

$$F(k,0,0) \equiv 0, \qquad G(k,0,0) \equiv 0 \quad \text{on } \mathbb{Z},$$

and the partial derivatives of F and G satisfy

$$\lim_{(x,y)\to(0,0)} D_{(2,3)}(F,G)(k,x,y) = 0 \quad \text{uniformly in } k \in \mathbb{Z}. \tag{2.3}$$

Remark 2.2 Let U be an open neighborhood of 0 in a Hilbert space Z. Suppose that the mapping $f : \mathbb{Z}\times U \to Z$ is of class C^m, $m \in \mathbb{N}$, in the state space variable and satisfies the following assumptions:

(i)′ $D_2 f(k,0) \in GL(Z)$ for $k \in \mathbb{Z}$, and the variational equation

$$z' = D_2 f(k,0) z \tag{2.4}$$

possesses an exponential dichotomy on $\mathbb{Z}$ with growth rates $\alpha \in (0,1)$, $\beta > \alpha$,

(ii)′ $f(k,0) \equiv 0$ on $\mathbb{Z}$ and $\lim_{z\to 0} D_2 f(k,z) = 0$ holds uniformly in $k \in \mathbb{Z}$.

Then any nonautonomous difference equation

$$z' = f(k,z)$$

can be brought into the "decoupled" form (2.1) such that Hypothesis 2.1 is fulfilled. This can be shown using methods from [4, Theorem 5] via a Lyapunov transformation. If, moreover, Z is finite dimensional, then the assumption $D_2 f(k,0) \in GL(Z)$, $k \in \mathbb{Z}$, can be dropped (cf. [12, p. 33, Satz 1.5.7]) and instead of (i)′ we can assume that the dichotomy spectrum (cf. [5]) of (2.4) is disjoint from $[\alpha, \beta]$. Finally, in case of an arbitrary Banach space Z, and if $C := D_2 f(k,0)$ does not depend on $k \in \mathbb{Z}$, it is sufficient to assume that the spectrum $\sigma(C) \subseteq \mathbb{C}$ of the operator $C \in L(Z)$ can be separated into a "stable" spectral part $\sigma_1 \subseteq B_\alpha(0)$, $0 < \alpha < 1$, and a disjoint "pseudo-unstable" spectral part σ_2 outside a circle with center 0 and radius $\beta > \alpha$ in the complex plane.

Our next aim is to introduce a nonautonomous counterpart of invariant manifolds. Thereto let $U_1 \subseteq X$, $U_2 \subseteq Y$ be open neighborhoods of 0 with $U_1 \times U_2 \subseteq U$ and let $\varphi = (\varphi_1, \varphi_2)$ denote the general solution of (2.1). Then the graph

$$C := \{(\kappa, c(\kappa,\eta), \eta) \in \mathbb{Z}\times X \times Y : \kappa \in \mathbb{Z}, \eta \in U_2\}$$

of a function $c : \mathbb{Z}\times U_2 \to U_1$ is called a *locally invariant fiber bundle* of the difference equation (2.1), if the implication

$$(\kappa, \xi, \eta) \in C \quad \Rightarrow \quad (k, \varphi(k; \kappa, \xi, \eta)) \in C$$

holds for all $k \in \mathbb{Z}_\kappa^+$ with the property $\varphi_2(k; \kappa, \xi, \eta) \in U_2$. Locally invariant fiber bundles evidently satisfy the following nonlinear functional equation, denoted as *invariance equation*

$$c\,(\kappa + 1, B(\kappa)\eta + G(\kappa, c(\kappa,\eta), \eta)) = A(\kappa) c(\kappa,\eta) + F(\kappa, c(\kappa,\eta), \eta) \tag{2.5}$$

for all $(\kappa, \eta) \in \mathbb{Z} \times U_2$ such that $B(\kappa)\eta + G(\kappa, c(\kappa, \eta), \eta) \in U_2$. Finally, a locally invariant fiber bundle C is called a *center fiber bundle* of (2.1), if the corresponding mapping c is continuously differentiable in $\eta \in U_2$ and if the relations

$$c(\kappa, 0) \equiv 0 \quad \text{on } \mathbb{Z}, \qquad \lim_{\eta \to 0} D_2 c(\kappa, \eta) = 0 \quad \text{uniformly in } \kappa \in \mathbb{Z} \tag{2.6}$$

are satisfied. We remark that the conditions (2.6) imply that C contains the zero solution of (2.1), and that C is fiber-wise tangent to the "pseudo-unstable" vector bundle $\mathbb{Z} \times \{0\} \times Y$. Such center fiber bundles are not unique in general, which can be seen from the succeeding

Example 2.3 Consider the two-dimensional autonomous difference equation

$$\begin{cases} x' = e^{-1} x \\ y' = y + \frac{y^2}{1-y} \end{cases}, \tag{2.7}$$

satisfying Hypothesis 2.1 with $U = \mathbb{R} \times (-\infty, 1)$, $K_1 = K_2 = 1$, $\alpha = e^{-1}$ and $\beta = 1$. Now it is easy to verify that

$$C_\gamma := \left\{ (\kappa, \xi, \eta) \in \mathbb{Z} \times \mathbb{R} \times (-\infty, 1) : \xi = \gamma e^{1/\eta} \text{ for } \eta < 0, \xi = 0 \text{ for } \eta \geq 0 \right\}$$

is a center fiber bundle of (2.7) for any parameter $\gamma \in \mathbb{R}$.

Nevertheless, center fiber bundles do exist under reasonable assumptions. In fact — and for purely technical reasons — we only need the additional assumption in Hypothesis 2.1 that X, Y are C^n-Banach spaces, $n \in \mathbb{N}$; i.e., the norms on X, Y have to be of class C^n away from 0. A characterization of such spaces, as well as examples, can be found in Kriegl and Michor [9, pp. 127–152, Section 13].

Theorem 2.4 (existence of center fiber bundles) *Suppose that the above Hypothesis 2.1 is satisfied under the gap condition $\alpha < \beta^n$ for some $n \in \{1, \ldots, m\}$, and that X, Y are C^n-Banach spaces. Then there exists a real number $\rho_0 > 0$ and a function $c : \mathbb{Z} \times B_{\rho_0}(0) \subseteq \mathbb{Z} \times Y \to B_{\rho_0}(0) \subseteq X$ with the following properties:*

(a) *$c : \mathbb{Z} \times B_{\rho_0}(0) \to X$ is continuous and $c(\kappa, \cdot) : B_{\rho_0}(0) \to X$ is n-times continuously differentiable for any $\kappa \in \mathbb{Z}$,*

(b) *the graph $C := \{(\kappa, c(\kappa, \eta), \eta) \in \mathbb{Z} \times X \times Y : \kappa \in \mathbb{Z}, \eta \in B_{\rho_0}(0)\}$ is a center fiber bundle of* (2.1)*,*

(c) *if the mappings A, B and F, G are periodic in k with period $\theta \in \mathbb{N}$, then*

$$c(\kappa + \theta, \eta) = c(\kappa, \eta) \quad \text{for } \kappa \in \mathbb{Z},\ \eta \in B_{\rho_0}(0),$$

and if equation (2.1) *is autonomous, then c is independent of $\kappa \in \mathbb{Z}$, i.e., the subset $C_0 := \{(c(\eta), \eta) \in X \times Y : \eta \in B_{\rho_0}(0)\}$ of the state space $X \times Y$ is an invariant manifold of* (2.1).

Proof One shows the existence of the mentioned mapping c by extending the nonlinearities F, G smoothly such that they are defined on $\mathbb{Z} \times X \times Y$ and have globally bounded derivatives there — in this cut-off technique the fact that X, Y are C^n-Banach spaces plays a decisive role, and the explicit construction can be found in [12, p. 73, Lemma 2.3.2]. Then it is possible to apply a general theorem on invariant fiber bundles (cf. [6, Theorem 4.1(b)]), where the above smoothness assertion follows from [13, Theorem 5.1(b)]. The fact that c satisfies the limit relation in (2.6) can be seen as in [12, p. 64, Korollar 2.2.15]. After all, the assertion (c) follows from [3, Corollary 4.2]. $\square$

3 Stability properties of center fiber bundles

From now on we assume that Hypothesis 2.1 is satisfied with $U = B_{\rho_0}(0)\times B_{\rho_0}(0)$ for some $\rho_0 > 0$, and that the difference equation (2.1) possesses a center fiber bundle C given by the mapping $c : \mathbb{Z}\times B_{\rho_0}(0)\to B_{\rho_0}(0)$. First of all, we observe that the mappings $\omega_F, \omega_G, \omega_c : [0, \rho_0) \to\mathbb{R}$,

$$\begin{aligned}
\omega_F(\rho) &:= \sup_{(k,x,y)\in\mathbb{Z}\times B_\rho(0)\times B_\rho(0)} \left\|D_{(2,3)}F(k, x, y)\right\|,\\
\omega_G(\rho) &:= \sup_{(k,x,y)\in\mathbb{Z}\times B_\rho(0)\times B_\rho(0)} \left\|D_{(2,3)}G(k, x, y)\right\|,\\
\omega_c(\rho) &:= \sup_{(\kappa,\eta)\in\mathbb{Z}\times B_\rho(0)} \|D_2c(\kappa, \eta)\|
\end{aligned}$$

are well-defined, increasing, and obtain by Hypothesis 2.1(ii) (cf. (2.3)) and (2.6) the limit relations

$$\lim_{\rho\searrow 0}\omega_F(\rho) = 0, \qquad \lim_{\rho\searrow 0}\omega_G(\rho) = 0, \qquad \lim_{\rho\searrow 0}\omega_c(\rho) = 0.$$

Lemma 3.1 *For any $\rho \in [0, \rho_0)$ we have*

$$\|c(\kappa, \eta)\| \le \omega_c(\rho)\, \|\eta\| \quad \textit{for } \kappa \in \mathbb{Z},\ \eta \in B_\rho(0). \tag{3.1}$$

Proof Using the mean value theorem (cf. [10, p. 341, Theorem 4.2]) we get

$$\|c(\kappa, \eta)\| \overset{(2.6)}{=} \|c(\kappa, \eta) - c(\kappa, 0)\| \le \left\|\int_0^1 D_2c(\kappa, h\eta)\, dh\right\| \|\eta\| \le \omega_c(\rho)\, \|\eta\|$$

for $\kappa \in \mathbb{Z}$ and $\eta \in B_\rho(0)$. □

Lemma 3.2 (Palmer's lemma) *Let $\kappa \le K$ be integers, let the real constant $\gamma \in (0, \min\{1, \beta\} - \alpha)$ be fixed and choose $\rho \in \left(0, \frac{\rho_0}{2}\right)$ so small that the estimates*

$$\begin{gathered}
\max\{2\omega_G(\rho), \omega_c(2\rho)\} < 1,\\
K_1\left(\omega_F(\rho) + \omega_c(\rho)\omega_G(\rho)\right) \le \gamma,\\
K_2\left[\omega_G(2\rho)(1 + \omega_c(2\rho)) + 2K_1\omega_G(\rho)\right] < \beta - \alpha - \gamma
\end{gathered} \tag{3.2}$$

hold. If $\nu = (\nu_1, \nu_2)$ is a solution of (2.1) defined on a discrete interval $[\kappa, K]_\mathbb{Z}$ and $\|\nu(k)\| < \rho$ for $k \in [\kappa, K]_\mathbb{Z}$, and if $\tilde\nu$ denotes a solution of the difference equation

$$y' = B(k)y + G(k, c(k, y), y), \tag{3.3}$$

with $\tilde\nu(K) = \nu_2(K)$, then the following holds:

(a) *$\tilde\nu$ is defined on the discrete interval $[\kappa, K]_\mathbb{Z}$,*
(b) *$\|\tilde\nu(k)\| < 2\rho$ for all $k \in [\kappa, K]_\mathbb{Z}$, and*
(c) *$\tilde\nu : [\kappa, K]_\mathbb{Z} \to Y$ satisfies the estimate*

$$\left\|\nu(k) - \begin{pmatrix} c(k, \tilde\nu(k)) \\ \tilde\nu(k)\end{pmatrix}\right\| \le \left(K_1 + \tfrac{1}{2}\right)\|\nu_1(\kappa) - c(\kappa, \nu_2(\kappa))\|\,(\alpha + \gamma)^{k-\kappa} \tag{3.4}$$

for all $k \in [\kappa, K]_\mathbb{Z}$.

Proof (I) We begin the present proof by deriving a preparatory estimate. Due to Lemma 3.1 the inequality

$$\|c(k, \nu_2(k))\| \overset{(3.1)}{\le} \omega_c(\rho)\, \|\nu_2(k)\| \le \omega_c(\rho)\rho \overset{(3.2)}{<} \rho \quad \text{for } k \in [\kappa, K]_\mathbb{Z} \tag{3.5}$$

holds, and taking the solution property of ν_2 into account, one has

$$\begin{aligned}
&\|B(k)\nu_2(k) + G(k, c(k, \nu_2(k)), \nu_2(k))\| \\
&\stackrel{(2.1)}{\leq} \quad \|\nu_2(k+1)\| + \|G(k, c(k, \nu_2(k)), \nu_2(k)) - G(k, \nu(k))\| \\
&\leq \quad \|\nu_2(k+1)\| + \omega_G(\rho) \|c(k, \nu_2(k)) - \nu_1(k)\| \\
&\leq \quad \rho + \omega_G(\rho) (\|c(k, \nu_2(k))\| + \|\nu_1(k)\|) \\
&\stackrel{(3.5)}{\leq} \quad \rho + 2\omega_G(\rho)\rho \stackrel{(3.2)}{<} 2\rho < \rho_0 \quad \text{for } k \in [\kappa, K-1]_{\mathbb{Z}}
\end{aligned} \tag{3.6}$$

by the mean value inequality (cf. [10, p. 342, Corollary 4.3]) applied to $G(k, \cdot)$. Using the mean value theorem (cf. [10, p. 341, Theorem 4.2]) and the invariance equation (2.5) we find the identity

$$\begin{aligned}
&A(k)\nu_1(k) + F(k, \nu(k)) - c\big(k+1, B(k)\nu_2(k) + G(k, c(k, \nu_2(k)), \nu_2(k))\big) \\
&\qquad +c\big(k+1, B(k)\nu_2(k) + G(k, c(k, \nu_2(k)), \nu_2(k))\big) \\
&\qquad -c\big(k+1, B(k)\nu_2(k) + G(k, \nu(k))\big) \\
&\stackrel{(2.5)}{=} \quad A(k)\big(\nu_1(k) - c(k, \nu_2(k))\big) + F(k, \nu(k)) - F(k, c(k, \nu_2(k)), \nu_2(k)) \\
&\qquad + \int_0^1 D_2c(k+1, \eta_k(h))\, dh\, [G(k, c(k, \nu_2(k)), \nu_2(k)) - G(k, \nu(k))]
\end{aligned}$$

for all $k \in [\kappa, K-1]_{\mathbb{Z}}$, and here we have abbreviated

$$\eta_k(h) := B(k)\nu_2(k) + G(k, \nu(k)) + h\, [G(k, c(k, \nu_2(k)), \nu_2(k)) - G(k, \nu(k))] .$$

Notice that the convexity of the ball $B_{\rho_0}(0) \subseteq Y$ and (3.6) implies $\eta_k(h) \in B_{\rho_0}(0)$ for $k \in [\kappa, K-1]_{\mathbb{Z}}$, $h \in [0,1]$. Then $\Delta_1(k) := \nu_1(k) - c(k, \nu_2(k))$ solves the linear inhomogeneous difference equation

$$\begin{aligned}
x' \quad = \quad & A(k)x + F(k, \nu(k)) - F(k, c(k, \nu_2(k)), \nu_2(k)) \\
& + \int_0^1 D_2c(k+1, \eta_k(h))\, dh\, [G(k, c(k, \nu_2(k)), \nu_2(k)) - G(k, \nu(k))]
\end{aligned}$$

on $[\kappa, K]_{\mathbb{Z}}$. Now the variation of constants formula (cf. [1, pp. 57–58]) gives us

$$\begin{aligned}
\Delta_1(k) \quad = \quad & \Phi(k, \kappa)\Delta_1(\kappa) + \sum_{n=\kappa}^{k-1} \Phi(k, n+1) \Big[F(n, \nu(n)) - F(n, c(n, \nu_2(n)), \nu_2(n)) \\
& + \int_0^1 D_2c(n+1, \eta_n(h))\, dh\, [G(n, c(n, \nu_2(n)), \nu_2(n)) - G(n, \nu(n))]\Big],
\end{aligned}$$

so that applying the mean value inequality (cf. [10, p. 342, Corollary 4.3]) yields

$$\|\Delta_1(k)\| \stackrel{(2.2)}{\leq} K_1\alpha^{k-\kappa} \|\Delta_1(\kappa)\| + K_1 (\omega_F(\rho) + \omega_c(\rho)\omega_G(\rho)) \sum_{n=\kappa}^{k-1} \alpha^{k-n-1} \|\Delta_1(n)\|$$

and consequently

$$\frac{\|\Delta_1(k)\|}{\alpha^k} \stackrel{(2.2)}{\leq} K_1\alpha^{-\kappa} \|\Delta_1(\kappa)\| + \frac{K_1 (\omega_F(\rho) + \omega_c(\rho)\omega_G(\rho))}{\alpha} \sum_{n=\kappa}^{k-1} \frac{\|\Delta_1(n)\|}{\alpha^n}$$

for $k \in [\kappa, K]_{\mathbb{Z}}$. By Gronwall's lemma (cf. [1, p. 183, Corollary 4.1.2]) we obtain

$$\|\Delta_1(k)\| \leq K_1 [\alpha + K_1(\omega_F(\rho) + \omega_c(\rho)\omega_G(\rho))]^{k-\kappa} \|\Delta_1(\kappa)\|$$

and using (3.2) it follows that

$$\|\nu_1(k) - c(k, \nu_2(k))\| \le K_1 (\alpha + \gamma)^{k-\kappa} \|\nu_1(\kappa) - c(\kappa, \nu_2(\kappa))\| \tag{3.7}$$

for $k \in [\kappa, K]_{\mathbb{Z}}$.

(II) Now let $k_0 \in [\kappa, K]_{\mathbb{Z}}$ be the least integer such that $\tilde{\nu}(k)$ is defined on the discrete interval $[k_0, K]_{\mathbb{Z}}$ and $\|\tilde{\nu}(k)\| < 2\rho$ there. Then $\Delta_2(k) := \nu_2(k) - \tilde{\nu}(k)$ solves the linear-inhomogeneous difference equation

$$y' = B(k)y + G(k, \nu(k)) - G(k, c(k, \tilde{\nu}(k)), \tilde{\nu}(k))$$

on $[k_0, K]_{\mathbb{Z}}$ and satisfies the initial condition $\Delta_2(K) = 0$. By the variation of constants formula in backward time (cf. [1, p. 58]) it follows

$$\begin{aligned} \Delta_2(k) \quad = \quad & -\sum_{n=k}^{K-1} \Psi(k, n+1) \big[G(n, \nu(n)) - G(n, c(n, \nu_2(n)), \nu_2(n)) \\ & + G(n, c(n, \nu_2(n)), \nu_2(n)) - G(n, c(n, \tilde{\nu}(n)), \nu_2(n)) \\ & + G(n, c(n, \tilde{\nu}(n)), \nu_2(n)) - G(n, c(n, \tilde{\nu}(n)), \tilde{\nu}(n)) \big], \end{aligned}$$

so that

$$\begin{aligned} \|\Delta_2(k)\| \quad \overset{(3.7)}{\le} \quad & \frac{K_1 K_2 \omega_G(\rho)}{\beta - \alpha - \gamma} (\alpha + \gamma)^{k-\kappa} \|\nu_1(\kappa) - c(\kappa, \nu_2(\kappa))\| \\ & + K_2 \omega_G(2\rho)(1 + \omega_c(2\rho)) \sum_{n=k+1}^{K} \beta^{k-n} \|\Delta_2(n-1)\| \end{aligned} \tag{3.8}$$

for $k \in [k_0, K]_{\mathbb{Z}}$. This yields

$$\begin{aligned} \frac{\|\Delta_2(k)\|}{\beta^k} \quad \le \quad & \frac{K_1 K_2 \omega_G(\rho)}{\beta - \alpha - \gamma} (\alpha + \gamma)^{-\kappa} \left(\frac{\alpha + \gamma}{\beta} \right)^k \|\nu_1(\kappa) - c(\kappa, \nu_2(\kappa))\| \\ & + \frac{K_2 \omega_G(2\rho)(1 + \omega_c(2\rho))}{\beta} \sum_{n=k}^{K-1} \frac{\|\Delta_2(n)\|}{\beta^n} \end{aligned}$$

and because of (3.2) we can apply Gronwall's lemma in backward time (cf. e.g. [14, pp. 68–69, Satz 2.1.3(b)]) to obtain

$$\|\Delta_2(k)\| \le \frac{K_1 K_2 \omega_G(\rho)}{\beta - \alpha - \gamma - K_2 \omega_G(2\rho)(1 + \omega_c(2\rho))} (\alpha + \gamma)^{k-\kappa} \|\nu_1(\kappa) - c(\kappa, \nu_2(\kappa))\|$$

for $k \in [k_0, K-1]_{\mathbb{Z}}$; it is easy to see from (3.8) that the above estimate also holds in case $k = K$, which implies

$$\|\nu_2(k) - \tilde{\nu}(k)\| \overset{(3.2)}{\le} \tfrac{1}{2} (\alpha + \gamma)^{k-\kappa} \|\nu_1(\kappa) - c(\kappa, \nu_2(\kappa))\| \tag{3.9}$$

for $k \in [k_0, K]_{\mathbb{Z}}$. This, in turn, leads to

$$\begin{aligned} \|\tilde{\nu}(k)\| \quad & \le \quad \|\nu_2(k)\| + \|\tilde{\nu}(k) - \nu_2(k)\| \le \\ & \overset{(3.9)}{\le} \quad \|\nu_2(k)\| + \tfrac{1}{2} (\|\nu_1(\kappa)\| + \|c(\kappa, \nu_2(\kappa))\|) \le \\ & \overset{(3.5)}{\le} \quad \rho + \tfrac{1}{2} (\rho + \omega_c(\rho)\rho) < 2\rho \quad \text{for } k \in [k_0, K]_{\mathbb{Z}}, \end{aligned} \tag{3.10}$$

in particular, $\|\tilde{\nu}(k_0)\| < 2\rho$ and so $k_0 = \kappa$. Hence, $\tilde{\nu}$ is defined on $[\kappa, K]_{\mathbb{Z}}$ and the inequalities (3.9), (3.10) hold for $k \in [\kappa, K]_{\mathbb{Z}}$. So we proved assertions (a) and (b).

(III) In order to show the remaining estimate (3.4) we get from (3.10), as well as from [10, p. 342, Corollary 4.3]

$$\begin{aligned}
&\left\| \nu(k) - \begin{pmatrix} c(k, \tilde{\nu}(k)) \\ \tilde{\nu}(k) \end{pmatrix} \right\| \\
&\stackrel{(1.1)}{=} \max \left\{ \|\nu_1(k) - c(k, \tilde{\nu}(k))\|, \|\nu_2(k) - \tilde{\nu}(k)\| \right\} \\
&\leq \max \left\{ \|\nu_1(k) - c(k, \nu_2(k))\| + \|c(k, \nu_2(k)) - c(k, \tilde{\nu}(k))\|, \|\nu_2(k) - \tilde{\nu}(k)\| \right\} \\
&\stackrel{(3.7)}{\leq} \max\Big\{ K_1 (\alpha+\gamma)^{k-\kappa} \|\nu_1(\kappa) - c(\kappa, \nu_2(\kappa))\| + \omega_c(2\rho) \|\nu_2(k) - \tilde{\nu}(k)\|, \\
&\qquad \|\nu_2(k) - \tilde{\nu}(k)\| \Big\} \\
&\stackrel{(3.9)}{\leq} \max \left\{ K_1 + \tfrac{1}{2}\omega_c(2\rho), \tfrac{1}{2} \right\} (\alpha+\gamma)^{k-\kappa} \|\nu_1(\kappa) - c(\kappa, \nu_2(\kappa))\| \\
&\stackrel{(3.2)}{\leq} \left(K_1 + \tfrac{1}{2}\right) (\alpha+\gamma)^{k-\kappa} \|\nu_1(\kappa) - c(\kappa, \nu_2(\kappa))\|
\end{aligned}$$

for all $k \in [\kappa, K]_{\mathbb{Z}}$. This concludes the proof. □

For the rest of the paper we fix $\gamma \in (0, \min\{1, \beta\} - \alpha)$ and choose $\rho \in \left(0, \frac{\rho_0}{2}\right)$ so small that the estimates (3.2) are fulfilled. As a first corollary of Lemma 3.2 we show that small bounded solutions of (2.1) must lie on the center fiber bundle.

Theorem 3.3 (asymptotic description of C) (a) *If the equation* (2.1) *has a solution* $\nu = (\nu_1, \nu_2) : \mathbb{Z} \to X \times Y$ *satisfying* $\|\nu(k)\| < \rho$ *for all* $k \in \mathbb{Z}$, *then* $\nu_1(k) = c(k, \nu_2(k))$ *holds for all* $k \in \mathbb{Z}$.

(b) *If* $S \subseteq \mathbb{Z} \times B_\rho(0) \times B_\rho(0) \subseteq \mathbb{Z} \times X \times Y$ *is a set such that for any* $(\kappa, \xi, \eta) \in S$ *the solution* $\varphi(\cdot; \kappa, \xi, \eta)$ *of* (2.1) *exists on* $\mathbb{Z}$ *and satisfies* $\|\varphi(k; \kappa, \xi, \eta)\| < \rho$ *there, then* $S \subseteq C$.

Proof (a) Putting $k = K$ in (3.4) we obtain from Lemma 3.2

$$\begin{aligned}
\|\nu_1(K) - c(K, \nu_2(K))\| &\leq \left(K_1 + \tfrac{1}{2}\right) \|\nu_1(\kappa) - c(\kappa, \nu_2(\kappa))\| (\alpha+\gamma)^{K-\kappa} \\
&\stackrel{(3.1)}{\leq} 2\rho \left(K_1 + \tfrac{1}{2}\right) (\alpha+\gamma)^{K-\kappa}
\end{aligned}$$

and due to $\alpha + \gamma < 1$, letting $\kappa \to -\infty$ gives us $\nu_1(K) = c(K, \nu_2(K))$. Since $K \in \mathbb{Z}$ was arbitrary, we get the assertion.

(b) This follows from the above assertion (a) applied to the individual solutions $\varphi(\cdot; \kappa, \xi, \eta)$ of (2.1) with $(\kappa, \xi, \eta) \in S$. □

The next result enables us to relate the asymptotic behavior of small solutions of equation (2.1) to solutions of (3.3), and guarantees that center fiber bundles are exponentially attractive.

Theorem 3.4 (asymptotic phase of C) *Let* Y *be finite dimensional,* $\kappa \in \mathbb{Z}$, *and let* $\nu : \mathbb{Z}_\kappa^+ \to X \times Y$ *be a solution of* (2.1) *satisfying* $\|\nu(k)\| < \rho$ *for all* $k \in \mathbb{Z}_\kappa^+$. *Then there exists a solution* $\tilde{\nu}_* : \mathbb{Z}_\kappa^+ \to Y$ *of* (3.3) *such that*

$$\left\| \nu(k) - \begin{pmatrix} c(k, \tilde{\nu}_*(k)) \\ \tilde{\nu}_*(k) \end{pmatrix} \right\| \leq \left(K_1 + \tfrac{1}{2}\right) \|\nu_1(\kappa) - c(\kappa, \nu_2(\kappa))\| (\alpha+\gamma)^{k-\kappa} \tag{3.11}$$

for all $k \in \mathbb{Z}_\kappa^+$.

Proof Let $\tilde{\nu}_m$, $m \in \mathbb{Z}_\kappa^+$, denote the solution of (3.3) satisfying $\tilde{\nu}_m(m) = \nu_2(m)$. Then Lemma 3.2 implies that $\tilde{\nu}_m$ is defined on $[\kappa, m]_{\mathbb{Z}}$ and $\|\tilde{\nu}_m(k)\| < 2\rho$ there. Moreover, we simply set $\tilde{\nu}_m(k) := 0$ for $k \in \mathbb{Z}_\kappa^+$, $k > m$. Hence, $(\tilde{\nu}_m(\kappa))_{m \in \mathbb{Z}_\kappa^+}$

is bounded and since the space Y is finite dimensional, there exists a convergent subsequence $(\tilde{\nu}_{m_n^{(\kappa)}}(\kappa))_{n\in\mathbb{Z}_\kappa^+}$ in Y with $\kappa+1 \le m_\kappa^{(\kappa)}$. In addition, $(\tilde{\nu}_{m_n^{(\kappa)}}(\kappa+1))_{n\in\mathbb{Z}_\kappa^+}$ is bounded and there exists a convergent subsequence $(\tilde{\nu}_{m_n^{(\kappa+1)}}(\kappa+1))_{n\in\mathbb{Z}_\kappa^+}$ in Y with $\kappa+2 \le m_\kappa^{(\kappa+1)}$. Iterating this construction, we obtain a sequence $(\tilde{\nu}_{m_n^{(l+1)}})_{n\in\mathbb{Z}_\kappa^+}$, which is a subsequence of $(\tilde{\nu}_{m_n^{(l)}})_{n\in\mathbb{Z}_\kappa^+}$, $l \in \mathbb{Z}_\kappa^+$, such that $l+1 \le m_\kappa^{(l)}$ and accordingly $(\tilde{\nu}_{m_n^{(l)}}(k))_{n\in\mathbb{Z}_\kappa^+}$ converges for $k \in [\kappa, l]_\mathbb{Z}$. Now we define $\bar{\nu}_n(k) := \tilde{\nu}_{m_n^{(n)}}(k)$, $k \in \mathbb{Z}_\kappa^+$, and $(\bar{\nu}_n(k))_{n\in\mathbb{Z}_\kappa^+}$ converges for any $k \in \mathbb{Z}_\kappa^+$, because beginning with the index $n = k$, $(\bar{\nu}_n(k))_{n\ge k}$ is a subsequence of the convergent sequence $(\tilde{\nu}_{m_n^{(k)}}(k))_{n\in\mathbb{Z}_\kappa^+}$. Now, by definition, $(\bar{\nu}_n)_{n\in\mathbb{Z}_\kappa^+}$ satisfies for any $n \in \mathbb{Z}_\kappa^+$

$$\bar{\nu}_n(k+1) \overset{(3.3)}{=} B(k)\bar{\nu}_n(k) + G(k, c(k, \bar{\nu}_n(k)), \bar{\nu}_n(k)) \quad \text{for } k \in \left[\kappa, m_n^{(n)} - 1\right]_\mathbb{Z},$$
$$\left\| \nu(k) - \begin{pmatrix} c(k, \bar{\nu}_n(k)) \\ \bar{\nu}_n(k) \end{pmatrix} \right\| \overset{(3.4)}{\le} \left(K_1 + \tfrac{1}{2}\right) \|\nu_1(\kappa) - c(\kappa, \nu_2(\kappa))\| \, (\alpha+\gamma)^{k-\kappa}$$

for $k \in \left[\kappa, m_n^{(n)}\right]_\mathbb{Z}$, and passing over to the limit $n \to \infty$ in these relations for k fixed, we see that $\tilde{\nu}_*(k) := \lim_{n\to\infty} \bar{\nu}_n(k)$, $k \in \mathbb{Z}_\kappa^+$, is a solution of the difference equation (3.3) on $\mathbb{Z}_\kappa^+$ satisfying (3.11). □

Having Theorem 3.4 available, it is not difficult to maintain stability properties (cf. e.g. [1, p. 240, Definition 5.4.1]) for (2.1) from the corresponding properties of the zero solution of the finite dimensional difference equation (3.3).

Theorem 3.5 (reduction principle) *Let Y be finite dimensional. If the zero solution of* (3.3) *is stable (uniformly stable, asymptotically stable, uniformly asymptotically stable, exponentially stable with rate $\tilde{\gamma}$, unstable, respectively), then also the zero solution of* (2.1) *is stable (uniformly stable, asymptotically stable, uniformly asymptotically stable, exponentially stable with rate* $\max\{\alpha+\gamma, \tilde{\gamma}\}$*, unstable, respectively).*

Proof If the zero solution of (3.3) is unstable, then by invariance of C, the zero solution of equation (2.1) is unstable, too. Let $\epsilon > 0$ and $\kappa \in \mathbb{Z}$ be given arbitrarily, but w.l.o.g. $\epsilon \le 2\rho_0$. We suppose now that the zero solution 0 is stable for (3.3); then there exists a $\delta > 0$ such that

$$\|\tilde{\nu}(k)\| < \tfrac{\epsilon}{2} \quad \text{for } k \in \mathbb{Z}_\kappa^+ \tag{3.12}$$

and any solution $\tilde{\nu} : \mathbb{Z}_\kappa^+ \to Y$ of the reduced equation (3.3) with $\|\tilde{\nu}(\kappa)\| < \delta$. Henceforth let $\nu = (\nu_1, \nu_2) : \mathbb{Z}_\kappa^+ \to X \times Y$ be an arbitrary solution of (2.1) with $\|\nu(k)\| < \rho$ for $k \in \mathbb{Z}_\kappa^+$ and $\|\nu(\kappa)\| < \min\left\{\frac{\delta}{2K_1+2}, \frac{\epsilon}{4K_1+3}\right\}$. Then due to Theorem 3.4 there exists a solution $\tilde{\nu}_* : \mathbb{Z}_\kappa^+ \to Y$ of (3.3) satisfying (3.11). This, in particular, yields

$$\begin{aligned} \|\tilde{\nu}_*(\kappa)\| &\le \|\nu_2(\kappa) - \tilde{\nu}_*(\kappa)\| + \|\nu_2(\kappa)\| \\ &\overset{(3.11)}{\le} \left(K_1 + \tfrac{1}{2}\right) \|\nu_1(\kappa) - c(\kappa, \nu_2(\kappa))\| + \|\nu_2(\kappa)\| \\ &\overset{(3.1)}{\le} \left(K_1 + \tfrac{1}{2}\right) (\|\nu_1(\kappa)\| + \|\nu_2(\kappa)\|) + \|\nu_2(\kappa)\| < \delta \end{aligned}$$

and consequently $\|\tilde{\nu}_*(k)\| < \frac{\epsilon}{2}$ for $k \in \mathbb{Z}_\kappa^+$ by (3.12). On the other hand, we have the estimate

$$\begin{aligned}
\|\nu(k)\| &\leq \left\| \nu(k) - \begin{pmatrix} c(k, \tilde{\nu}_*(k)) \\ \tilde{\nu}_*(k) \end{pmatrix} \right\| + \left\| \begin{pmatrix} c(k, \tilde{\nu}_*(k)) \\ \tilde{\nu}_*(k) \end{pmatrix} \right\| \\
&\overset{(3.11)}{\leq} \left(K_1 + \tfrac{1}{2}\right) \|\nu_1(\kappa) - c(\kappa, \nu_2(\kappa))\| \, (\alpha + \gamma)^{k-\kappa} + \left\| \begin{pmatrix} c(k, \tilde{\nu}_*(k)) \\ \tilde{\nu}_*(k) \end{pmatrix} \right\| \\
&\overset{(3.1)}{\leq} \left(K_1 + \tfrac{1}{2}\right) (\|\nu_1(\kappa)\| + \|\nu_2(\kappa)\|) \, (\alpha + \gamma)^{k-\kappa} + \|\tilde{\nu}_*(k)\| < \epsilon
\end{aligned}$$

for all $k \in \mathbb{Z}_\kappa^+$ and therefore, the solution ν is stable. Finally, one can show the assertions about the remaining stability properties along the same lines. □

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