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Reduction principle at work

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Abstract: The purpose of this informal paper is three-fold: First, filling a gap in the literature, we provide a (necessary and sufficient) principle of linearized stability for nonautonomous difference equations in Banach spaces based on the dichotomy spectrum. Second, complementing the above, we survey and exemplify an ambient nonautonomous and infinite-dimensional center manifold reduction, that is Pliss's reduction principle suitable for critical stability situations. Third, these results are applied to integrodifference equations of Hammerstein- and Urysohn-type both in C- and L^p-spaces. Specific features of the nonautonomous case are underlined. Yet, for the simpler situation of periodic time-dependence even explicit computations are feasible.

Keywords: nonautonomous difference equation, periodic difference equation, linearized stability, dichotomy spectrum, center manifold reduction, integrodifference equation

MSC(2010): 39A30, 37C60

1 Stability and dispersal in discrete time

Determining local stability properties of fixed points u^* to autonomous difference equations $u_{t+1} = \mathcal{F}(u_t)$ (maps) by linearization is a fairly classical and textbook matter: If the spectrum $\sigma(D\mathcal{F}(u^*))$ is contained in the open unit disk of the complex plane, then u^* is exponentially stable (see e.g. [13, p. 2, Thm. 1]), whereas a component of $\sigma(D\mathcal{F}(u^*))$ outside the closed unit disk guarantees instability (see [13, p. 3, Thm. 2]). This situation changes when the equilibrium u^* is replaced by a nonconstant solution u_t^* or the difference equation is time-variant in advance, i.e. $u_{t+1} = \mathcal{F}_t(u_t)$. Here, unless for rather slow time-dependencies, the elements of $\sigma(D\mathcal{F}_t(u_t^*))$ have no relevance in stability theory and effectively become useless.

However, by virtue of the dichotomy (also called Sacker-Sell, cf. [27] for ODEs) spectrum, the above statements canonically generalize to the time-dependent situation. This dynamical spectrum is a subset of the positive real line. It proved to be an efficient tool in the geometric theory of nonautonomous dynamical systems when it comes to the construction of invariant manifolds and foliations, as well as for topological or smooth linearization questions (see [21] for related references). Given this, it is rather surprising that the crucial role of the dichotomy spectrum in stability criteria based on linearization is not explicitly present in the literature to our best knowledge. Indeed, corresponding results are not only convenient, but also essential for instance in nonautonomous bifurcation theory. For this reason, we provide a natural theorem with self-contained proof allowing to infer stability properties of arbitrary solutions to time-variant problems in general Banach spaces based on the spectrum of their variational equation. This result can be seen as a nonautonomous version of the recent [9, Thm. 1]. It states that a dichotomy spectrum contained in $(0, 1)$ is necessary and sufficient for uniform exponential stability, while a spectral component in $(1, \infty)$ implies instability.

If the stability boundary 1 is contained in the spectrum, then stability cannot be determined via linearization, because nonlinear terms matter. Due to Pliss's reduction principle (see [19] for ODEs and [13, pp. 131ff, Chapt. 4] for maps between Banach spaces) one has to investigate the equation reduced to a lower-dimensional center manifold instead. A related nonautonomous theory in discrete time can be traced back to [30], although we follow the more recent contributions [1, 24] and [21, pp. 187ff, Chapt. 4]. Then hyperbolicity is not a generic property anymore and therefore center manifold reduction is even more important for equations

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featuring general time-dependence. The theory is nevertheless rather involved, Taylor coefficients of center manifolds are determined by dynamical (rather than algebraic) properties [24] and typically assumptions can only be verified approximately using numerical methods.

Probably due to this reason we are not aware of an explicit application of the tools described above in a discrete time, infinite-dimensional context. This motivates us to study a corresponding relevant class of problems, namely nonautonomous integrodifference equations of *Urysohn type*

$$u_{t+1}(x) = \int_\Omega f_t(x, y, u_t(y))\,\mathrm{d}y \quad \text{for all } x \in \Omega$$

and of *Hammerstein type*

$$u_{t+1}(x) = \int_\Omega k_t(x, y) g_t(y, u_t(y))\,\mathrm{d}y \quad \text{for all } x \in \Omega.$$

Initiated by [15] such recursions became popular and sucessful models in theoretical ecology to describe the dispersal of populations with nonoverlapping generations (see [18]). Formally, they are difference equations in the space of continuous or integrable functions over a compact set $\Omega \subset \mathbb{R}^\kappa$ (the habitat). The growth period is determined in terms of a typical growth function $g_t : \Omega \times \mathbb{R} \to \mathbb{R}$ of e.g. Beverton-Holt or Ricker type. The kernel $k_t : \Omega \times \Omega \to \mathbb{R}$ describes the dispersal and $k_t(x, y)$ can be interpreted as probability to move from position y to x in the habitat Ω at time t. These functions can vary in time, since seasonal, but also aperiodic external influences are well-motivated from applications. For instance, [14] investigate persistence questions in river environments.

We provide examples of nonautonomous integrodifference equations allowing an analytical stability analysis without relying on numerical techniques to a large extend. Admittedly these examples are simple but have their merit and are still involved to a certain degree. Due to their multiplicative time dependence the dichotomy spectrum and the relevant invariant subspaces can be computed directly. In the Hammerstein case they for instance rely on the Laplace kernel k_t, which is often met in applications (cf. [25] or [18, pp. 17ff, Sect. 2.3]) but allows for explicit solutions to related Fredholm integral equations. In critical stability situations we even retreat to periodic time-dependence.

The structure of this paper is as follows: By means of the central Thm. 2.1 we first provide a stability characterization based on linearization and a nonautonomous spectral theory recently developed in [26]. Despite its rather evident nature, we are not aware of another reference in the literature connecting stability properties with the dichotomy spectrum. To address complementary critical stability situations, the required center manifold theory (in the nonautonomous case one speaks of *center fiber bundles*) is illustrated in Sect. 3, which includes generalizations of Pliss's reduction principle (see Thm. 3.1) and a scheme to compute Taylor approximations of center fiber bundles exemplifying [24]. With regard to our applications and in order to limit the technical effort, we restrict to 1-dimensional bundles. For the sake of our applications, basics on Hammerstein- and Urysohn-integrodifference equations in the Banach spaces $L^p(\Omega)$ and $C(\Omega)$ are provided in Sect. 4. Our final Sect. 5 is devoted to several specific time-variant integrodifference equations and their stability. In order to keep the presentation self-contained, we close with three appendices. They tackle critical stability cases for time-periodic scalar difference equations, provide solutions to Fredholm integral equations of the second kind and give explicit constants for a stability example in the text.

Notation

Suppose $\mathbb{I}$ is an unbounded set of consecutive integers (one speaks of a *discrete interval*), $\mathbb{I}' := \{t \in \mathbb{I} : t + 1 \in \mathbb{I}\}$ and $\mathbb{Z}_\tau^+ := \{t \in \mathbb{Z} : \tau \leq t\}$ for some $\tau \in \mathbb{Z}$. We write δ_{ij} for the *Kronecker symbol.*

Let $(X, \|\cdot\|)$ be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ having the dual space X' and $\langle x, x' \rangle := x'(x)$ stands for the duality pairing of $x \in X$ with $x' \in X'$. The open ball in X with radius $r > 0$ and center x is denoted as $B_r(x)$. For the set of continuous n-linear mappings $T : X^n \to X$, $n \in \mathbb{N}$, we write $L_n(X)$ and abbreviate $L(X) := L_1(X)$, $L_0(X) := X$; I_X is the identity map on X. Given $S \in L(X)$, $R(S) := SX$ is the *range* and $N(S) := S^{-1}(0)$ the *kernel* of S. Moreover, we write $\sigma(S)$ for the *spectrum* and $\sigma_p(S)$ for the *point spectrum.*

A subset $\mathcal{V} \subseteq \mathbb{I} \times X$ is called *nonautonomous set* with *t-fiber* $\mathcal{V}(t) := \{x \in X : (t, x) \in \mathcal{V}\}$.

2 Linearized stability in nonautonomous difference equations

We investigate the behavior of nonautonomous difference equations

$$\boxed{u_{t+1} = \mathcal{F}_t(u_t)} \tag{Δ}$$

in X near a given reference solution $\phi^* = (\phi_t^*)_{t\in\mathbb{I}}$ on a discrete interval $\mathbb{I}$. Assume thereto that C^m-mappings $\mathcal{F}_t : U_t \to X$, $t \in \mathbb{I}'$, are defined on open convex sets $U_t \subseteq X$ and $m \in \mathbb{N}$. Then the *general solution* to (Δ) is defined via the compositions

$$\varphi(t;\tau,u_\tau) := \begin{cases} u_\tau, & t=\tau, \\ \mathcal{F}_{t-1}\circ\ldots\circ\mathcal{F}_\tau(u_\tau), & \tau<t \end{cases} \quad \text{for all } \tau\in\mathbb{I},\ u_\tau\in U_\tau \tag{2.1}$$

of the right-hand sides, as long as the inclusion $\varphi(t;\tau,u_\tau) \in U_t$ holds.

One denotes (Δ) as *periodic*, if there exists a $\theta \in \mathbb{N}$ such that $\mathcal{F}_t = \mathcal{F}_{t+\theta}$ holds for all $t \in \mathbb{I} = \mathbb{Z}$; *autonomous* eqns. ($\Delta$) correspond to the case $\theta = 1$, i.e. when the right-hand sides do not depend on t.

A solution ϕ^* to (Δ) is called *permanent*, provided it uniformly stays away from the boundary of the domain, that is

$$\inf_{t\in\mathbb{I}} \operatorname{dist}(\phi_t^*, \partial U_t) > 0. \tag{2.2}$$

For the reader's convenience, we repeat the stability notions mentioned in this paper. Provided the interval $\mathbb{I}$ is unbounded above, a solution ϕ^* is denoted as

- *stable*, if for all $\varepsilon > 0$, $\tau \in \mathbb{I}$ there exists a $\delta > 0$ so that $\varphi(\cdot;\tau,u_\tau)$ exists and satisfies $\|\varphi(t;\tau,u_\tau) - \phi_t^*\| < \varepsilon$ for every $\tau \le t$, $u_\tau \in B_\delta(\phi_\tau^*)$ and *uniformly stable*, if δ does not depend on τ,
- *asymptotically stable*, if it is stable and for every $\tau \in \mathbb{I}$ there exists a $\rho > 0$ so that $\varphi(\cdot;\tau,u_\tau)$ exists and satisfies $\lim_{t\to\infty} \|\varphi(t;\tau,u_\tau) - \phi_t^*\| = 0$ for every $u_\tau \in B_\rho(\phi_\tau^*)$; a *uniformly asymptotically stable* solution is uniformly stable and $\rho > 0$ is independent of τ,
- *uniformly exponentially stable*, if there exist $K \ge 1$, $\alpha \in (0,1)$, $\delta > 0$ so that $\varphi(\cdot;\tau,u_\tau)$ exists and satisfies $\|\varphi(t;\tau,u_\tau) - \phi_t^*\| \le K\alpha^{t-\tau}\|u_\tau - \phi_\tau^*\|$ for every $\tau \le t$, $u_\tau \in B_\delta(\phi_\tau^*)$

and an *unstable* solution ϕ^* is not stable. These stability notions are related as follows

$$\begin{array}{ccccc} UES & \Rightarrow & UAS & \Rightarrow & US \\ & & \Downarrow & & \Downarrow \\ & & AS & \Rightarrow & S \end{array}$$

and for periodic solutions ϕ^* to periodic eqns. (Δ), stability resp. asymptotic stability is always uniform.

Using the mean value theorem [31, p. 148–149, Thm. 4.A(b) for $n = 1$] the *difference equation of perturbed motion* $u_{t+1} = \mathcal{F}_t(u_t + \phi_t^*) - \mathcal{F}_t(\phi_t^*)$ becomes

$$u_{t+1} = D\mathcal{F}_t(\phi_t^*)u_t + \mathcal{R}_t(u_t) \tag{2.3}$$

with the nonlinearity $\mathcal{R}_t(u) := \int_0^1 [D\mathcal{F}_t(\phi_t^* + hu) - D\mathcal{F}_t(\phi_t^*)]\,\mathrm{d}h\,u$. The *variational difference equation* of (Δ) along ϕ^* reads as

$$\boxed{u_{t+1} = D\mathcal{F}_t(\phi_t^*)u_t.} \tag{V_{ϕ^*}}$$

We define the *transition operator* to (V_{ϕ^*}) by

$$\Phi(t,\tau) := \begin{cases} D\mathcal{F}_{t-1}(\phi_{t-1}^*)\cdots D\mathcal{F}_\tau(\phi_\tau^*), & \tau<t, \\ I_X, & t=\tau. \end{cases}$$

It is well-known that the time-dependent spectrum $\sigma(D\mathcal{F}_t(\phi_t^*))$, $t \in \mathbb{I}'$, provides no stability information unless for periodic or slow temporal variation (see [10, pp. 177ff]). Yet, a feasible approach is given as follows: The variational eqn. (V_{ϕ^*}) is said to have an *exponential dichotomy* (ED for short) on $\mathbb{I}$, if there exists a projection-valued sequence $(P_t^+)_{t\in\mathbb{I}}$ in $L(X)$ with

$$P_{t+1}^+ D\mathcal{F}_t(\phi_t^*) = D\mathcal{F}_t(\phi_t^*)P_t^+, \qquad D\mathcal{F}_t(\phi_t^*) : N(P_t^+) \to N(P_{t+1}^+) \text{ is an isomorphism for all } t \in \mathbb{I}' \tag{2.4}$$

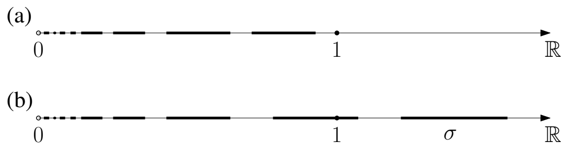

Fig. 1: Dichotomy spectra $\Sigma(\phi^*)$ indicating (a) uniform exponential stability of ϕ^* and (b) instability of ϕ^*

and reals $\alpha \in (0,1)$, $K \geq 1$ such that

$$\left\|\Phi(t,s)P_s^+\right\| \leq K\alpha^{t-s}, \qquad \left\|\Phi(s,t)P_t^-\right\| \leq K\alpha^{t-s} \quad \text{for all } s \leq t$$

with the complementary projector $P_t^- := I_X - P_t^+$. An ED on $\mathbb{I}$ with $P_t^+ \equiv I_X$ on $\mathbb{I}$ describes a uniformly exponentially stable variational eqn. (V_{ϕ^*}).

The *dichotomy spectrum* (see [26] and references therein) of a solution ϕ^* is

$$\Sigma(\phi^*) := \left\{\rho > 0 : \, u_{t+1} = \tfrac{1}{\rho} D\mathcal{F}_t(\phi_t^*)u_t \text{ has no ED on } \mathbb{I}\right\}.$$

This closed set $\Sigma(\phi^*)$ is a union of *spectral intervals* among which the most right-hand one (see Fig. 1) is called *dominant*. Each spectral interval has an associated *spectral bundle* — a nonautonomous set whose fibers are linear subspaces being invariant w.r.t. (V_{ϕ^*}). Their dimension is called *multiplicity* of the spectral interval. If the associate spectral bundle is one-dimensional, then one speaks of a *simple* spectral interval.

A survey on the dichotomy spectra of difference equations in infinite dimensions is given in [22]. In the simplest time-variant situation, the periodic case one obtains:

Example 2.1 (periodic and autonomous case)**.** *For θ-periodic variational eqns. (V_{ϕ^*}) the spectrum is*

$$\Sigma(\phi^*) = \sqrt[\theta]{|\sigma(\Phi(\tau+\theta,\tau))|} \setminus \{0\};$$

this quantity is independent of $\tau \in \mathbb{Z}$ due to [26, p. 42, Prop. 3.1(a)]. In particular, a fixed point u^* of an autonomous difference equation

$$\boxed{u_{t+1} = \mathcal{F}(u_t)} \tag{Δ'}$$

has the dichotomy spectrum $\Sigma(u^*) = |\sigma(D\mathcal{F}(u^*))| \setminus \{0\}$ and the spectral intervals correspond to the moduli of spectral points.

Our main result tackling stability is as follows:

Theorem 2.1 (linearized stability)**.** *If $\phi^* = (\phi_t^*)_{t\in\mathbb{I}}$ is a permanent solution of (Δ) on an interval $\mathbb{I}$ unbounded above and the relations*

$$\sup_{t\in\mathbb{I}} \left\|D\mathcal{F}_t(\phi_t^*)\right\| < \infty, \qquad \lim_{u\to 0} \sup_{t\in\mathbb{I}} \left\|D\mathcal{F}_t(\phi_t^* + u) - D\mathcal{F}_t(\phi_t^*)\right\| = 0 \tag{2.5}$$

are satisfied, then the following holds:

(a) *If $\Sigma(\phi^*) \subseteq (0,1)$, then ϕ^* is uniformly exponentially stable and on an interval $\mathbb{I}$ bounded below also the converse holds (cf. Fig. 1(a)).*

(b) *If $(1,\infty)$ contains a spectral interval of $\Sigma(\phi^*)$, then ϕ^* is unstable (cf. Fig. 1(b)).*

Remark 2.1. *(1) Since Fréchet differentiability is a rather strong assumption in certain applications, the subsequent proof shows that the assertions of Thm. 2.1 remain true for the zero solution of semilinear difference equations $u_{t+1} = \mathcal{K}_t u_t + \mathcal{R}_t(u_t)$ under corresponding assertions on the dichotomy spectrum $\Sigma(\mathcal{K})$ and nonlinearities $\mathcal{R}_t : U_t \to X$ satisfying $\mathcal{R}_t(0) = 0 \in U_t$ and $\lim_{u,\bar{u}\to 0} \frac{\|\mathcal{R}_t(u)-\mathcal{R}_t(\bar{u})\|}{\|u-\bar{u}\|} = 0$ uniformly in $t \in \mathbb{I}$.*

(2) Both scalar difference eqns. $u_{t+1} = u_t + u_t^2$ and $u_{t+1} = u_t - u_t^3$ have the trivial solution with $\Sigma(0) = \{1\}$. Due to Thm. A.1(a) the zero fixed point of the first equation is unstable, which shows that the converse of statement (b) does not hold. Thm. A.1(b) implies that the zero solution of the second equation is (uniformly) asymptotically stable and thus exponential stability in statement (a) cannot be replaced by asymptotic stability.

Yet, in the light of Exam. 2.1 the above result generalizes the classical autonomous situation [13, 9] with 1 as *stability boundary*. In general, however, $\Sigma(\phi^*)$ has to be approximated numerically (see [12] for $\dim X < \infty$). The next example shows that the boundedness assumption on the coefficients of (V_{ϕ^*}) is not technical:

Example 2.2. *The linear difference equation*

$$u_{t+1} = \begin{pmatrix} |t|+1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} u_t \tag{2.6}$$

in $X = \mathbb{R}^2$ has the dichotomy spectrum $\left\{\frac{1}{2}\right\} \subset (0,1)$. Its transition operator

$$\Phi(t,s) = \begin{pmatrix} \prod_{r=s}^{t-1}(|r|+1) & 0 \\ 0 & 2^{s-t} \end{pmatrix} \quad \textit{for all } s \le t$$

shows that unbounded solutions to (2.6) *exist. Therefore, eqn.* (2.6) *is unstable.*

In case $\mathbb{I} = \mathbb{Z}$ the assertion (b) follows from the existence of an unstable manifold (fiber bundle) associated to ϕ^* (see [21, p. 259, Thm. 4.6.4(b)]). However, our subsequent proof merely assumes (V_{ϕ^*}) to be given on a half-line unbounded above.

Proof (of Thm. 2.1). First, we restrict to stability properties for the trivial solution of the eqn. (2.3) of perturbed motion, which is defined in a neighborhood of 0 uniformly in $t \in \mathbb{I}$ (thanks to permanence (2.2)). Moreover, let $\tau \in \mathbb{I}$ and $\hat\varphi$ stands for the general solution to (2.3). Note that $\hat\varphi(\cdot;\tau,u_\tau)$ is also a solution to the linearly inhomogeneous difference eqn. $u_{t+1} = D\mathcal{F}_t(\phi_t^*)u_t + \mathcal{R}_t(\hat\varphi(t;\tau,u_\tau))$ and the variation of constants formula (see [21, p. 100, Thm. 3.1.16(a)]) yields

$$\hat\varphi(t;\tau,u_\tau) = \Phi(t,\tau)u_\tau + \sum_{s=\tau}^{t-1} \Phi(t,s+1)\mathcal{R}_s(\hat\varphi(s;\tau,u_\tau)) \tag{2.7}$$

as long as $\hat\varphi(\cdot;\tau,u_\tau)$ exists. Second, due to (2.5), for all $M > 0$ there is a $\rho > 0$ with

$$\|\mathcal{R}_t(u)\| \le M\|u\| \quad \text{for all } t \in \mathbb{I},\ u \in \bar{B}_\rho(0). \tag{2.8}$$

(a) ($\Rightarrow$) Due to $\Sigma(\phi^*) \subseteq (0,1)$ there exists reals $K \ge 1$ and $\alpha \in (0,1)$ such that

$$\|\Phi(t,\tau)\| \le K\alpha^{t-\tau} \quad \text{for all } \tau \le t. \tag{2.9}$$

We choose $M \in \left[0, \frac{1-\alpha}{K}\right)$ and obtain $\alpha + KM \in [0,1)$. Finally, given an initial value $u_\tau \in B_\rho(0)$ we define $T^*(u_\tau) := \sup\left\{\theta \in \mathbb{Z}_\tau^+ : \|\hat\varphi(t;\tau,u_\tau)\| \le \rho \text{ for all } \tau \le t \le \theta\right\}$ as exit time at which the solution $\hat\varphi(\cdot;\tau,u_\tau)$ leaves the ρ-neighborhood of the trivial solution for the first time. This definition includes $T^*(u_\tau) = \infty$, when the solution stays in $B_\rho(0)$.

(I) We show that every initial value $u_\tau \in B_\rho(0)$ yields an estimate

$$\|\hat\varphi(t;\tau,u_\tau)\| \le K(\alpha+KM)^{t-\tau}\|u_\tau\| \quad \text{for all } \tau \le t \le T^*(u_\tau). \tag{2.10}$$

Thereto, on this discrete interval, (2.7) brings us to the estimate

$$\begin{aligned}
\|\hat\varphi(t;\tau,u_\tau)\| &\overset{(2.9)}{\le} K\alpha^{t-\tau}\|u_\tau\| + K\sum_{s=\tau}^{t-1}\alpha^{t-s-1}\|\mathcal{R}_s(\hat\varphi(s;\tau,u_\tau))\| \\
&\overset{(2.8)}{\le} K\alpha^{t-\tau}\|u_\tau\| + \frac{KM}{\alpha}\sum_{s=\tau}^{t-1}\alpha^{t-s}\|\hat\varphi(s;\tau,u_\tau)\|
\end{aligned}$$

and multiplication with $\alpha^{\tau-t}$ implies the inequality

$$\alpha^{\tau-t}\|\hat\varphi(t;\tau,u_\tau)\| \le K\|u_\tau\| + \frac{KM}{\alpha}\sum_{s=\tau}^{t-1}\alpha^{\tau-s}\|\hat\varphi(s;\tau,u_\tau)\| \quad \text{for all } \tau \le t \le T^*(u_\tau).$$

This relation and the Grönwall lemma [21, p. 348, Prop. A.2.1(a)] yield

$$\alpha^{\tau-t}\|\hat\varphi(t;\tau,u_\tau)\| \le K\left(1+\tfrac{KM}{\alpha}\right)^{t-\tau}\|u_\tau\| \quad \text{for all } \tau\le t\le T^*(u_\tau),$$

which is obviously equivalent to (2.10).

(II) In order to conclude the proof, we exploit (2.10) where $(\alpha+KM)^{t-\tau}\in[0,1)$ is strictly decreasing in $t\in\mathbb{Z}_\tau^+$. First, given initial values $u_\tau\in B_{\rho/K}(0)$ we obtain that $\|\hat\varphi(t;\tau,u_\tau)\|\le(\alpha+KM)^{t-\tau}\rho\le\rho$ and hence $T^*(u_\tau)=\infty$. Thus, (2.10) holds for $t\in\mathbb{Z}_\tau^+$ and the trivial solution to (2.3) is uniformly exponentially stable.

($\Leftarrow$) Let $\mathbb{I}$ be bounded below. By assumption there exist reals $K\ge1$, $\alpha\in(0,1)$ and $\delta>0$ with

$$\|\hat\varphi(t;\tau,u_\tau)\|\le K\alpha^{t-\tau}\|u_\tau\| \quad \text{for all } \tau\le t,\ u_\tau\in B_\delta(0) \tag{2.11}$$

and we follow the finite-dimensional case considered in [11]. For fixed $\beta\in(\alpha,1)$ choose $M\in\left(0,\frac{\beta-\alpha}{K}\right)$ and using (2.5) there exist a $\rho>0$ such that (2.8) holds. If we take $\rho_0\in(0,\min\left\{\delta,\frac{\rho}{K}\right\})$, then (2.11) implies $\|\hat\varphi(t;\tau,u_\tau)\|\le K\|u_\tau\|\le K\rho_0<\rho$ for all $\tau\le t$, $u_\tau\in B_{\rho_0}(0)$ and

$$\begin{aligned}\|\Phi(t,\tau)u_\tau\| &\overset{(2.7)}{\le} \|\hat\varphi(t;\tau,u_\tau)\|+M\sum_{s=\tau}^{t-1}\|\Phi(t,s+1)\|\,\|\hat\varphi(s;\tau,u_\tau)\|\\ &\overset{(2.11)}{\le} K\alpha^{t-\tau}\rho_0+MK\rho_0\sum_{s=\tau}^{t-1}\alpha^{s-\tau}\|\Phi(t,s+1)\|.\end{aligned}$$

Because of $\alpha<\beta$ this guarantees

$$\beta^{\tau-t}\|\Phi(t,\tau)u_\tau\|\le K\rho_0+\frac{MK\rho_0}{\beta}\sum_{s=\tau}^{t-1}\left(\frac{\alpha}{\beta}\right)^{s-\tau}\beta^{(s+1)-t}\|\Phi(t,s+1)\|,$$

with $\omega_t:=\max_{s=\tau}^t\beta^{s-t}\|\Phi(t,s)\|$ it results

$$\beta^{\tau-t}\|\Phi(t,\tau)u_\tau\|\le K\rho_0+\frac{MK\rho_0}{\beta}\omega_t\sum_{s=\tau}^{t-1}\left(\frac{\alpha}{\beta}\right)^{s-\tau}\le K\rho_0+\frac{MK\rho_0}{\beta-\alpha}\omega_t$$

and consequently

$$\beta^{\tau-t}\|\Phi(t,\tau)\|=\beta^{\tau-t}\sup_{\|u\|\le1}\|\Phi(t,\tau)u\|=\frac{1}{\rho_0}\sup_{\|u\|\le\rho_0}\|\Phi(t,\tau)u\|\le K+\frac{KM}{\beta-\alpha}\omega_t$$

for all $\tau\le t$. Hence, $\omega_t\le K+\frac{MK}{\beta-\alpha}\omega_t$ by passing to the maximum over $\tau\in\{0,\dots,t\}$ on the left-hand side. Due to $\frac{MK}{\beta-\alpha}<1$ and the choice of M, this inequality implies that $\beta^{\tau-t}\|\Phi(t,\tau)\|\le\frac{K(\beta-\alpha)}{\beta-\alpha-MK}$. Thanks to $\beta<1$ this means that (V_{ϕ^*}) has an ED with projector $P_t^+\equiv I_X$ and therefore $\Sigma(\phi^*)\subseteq(0,1)$.

(b) Let σ be a spectral interval in $(1,\infty)$ (see Fig. 1(b)). We start with the substitution $w_t:=\gamma^{-t}u_t$ for some $\gamma\in(0,\infty)\setminus\Sigma(\phi^*)\cap[1,\min\sigma)$ yielding the equation

$$w_{t+1}=\tfrac{1}{\gamma}D\mathcal{F}_t(\phi_t^*)w_t+\tfrac{1}{\gamma^{t+1}}\mathcal{R}_t(\gamma^t w_t) \tag{2.12}$$

and the general solutions $\hat\varphi$ of (2.3) and ψ to (2.12) are related by

$$\psi(t;\tau,w_\tau)=\gamma^{-t}\hat\varphi(t;\tau,u_\tau). \tag{2.13}$$

Thanks to $\min\sigma>1$ the scaled variational equation

$$w_{t+1}=\tfrac{1}{\gamma}D\mathcal{F}_t(\phi_t^*)w_t \tag{2.14}$$

has an ED on $\mathbb{I}$ with projectors $P_t^+\ne I_X$ for all $t\in\mathbb{I}$, i.e. there exist reals $K\ge1$, $\alpha\in(0,1)$ such that

$$\left\|\Phi_\gamma(t,s)P_s^+\right\|\le K\alpha^{t-s}, \qquad \left\|\Phi_\gamma(s,t)P_t^-\right\|\le K\alpha^{t-s} \quad \text{for all } s\le t, \tag{2.15}$$

where Φ_γ is the transition operator to (2.14). Hence, we obtain

$$\begin{aligned}&\sum_{s=\tau}^{t-1}\left\|\Phi_\gamma(t,s+1)P_{s+1}^+\right\|+\sum_{s=t}^{\infty}\left\|\Phi_\gamma(t,s+1)P_{s+1}^-\right\|\\&\stackrel{(2.15)}{\leq} K\sum_{s=\tau}^{t-1}\alpha^{t-s-1}+K\sum_{s=t}^{\infty}\alpha^{s+1-t}\leq K\frac{1+\alpha}{1-\alpha}=:\tilde K\quad\text{for all }\tau\leq t\end{aligned}\tag{2.16}$$

and choosing $M \in \left(0, \tilde{K}^{-1}\right)$ in (2.8) implies $1-\tilde{K}M \in (0,1)$.

(I) We show that if $\hat\varphi(\cdot;\tau,u_\tau)$ satisfies $\|\hat\varphi(t;\tau,u_\tau)\| \leq \rho$ for all $\tau \leq t$, then

$$\|\hat\varphi(t;\tau,u_\tau)\| \leq \gamma^{t-\tau}(1-\tilde KM)^{-1}K\left\|P_\tau^+u_\tau\right\|\quad\text{for all }\tau\leq t.\tag{2.17}$$

Indeed, the bound on $\hat\varphi(t;\tau,u_\tau)$ yields

$$\left\|\tfrac{1}{\gamma^{t+1}}\mathcal{R}_t(\hat\varphi(t;\tau,u_\tau))\right\|\stackrel{(2.8)}{\leq}\tfrac{M}{\gamma^{t+1}}\|\hat\varphi(t;\tau,u_\tau)\|\leq\tfrac{M}{\gamma^\tau}\rho\quad\text{for all }\tau\leq t.$$

Combined with (2.16) this guarantees that

$$\begin{aligned}\eta_t:=\psi(t;\tau,w_\tau)-\Phi_\gamma(t,\tau)P_\tau^+w_\tau+\sum_{s=\tau}^{t-1}\tfrac{1}{\gamma^{s+1}}\Phi_\gamma(t,s+1)P_{s+1}^+\mathcal{R}_s(\hat\varphi(s;\tau,u_\tau))\\-\sum_{s=t}^{\infty}\tfrac{1}{\gamma^{s+1}}\Phi_\gamma(t,s+1)P_{s+1}^-\mathcal{R}_s(\hat\varphi(s;\tau,u_\tau))\end{aligned}$$

defines a bounded sequence $\eta=(\eta_t)_{\tau\leq t}$. Moreover, the curious reader might verify that η solves (2.14) and satisfies $P_\tau^+\eta_\tau=0$. Since (2.14) admits an ED on $\mathbb{Z}_\tau^+$, this is only possible if $\eta_t\equiv 0$ on $\mathbb{Z}_\tau^+$ (cf. [21, p. 140, Cor. 3.4.21(a)]). Thus, we obtain

$$\begin{aligned}\psi(t;\tau,w_\tau)=\Phi_\gamma(t,\tau)P_\tau^+w_\tau-\sum_{s=\tau}^{t-1}\tfrac{1}{\gamma^{s+1}}\Phi_\gamma(t,s+1)P_{s+1}^+\mathcal{R}_s(\hat\varphi(s;\tau,u_\tau))\\+\sum_{s=t}^{\infty}\tfrac{1}{\gamma^{s+1}}\Phi_\gamma(t,s+1)P_{s+1}^-\mathcal{R}_s(\hat\varphi(s;\tau,u_\tau)),\end{aligned}$$

according to (2.8) this implies

$$\begin{aligned}\|\psi(t;\tau,w_\tau)\|\leq K\alpha^{t-\tau}\left\|P_\tau^+w_\tau\right\|+M\sum_{s=\tau}^{t-1}\left\|\Phi_\gamma(t,s+1)P_{s+1}^+\right\|\frac{\|\hat\varphi(s;\tau,u_\tau)\|}{\gamma^{s+1}}\\+M\sum_{s=t}^{\infty}\left\|\Phi_\gamma(t,s+1)P_{s+1}^-\right\|\frac{\|\hat\varphi(s;\tau,u_\tau)\|}{\gamma^{s+1}},\end{aligned}$$

a combination of (2.13), (2.16) and $\gamma\geq 1$ gives

$$\|\psi(t;\tau,w_\tau)\|\leq K\alpha^{t-\tau}\left\|P_\tau^+w_\tau\right\|+\tilde KM\sup_{\tau\leq s}\|\psi(s;\tau,w_\tau)\|\quad\text{for all }\tau\leq t$$

and finally the estimate $\|\psi(t;\tau,w_\tau)\|\leq\gamma^{-\tau}(1-\tilde KM)^{-1}K\left\|P_\tau^+u_\tau\right\|$ holds. Due to (2.13) we arrive at (2.17).

(II) Suppose that the trivial solution to (2.3) is stable, i.e. for $\varepsilon>0$ there is a $\delta>0$ so that $u_\tau\in B_\delta(0)$ guarantees $\|\hat\varphi(t;\tau,u_\tau)\|<\varepsilon$ for all $\tau\leq t$. Since $P_\tau^+\neq I_X$, one can choose a nonzero $u_\tau\in N(P_\tau^+)$ and (2.17) implies $0<\|u_\tau\|=\|\hat\varphi(\tau;\tau,u_\tau)\|=0$, which is a contradiction. □

Concrete applications of Thm. 2.1 will be given in Sect. 5.

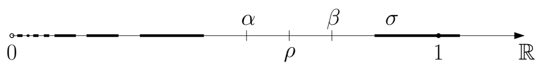

Fig. 2: Dichotomy spectrum requiring a reduction to a center-unstable fiber bundle $\mathcal{W}^-$ and choice of $\alpha < \beta$

3 Critical stability situations

This section complements Thm. 2.1. Indeed, given a permanent solution $(\phi_t^*)_{t\in\mathbb{I}}$ of (Δ) we assume that the stability boundary 1 is contained in the dominant spectral interval of the variational eqn. (V_{ϕ^*}). More precisely, we suppose $\mathbb{I}$ is unbounded below and

(σ) $\Sigma(\phi^*)$ has at least two components and 1 is contained in the dominant spectral interval σ. Moreover, choose reals $0 < \alpha < \beta < 1$ such that $(\alpha, \beta) \cap \Sigma(\phi^*) = \emptyset$.

There is a $\rho \in (\alpha, \beta)$ as in Fig. 2, i.e. in the spectral gap just left of the dominant interval σ. By construction, the scaled variational eqn. $u_{t+1} = \frac{1}{\rho} D\mathcal{F}_t(\phi_t^*)u_t$ has an ED on $\mathbb{I}$, whose projector may have the (unique) complementary projector $(P_t^-)_{t\in\mathbb{I}}$.

Now stability properties of the reference solution ϕ^* need not to be determined by the linearization (V_{ϕ^*}). They rather depend on the nonlinearity $\mathcal{R}_t$ in (2.3) and our further analysis requires some preparations. The *center-unstable vector bundle*

$$\mathcal{V}^- := \left\{(t, x) \in \mathbb{I} \times X : x \in R(P_t^-)\right\}$$

of the variational eqn. (V_{ϕ^*}) is invariant and (2.4) guarantees that all fibers $\mathcal{V}^-(t)$ are isomorphic; in particular, they have the same dimension, which is denoted as *multiplicity* of the dominant spectral interval σ. Addressing the nonlinear eqn. (Δ), the set $\mathcal{V}^-$ persists as locally invariant *center-unstable fiber bundle*

$$\mathcal{W}^- = \phi^* + \left\{(\tau, \xi + w_\tau^-(\xi)) \in \mathbb{I} \times X : \xi \in \mathcal{V}^-(\tau) \cap B_r(0)\right\}$$

of the solution ϕ^*: This means there exists a $r > 0$ so that each fiber $\mathcal{W}^-(t)$ is graph of a Lipschitzian function $w_t^- : \mathcal{V}^-(t) \cap B_r(0) \to R(P_t^+)$, $t \in \mathbb{I}$, satisfying $w_t^-(0) \equiv 0$ on $\mathbb{I}$ (cf. [21, pp. 259–260, Thm. 4.6.4(b)]). The nonautonomous set $\mathcal{W}^-$ contains all solutions to (Δ) which exist in backward time and have a bounded distance to ϕ^*. Thus, $\mathcal{W}^-$ captures the essential dynamics of (Δ) near ϕ^* in terms of

Theorem 3.1 (reduction principle)**.** *Suppose (σ) and* (2.5) *hold on $\mathbb{I} = \mathbb{Z}$. A permanent solution ϕ^* of (Δ) is (uniformly, asymptotically, uniformly asymptotically, uniformly exponentially) stable, or unstable, if and only if the zero solution of the* reduced difference equation

$$v_{t+1} = D\mathcal{F}_t(\phi_t^*)v_t + P_{t+1}^- \mathcal{R}_t(v_t + w_t^-(v_t)) \tag{3.1}$$

in the center-unstable vector bundle $\mathcal{V}^-$ has the respective stability property.

Proof. The trivial solution to the eqn. (2.3) inherits the stability properties of ϕ^*. Furthermore, [21, p. 267, Thm. 4.6.15] applies to (2.3) and yields the claim. □

For compact operators $D\mathcal{F}_t(\phi_t^*) \in L(X)$, $t \in \mathbb{I}'$, more can be said on the structure of $\Sigma(\phi^*)$ and [26] gives a detailed classification. In particular, the center-unstable bundle $\mathcal{V}^-$ has fibers of constant and finite dimension $c \in \mathbb{N}$. Thus, the reduced eqn. (3.1) can be transformed to a difference equation in $\mathbb{K}^c$ as follows: For the sake of a (significant) simplification, we suppose the (complementary) projectors $P_t^\pm \equiv P^\pm$ do not depend on t and choose a basis $\{e_1, \dots, e_c\}$ of $R(P^-)$. By means of the Hahn-Banach theorem we can complement it with elements $\left\{e_1', \dots, e_c'\right\} \subset X'$ to a biorthogonal system, i.e.

$$\left\langle e_i, e_j' \right\rangle = \delta_{ij} \quad \text{for all } 1 \le i, j \le c. \tag{3.2}$$

With $v_t = \sum_{j=1}^c \xi_t^j e_j$ and $P^- v := \sum_{i=1}^c \left\langle v, e_i' \right\rangle e_i$ the reduced eqn. (3.1) becomes

$$\sum_{j=1}^c \xi_{t+1}^j e_j = D\mathcal{F}_t(\phi_t^*) \sum_{j=1}^c \xi_t^j e_j + \sum_{j=1}^c \left\langle \mathcal{R}_t\left(\sum_{i=1}^c \xi_t^i e_i + w_t^-\left(\sum_{i=1}^c \xi_t^i e_i\right)\right), e_j' \right\rangle e_j,$$

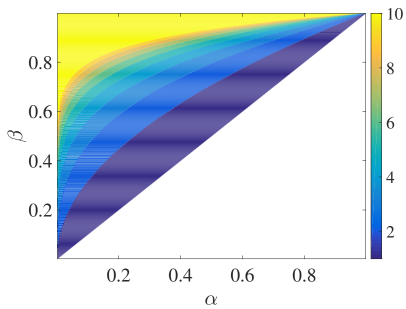

Fig. 3: The α-β-plane illustrating the maximal degree of differentiability $m < \frac{\ln\alpha}{\ln\beta}$ (encoded via the color bar) for the center-unstable bundle $\mathcal{W}^-$ according to the spectral gap condition (G_m) for different values of the reals $0 < \alpha < \beta < 1$

applying $\langle \cdot, e_k' \rangle$ on both sides yields

$$\xi_{t+1}^k \overset{(3.2)}{=} \sum_{j=1}^c \langle D\mathcal{F}_t(\phi_t^*)e_j, e_k'\rangle \xi_t^j + \left\langle \mathcal{R}_t\left(\sum_{i=1}^c \xi_t^i e_i + w_t^-\left(\sum_{i=1}^c \xi_t^i e_i\right)\right), e_k'\right\rangle \quad \text{for all } 1 \le k \le c$$

and we finally arrive at the finite-dimensional equation

$$\boxed{\xi_{t+1} = f_t(\xi_t) := C_t\xi_t + R_t(\xi_t)} \tag{3.3}$$

in $\mathbb{K}^c$ with $\xi = (\xi^1, \dots, \xi^c)$, $C_t := (\langle D\mathcal{F}_t(\phi_t^*)e_j, e_k'\rangle)_{j,k=1}^c$ and the nonlinearity

$$R_t(\xi) := \left(\left\langle \mathcal{R}_t\left(\sum_{i=1}^c \xi^i e_i + w_t^-\left(\sum_{i=1}^c \xi^i e_i\right)\right), e_k'\right\rangle\right)_{k=1}^c \quad \text{for all } t \in \mathbb{I}'.$$

3.1 Smoothness and approximation of center-unstable fiber bundles

As mentioned above, the fibers of a center-unstable bundle $\mathcal{W}^-$ can be represented as graphs of Lipschitzian mappings $w_t^- : \mathcal{V}^-(t) \cap B_r(0) \to R(P^+)$, $t \in \mathbb{I}$. In order to approximate w_t^-, a higher-order smoothness is desirable. In the general nonautonomous situation this is based on the following technical assumption:

– X is a C^m-*Banach space*, i.e. the norm $\|\cdot\| : X \to \mathbb{R}$ is m-times continuously differentiable away from 0. Concrete examples and information on such spaces are given in [21, pp. 364ff, Sect. C.2].

Then, [21, pp. 259–260, Thm. 4.6.4(b)] shows that the functions w_t^-, $t \in \mathbb{I}$, are of class C^1 and satisfy

$$w_t^-(0) \equiv 0, \qquad Dw_t^-(0) \equiv 0 \quad \text{on } \mathbb{I}. \tag{3.4}$$

Concerning higher-order differentiability, for nonautonomous problems it is not guaranteed that $\mathcal{W}^-$ inherits the smoothness from the right-hand side of eqn. (Δ):

Remark 3.1 (spectral gap condition)**.** *The additional* spectral gap condition

$$\alpha < \beta^m \tag{G_m}$$

yields that also the bundle $\mathcal{W}^-$ is m-times continuously differentiable. For dominant spectral intervals σ satisfying $\min\sigma = 1$, which particularly holds in the compact periodic situation, it is always possible to fulfill (G_m) by choosing $\beta < 1$ sufficiently close to 1. If $\min\sigma < 1$, then (G_m) is an actual restriction on the differentiability order and Fig. 3 illustrates the maximal value m such that (G_m) holds for $0 < \alpha < \beta < 1$.

For simplicity we suppose from now on that the dominant spectral interval containing 1 has multiplicity 1. Hence, also $\mathcal{W}^-$ possesses dimension $c = 1$. Given a sufficient differentiability order $m \in \mathbb{N}$, the derivatives of the nonlinearity $R_t(\xi) = \left\langle \mathcal{R}_t(\xi e_1 + w_t^-(\xi e_1)), e_1' \right\rangle$ become

$$\begin{aligned}
R_t'(\xi) =& \left\langle D\mathcal{R}_t(\xi e_1 + w_t^-(\xi e_1))[e_1 + Dw_t^-(\xi e_1)e_1], e_1' \right\rangle, \\
R_t''(\xi) =& \langle D^2\mathcal{R}_t(\xi e_1 + w_t^-(\xi e_1))[e_1 + Dw_t^-(\xi e_1)e_1]^2 + D\mathcal{R}_t(\xi e_1 + w_t^-(\xi e_1))D^2 w_t^-(\xi e_1)e_1^2, e_1' \rangle, \\
R_t'''(\xi) =& \langle D^3\mathcal{R}_t(\xi e_1 + w_t^-(\xi e_1))[e_1 + Dw_t^-(\xi e_1)e_1]^3 \\
&+ 3D^2\mathcal{R}_t(\xi e_1 + w_t^-(\xi e_1)) \left[D^2 w_t^-(\xi e_1)e_1^2\right] [e_1 + Dw_t^-(\xi e_1)e_1] \\
&+ D\mathcal{R}_t(\xi e_1 + w_t^-(\xi e_1))D^3 w_t^-(\xi e_1)e_1^3, e_1' \rangle
\end{aligned}$$

and consequently, due to (3.4), we arrive at $R_t'(0) = 0$,

$$f_t''(0) = \left\langle D^2\mathcal{R}_t(0)e_1^2, e_1' \right\rangle, \qquad f_t'''(0) = \left\langle D^3\mathcal{R}_t(0)e_1^3 + 3D^2\mathcal{R}_t(0)[D^2 w_t^-(0)e_1^2]e_1, e_1' \right\rangle \tag{3.5}$$

for all $t \in \mathbb{I}'$, provided (G_2) resp. (G_3) holds. Hence, an explicit knowledge of Taylor coefficients to the center-unstable bundle $\mathcal{W}^-$ is only required for 3rd and higher order approximations of (3.1). Our approach is based on the fact that the functions w_t^- satisfy the *invariance equation*

$$\begin{aligned}
&P^+ D\mathcal{F}_t(\phi_t^*)w_t^-(\xi e_1) + P^+\mathcal{R}_t(\xi e_1 + w_t^-(\xi e_1)) \\
&\qquad = w_{t+1}\big(\xi D\mathcal{F}_t(\phi_t^*)e_1 + P^-\mathcal{R}_{t+1}(\xi e_1 + w_t^-(\xi e_1))\big) \quad \text{for all } t \in \mathbb{I}',\ \xi \in \mathbb{R}
\end{aligned} \tag{3.6}$$

as long as the inclusion $\xi D\mathcal{F}_t(\phi_t^*)e_1 + P^-\mathcal{R}_{t+1}(\xi e_1 + w_t^-(\xi e_1)) \in B_r(0)$ holds. If $m > 1$, then differentiating (3.6) twice w.r.t. ξ yields

$$\begin{aligned}
&P^+ D\mathcal{F}_t(\phi_t^*)D^2 w_t^-(\xi e_1)e_1^2 + P^+\left[D^2\mathcal{R}_t(\xi e_1 + w_t^-(\xi e_1))[e_1 + Dw_t^-(\xi e_1)e_1]^2 + D\mathcal{R}_t(\xi e_1 + w_t^-(\xi e_1))D^2 w_t^-(\xi e_1)e_1^2\right] \\
&= D^2 w_{t+1}\big(\xi D\mathcal{F}_t(\phi_t^*)e_1 + P^-\mathcal{R}_{t+1}(\xi e_1 + w_t^-(\xi e_1))\big)\cdot \\
&\quad \cdot [D\mathcal{F}_t(\phi_t^*)e_1 + P^- D\mathcal{R}_{t+1}(\xi e_1 + w_t^-(\xi e_1))[e_1 + Dw_t^-(\xi e_1)e_1]]^2 \\
&+ Dw_{t+1}\big(\xi D\mathcal{F}_t(\phi_t^*)e_1 + P^-\mathcal{R}_{t+1}(\xi e_1 + w_t^-(\xi e_1))\big) P^-[D^2\mathcal{R}_{t+1}(\xi e_1 + w_t^-(\xi e_1))[e_1 + Dw_t^-(\xi e_1)e_1]^2 \\
&\qquad + D\mathcal{R}_{t+1}(\xi e_1 + w_t^-(\xi e_1))D^2 w_t^-(\xi e_1)e_1^2].
\end{aligned}$$

Due to (3.5), setting $\xi = 0$ implies a linearly inhomogeneous difference equation

$$P^+ D\mathcal{F}_t(\phi_t^*)D^2 w_t^-(0)e_1^2 + P^+ D^2\mathcal{R}_t(0)e_1^2 = D^2 w_{t+1}^-(0)[D\mathcal{F}_t(\phi_t^*)e_1]^2 \quad \text{for all } t \in \mathbb{I}'.$$

Keeping our assumption of constant projectors in mind, $R(P^-)$ is invariant w.r.t. (V_{ϕ^*}) and there exist scalars $\eta_t \neq 0$ such that $\eta_t e_1 = D\mathcal{F}_t(\phi_t^*)e_1$. Indeed, (3.2) implies $\eta_t = \left\langle D\mathcal{F}_t(\phi_t^*)e_1, e_1' \right\rangle$ for all $t \in \mathbb{I}'$. This means that the sequence $t \mapsto D^2 w_t^-(0)e_1^2$ in $R(P^+)$, bounded due to [24, Thm. 3.2(b)], solves the *homological equation*

$$\eta_t^2 w_{t+1} = P^+ D\mathcal{F}_t(\phi_t^*)w_t + P^+ D^2\mathcal{R}_t(0)e_1^2 \tag{3.7}$$

and, for nonzero coefficients η_t, is therefore of the form (cf. [24])

$$D^2 w_t^-(0)e_1^2 = \sum_{s=-\infty}^{t-1} \left(\prod_{r=s+1}^{t-1} \frac{1}{\eta_r^2} \right) \Phi(t, s+1)P^+ D^2\mathcal{R}_s(0)e_1^2 \quad \text{for all } t \in \mathbb{I}.$$

3.2 Periodic and autonomous equations

The general nonautonomous situation simplifies for θ-periodic eqns. (Δ) and solutions ϕ^* such that the variational eqn. (Δ) has compact coefficients. First, the dichotomy spectrum is discrete. Second, as a consequence, the gap condition (G_m) can always be fulfilled. Third, the center-unstable vector bundle $\mathcal{V}^-$ is finite-dimensional and the technical assumption of X being a C^m-Banach space can be avoided. In fact, an ambient center manifold theorem for maps is due to [16, p. 189, Thm. III.1] for $m = 1$ and [6] for $m \in \mathbb{N}$. These results carry over to θ-periodic difference eqns. (Δ) and solutions ϕ^* as follows: They apply to the period maps

$$\pi_\tau := \mathcal{F}_{\tau+\theta-1} \circ \cdots \circ \mathcal{F}_\tau : U_\tau \to X$$

and each $\tau \in \mathbb{Z}$ yields a center-unstable manifold $\mathcal{W}^-(\tau)$ for the autonomous system $u_{t+1} = \pi_\tau(u_t)$. By the π_τ-invariance of $\mathcal{W}^-(\tau)$, the fibers $\mathcal{W}^-(t) := \varphi(t; \tau, \mathcal{W}^-(\tau))$, as well as the functions w_t^- are θ-periodic and define the center-unstable fiber bundle of (Δ). Thus, one has to solve the homological eqn. (3.7) for θ-periodic sequences. This results in the θ cyclic linear equations

$$\begin{cases} \eta_t^2 w_{t+1} = P^+ D\mathcal{F}_t(\phi_t^*) w_t + P^+ D^2 \mathcal{R}_t(0) e_1^2 & \text{for all } 0 \le t < \theta - 1, \\ \eta_{\theta-1}^2 w_0 = P^+ D\mathcal{F}_{\theta-1}(\phi_{\theta-1}^*) w_{\theta-1} + P^+ D^2 \mathcal{R}_{\theta-1}(0) e_1^2, \end{cases}$$

whose solution $(w_0, \ldots, w_{\theta-1}) \in R(P^+)^\theta$ yields the coefficients $D^2 w_t^-(0) e_1^2 = w_t$ for $0 \le t < \theta$.

For a fixed point u^* of an autonomous eqn. (Δ') and $D\mathcal{F}(u^*) e_1 = \eta e_1$ one solves

$$[\eta^2 I_X - P^+ D\mathcal{F}(u^*)] D^2 w^-(0) e_1^2 = P^+ D^2 \mathcal{R}(0) e_1^2 \quad \text{in } R(P^+);$$

that is a single linear-inhomogeneous equation.

4 Integrodifference equations

Our next aim is to apply the above general methods to concrete integrodifference equations (abbreviated as IDEs). Thereto, suppose $(\Omega, \mathfrak{A}, \mu)$ is a measure space with nonempty bounded $\Omega \subset \mathbb{R}^\kappa$ and $\mu(\Omega) < \infty$.

4.1 Hammerstein equations in $X = L^p(\Omega)$

A first possible state space are the p-integrable functions

$$L_d^p(\Omega, \mu) := \left\{ u : \Omega \to \mathbb{K}^d \,\middle|\, u \text{ is } \mu\text{-measurable}, \int_\Omega |u|^p \, \mathrm{d}\mu < \infty \right\}$$

equipped with the canonical norm $\|u\|_p := \left(\int_\Omega |u|^p \, \mathrm{d}\mu\right)^{1/p}$ for $p \ge 1$. It is well-known that $(L_d^p(\Omega, \mu))_{p \ge 1}$ forms a strictly decreasing scale of Banach spaces, which in the terminology of [3, p. 43], are pairwise compatible. Moreover, $L_d^p(\Omega, \mu)$ is a C^{m_p}-Banach space with (see [4, p. 184, Thm. 1.1])

$$m_p := \begin{cases} p-1, & p \in \mathbb{N} \text{ is odd}, \\ [p], & p \in [1, \infty) \setminus \mathbb{N}, \\ \infty, & p \in \mathbb{N} \text{ is even}. \end{cases}$$

In particular, $L_d^2(\Omega, \mu)$ is a Hilbert space with $\langle u, v \rangle = \int_\Omega \sum_{j=1}^d u_j(x) \overline{v_j(x)} \, \mathrm{d}\mu(x)$ as inner product. We abbreviate $L^p(\Omega, \mu) := L_1^p(\Omega, \mu)$ and $L^p(\Omega) := L^p(\Omega, \mu_\kappa)$ when the Lebesgue measure $\mu = \mu_\kappa$ on $\mathbb{R}^\kappa$ is used; here we write $\int_\Omega u := \int_\Omega u(y) \, \mathrm{d}y = \int_\Omega u \, \mathrm{d}\mu_\kappa$.

Let us consider *Hammerstein integrodifference equations* of the form

$$u_{t+1} = \int_\Omega k_t(\cdot, y) g_t(y, u_t(y)) \, \mathrm{d}\mu(y) + h_t \tag{H}$$

with inhomogeneity $h_t \in L_d^p(\Omega, \mu)$. Our analysis of (H) requires to represent the right-hand side

$$\mathcal{F}_t(u) = \mathcal{K}_t \circ \mathcal{G}_t(u) + h_t$$

of (Δ) as composition of a linear integral operator and a substitution operator

$$(\mathcal{K}_t v)(x) := \int_\Omega k_t(x,y)v(y)\,\mathrm{d}\mu(y), \qquad (\mathcal{G}_t(v))(x) := g_t(x,v(x)) \quad \text{for all } x \in \Omega.$$

Their properties are subject to our following analysis, in which $t \in \mathbb{I}'$ is kept fixed: For the kernel functions $k_t : \Omega \times \Omega \to L(\mathbb{K}^d)$ we suppose *Hille-Tamarkin conditions*: There exist $p, q \geq 1$ such that

(ht1) k_t is $\mu \otimes \mu$-measurable,

(ht2) for all $u \in L^q_d(\Omega,\mu)$ there exists a μ-zero set N_u so that $k_t(x,\cdot)u$ is μ-measurable on Ω for all $x \in \Omega \setminus N_u$ and the following function is p-integrable

$$\hat{u}(x) := \begin{cases} \int_\Omega k_t(x,y)u(y)\,\mathrm{d}\mu(y), & x \in \Omega \setminus N_u, \\ 0, & x \in N_u, \end{cases}$$

(ht3) if $q' \geq 1$ is determined by $\frac{1}{q} + \frac{1}{q'} = 1$, then $k'(x) := \|k_t(x,\cdot)\|_{q'} < \infty$ for μ-almost all $x \in \Omega$ and $k' \in L^p(\Omega,\mu)$ holds.

Lemma 4.1. *If (ht1-ht2) hold, then $\mathcal{K}_t \in L(L^q_d(\Omega,\mu), L^p_d(\Omega,\mu))$ is well-defined. Assuming additionally $p > 1$, then (ht3) implies compactness.*

Proof. See [7, p. 288, Satz 1 and p. 293, Satz 2]. □

Concerning the growth function $g_t : \Omega \times \mathbb{K}^d \to \mathbb{K}^d$ we suppose that the partial derivatives $D_2^l g_t$ exist and satisfy *Carathéodory conditions* for all $0 \leq l \leq m$, $m \in \mathbb{N}$:

(c1) $D_2^l g_t(\cdot,z) : \Omega \to L_l(\mathbb{K}^d)$ is μ-measurable on Ω for all $z \in \mathbb{K}^d$,

(c2) $D_2^l g_t(x,\cdot) : \mathbb{K}^d \to L_l(\mathbb{K}^d)$ is continuous for μ-almost all $x \in \Omega$.

Lemma 4.2. *Let $0 \leq l \leq m$, $c_0 \in L^q(\Omega,\mu)$ for $q \geq 1$, $c_1 \geq 0$ and suppose (c1-c2) are satisfied. If $mq < p$ and the growth conditions*

$$\left|D_2^l g_t(x,z)\right| \leq c_0(x) + c_1 |z|^{\frac{p-lq}{q}} \quad \text{for } \mu\text{-a.a. } x \in \Omega \text{ and all } z \in \mathbb{K}^d \tag{4.1}$$

hold, then $\mathcal{G}_t : L^p_d(\Omega,\mu) \to L^q_d(\Omega,\mu)$ is well-defined and of class C^m with derivatives

$$[D^l\mathcal{G}_t(u)v_1\cdots v_l](x) = D_2^l g_t(x,u(x))v_1(x)\cdots v_l(x) \quad \text{for all } x \in \Omega \text{ and all } u, v_1, \dots, v_m \in L^p_d(\Omega,\mu).$$

Proof. A proof can be modeled after [5, p. 372, Prop. 7.57]. □

Proposition 4.1. *Let $c_0 \in L^q(\Omega,\mu)$ for $q \geq 1$, $c_1 \geq 0$, suppose (ht1-ht2) and (c1-c2) are satisfied. If $mq < p$ and (4.1) hold, then $\mathcal{K}_t \circ \mathcal{G}_t : L^p_d(\Omega,\mu) \to L^p_d(\Omega,\mu)$ is well-defined and of class C^m with derivatives $D^l(\mathcal{K}_t \circ \mathcal{G}_t)(u) = \mathcal{K}_t D^l \mathcal{G}_t(u)$ for all $0 \leq l \leq m$ and $u \in L^p_d(\Omega,\mu)$.*

Proof. The linear operator $\mathcal{K}_t \in L(L^q_d(\Omega,\mu), L^p_d(\Omega,\mu))$ is well-defined by Lemma 4.1. Then Lemma 4.2 and the chain rule applied to $\mathcal{K}_t \circ \mathcal{G}_t$ yield the claim. □

4.2 Urysohn equations in $X = C(\Omega)$

Let us focus on compact sets $\Omega \subset \mathbb{R}^\kappa$ and equip

$$C_d(\Omega) := \left\{ u : \Omega \to \mathbb{K}^d \,|\, u \text{ is continuous} \right\}$$

with the natural norm $\|u\| := \sup_{x\in\Omega} |u(x)|$. The Banach spaces $C_d(\Omega)$ and $L^p_d(\Omega)$, $p \geq 1$, are compatible as understood in [3, p. 43] and we abbreviate $C(\Omega) := C_1(\Omega)$. However, $C_d(\Omega)$ typically has no smooth norm.

Consider an *Urysohn integrodifference equation*

$$u_{t+1} = \int_\Omega f_t(\cdot, y, u_t(y))\,\mathrm{d}\mu(y) + h_t \tag{U}$$

with an inhomogeneity $h_t \in C_d(\Omega)$ and a function $f_t : \Omega \times \Omega \times Z_t \to \mathbb{K}^d$ defined on a nonempty open sets $Z_t \subseteq \mathbb{K}^d$ satisfying for each fixed $t \in \mathbb{I}'$ that:
(u1) $D_3^l f_t : \Omega \times \Omega \times Z_t \to L_l(\mathbb{K}^d)$, $0 \le l \le m$, exist as continuous functions.

Proposition 4.2 (see [23]). *If (u1) holds, then $\mathcal{F}_t : C_d(\Omega, Z_t) \to C_d(\Omega)$ is well-defined and of class C^m with derivatives*

$$D^l\mathcal{F}_t(u)v_1 \cdots v_l = \int_\Omega D_3^l f_t(\cdot, y, u(y))v_1(y)\cdots v_l(y)\,\mathrm{d}\mu(y) \text{ for all } 0 \le l \le m,\ u \in C(\Omega, Z_t),\ v_1, \dots, v_m \in C_d(\Omega).$$

4.3 Linear integral operators

We equip $\mathbb{R}^\kappa$ with the Lebesgue measure $\mu = \mu_\kappa$ and suppose $\Omega \subset \mathbb{R}^\kappa$ is compact. Given a kernel function $k : \Omega \times \Omega \to L(\mathbb{K}^d)$ consider the Fredholm integral operator

$$\mathcal{K}v := \int_\Omega k(\cdot, y)v(y)\,\mathrm{d}y. \tag{4.2}$$

If $\mathcal{K}$ is (power) compact, then the Riesz-Schauder theory [7, 8] guarantees that $\sigma(\mathcal{K})\setminus\{0\}$ consists of eigenvalues λ_n, $n \in N$, having finite multiplicity with 0 as only possible accumulation point and a countable set N. As a convention, the λ_n are numbered according to

$$\ldots \le |\lambda_3| \le |\lambda_2| \le |\lambda_1|\,.$$

Proposition 4.3. *Let $p > 1$ and $X \in \{L_d^p(\Omega), C_d(\Omega)\}$. If $k : \Omega \times \Omega \to L(\mathbb{K}^d)$ is continuous, then $\mathcal{K} \in L(X)$ defined in* (4.2) *is well-defined and compact. Moreover, the eigenpairs $((\lambda_n, e_n))_{n\in N}$ of $\mathcal{K}$ are independent of X and the following holds true:*
(a) *If $N(\mathcal{K}) = \{0\}$ and $k(x, y) = \overline{k(y, x)}$ for $x, y \in \Omega$, then N is countably infinite and the (normed) eigenfunctions $(e_n)_{n\in N}$ are an orthonormal basis of $L_d^2(\Omega)$.*
(b) *If $d = 1$, $\mathbb{K} = \mathbb{R}$ and $k(x, y) > 0$ for all $x, y \in \Omega$, then $e_1(x) > 0$ for all $x \in \Omega$.*

The assumptions of (a) rule out that $\mathcal{K}$ has a degenerate kernel.

Proof. Being continuous on the compact set Ω^2, the kernel $k : \Omega^2 \to L(\mathbb{K}^d)$ fulfills (ht1-ht3) with $\mu = \mu_\kappa$. Therefore, Lemma 4.1 shows that $\mathcal{K} \in L(L_d^q(\Omega))$ is well-defined and compact. On the space $X = C_d(\Omega)$ these properties are due to [7, p. 247, Satz 4]. Since $C_d(\Omega)$ and $L_d^p(\Omega)$, $p > 1$, are compatible Banach spaces, [3, pp. 109–110, Thm. 4.2.14] implies eigenpairs $((\lambda_n, e_n))_{n\in\mathbb{N}}$ of $\mathcal{K} \in L(X)$ being independent of X.

(a) By assumption, $\mathcal{K} \in L(L_d^2(\Omega))$ is self-adjoint and [8, p. 200, Satz 3] implies that the (normed) eigenfunctions of $\mathcal{K}$ are a complete orthonormal system in $L_d^2(\Omega)$, i.e. an orthonormal basis.

(b) results immediately from the Krein-Rutman Theorem [31, p. 290, Thm. 7.C] applied to $\mathcal{K} \in L(C(\Omega))$ and the solid cone of nonnegative continuous functions. □

In what follows, we suppose $d = 1$ and conveniently introduce the numbers

$$\kappa_i := \int_\Omega \int_\Omega k(x, y)e_1(y)^i\,\mathrm{d}y\overline{e_1(x)}\,\mathrm{d}x \quad \text{for all } i \in \mathbb{N}.$$

Example 4.1 (degenerate kernel). *Let $p \geq 1$ and $X \in \{L^p(\Omega), C(\Omega)\}$. Let us assume that $a_j \in C(\Omega)$, $b_j \in X$, $1 \leq j \leq J$, are functions such that*

$$k(x,y) = \sum_{j=1}^{J} a_j(y) b_j(x) \quad \text{for all } x, y \in \Omega$$

and $\{b_1, \ldots, b_J\} \subseteq X$ is linearly independent. We define the matrix

$$K := (k_{ij})_{i,j=1}^{J}, \qquad k_{ij} := \int_\Omega a_i(y) b_j(y)\,\mathrm{d}y$$

and obtain $\sigma(\mathcal{K}) \setminus \{0\} = \sigma(K) \setminus \{0\}$. If $v \in \mathbb{K}^J$ is an eigenvector of K, then the corresponding eigenvector of the operator $\mathcal{K}$ with degenerate kernel is $v := \sum_{j=1}^{n} v_j b_j$.

The habitat in the next example is an interval $\Omega := \left[-\frac{L}{2}, \frac{L}{2}\right]$ of length $L > 0$, equipped with the Lebesgue measure $\mu = \mu_1$. Let us begin with a prototypical kernel relevant in applications, but allowing an analysis based on few numerical tools.

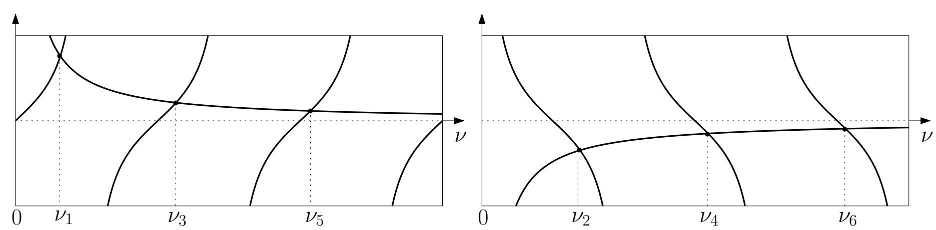

Fig. 4: Solutions $(\nu_j)_{j\in\mathbb{N}}$ of the transcendental equations $\tan(\frac{aL}{2}\nu) = \frac{1}{\nu}$ (left) or $\cot(\frac{aL}{2}\nu) = -\frac{1}{\nu}$ (right) from Exam. 4.2 as intersection of the corresponding graphs. See Tab. 1 for numerical values in case $aL = 2$

Example 4.2 (Laplace kernel). *Let $L > 0$ and $\Omega = [-\frac{L}{2}, \frac{L}{2}]$. For reals $a > 0$ the integral operator (4.2) with kernel*

$$k(x,y) := \tfrac{a}{2} e^{-a|x-y|} \quad \text{for all } x, y \in \Omega \tag{4.3}$$

is well-defined and compact. Due to [25, Appendix 2] its spectrum is obtained as follows: Provided (see Fig. 4)
- *$\tan(\frac{aL}{2}\nu) = \frac{1}{\nu}$ has the positive solutions $\nu_1 < \nu_3 < \ldots$,*
- *$\cot(\frac{aL}{2}\nu) = -\frac{1}{\nu}$ has the positive solutions $\nu_2 < \nu_4 < \ldots$,*

then $\sigma_p(\mathcal{K}) = \{\lambda_n \in \mathbb{R} : n \in \mathbb{N}\} \subset (0,1)$ with the strictly decreasing eigenvalue sequence $\lambda_n = \frac{1}{1+\nu_n^2}$ and

$$e_n(x) := \sqrt{\frac{2a\nu_n}{La\nu_n - (-1)^n \sin(aL\nu_n)}} \begin{cases} \cos(a\nu_n x), & n \text{ is odd}, \\ \sin(a\nu_n x), & n \text{ is even} \end{cases} \quad \text{for all } n \in \mathbb{N}$$

as associate eigenfunctions. Moreover, $e_1(x) > 0$ for all $x \in \Omega$ and thus the reals

$$\kappa_i = \frac{a}{2} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} e^{-a|x-y|} e_1(y)^i \,\mathrm{d}y e_1(x)\,\mathrm{d}x > 0 \quad \text{for all } i, j \in \mathbb{N}_0$$

are positive by Prop. 4.3(b). We refer to Tab. 1 for numerical approximations of the eigenvalues λ_n of $\mathcal{K}$ (for $aL = 2$) and note that they behave asymptotically as

$$\lambda_{2n} \sim \frac{a^2L^2}{a^2L^2 + \pi^2(1+2n)^2}, \qquad \lambda_{2n-1} \sim \frac{a^2L^2}{a^2L^2 + \pi^2(2n)^2} \quad \text{as } n \to \infty.$$

Since all eigenvalues are positive, $\mathcal{K}$ is positive-definite and consequently $N(\mathcal{K}) = \{0\}$ implies that $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ due to Prop. 4.3(a).

n	ν_n	λ_n	$\frac{\lambda_n}{\lambda_{n+1}}$
1	0.860334	0.574655	2.93985
2	2.02876	0.195471	2.48929
3	3.42562	0.0785245	1.97406
4	4.91318	0.0397783	1.68814
5	6.4373	0.0235633	1.52358
6	7.97867	0.0154657	1.41988
7	9.52933	0.0108923	1.34943

Tab. 1: The solutions $\nu_n > 0$ and the resulting eigenvalues λ_n, $1 \leq n \leq 7$, from Exam. 4.2 in case $aL = 2$. See Fig. 4 for an illustration

Example 4.3 (periodic kernels). *Provided $k_0 : \mathbb{R} \to \mathbb{C}$ is a continuous, L-periodic function, a convolution kernel $k(x, y) := k_0(x - y)$ for all $x, y \in \Omega$ yields a Fredholm integral operator* (4.2)*, whose eigenvalues are given by the Fourier coefficients*

$$\lambda_n = \frac{1}{L} \int_{-L/2}^{L/2} k_0(y) e^{-\frac{2\pi \iota n}{L} y} \, \mathrm{d}y$$

with corresponding eigenfunctions $e_n : \Omega \to \mathbb{C}$, $e_n(x) := \sqrt{\frac{1}{L}} e^{\frac{2\pi \iota n}{L} x}$ for all $n \in \mathbb{Z}$. That is, the eigenfunctions coincide with the trigonometric system $(e_n)_{n \in \mathbb{Z}}$ and form an orthonormal basis of $L^2(\Omega)$, as well as a Schauder basis of $L^p(\Omega)$ for $1 < p < \infty$.

5 Examples

Let $(\gamma_t)_{t \in \mathbb{I}}$ be a nonzero sequence in $\mathbb{K}$ satisfying

$$\sup_{t \in \mathbb{I}} \max \left\{ |\gamma_t|, |\gamma_t|^{-1} \right\} < \infty \tag{5.1}$$

and we define its *upper* resp. *lower Bohl exponent*

$$\overline{\beta}(\gamma) := \lim_{L \to \infty} \sup_{t \in \mathbb{I}} \sqrt[L]{\left| \prod_{s=t}^{t+L-1} \gamma_s \right|}, \qquad \underline{\beta}(\gamma) := \lim_{L \to \infty} \inf_{t \in \mathbb{I}} \sqrt[L]{\left| \prod_{s=t}^{t+L-1} \gamma_s \right|}$$

on a discrete interval $\mathbb{I}$ unbounded above. For instance, θ-periodic sequences have the geometric mean as Bohl exponents $\overline{\beta}(\gamma) = \underline{\beta}(\gamma) = \sqrt[\theta]{|\gamma_\theta \cdots \gamma_1|}$.

We are interested in linear nonautonomous IDEs

$$\boxed{u_{t+1} = \gamma_t \mathcal{K} u_t,} \tag{5.2}$$

where $\mathcal{K} \in L(X)$ denotes a Fredholm operator as in (4.2) satisfying the assumptions of Prop. 4.3. Let us define a sequence of positive reals $\ldots < \rho_2 < \rho_1$ by means of

$$\left\{ \rho_n > 0 : n \in N' \right\} = |\sigma(\mathcal{K})| \setminus \{0\}$$

and suppose the spectrum of (5.2) is of the form $\Sigma = \bigcup_{j \in N'} [\rho_j \underline{\beta}(\gamma), \rho_j \overline{\beta}(\gamma)]$.

Example 5.1 (Laplace kernel). *The dichotomy spectrum Σ of a linear nonautonomous IDE* (5.2) *with the Laplace kernel* (4.3) *has the following properties (cf. Fig. 5):*

(a) *If $\underline{\beta}(\gamma) = \overline{\beta}(\gamma)$, then $\Sigma = \bigcup_{j \in \mathbb{N}} \left\{ \lambda_j \overline{\beta}(\gamma) \right\}$ is discrete with spectral intervals of multiplicity* 1 *and constant spectral manifolds $\mathcal{V}_j = \mathbb{I} \times \operatorname{span} \left\{ e_j \right\}$, $j \in \mathbb{N}$,*

(b) *if $\underline{\beta}(\gamma) < \overline{\beta}(\gamma)$, then $\Sigma = (0, \lambda_{J+1} \overline{\beta}(\gamma)] \cup \bigcup_{j=1}^{J} [\lambda_j \underline{\beta}(\gamma), \lambda_j \overline{\beta}(\gamma)]$ for some $J \in \mathbb{N}_0$, with constant spectral manifolds $\mathcal{V}_j = \mathbb{I} \times \operatorname{span} \left\{ e_j \right\}$, $1 \leq j \leq J$,*

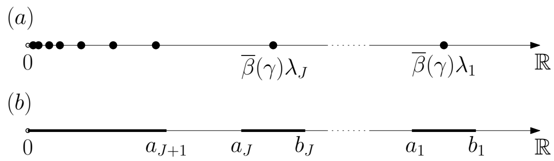

Fig. 5: The dichotomy spectra from Exam. 5.1: (a) Countably many singletons accumulating at 0 as spectral intervals for $\underline{\beta}(\gamma) = \overline{\beta}(\gamma)$ and (b) $J+1$ spectral intervals for $\underline{\beta}(\gamma) < \overline{\beta}(\gamma)$

(c) *if* $\overline{\beta}(\gamma)\lambda_2 < \underline{\beta}(\gamma)\lambda_1$, *then the dominant spectral interval is simple,*

where the real and positive eigenvalues λ_n *were determined in Exam. 4.2. This is shown in [22] and particularly [22, Thm. 1, Exam. 12] for (b). Concerning (c) the assumption implies that the dominant spectral interval has positive distance from the remaining spectrum and the spectral manifold* $\mathbb{I} \times \operatorname{span}\{e_1\}$ *has dimension* 1.

From now on, suppose $(\gamma_t)_{t\in\mathbb{I}'}$ is a sequence of positive reals satisfying (5.1).

5.1 Beverton-Holt integrodifference equation

Let us understand the scalar IDE

$$u_{t+1} = \gamma_t \int_\Omega k(\cdot, y)\frac{u_t(y)}{1+u_t(y)}\,\mathrm{d}y \tag{5.3}$$

as of Urysohn type (U) with the function $f_t : \Omega \times \Omega \times (-1, \infty) \to \mathbb{R}$ given by $f_t(x, y, z) := \gamma_t k(x, y)\frac{z}{1+z}$, $t \in \mathbb{I}'$, satisfying (u1) due to

$$D_3^l f_t(x, y, z) = \gamma_t k(x, y)\tfrac{(-1)^{l+1} l!}{(1+z)^{l+1}} \quad \text{for all } l \in \mathbb{N}.$$

Prop. 4.2 shows that the right-hand side $\mathcal{F}_t : U_t \to C(\Omega)$ of (5.3) is of class C^∞ on the constant open sets $U_t := \{u \in C(\Omega) : -1 < \inf_{x\in\Omega} u(x)\}$. We interpret (5.3) as difference equation in $C(\Omega)$ and in the equation of perturbed motion (2.3) (corresponding to the trivial solution $\phi_t^* \equiv 0$) one has

$$D\mathcal{F}_t(0) = \gamma_t\mathcal{K}, \qquad \mathcal{R}_t(u) = -\gamma_t \int_\Omega k(\cdot, y)\frac{u(y)^2}{1+u(y)}\,\mathrm{d}y \quad \text{for all } t \in \mathbb{I}'.$$

In order to obtain stability properties of the trivial solution, consider the associate variational eqn. (5.2) with the operator $\mathcal{K}$ from (4.3). Thus, due to Thm. 2.1 the trivial solution to (5.3) is

- uniformly exponentially stable, if and only if $\rho_1\overline{\beta}(\gamma) < 1$ (see Fig. 1(a)),
- unstable, if $\rho_2\overline{\beta}(\gamma) < \rho_1\underline{\beta}(\gamma)$ (guaranteeing a spectral gap left of the dominant interval) and $1 < \rho_1\underline{\beta}(\gamma)$ (see Fig. 1(b)).

As alternative spectral constellation (see Fig. 2) we now assume

(σ1) The eigenvalue λ_1 of $\mathcal{K}$ is simple and dominant (i.e. $\rho_1 = |\lambda_1|$) with eigenfunction $e_1 : \Omega \to \mathbb{K}$

(σ2) $\Sigma(\phi^*)$ has at least two components, where

$$\rho_1\underline{\beta}(\gamma) \le 1 \le \rho_1\overline{\beta}(\gamma), \qquad \rho_2\overline{\beta}(\gamma) < \rho_1\underline{\beta}(\gamma). \tag{5.4}$$

Note that $\rho_2\overline{\beta}(\gamma) < \rho_1^m\underline{\beta}(\gamma)^m$ is sufficient for the spectral gap condition (G_m).

In the following, given $v \in C(\Omega)$ we use

$$\langle v, e_1' \rangle := \int_\Omega v(x)\overline{e_1(x)}\,\mathrm{d}x$$

as duality pairing and define constant, complementary projectors

$$P^- v := \int_\Omega v(x)\overline{e_1(x)}\,\mathrm{d}x e_1, \qquad P^+ v := v - P^- v. \tag{5.5}$$

Under the above spectral constellation a reduction to the center-unstable fiber bundle $\mathcal{W}^-$ of (5.3) is due. The reduced eqn. (3.3) has the right-hand side

$$f_t(\xi) = \gamma_t \lambda_1 \xi - \gamma_t \int_\Omega \int_\Omega k(x,y) \frac{(\xi e_1(y) + w_t^-(\xi e_1)(y))^2}{1 + \xi e_1(y) + w_t^-(\xi e_1)(y)} \,\mathrm{d}y \overline{e_1(x)} \,\mathrm{d}x$$

and $D^2\mathcal{R}_t(0)e_1^2 = -2\gamma_t \int_\Omega k(\cdot,y)e_1(y)^2\,\mathrm{d}y$. Let us now refrain from the general nonautonomous case due to the following reasons: First, since $C(\Omega)$ is only a C^0-Banach space, the reduced difference eqn. (3.1) is merely Lipschitz. Second, we are not aware of tools to investigate critical time-variant difference equations (note that e.g. [24, Prop. 5.4] does not apply). We supplement (σ1-σ2) by the assumption

(σ3) The dominant eigenvalue λ_1 of $\mathcal{K}$ is real and positive with real and positive eigenfunction $e_1 : \Omega \to \mathbb{R}$.

Example 5.2 (periodic and autonomous case). *Let (σ1-σ3) hold and suppose $(\gamma_t)_{t\in\mathbb{Z}}$ is a θ-periodic sequence. Thanks to Exam. 2.1 the dichotomy spectrum is discrete. The Beverton-Holt IDE* (5.3) *and its reduced eqn.* (3.3) *become θ-periodic. Whence, upper and lower Bohl exponent of $(\gamma_t)_{t\in\mathbb{I}}$ agree and* (5.4) *reduces to*

$$\lambda_1 \sqrt[\theta]{\gamma_\theta \cdots \gamma_1} = 1, \qquad |\lambda_2| < \lambda_1.$$

By choosing β sufficiently close to 1, one can always fulfill (G_m). In particular, for $m = 2$ the center-unstable bundle $\mathcal{W}^-$ is given by graphs of C^2-functions w_t^-, also f_t is of class C^2 in a (uniform) neighborhood of the origin. From the coefficients

$$f_t'(0) = \lambda_1 \gamma_t, \qquad f_t''(0) = -2\gamma_t \kappa_2 \quad \textit{for all } t \in \mathbb{Z} \tag{5.6}$$

we have $\phi_0(t,\tau) = \lambda_1^{t-\tau} \prod_{s=\tau}^{t-1} \gamma_s > 0$. Thus, (5.4) *implies $\phi_0(\theta, 0) = 1$. We apply Thm. A.1 based on*

$$d_2(\theta) = \sum_{s_2=0}^{\theta-1} \Phi(\theta, s_2+1) f_{s_2}''(0) \Phi(s_2,0)^2 = -\frac{2\kappa_2}{\lambda_1} \sum_{s_2=0}^{\theta-1} \prod_{s=0}^{s_2-1} (\lambda_1 \gamma_s) < 0$$

and consequently the trivial solution of (5.3) *is unstable.*

5.2 Ricker integrodifference equation

An analogous approach as above applies to another Hammerstein-type eqn. (H), but now with *Ricker nonlinearity* $g_t(x,z) := \gamma_t z e^{-z}$. Yet, the scalar IDE

$$u_{t+1} = \gamma_t \int_\Omega k(\cdot,y) u_t(y) e^{-u_t(y)} \,\mathrm{d}y \tag{5.7}$$

is still understood as Urysohn equation with kernel function $f_t : \Omega \times \Omega \times \mathbb{R} \to \mathbb{R}$, $f_t(x,y,z) := k(x,y)ze^{-z}$, $t \in \mathbb{I}'$. Mathematical induction yields

$$D_3^l f_t(x,y,z) = (-1)^l \gamma_t k(x,y)(z-l)e^{-z} \quad \text{for all } l \in \mathbb{N}_0$$

and thus (u1) holds. In conclusion, Prop. 4.2 yields right-hand sides $\mathcal{F}_t : U_t \to C(\Omega)$ of class C^∞ with constant domains $U_t = C(\Omega)$. In the equation of perturbed motion (2.3) (associated to the zero solution) one has

$$D\mathcal{F}_t(0) = \gamma_t \mathcal{K}, \qquad \mathcal{R}_t(u) = \gamma_t \int_\Omega k(\cdot,y) u(y)(e^{-u(y)} - 1)\,\mathrm{d}y \quad \text{for all } t \in \mathbb{I}'.$$

Hence, stability criteria are literally as in Sect. 5.1 and we focus on the constellation (σ1-σ2), requiring a center fiber bundle reduction. The reduced eqn. (3.3) possesses

$$f_t(\xi) = \lambda_1 \gamma_t \xi + \gamma_t \int_\Omega \int_\Omega k(x,y)(\xi e_1(y) + w_t^-(\xi e_1)(y))(e^{-\xi e_1(y) - w_t^-(\xi e_1)(y)} - 1)\,\mathrm{d}y e_1(x)\,\mathrm{d}x$$

as right-hand side. For the same reason as above, we retreat to

Example 5.3 (periodic and autonomous case). *Let (σ1-σ3) hold and suppose the sequence $(\gamma_t)_{t\in\mathbb{Z}}$ is θ-periodic. We again obtain the coefficients* (5.6) *and as in Exam. 5.2 the trivial solution to* (5.7) *is unstable.*

5.3 Logistic integrodifference equation

From now on, let us actually work with Hammerstein difference eqns. (H), namely

$$u_{t+1} = \gamma_t \int_\Omega k(\cdot, y) u_t(y)(1 - u_t(y))\, \mathrm{d}y \tag{5.8}$$

having a *logistic nonlinearity* $g_t(x, z) := \gamma_t z(1 - z)$, $t \in \mathbb{I}'$. We understand the right-hand side as composition $\mathcal{F}_t = \mathcal{K} \circ \mathcal{G}_t : U_t \to L^p(\Omega)$ of two operators with $U_t = L^p(\Omega)$:

- The linear operator $\mathcal{K} \in L(L^1(\Omega), L^p(\Omega))$ is well-defined and compact for every $p > 1$ due to Lemma 4.1.
- $\mathcal{G}_t : L^p(\Omega) \to L^1(\Omega)$ is of class C^2 due to Lemma 4.2. Indeed, one has

$$D_2 g_t(x, z) = \gamma_t(1 - 2z), \qquad D_2^2 g_t(x, z) = -2\gamma_t \quad \text{for all } t \in \mathbb{I}'$$

  and thus (c1-c2) hold with appropriate c_0, c_1 in (4.1), provided $p > 2$.

Hence, Prop. 4.1 applies for $p > 2$ and yields right-hand sides $\mathcal{F}_t : L^p(\Omega) \to L^p(\Omega)$ of class C^2. We arrive at an equation of perturbed motion (2.3) with

$$D\mathcal{F}_t(0) = \gamma_t \mathcal{K}, \qquad \mathcal{R}_t(u) = -\gamma_t \int_\Omega k(\cdot, y) u(y)^2\, \mathrm{d}y \quad \text{for all } t \in \mathbb{I}'.$$

Linearizing (5.8) along the trivial solution yields the variational eqn. (5.2) on the state space $L^p(\Omega)$, while stability criteria are as in Sect. 5.1.

We next study the alternative constellation (σ1-σ2) requiring reduction to a center fiber bundle. For this,

$$\langle u, e_1' \rangle := \int_\Omega u(y) \overline{e_1(y)}\, \mathrm{d}y \quad \text{for all } u \in L^p(\Omega)$$

serves as duality pairing and we define projectors as in (5.5). Therefore, the reduced eqn. (3.3) possesses

$$f_t(\xi) = \lambda_1 \gamma_t \xi - \gamma_t \int_\Omega \int_\Omega k(x, y)(\xi e_1(y) + w_t^-(\xi e_1)(y))^2\, \mathrm{d}y e_1(x)\, \mathrm{d}x$$

as right-hand side. Because $L^p(\Omega)$ is a C^2-Banach space, for $p > 2$ the center-unstable bundle $\mathcal{W}^-$ is given by graphs of (at least) C^2-functions and f_t is of class C^2 in a (uniform) neighborhood of the origin, provided the spectral gap condition (G_2) can be fulfilled.

In the θ-periodic situation, this anew leads to the coefficients (5.6) and an analysis as in Sect. 5.1.

5.4 Toy model 1

We continue with a more artificial example, whose stability behavior is rather subtle in the critical case. Consider the scalar, inhomogeneous IDE

$$u_{t+1} = \gamma_t \int_\Omega k(\cdot, y) \sin u_t(y)\, \mathrm{d}y + \pi h_t, \tag{5.9}$$

where $(h_t)_{t \in \mathbb{I}'}$ is a bounded sequence of integers interpreted as constant functions in $L^p(\Omega)$, $p \geq 1$. The sequence $\phi_t^*(x) :\equiv \pi h_{t-1}$ on Ω defines a bounded solution $(\phi_t^*)_{t \in \mathbb{I}'}$ due to the following identity on $\mathbb{I}'$,

$$\phi_{t+1}^* \equiv \gamma_t \int_\Omega k(\cdot, y) \underbrace{\sin(\pi h_{t-1})}_{=0}\, \mathrm{d}y + \pi h_t \equiv \gamma_t \int_\Omega k(\cdot, y) \sin \phi_t^*(y)\, \mathrm{d}y + \pi h_t.$$

The eqn. (5.9) is of Hammerstein type (H) with growth function $g_t : \Omega \times \mathbb{R} \to \mathbb{R}$, $g_t(y, z) := \gamma_t \sin z$ satisfying

$$\left| D_2^l g_t(x, z) \right| \leq \gamma_t \quad \text{for all } x \in \Omega,\, z \in \mathbb{R},\, l \in \mathbb{N}_0,\, t \in \mathbb{I}'.$$

Thus, due to Prop. 4.1 we can understand (5.9) as equation in $L^p(\Omega)$, $U_t := L^p(\Omega)$ and for $p > 1$ the right-hand side of (5.9) is continuously differentiable. We represent the equation of perturbed motion corresponding to ϕ^* as (2.3) with

$$D\mathcal{F}_t(\phi_t^*) = \varrho_t\gamma_t\mathcal{K}, \qquad \mathcal{R}_t(u) = \varrho_t\gamma_t \int_\Omega k(\cdot, y)\,(\sin u(y) - u(y))\,\mathrm{d}y \quad \text{for all } t \in \mathbb{I}'$$

and

$$\varrho_t := \cos(\pi h_{t-1}) = \begin{cases} -1, & h_{t-1} \text{ is odd,} \\ 1, & h_{t-1} \text{ is even.} \end{cases}$$

Since the variational difference eqn. $u_{t+1} = \varrho_t\gamma_t\mathcal{K}u_t$ fits in the setting of (5.2) and the Bohl exponents satisfy $\underline{\beta}(\varrho\gamma) = \underline{\beta}(\gamma)$, $\overline{\beta}(\varrho\gamma) = \overline{\beta}(\gamma)$, Thm. 2.1 shows that ϕ^* is

- uniformly exponentially stable, if and only if $\rho_1\overline{\beta}(\gamma) < 1$ (see Fig. 1(a)),
- unstable, if $\rho_2\overline{\beta}(\gamma) < \rho_2\underline{\beta}(\gamma)$ and $1 < \rho_1\underline{\beta}(\gamma)$ (see Fig. 1(b)).

A stability analysis in the critical spectral setting (σ1-σ3) is more interesting and based on the reduced eqn. (3.3) with right-hand side

$$f_t(\xi) = \varrho_t\gamma_t\lambda_1\xi + \varrho_t\gamma_t \int_\Omega\int_\Omega k(x,y)(\sin(\xi e_1(y) + w_t^-(\xi e_1)(y)) - \xi e_1(y) - w_t^-(\xi e_1)(y))\,\mathrm{d}y e_1(x)\,\mathrm{d}x.$$

If even $p > 3$ holds, then (5.9) is of class C^3 (see Prop. 4.1), and since $L^p(\Omega)$ is a C^3-Banach space, also the resulting 1-dimensional center-unstable fiber bundle $\mathcal{W}^-$ is of class C^3, when the spectral gap condition (G_3) holds. This leads to the coefficients

$$f_t'(0) = \varrho_t\gamma_t\lambda_1, \qquad f_t''(0) = 0, \qquad f_t'''(0) = -\varrho_t\gamma_t\kappa_3 \neq 0 \quad \text{for all } t \in \mathbb{I}'.$$

Example 5.4 (periodic and autonomous case)**.** *Let (σ1-σ3) hold and suppose the sequences $(\gamma_t)_{t\in\mathbb{Z}}$, $(h_t)_{t\in\mathbb{Z}}$ are θ-periodic. The stability properties of the θ-periodic solution ϕ^* to* (5.9) *in critical situations $\phi_0(\theta, 0) = \pm 1$ are determined by the coefficient $b := h_1 + \cdots + h_\theta$. Indeed, from the elementary trigonometric identity*

$$\cos(\pi k)\cos(\pi l) = \cos(\pi k)\cos(\pi l) - \sin(\pi k)\sin(\pi l) = \cos(\pi(k + l)) \quad \textit{for all } k, l \in \mathbb{Z},$$

we compute

$$\prod_{s=0}^{\theta-1} \varrho_s = \prod_{s=0}^{\theta-1} \cos(\pi h_{s-1}) = \cos(\pi(h_{-1} + \cdots + h_{\theta-2})) = \begin{cases} 1, & b \textit{ is even,} \\ -1, & b \textit{ is odd} \end{cases}$$

and this yields

$$\phi_0(\theta, 0) = \prod_{s=0}^{\theta-1}(\varrho_s\gamma_s\lambda_1) = \left(\prod_{s=0}^{\theta-1}\varrho_s\right)\prod_{s=0}^{\theta-1}(\gamma_s\lambda_1) \begin{cases} > 0, & b \textit{ is even,} \\ < 0, & b \textit{ is odd.} \end{cases}$$

From $f_t''(0) = 0$ we see that $d_2(t) = 0$, but

$$\begin{aligned} d_3 &= \sum_{s_3=0}^{\theta-1} \phi_0(\theta, s_3 + 1) f_{s_3}'''(0)\phi_0(s_3, 0)^3 = -\frac{\kappa_3}{\lambda_1}\sum_{s_3=0}^{\theta-1}\phi_0(\theta, s_3 + 1)\varrho_{s_3}\gamma_{s_3}\lambda_1\phi_0(s_3, 0)^3 \\ &= -\frac{\kappa_3}{\lambda_1}\phi_0(\theta, 0)\sum_{s_3=0}^{\theta-1}\phi_0(s_3, 0)^2 \begin{cases} < 0, & b \textit{ is even,} \\ > 0, & b \textit{ is odd.} \end{cases} \end{aligned}$$

In conclusion, for $\phi_0(\theta, 0) = 1$ Thm. A.1 shows that ϕ^* is uniformly asymptotically stable for an even sum b and unstable for odd b. In case $\phi_0(\theta, 0) = -1$ the situation reverses and Thm. A.2 implies that ϕ^* is uniformly asymptotically stable for an odd sum b and unstable for even b.

5.5 Toy model 2

Our final example allows an explicit analysis at least in the autonomous situation. Yet, let us begin with the nonautonomous, scalar and inhomogeneous IDE

$$u_{t+1} = \gamma_t \int_\Omega k(\cdot, y) u_t(y) \cos u_t(y) \, \mathrm{d}y + \tfrac{\pi}{2} \tag{5.10}$$

having the stationary solution $\phi_t^*(x) \equiv \frac{\pi}{2}$, because of the solution identity

$$\phi_{t+1}^* \equiv \gamma_t \int_\Omega k(\cdot, y) \tfrac{\pi}{2} \underbrace{\cos \tfrac{\pi}{2}}_{=0} \mathrm{d}y + \tfrac{\pi}{2} \equiv \gamma_t \int_\Omega k(\cdot, y) \phi_t^*(y) \cos \phi_t^*(y) \, \mathrm{d}y + \tfrac{\pi}{2} \quad \text{on } \mathbb{I}'.$$

For the growth function $g_t(x, z) := \gamma_t g(z)$, $g(z) := z \cos z$ we obtain the derivatives

$$g'(z) = \cos z - z \sin z, \qquad g''(z) = -2 \sin z - z \cos z, \qquad g'''(z) = -3 \cos z + z \sin z$$

and thus $\left| D_2^l g_t(x, z) \right| \le \sup_{t \in \mathbb{I}'} \gamma_t (l + |z|)$ for all $x \in \Omega$, $z \in \mathbb{R}$ and $0 \le l \le 3$. In order to apply Prop. 4.1 we choose $p > 3$, $q = 1$, $c_0(x) :\equiv 3$ and $c_1 := 1$, resulting in a C^3-right-hand side $\mathcal{F}_t : L^p(\Omega) \to L^p(\Omega)$ of (5.10). In particular, we view (5.10) as difference equation in $L^p(\Omega)$.

In the associate equation of perturbed motion (2.3) one has

$$D\mathcal{F}_t(\phi_t^*) = -\tfrac{\pi}{2} \gamma_t \mathcal{K}, \qquad \mathcal{R}_t(u) = \gamma_t \int_\Omega k(\cdot, y) \left(\tfrac{\pi}{2} u(y) - (u(y) + \tfrac{\pi}{2}) \sin u(y) \right) \mathrm{d}y \quad \text{for all } t \in \mathbb{I}'$$

leading to $D^2\mathcal{R}_t(0) e_1^2 = -\gamma_t \int_\Omega k(\cdot, y) e_1(y)^2 \, \mathrm{d}y$. Directly from the relations (3.5) it results $f_t'(0) = -\frac{\pi}{2} \gamma_t \lambda_1$, and under the gap conditions (G_2) resp. (G_3),

$$\begin{aligned}
f_t''(0) &= -2\gamma_t \int_\Omega \int_\Omega k(x, y) e_1(y)^2 \, \mathrm{d}y e_1(x) \, \mathrm{d}x = -2\gamma_t \kappa_2, \\
f_t'''(0) &= \frac{\pi \gamma_t}{2} \int_\Omega \int_\Omega k(x, y) e_1(y)^3 \, \mathrm{d}y e_1(x) \, \mathrm{d}x - 6\gamma_t \int_\Omega \int_\Omega k(x, y) (D^2 w_t^-(0) e_1^2)(y) e_1(y) \, \mathrm{d}y e_1(x) \, \mathrm{d}x \\
&= \frac{\pi \gamma_t}{2} \kappa_3 - 6\gamma_t \int_\Omega \int_\Omega k(x, y) (D^2 w_t^-(0) e_1^2)(y) e_1(y) \, \mathrm{d}y e_1(x) \, \mathrm{d}x.
\end{aligned}$$

Therefore, we have to compute the Taylor coefficients $D^2 w_t^-(0) e_1^2 \in L^p(\Omega)$, $t \in \mathbb{I}$, of the functions parametrizing the center-unstable fiber bundle $\mathcal{W}^-$ of ϕ^*. It solves the homological eqn. (3.7) which now reads as

$$\eta_t^2 w_{t+1} = -\tfrac{\pi}{2} \gamma_t \mathcal{K} w_t + \gamma_t \kappa_2 e_1 - \gamma_t \int_\Omega k(\cdot, y) e_1(y)^2 \, \mathrm{d}y.$$

Due to the eigenvalue property it follows $\eta_t = \left\langle D\mathcal{F}_t(\phi_t^*) e_1, e_1' \right\rangle = -\frac{\pi}{2} \gamma_t \lambda_1$ and the unique bounded solution of the homological equation becomes

$$D^2 w_t^-(0) e_1^2 = \sum_{s=-\infty}^{t-1} \left(\tfrac{2}{\pi \lambda_1} \right)^{2(t-s-1)} \gamma_s \left(\prod_{r=s+1}^{t-1} \tfrac{1}{\gamma_r} \right) \mathcal{K}^{t-s-1} \left(\kappa_2 e_1 - \int_\Omega k(\cdot, y) e_1(y)^2 \, \mathrm{d}y \right) \quad \text{for all } t \in \mathbb{I}.$$

The powers of the Fredholm operator $\mathcal{K}$ compute as t-fold integral

$$\mathcal{K}^t v = \int_{\Omega^t} k(\cdot, y_t) \left(\prod_{r=1}^{t-1} k(y_{r+1}, y_r) \right) v(y_1) \, \mathrm{d}(y_1, \dots, y_t) \quad \text{for all } t \in \mathbb{Z}_1^+,$$

which results from Fubini's theorem, and we arrive at the formula

$$D^2 w_t^-(0)e_1^2 = \sum_{s=-\infty}^{t-1} \left(\tfrac{2}{\pi\lambda_1}\right)^{2(t-s-1)} \gamma_s \left(\prod_{r=s+1}^{t-1} \tfrac{1}{\gamma_r}\right) \cdot \left(\kappa_2 \int_{\Omega^{t-s-1}} k(\cdot, y_{t-s-1})\cdots k(y_2,y_1)e_1(y_1)\,\mathrm{d}(y_1,\dots,y_{t-s-1}) - \int_{\Omega^{t-s}} k(\cdot,y_{t-s-1})\cdots k(y_1,y_0)e_1(y_0)^2\,\mathrm{d}(y_0,\dots,y_{t-s-1})\right) \quad \text{for all } t \in \mathbb{I}.$$

Since it is apparently problematic to evaluate this expression, we again retreat to

Example 5.5 (periodic and autonomous case)**.** *Let (σ1-σ3) hold and suppose $(\gamma_t)_{t\in\mathbb{Z}}$ is a θ-periodic sequence. Thanks to*

$$\phi_0(\theta,0) = \left(-\tfrac{\pi}{2}\lambda_1\right)^\theta \gamma_\theta\cdots\gamma_1,$$

we need to distinguish two cases:
(I) θ is even: One has $\phi_0(\theta,0)=1$ and in order to apply Thm. A.1 we note

$$d_2(\theta) = \sum_{s_2=0}^{\theta-1} \phi_0(\theta,s_2+1) f''_{s_2}(0)\phi_0(s_2,0)^2 = \frac{4\kappa_2}{\pi\lambda_1}\sum_{s_2=0}^{\theta-1}\phi_0(s_2,0).$$

For $d_2(\theta)\neq 0$ we derive from Thm. A.1 that ϕ^* is unstable. In the degenerate case $d_2(\theta)=0$ results

$$\begin{aligned} d_2(t) &= \frac{4\kappa_2}{\pi\lambda_1}\phi_0(t,0)\sum_{s_2=0}^{t-1}\phi_0(s_2,0),\\ d_3 &= \sum_{s_3=0}^{\theta-1}\phi_0(\theta,s_3+1)f'''_{s_3}(\phi^*_{s_3})\phi_0(s_3,0)^3 + \frac{12\kappa_2}{\pi\lambda_1}\sum_{s_3=1}^{\theta-1} d_2(s_3)\\ &= \frac{\kappa_3}{\lambda_1}\sum_{s_3=0}^{\theta-1}\phi_0(s_3,0)^2 + \frac{12}{\pi\lambda_1}\sum_{s_3=0}^{\theta-1}\int_\Omega\int_\Omega k(x,y)(D^2 w_{s_3}^-(0)e_1^2)(y)e_1(y)\,\mathrm{d}y e_1(x)\,\mathrm{d}x\phi_0(s_3,0)^2\\ &\quad + \frac{12\kappa_2}{\pi\lambda_1}\sum_{s_3=1}^{\theta-1} d_2(s_3), \end{aligned}$$

where the vectors $D^2 w_{s_3}^-(0)e_1^2 \in L^p(\Omega)$, $0 \le s_3 < \theta$ are to be computed from the cyclic θ coupled linear Fredholm integral equations

$$\begin{cases} \eta_t^2 w_{t+1} = -\frac{\pi}{2}\gamma_t \int_\Omega k(\cdot,y)w_t(y)\,\mathrm{d}y + \gamma_t\kappa_2 e_1 - \gamma_t\int_\Omega k(\cdot,y)e_1(y)^2\,\mathrm{d}y & \text{for all } 0\le t<\theta-1,\\ \eta_{\theta-1}^2 w_0 = -\frac{\pi}{2}\gamma_{\theta-1}\int_\Omega k(\cdot,y)w_{\theta-1}(y)\,\mathrm{d}y + \gamma_{\theta-1}\kappa_2 e_1 - \gamma_{\theta-1}\int_\Omega k(\cdot,y)e_1(y)^2\,\mathrm{d}y. \end{cases}$$

In the autonomous case this reduces to the Fredholm equation of the second kind

$$\eta_0^2 w_0 = -\frac{\pi}{2}\gamma\int_\Omega k(\cdot,y)w_0(y)\,\mathrm{d}y + \gamma\kappa_2 e_1 - \gamma\int_\Omega k(\cdot,y)e_1(y)^2\,\mathrm{d}y,$$

while for instance the 2-periodic situation requires

$$\begin{cases} \eta_1^2 w_0 = -\frac{\pi}{2}\gamma_1\int_\Omega k(\cdot,y)w_1(y)\,\mathrm{d}y + \gamma_1\kappa_2 e_1 - \gamma_1\int_\Omega k(\cdot,y)e_1(y)^2\,\mathrm{d}y,\\ \eta_0^2 w_1 = -\frac{\pi}{2}\gamma_0\int_\Omega k(\cdot,y)w_0(y)\,\mathrm{d}y + \gamma_0\kappa_2 e_1 - \gamma_0\int_\Omega k(\cdot,y)e_1(y)^2\,\mathrm{d}y. \end{cases}$$

Eventually, explicit computations are feasible for autonomous IDEs (5.10) with the Laplace kernel (4.3) on the interval $\Omega = [-\frac{L}{2},\frac{L}{2}]$ and thus $\eta_0 = -\frac{\pi}{2}\gamma\lambda$. Suppose $\nu>0$ denotes the smallest positive solution of the

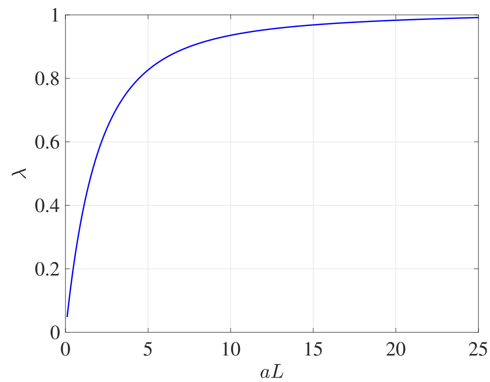

Fig. 6: Values of the dominant eigenvalue λ of the integral operator (4.2) with Laplace kernel (4.3) depending on the product aL

equation $\tan\left(\frac{aL}{2}\nu\right) = \frac{1}{\nu}$, *which by Exam. 4.2 is related to the dominant eigenvalue to* $\mathcal{K}$ *via* $\lambda = \frac{1}{1+\nu^2} \in (0,1)$ *(see Fig. 6) with corresponding eigenfunction*

$$e_1(x) = C\cos(a\nu x), \qquad C := \sqrt{\tfrac{2a\nu}{La\nu+\sin(La\nu)}}.$$

The nonhyperbolic situation holds for $\frac{\pi}{2}\gamma\lambda = 1$, *i.e.* $\gamma = \frac{2}{\pi\lambda} = \frac{2(1+\nu^2)}{\pi} = \frac{2\omega^2}{\pi}$, *where we conveniently abbreviate* $\omega := \sqrt{1+\nu^2}$. *This implies the homological equation*

$$w_0(x) + \frac{a\omega^2}{2}\int_{-L/2}^{L/2} e^{-a|x-y|}w_0(y)\,\mathrm{d}y = \frac{2\kappa_2\omega^2}{\pi}e_1(x) - \frac{a\omega^2}{\pi}\int_{-L/2}^{L/2} e^{-a|x-y|}e_1(y)^2\,\mathrm{d}y \quad \text{for all } x \in [-\tfrac{L}{2},\tfrac{L}{2}],$$

which is a Fredholm integral eqn. (B.1) *of the second kind with* $\delta = -\omega^2$, *the inhomogeneity*

$$b(x) = \frac{2\kappa_2\omega^2}{\pi}e_1(x) - \frac{a\omega^2}{\pi}\int_{-L/2}^{L/2} e^{-a|x-y|}e_1(y)^2\,\mathrm{d}y = \sum_{j=0}^{3}\varrho_j h_j(x), \tag{5.11}$$

the linearly independent functions $h_0,\ldots,h_3 : [-\frac{L}{2},\frac{L}{2}] \to \mathbb{R}$,

$$h_0(x) :\equiv 1, \qquad h_1(x) := e_1(x), \qquad h_2(x) := \cos(2a\nu x), \qquad h_3(x) := \cosh(ax)$$

and coefficients $\varrho_0,\ldots,\varrho_3$ *(all depending on* a, L*) from* (C.1). *Due to* $\delta < 1$ *the solution from Sect. B implies*

$$[D^2w^-(0)e_1^2](x) = c_1\cosh(\vartheta x) + c_2\sinh(\vartheta x) + b(x) + a\omega\int_{-L/2}^{x} b(y)\sinh(\vartheta(x-y))\,\mathrm{d}y \quad \text{for all } x \in [-\tfrac{L}{2},\tfrac{L}{2}]$$

with $\vartheta := a\omega$ *and coefficients*

$$c_1 = -\frac{a\omega\int_{-L/2}^{L/2} b(y)\left(\sinh\left(\vartheta\left(\frac{L}{2}-y\right)\right) + \omega\cosh\left(\vartheta\left(\frac{L}{2}-y\right)\right)\right)\mathrm{d}y}{2\left(\cosh\frac{aL\omega}{2} + \omega\sinh\frac{aL\omega}{2}\right)} = \frac{B_0\varrho_0 + B_1\varrho_1 + B_2\varrho_2 + B_3\varrho_3}{\cosh\frac{aL\omega}{2} + \omega\sinh\frac{aL\omega}{2}},$$

$$c_2 = -\frac{a\omega\int_{-L/2}^{L/2} b(y)\left(\sinh\left(\vartheta\left(\frac{L}{2}-y\right)\right) + \omega\cosh\left(\vartheta\left(\frac{L}{2}-y\right)\right)\right)\mathrm{d}y}{2\left(\sinh\frac{aL\omega}{2} + \omega\cosh\frac{aL\omega}{2}\right)} = \frac{B_0\varrho_0 + B_1\varrho_1 + B_2\varrho_2 + B_3\varrho_3}{\sinh\frac{aL\omega}{2} + \omega\cosh\frac{aL\omega}{2}},$$

where the real coefficients $B_0,\ldots,B_3$ *are defined in* (C.2). *This leads to* $D^2w^-(0)e_1^2 = \sum_{j=1}^{5}\omega_j h_j$ *with*

$$h_4, h_5 : [-\tfrac{L}{2},\tfrac{L}{2}] \to \mathbb{R}, \qquad h_4(x) := e^{a\omega x}, \qquad h_5(x) := e^{-a\omega x}$$

as additional coefficient functions and the reals $\omega_1, \dots, \omega_5$ *given in* (C.3), (C.4). *Based on these preparations, we compute* $f'(0) = -1$, $f''(0) = -\frac{4}{\pi\lambda_1}\kappa_2$ *and finally*

$$\begin{aligned} f'''(0) &= \frac{\kappa_3}{\lambda_1} - \frac{6a}{\pi\lambda_1} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} e^{-a|x-y|} (D^2 w^-(0)e_1^2)(y) e_1(y)\, \mathrm{d}y e_1(x)\, \mathrm{d}x \\ &= (1+\nu^2) \left(\kappa_3 - \frac{6a}{\pi} \sum_{j=1}^{5} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} e^{-a|x-y|} h_j(y) e_1(y)\, \mathrm{d}y e_1(x)\, \mathrm{d}x \right). \end{aligned}$$

The value $f''(0)$ *depends only on* $a, L > 0$ *and Fig. 7 (left) indicates* $f''(0) < 0$ *for parameters* $0 < a, L < 5$. *Whence, the trivial solution of the reduced equation is unstable due to Thm. A.1 and the reduction principle in Thm. 3.1 ensures that the constant solution* $\phi^*(x) \equiv \frac{\pi}{2}$ *of* (5.10) *is unstable as well.*

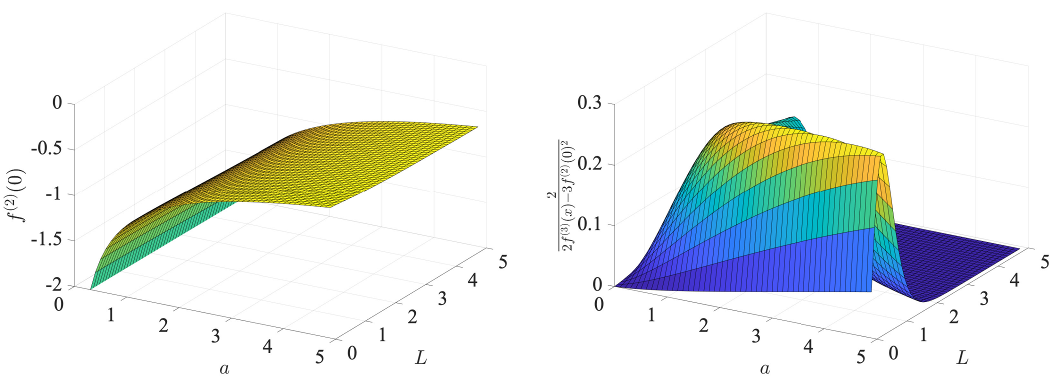

Fig. 7: Values of $f''(0)$ (left) and of (the reciprocal of) the stability indicator $f'''(0) - \frac{3}{2}f''(0)^2$ (right) from Exam. 5.5 depending on the parameters a, L; the reciprocals are shown due to large values.

(II) <u>*θ is odd*</u>*: Now* $\phi_0(\theta, 0) = -1$ *requires to apply Thm. A.2. Both derivatives* $f''(0)$ *and* $f'''(0)$ *depend only on* $a, L > 0$, *and so does the stability indicator* $f'''(0) + \frac{3}{2}f''(0)^2$. *Fig. 7 (right) indicates its positivity, at least as long as* $a, L \in [0, 5]$ *and therefore asymptotic stability. Again, the reduction principle Thm. 3.1 shows that* $\phi^*(x) \equiv \frac{\pi}{2}$ *is an asymptotically stable solution of the IDE* (5.10).

6 Perspectives

We conclude this paper with some open questions and possible perspectives:

– It is not clear what can be said on the stability of a critical solution ϕ^*, when center-unstable bundles $\mathcal{W}^-$ are not at hand (and one cannot pull a Lyapunov function out of the hat)? First, the construction of $\mathcal{W}^-$ via the Lyapunov-Perron method requires that (Δ) is defined on a discrete interval unbounded below. Hence, equations given in forward time need an ambient backward extension. Second, what if (V_{ϕ^*}) has a dichotomy spectrum as in Fig. 8, that is, it consists of a single spectral interval containing the stability boundary 1? However, we point out that $\Sigma(\phi^*)$ associated to EDs on a positive half-line $\mathbb{I}$ can be strictly smaller than the dichotomy spectrum associated to the entire integer line. As a result, Thm. 2.1 applied with the half-line spectrum might yield information. In an autonomous, continuous time setting it is shown by [29, Thm. 1] that a spectrum merely meeting the positive half plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ already guarantees instability.

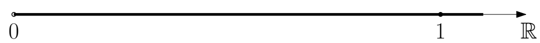

Fig. 8: Dichotomy spectrum consisting of a single interval $(0, \rho]$ with $\rho > 1$

- When the spectral gap condition (G_m) is violated, the center-unstable fiber bundle is merely continuously differentiable and higher-order terms in the reduced equation cannot be computed. Thus, stability criteria not based on Taylor coefficients of order > 1 are needed (see, for instance, [28, pp. 13ff, Chapt. 2]).
- As already indicated above, very little is known on the stability for, say the trivial solution of scalar nonautonomous equations $x_{t+1} = f_t(x_t)$ when the variational eqn. $x_{t+1} = f_t'(0)x_t$ has a dichotomy spectrum containing 1. Although this question is elementary, the nonautonomous reduction principle and an analysis of the reduced equation crucially depends on corresponding answers.

A solution to the above problems might be a dynamical spectrum different from the dichotomy spectrum. For instance, the Lyapunov spectrum is finer, but problematic for equations, which are not regular (see [17]). One could also think of alternative spectra measuring subexponential or polynomial growth.

Appendices

A Stability of nonhyperbolic periodic solutions

Let $\mathbb{I}$ be a discrete interval and $m \in \mathbb{N}$. We consider scalar (real) difference equations

$$\boxed{x_{t+1} = f_t(x_t)} \tag{A.1}$$

having a solution $(\phi_t^*)_{t \in \mathbb{I}}$ and a C^m-right-hand side f_t being defined in a neighborhood of ϕ_t^* uniformly in $t \in \mathbb{I}'$. Furthermore, for the transition operator we abbreviate

$$\phi_0(t, \tau) := \begin{cases} \prod_{s_1=\tau}^{t-1} f_{s_1}'(\phi_{s_1}^*), & \tau < t, \\ 1, & t = \tau. \end{cases}$$

Lemma A.1. *If $\tau \le t$, then the general solution $\varphi(t; \tau, \cdot)$ of* (A.1) *is of class C^m and for $m = 1, 2$ resp. 3, the derivatives satisfy $D_3\varphi(t; \tau, \phi_\tau^*) = \phi_0(t, \tau)$,*

$$\begin{aligned} D_3^2\varphi(t; \tau, \phi_\tau^*) &= \sum_{s_2=\tau}^{t-1} \phi_0(t, s_2 + 1) f_{s_2}''(\phi_{s_2}^*) D_3\varphi(s_2; \tau, \phi_\tau^*)^2, \\ D_3^3\varphi(t; \tau, \phi_\tau^*) &= \sum_{s_3=\tau}^{t-1} \phi_0(t, s_3 + 1) f_{s_3}'''(\phi_{s_3}^*) D_3\varphi(s_3; \tau, \phi_\tau^*)^3 \\ &\quad + 3 \sum_{s_3=\tau+1}^{t-1} \phi_0(t, s_3 + 1) f_{s_3}''(\phi_{s_3}^*) D_3\varphi(s_3; \tau, \phi_\tau^*) D_3^2\varphi(s_3; \tau, \phi_\tau^*). \end{aligned}$$

Proof. Assume $\tau \le t$ throughout. The smoothness of $\varphi(t; \tau, \cdot)$ is an immediate consequence of the chain rule and (2.1). It is convenient to transform $(\phi_t^*)_{t \in \mathbb{I}}$ to the trivial solution by passing over to the associate equation of perturbed motion

$$\boxed{x_{t+1} = f_t^*(x_t) := f_t(x_t + \phi_t^*) - f_t(\phi_t^*),} \tag{A.2}$$

whose general solution is denoted as $\varphi_{t,\tau}(\xi)$ for reals ξ. Differentiating the properties $\varphi_{t+1,\tau}(\xi) = f_t^*(\varphi_{t,\tau}(\xi))$ and $\varphi_{\tau,\tau}(\xi) = \xi$ w.r.t. ξ yields

$$\varphi_{t+1,\tau}'(\xi) = Df_t^*(\varphi_{t,\tau}(\xi))\varphi_{t,\tau}'(\xi), \qquad \varphi_{\tau,\tau}'(\xi) = 1. \tag{A.3}$$

Consequently, since (A.2) has the trivial solution, i.e. $\varphi_{t,\tau}(0) = 0$, we get

$$\varphi'_{t,\tau}(0) = \prod_{s_1=\tau}^{t-1} Df^*_{s_1}(0) = \phi_0(t,\tau) \quad \text{for all } \tau \le t.$$

Differentiating (A.3) again leads to

$$\varphi''_{t+1,\tau}(\xi) = Df^*_t(\varphi_{t,\tau}(\xi))\varphi''_{t,\tau}(\xi) + D^2 f^*_t(\varphi_{t,\tau}(\xi))\varphi'_{t,\tau}(\xi)^2, \qquad \varphi''_{\tau,\tau}(\xi) = 0$$

and therefore $\varphi''_{\cdot,\tau}(\xi)$ solves the linearly inhomogeneous difference equation

$$x_{t+1} = Df^*_t(\varphi_{t,\tau}(\xi))x_t + D^2 f^*_t(\varphi_{t,\tau}(\xi))\varphi'_{t,\tau}(\xi)^2$$

to the initial condition $x_\tau = 0$. Thus, the variation of constants formula [21, p. 100, Thm. 3.1.16(a)] yields $\varphi''_{t,\tau}(0) = \sum_{s_2=\tau}^{t-1} \phi_0(t, s_2+1) D^2 f^*_{s_2}(0)\varphi'_{s_2,\tau}(0)^2$ for $\tau \le t$. Eventually, differentiating (A.3) twice implies

$$\begin{aligned}\varphi'''_{t+1,\tau}(\xi) =& Df^*_t(\varphi_{t,\tau}(\xi))\varphi'''_{t,\tau}(\xi) + D^3 f^*_t(\varphi_{t,\tau}(\xi))\varphi'_{t,\tau}(\xi)^3, \qquad \varphi'''_{\tau,\tau}(\xi) = 0\\ &+ 3D^2 f^*_t(\varphi_{t,\tau}(\xi))\varphi'_{t,\tau}(\xi)\varphi''_{t,\tau}(\xi),\end{aligned}$$

and by the same argument as above, because of $\varphi''_{\tau,\tau}(0) = 0$ the variation of constants formula leads to

$$\varphi'''_{t,\tau}(0) = \sum_{s_3=\tau}^{t-1} \phi_0(t, s_3+1) D^3 f^*_{s_3}(0)\varphi'_{s_3,\tau}(0)^3 + 3\sum_{s_3=\tau+1}^{t-1} \phi_0(t, s_3+1) D^2 f^*_{s_3}(0)\varphi'_{s_3,\tau}(0)\varphi''_{s_3,\tau}(0).$$

Since the general solutions $\varphi(t;\tau,\xi)$ of (A.1) and $\varphi_{t,\tau}(\xi)$ to (A.2) are related by $\varphi_{t,\tau}(\xi) = \varphi(t;\tau,\xi+\phi^*_\tau) - \phi^*_t$ for $\tau \le t$ the claim follows, if we differentiate this relation w.r.t. ξ and set $\xi := 0$. □

Let $\theta \in \mathbb{N}$. We next retreat to θ-periodic eqns. (A.1), i.e. $\mathbb{I} = \mathbb{Z}$, $f_{t+\theta} = f_t$ for all $t \in \mathbb{Z}$ and θ-periodic sequences ϕ^*. Furthermore, we introduce the real numbers

$$\begin{aligned} d_2(t) &:= \sum_{s_2=0}^{t-1} \phi_0(t, s_2+1) f''_{s_2}(\phi^*_{s_2})\phi_0(s_2, 0)^2 \quad \text{for all } 1 \le t \le \theta,\\ d_3 &:= \sum_{s_3=0}^{\theta-1} \phi_0(\theta, s_3+1) f'''_{s_3}(\phi^*_{s_3})\phi_0(s_3,0)^3 + 3\sum_{s_3=1}^{\theta-1} \phi_0(\theta, s_3+1) f''_{s_3}(\phi^*_{s_3})\phi_0(s_3,0) d_2(s_3).\end{aligned}$$

In the autonomous case $\theta = 1$ of constant sequences $f = f_t$ and $\phi^*_t = u^*$ one obtains

$$d_2(1) = f''(u^*), \qquad d_3 = f'''(u^*).$$

Theorem A.1 (nonhyperbolic solution I)**.** *Suppose both the solution ϕ^* and the difference eqn.* (A.1) *are θ-periodic with $\phi_0(\theta, 0) = 1$.*
(a) *If $m = 2$ and $d_2(\theta) \ne 0$, then ϕ^* is unstable,*
(b) *if $m = 3$ and $d_2(\theta) = 0$, $d_3 < 0$, then ϕ^* is uniformly asymptotically stable,*
(c) *if $m = 3$ and $d_2(\theta) = 0$, $d_3 > 0$, then ϕ^* is unstable.*

Proof. Stability properties of ϕ^* coincide with those of the zero solution to the equation of perturbed motion (A.2), which, in turn, are determined by the stability of the fixed point ϕ^*_0 to the period map $\pi_0 := \varphi(\theta; 0, \cdot)$. The claim follows by [2, Thm. 2.3i] applied to π_0, whose derivatives, by Lemma A.1 are given by $d_2(\theta), d_3$. □

Theorem A.2 (nonhyperbolic solution II)**.** *Let $m = 3$. Suppose both the solution ϕ^* and the difference eqn.* (A.1) *are θ-periodic with $\phi_0(\theta, 0) = -1$.*
(a) *If $d_3 + \frac{3}{2} d_2(\theta)^2 > 0$, then ϕ^* is uniformly asymptotically stable,*
(b) *if $d_3 + \frac{3}{2} d_2(\theta)^2 < 0$, then ϕ^* is unstable.*

Proof. We borrow the notation and arguments from the above proof of Thm. A.1. By assumption $\pi_0'(\phi_0^*) = -1$ (cf. Lemma A.1), the Schwarzian derivative becomes

$$S\pi_0(\phi_0^*) = \frac{\pi_0'''(\phi_0^*)}{\pi_0'(\phi_0^*)} - \frac{3}{2}\left(\frac{\pi_0''(\phi_0^*)}{\pi_0'(\phi_0^*)}\right)^2 = -\pi_0'''(\phi_0^*) - \tfrac{3}{2}\pi_0''(\phi_0^*)^2 = -\left(d_3 + \tfrac{3}{2}d_2(\theta)^2\right)$$

and the assertion finally results from [2, Thm. 2.3ii]. □

B Integral equations with Laplace kernel

Given $a, L > 0$, let us consider the Fredholm integral operator

$$\mathcal{K}u(x) := \frac{a}{2}\int_{-L/2}^{L/2} e^{-a|x-y|}u(y)\,\mathrm{d}y \quad \text{for all } x \in [-\tfrac{L}{2}, \tfrac{L}{2}]$$

on the space $C[-\frac{L}{2}, \frac{L}{2}]$. For C^2-inhomogeneities $b : [-\frac{L}{2}, \frac{L}{2}] \to \mathbb{R}$ we provide twice continuously differentiable solutions $u : [-\frac{L}{2}, \frac{L}{2}] \to \mathbb{R}$ to the integral equation

$$u - \delta\mathcal{K}u = b \tag{B.1}$$

of the second kind. Following [20, p. 324, 15.], the structure of these solutions depends on the parameter $\delta \in \mathbb{R} \setminus \{0\}$ satisfying $\frac{1}{\delta} \notin \sigma(\mathcal{K})$:

- If $\delta < 1$, then

  $$u(x) = c_1\cosh(\vartheta x) + c_2\sinh(\vartheta x) + b(x) - \frac{a^2\delta}{\vartheta}\int_{-L/2}^{x} b(y)\sinh(\vartheta(x-y))\,\mathrm{d}y$$

  with $\vartheta := a\sqrt{1-\delta}$ and coefficients

  $$c_1 := \frac{a^2\delta\int_{-L/2}^{L/2} b(y)\left(a\sinh\left(\vartheta\left(\frac{L}{2}-y\right)\right) + \vartheta\cosh\left(\vartheta\left(\frac{L}{2}-y\right)\right)\right)\mathrm{d}y}{2\vartheta\left(a\cosh\frac{\vartheta L}{2} + \vartheta\sinh\frac{\vartheta L}{2}\right)},$$

  $$c_2 := \frac{a^2\delta\int_{-L/2}^{L/2} b(y)\left(a\sinh\left(\vartheta\left(\frac{L}{2}-y\right)\right) + \vartheta\cosh\left(\vartheta\left(\frac{L}{2}-y\right)\right)\right)\mathrm{d}y}{2\vartheta\left(a\sinh\frac{\vartheta L}{2} + \vartheta\cosh\frac{\vartheta L}{2}\right)}.$$

- If $\delta = 1$, then $u(x) = c_1 + c_2 x + b(x) - a^2\int_{-L/2}^{x} b(y)(x-y)\,\mathrm{d}y$, where

  $$c_1 := \frac{a}{2}\int_{-L/2}^{L/2} b(y)\left(1 + a\left(\tfrac{L}{2}-y\right)\right)\mathrm{d}y, \qquad c_2 := \frac{a^2}{aL+2}\int_{-L/2}^{L/2} b(y)\left(1 + a\left(\tfrac{L}{2}-y\right)\right)\mathrm{d}y.$$

- Finally, for $\delta > 1$ the solutions are of the form

  $$u(x) = c_1\cos(\vartheta x) + c_2\sin(\vartheta x) + b(x) - \frac{a^2\delta}{\vartheta}\int_{-L/2}^{x} b(y)\sin(\vartheta(x-y))\,\mathrm{d}y$$

  with $\vartheta := a\sqrt{\delta-1}$ and the constants

  $$c_1 := \frac{a^2\delta\int_{-L/2}^{L/2} b(y)\left(a\sin\left(\vartheta\left(\frac{L}{2}-y\right)\right) + \vartheta\cos\left(\vartheta\left(\frac{L}{2}-y\right)\right)\right)\mathrm{d}y}{2\vartheta\left(a\cos\frac{\vartheta L}{2} - \vartheta\sin\frac{\vartheta L}{2}\right)},$$

  $$c_2 := \frac{a^2\delta\int_{-L/2}^{L/2} b(y)\left(a\sin\left(\vartheta\left(\frac{L}{2}-y\right)\right) + \vartheta\cos\left(\vartheta\left(\frac{L}{2}-y\right)\right)\right)\mathrm{d}y}{2\vartheta\left(a\sin\frac{\vartheta L}{2} + \vartheta\cos\frac{\vartheta L}{2}\right)}.$$

C Coefficients of $f'''(0)$ in Subsect. 5.5

The real coefficients in (5.11) are $\varrho_0 := -\frac{C^2(1+\nu^2)}{\pi}$,

$$\begin{aligned}
\varrho_1 :=& \frac{C^3 e^{-aL}}{6\pi a\nu\,(1+4\nu^2)}\Big(2\left(\left(5\nu^2+2\right)e^{aL}-9\nu^2\right)\cos(aL\nu)\sin\tfrac{aL\nu}{2}\\
&+\left(\left(6\nu^4+65\nu^2+20\right)e^{aL}+6\nu^4-9\nu^2\right)\sin\tfrac{aL\nu}{2}\\
&-3\nu\left((1+2\nu^2)e^{aL}+2\nu^2-1\right)\cos\tfrac{3aL\nu}{2}\Big),
\end{aligned}\tag{C.1}$$

$$\varrho_2 := -\frac{C^2\omega^2}{\pi(1+4\nu^2)},$$

$$\varrho_3 := \frac{C^2\omega^2}{\pi(1+4\nu^2)}e^{-\frac{aL}{2}}(1+4\nu^2-2\nu\sin(aL\nu)+\cos(aL\nu)),$$

the real coefficients determining c_1, c_2 read as $B_0 = 1-\omega\sinh(aL\omega)-\cosh(aL\omega)$,

$$\begin{aligned}
B_1 =& -\frac{2C\omega}{1+2\nu^2}\cos\tfrac{aL\nu}{2}\left(\sinh\tfrac{aL\omega}{2}+\omega\cosh\tfrac{aL\omega}{2}\right)\left(\omega\sinh\tfrac{aL\omega}{2}+\cosh\tfrac{aL\omega}{2}\right),\\
B_2 =& -\frac{\omega(2\nu\sin(aL\nu)}{1+5\nu^2}\big((\sinh(aL\omega)+\omega)+\omega\cos(aL\nu)(\omega\sinh(aL\omega)-1)\\
&+\omega\cosh(aL\omega)(2\nu\sin(aL\nu)+\cos(aL\nu)))\big),\\
B_3 =& \frac{\omega}{\nu^2}\left(\sinh\tfrac{aL}{2}(\sinh(aL\omega)+\omega\cosh(aL\omega)+\omega)-\omega\cosh\tfrac{aL}{2}\left(2\sinh^2\tfrac{aL\omega}{2}+\omega\sinh(aL\omega)\right)\right)
\end{aligned}\tag{C.2}$$

and finally one has

$$\omega_1 = \frac{\varrho_1\nu^2}{1+2\nu^2}, \qquad \omega_2 = \frac{4\varrho_2\nu^2}{1+5\nu^2}, \qquad \omega_3 = -\frac{\varrho_3}{\nu^2}, \tag{C.3}$$

as well as

$$\begin{aligned}
\omega_4 =& \frac{c_1+c_2}{2}+e^{\frac{aL\omega}{2}}\left[\frac{\varrho_0}{2}+\omega\Big(\frac{C\varrho_1\omega}{1+2\nu^2}+\frac{\varrho_2\omega}{2(1+5\nu^2)}+\frac{\varrho_2}{1+5\nu^2}\Big)\cos\tfrac{aL\nu}{2}+\frac{\varrho_3\omega}{4}\left(\frac{e^{\frac{aL}{2}}}{\omega+1}+\frac{e^{-\frac{aL}{2}}}{\omega-1}\right)\right],\\
\omega_5 =& \frac{c_1-c_2}{2}+e^{-\frac{aL\omega}{2}}\left[\frac{\varrho_0}{2}+\frac{\varrho_2\omega}{1+5\nu^2}\left(\frac{\omega}{2}\cos(aL\nu)-\nu\sin(aL\nu)\right)+\frac{\varrho_3\omega}{4}\left(\frac{e^{-\frac{aL}{2}}}{\omega+1}+\frac{e^{\frac{aL}{2}}}{\omega-1}\right)\right].
\end{aligned}\tag{C.4}$$

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