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TOPOLOGICAL DECOUPLING AND LINEARIZATION OF NONAUTONOMOUS EVOLUTION EQUATIONS

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Abstract. Topological linearization results typically require solution flows rather than merely semiflows. An exception occurs when the linearization fulfills spectral assumptions met e.g. for scalar reaction-diffusion equations. We employ tools from the geometric theory of nonautonomous dynamical systems in order to extend earlier work by Lu [12] to time-variant evolution equations under corresponding conditions on the Sacker-Sell spectrum of the linear part. Our abstract results are applied to nonautonomous reaction-diffusion and convection equations.

1. **Introduction.** One of the vital pillars in the theory of dynamical systems is the Hartman-Grobman theorem dating back to [9, 10]. Its simplest form states that generically the flow of an ordinary differential equation (ODE for short) in the vicinity of an equilibrium is topologically conjugated to its linearization. Thus, the local phase portraits near equilibria are homeomorphic. The generic property under which the Hartman-Grobman theorem holds is hyperbolicity, i.e. a linearization without spectrum on the imaginary axis. This underlines the importance of such results in bifurcation theory and for the concept of structural stability.

A Hartman-Grobman theorem in the discrete time setting of difference equations or mappings is more subtle. Here, besides hyperbolicity (no spectrum on the unit circle in $\mathbb{C}$) one also needs invertibility (cf. [17, p. 334, Ex. 5.2.8]). This necessity affects possible applications to evolution equations via the time-1-map of their solution semiflow. In the invertible case, [13] obtain a C^1-linearization result for semilinear wave equations. Otherwise, while [1, 20, 21] give conditions for at least partial linearization of general maps, it is remarkable that Lu [12] derived a topological linearization result for scalar reaction-diffusion equations under Dirichlet boundary conditions. Later his theory was extended to the nonhyperbolic situation in [3], where the semiflow is conjugated to an infinite-dimensional saddle times an ODE on the finite-dimensional center manifold. Both [12, 3] substantially require an appropriate spectrum of the linearization: The projection of its resolvent set onto the real axis must contain an infinite sequence with sufficiently rapid decay to $-\infty$ and components of lengths being bounded away from 0.

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The paper at hand establishes a generalization of the main result in [12] to nonautonomous abstract evolution equations. They are not assumed to be dissipative or to possess an inertial manifold. Explicitly time-dependent right-hand sides are well-motivated when studying the behavior near non-constant reference solutions, compact invariant sets or under time-varying parameters. When dealing with nonautonomous evolution equations the crucial hyperbolicity concept, although not generically given, is an exponential dichotomy with the Sacker-Sell theory [22] yielding an ambient spectral notion. For the Sacker-Sell (also called dichotomy) spectrum in infinite dimensions we refer to [7, 18, 19]. This allows us to closely follow the strategy from [12]: Under appropriate compactness properties of the linear part the unstable vector bundles (subspaces) become finite-dimensional. Restricting our nonautonomous evolution equation to these sets yields a hierarchy of finite-dimensional ODEs. Here, well-established linearization tools (see [15, 2, 16, 24] in a nonautonomous setting) based on invariant foliations and asymptotic phases apply. They permit to first decouple and afterwards to linearize these finite-dimensional problems. Then an infinite composition of the related topological conjugations applies to the general evolution equation and convergence is guaranteed by our spectral assumptions on the linear part. The resulting central Thm. 5.2 even captures the nonhyperbolic situation and allows a topological linearization except from the flow on the finite-dimensional center manifold. In addition, Cor. 5.3 essentially contains the Hartman-Grobman result from [12] as special case. We illustrate the applicability of our approach to nonautonomous reaction-diffusion and convection equations in Sect. 6. An outlook and appendix (two minor technical tools) close the paper.

Let us finally put this paper into the context of previous work: As pointed out above, [3, 12] study time-invariant evolutionary PDEs. Concerning Hartman-Grobman-like results for autonomous retarded functional differential equations (for short, FDEs) we refer to [26] (establishing a conjugation on the global attractor) and [8] (showing conjugation on the center-unstable manifold). Hence, both contributions are essentially finite-dimensional. Related integral manifold, decoupling and linearization results for nonautonomous differential equations in Banach spaces are due to [2, 5, 6, 16]. They led to a Palmer-Šošitaĭšvili-type theorem (see additionally [15, 17, 24, 25]). On the other hand, a theory of invariant foliations in the field of infinite-dimensional random dynamical systems was developed recently in [11]. We extensively benefit from these preparations and their proofs concerning integral manifolds, foliations and asymptotic phases basically carry over to the present set-up. One finds them summarized in Sect. 3. This enables us to restrict to remarks handling the case of unbounded operators and certain required further statements. The construction of the topological conjugation resembles [12] with certain modifications and additions due to the time-dependent set-up.

Our terminology is as follows: Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space, $L(X)$ the algebra of bounded linear operators on X and id_X be the identity mapping on X. Then $R(T) := TX$ is the range and $N(T) := T^{-1}(0)$ the kernel of a bounded linear operator $T \in L(X)$. The *diameter* of a subset $A \subseteq X$ is defined as $\operatorname{diam} A := \sup_{x,y\in A} \|x - y\|$. On the cartesian product $X^2 = X \times X$ we use the norm $\|(x, y)\| := \max\{\|x\|, \|y\|\}$.

Nonautonomous sets. Any subset $\mathcal{A} \subseteq \mathbb{R}\times X$ is called a *nonautonomous set* (marked by calligraphic letters throughout) and its *t-fibers* are denoted by

$$\mathcal{A}(t) := \{x \in X : (t, x) \in \mathcal{A}\} \quad \text{for all } t \in \mathbb{R}.$$

In case $\mathcal{A}(t) \subseteq X$ is a linear space or a manifold, we speak of a *vector bundle* resp. a *fiber bundle*. If each fiber $\mathcal{A}(t)$, $t \in \mathbb{R}$, of a vector bundle has the same dimension, then $\dim \mathcal{A} := \dim \mathcal{A}(t)$ is called *dimension* of $\mathcal{A}$. The same terminology applies to fiber bundles when the dimension of the manifolds $\mathcal{A}(t)$ does not change. Given invariant vector bundles $\mathcal{A}, \mathcal{B}$ their *Whitney sum* resp. their *cartesian product* is

$$\mathcal{A} \oplus \mathcal{B} := \{(t,x) \in \mathbb{R} \times X : x \in \mathcal{A}(t) \oplus \mathcal{B}(t)\},$$
$$\mathcal{A} \times \mathcal{B} := \{(t,x,y) \in \mathbb{R} \times X \times X : x \in \mathcal{A}(t), y \in \mathcal{B}(t)\}$$

and other operations between nonautonomous sets are defined fiber-wise. A family of nonautonomous sets $\mathcal{A}_p$ is said to form a *foliation* of $\mathbb{R} \times X$ over a set P, if

$$X = \bigcup_{p \in P} \mathcal{A}_p(t), \quad \mathcal{A}_{p_1}(t) \cap \mathcal{A}_{p_2}(t) = \emptyset \quad \text{for all } p_1 \neq p_2,\ p_1, p_2 \in P \text{ and } t \in \mathbb{R};$$

in this case every $\mathcal{A}_p$ is called a *leaf* of the foliation.

Exponentially bounded functions. With a growth rate $\gamma \in \mathbb{R}$ and a fixed $\tau \in \mathbb{R}$, we say a continuous function

- $\phi : [\tau, \infty) \to X$ is γ^+*-bounded*, if $\sup_{t \geq \tau} \|\phi(t)\| e^{\gamma(\tau - t)} < \infty$
- $\phi : (-\infty, \tau] \to X$ is γ^-*-bounded*, if $\sup_{t \leq \tau} \|\phi(t)\| e^{\gamma(\tau - t)} < \infty$
- $\phi : \mathbb{R} \to X$ is γ*-bounded*, if $\sup_{t \in \mathbb{R}} \|\phi(t)\| e^{\gamma(\tau - t)} < \infty$.

The sets $B^+_{\tau,\gamma}$ and $B^-_{\tau,\gamma}$ of γ^+-bounded resp. γ^--bounded functions, as well as the γ-bounded functions B_γ become normed spaces w.r.t. the norms

$$\|\phi\|^+_{\tau,\gamma} := \sup_{\tau \leq t} e^{\gamma(\tau - t)} \|\phi(t)\|, \qquad \|\phi\|^-_{\tau,\gamma} := \sup_{t \leq \tau} e^{\gamma(\tau - t)} \|\phi(t)\|,$$
$$\|\phi\|_{\tau,\gamma} := \sup_{t \in \mathbb{R}} e^{\gamma(\tau - t)} \|\phi(t)\|.$$

These sets form a scale of Banach spaces allowing the continuous embeddings

$$\gamma \leq \delta \quad \Rightarrow \quad B^+_{\tau,\gamma} \hookrightarrow B^+_{\tau,\delta} \text{ and } B^-_{\tau,\delta} \hookrightarrow B^-_{\tau,\gamma}. \tag{1.1}$$

2. **Semilinear evolution equations.** For the sake of a transparent presentation it has advantages to abstractly develop our theory for semilinear evolution equations

$$\boxed{\dot{u} = A(t)u + F(t,u)} \tag{E}$$

on a Banach space X first. In particular, for nonlinearities $F : \mathbb{R} \times X \to X$ fulfilling an ambient global smallness condition we are able to derive largely explicit results.

2.1. **Linear theory.** We begin by stating our assumptions on the linear part, which are collectively denoted by (**L**): For unbounded operators $A(t) : D(A(t)) \subset X \to X$, $t \in \mathbb{R}$, let us suppose that the linear evolution equation

$$\boxed{\dot{u} = A(t)u} \tag{L}$$

in X generates an *evolution family* $U : \{(t,s) \in \mathbb{R} \times \mathbb{R} : s \leq t\} \to L(X)$ (see [14]), i.e. $(t,s) \mapsto U(t,s)x$ is continuous for all $x \in X$ and furthermore fulfills

($\mathbf{L_1}$) $U(t,t) = \mathrm{id}_X$ and $U(t,s)U(s,\tau) = U(t,\tau)$ for all $\tau \leq s \leq t$

($\mathbf{L_2}$) there exist reals $K_0 \geq 1$, $\alpha_0 \in \mathbb{R}$ such that $\|U(t,s)\| \leq K_0 e^{\alpha_0(t-s)}$ for all $s \leq t$.

We say that (L) or the evolution family U admits an *exponential dichotomy* (ED for short), if there exists a projector $P : \mathbb{R} \to L(X)$ and $K \geq 1$, $\alpha > 0$ such that

- $U(t,s)P(s) = P(t)U(t,s)$ for all $s \leq t$

- the restriction $\bar{U}(t,s) := U(t,s)|_{N(P(s))} : N(P(s)) \to N(P(t))$ is a topological isomorphism for every $s < t$
- $\|U(t,s)P(s)\| \le Ke^{-\alpha(t-s)}$ and $\left\|\bar{U}(s,t)\,[\mathrm{id}_X - P(t)]\right\| \le Ke^{\alpha(s-t)}$ for $s \le t$.

The mapping $t \mapsto P(t)$ is denoted as *regular projector* and [14, Lemma 4.2] establishes its strong continuity. For θ-periodic or autonomous eqns. (L) the projectors can be chosen accordingly having the same period $\theta > 0$.

With $\gamma \in \mathbb{R}$ we write $U_\gamma(t,s) := e^{\gamma(s-t)}U(t,s)$ for the associated *scaled evolution family*. If U_γ admits an ED, then the inequalities

$$\|U(t,s)P(s)\| \le Ke^{(\gamma-\alpha)(t-s)}, \quad \left\|\bar{U}(s,t)\,[\mathrm{id}_X - P(t)]\right\| \le Ke^{(\gamma+\alpha)(s-t)} \quad \text{for } s \le t$$

are equivalent to the above dichotomy estimates. On this basis, the *dichotomy spectrum* $\Sigma(A)$ of (L) is defined as the closed set

$$\Sigma(A) = \{\gamma \in \mathbb{R} : U_\gamma \text{ admits no ED on } \mathbb{R}\}.$$

In the following, we assume that the dichotomy spectrum $\Sigma(A)$ of (L) satisfies

($\mathbf{L_3}$) $\Sigma(A) = \bigcup_{n\in\mathbb{N}} [\lambda_n^-, \lambda_n^+]$ with two sequences $(\lambda_n^+)_{n\in\mathbb{N}}, (\lambda_n^-)_{n\in\mathbb{N}}$, such that

$$\alpha_n < \beta_n < \lambda_n^- \le \lambda_n^+ < \alpha_{n-1} \quad \text{for all } n \in \mathbb{N}$$

and strictly decreasing real sequences $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}}$ (cf. Fig. 1). Moreover, the spectral projectors $P_n : \mathbb{R} \to L(X)$ associated to the spectral intervals $[\lambda_n^-, \lambda_n^+]$ are *complete*, i.e.

$$\sum_{n\in\mathbb{N}} P_n(t)x = x \quad \text{for all } x \in X \text{ uniformly in } t \in \mathbb{R}.$$

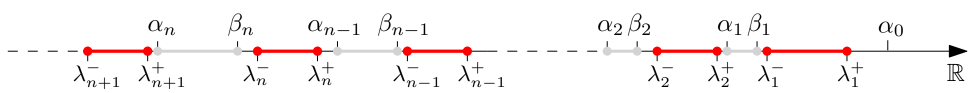

FIGURE 1. Dichotomy spectrum $\Sigma(A)$ of (L) consisting of the spectral intervals $[\lambda_n^-, \lambda_n^+]$ (in red) and the gap intervals $[\alpha_n, \beta_n]$, $n \in \mathbb{N}$

This yields that $\Sigma(A)$ is bounded above and ($\mathbf{L_2}$) implies $\Sigma(A) \subseteq (-\infty, \alpha_0]$. The union $\bigcup_{n\in\mathbb{N}}(\beta_n, \alpha_{n-1})$ is an open cover of $\Sigma(A)$ and every *gap interval* $[\alpha_n, \beta_n]$ is located left of the spectral interval $[\lambda_n^-, \lambda_n^+]$ in the spectral gap $(\lambda_{n+1}^+, \lambda_n^-)$.

For any given $n \in \mathbb{N}$ let us define the *n-stable vector bundle*

$$\mathcal{X}_n^+ := \left\{(\tau,\xi) \in \mathbb{R} \times X : U(\cdot,\tau)\xi \text{ is } \gamma^+\text{-bounded for each } \gamma \in (\lambda_{n+1}^+, \lambda_n^-)\right\},$$

as well as the associated *n-unstable vector bundle*

$$\mathcal{X}_n^- := \left\{ (\tau,\xi) \in \mathbb{R} \times X : \begin{array}{c} (L) \text{ has a } \gamma^-\text{-bounded solution} \\ \phi : (-\infty,\tau] \to X \text{ with } \phi(\tau) = \xi \\ \text{for each } \gamma \in (\lambda_{n+1}^+, \lambda_n^-) \end{array} \right\};$$

we supplement this with the convention $\mathcal{X}_0^+ := \mathbb{R} \times X$. Due to ($\mathbf{L_3}$) there exists a regular invariant projector P_n^+ allowing the representation

$$\begin{aligned} \mathcal{X}_n^+ &= \left\{(t,x) \in \mathbb{R} \times X : x \in R(P_n^+(t))\right\}, \\ \mathcal{X}_n^- &= \left\{(t,x) \in \mathbb{R} \times X : x \in N(P_n^+(t))\right\}. \end{aligned} \tag{2.1}$$

Moreover, there are reals $K_n \ge 1$ such that the crucial estimates

$$\left\|U(t,s)P_n^+(s)\right\| \le K_n e^{\alpha_n(t-s)}, \quad \left\|\bar{U}(s,t)P_n^-(t)\right\| \le K_n e^{\beta_n(s-t)} \quad \text{for } s \le t \tag{2.2}$$

with the complementary projector $P_n^-(t) := \mathrm{id}_X - P_n^+(t)$ hold. In addition, we have to require that U is compactifying, which in turn means

($\mathbf{L_4}$) There exists a *compactification time* $T \geq 0$ such that $U(t,s)B \subseteq X$ is relatively compact for all $t - s > T$ and bounded $B \subseteq X$.

Proposition 2.1. *For all* $n \in \mathbb{N}$ *we have* $\dim \mathcal{X}_n^- < \infty$.

Proof. Let $n \in \mathbb{N}$, $t_0 \in \mathbb{R}$ be arbitrarily fixed and set $S_0 := \{x \in \mathcal{X}_n^-(t_0) : \|x\| \leq 1\}$ for the closed unit ball in $\mathcal{X}_n^-(t_0)$. Since

$$\bar{U}(t,s) : \mathcal{X}_n^-(s) \to \mathcal{X}_n^-(t) \quad \text{for all } s,t \in \mathbb{R} \tag{2.3}$$

is an isomorphism we can define $S(t) := \bar{U}(t,t_0)S_0$ for $t \leq t_0$ and obtain from

$$\operatorname{diam} S(t) = \operatorname{diam} \bar{U}(t,t_0)S(t_0) \overset{(2.2)}{\leq} 2K_n e^{\beta_n(t-t_0)} \quad \text{for all } t \leq t_0$$

that every $S(t)$ is bounded. The compactification condition ($\mathbf{L_4}$) allows us to choose $t > t_0 + T$ so that the closed $S(t_0) = U(t_0,t)\bar{U}(t,t_0)S_0 = U(t_0,t)S(t) \subseteq \mathcal{X}_n^-(t_0)$ is compact; thus, $\dim \mathcal{X}_n^-(t_0) < \infty$. The isomorphism property (2.3) implies that every $\mathcal{X}_n^-(t)$, $t \in \mathbb{R}$, has the same finite dimension and the claim follows. $\square$

To each spectral interval $[\lambda_n^-, \lambda_n^+]$ we associate a *spectral bundle* (see Fig. 2)

$$\mathcal{X}_n := \mathcal{X}_n^- \cap \mathcal{X}_{n-1}^+$$

having the *multiplicity* $\dim \mathcal{X}_n := \dim \mathcal{X}_n(t) =: d_n$.

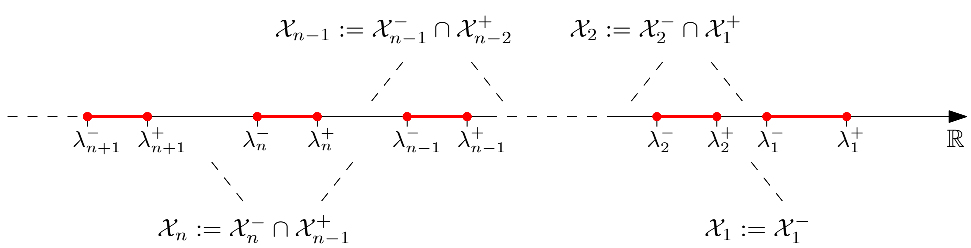

FIGURE 2. Dichotomy spectrum $\Sigma(A)$ of (L) (in red) and the associated d_n-dimensional spectral bundles $\mathcal{X}_n$

By Prop. 2.1 the spectral bundle $\mathcal{X}_n$ is finite-dimensional with the representation

$$\mathcal{X}_n = \{(t,x) \in \mathbb{R} \times X : x \in R(P_n(t))\}$$

for a projector $P_n : \mathbb{R} \to L(X)$ from ($\mathbf{L_3}$). Furthermore, there is a complementary subbundle $\mathcal{X}_n^\perp$ of $\mathbb{R} \times X$ establishing the Whitney sum $\mathbb{R} \times X = \mathcal{X}_n \oplus \mathcal{X}_n^\perp$. Writing

$$\mathcal{X}_n^m := \mathcal{X}_n \oplus \cdots \oplus \mathcal{X}_m \quad \text{for all } n \leq m,$$

we finally obtain for every $n \in \mathbb{N}$ that

$$\mathbb{R} \times X = \mathcal{X}_1^n \oplus \mathcal{X}_n^+, \qquad \mathcal{X}_n^- = \mathcal{X}_1^n, \qquad n \leq \dim \mathcal{X}_n^- = d_1 + \cdots + d_n.$$

Note that [18] provides concrete information on the dichotomy spectrum and the spectral bundles for nonautonomous parabolic PDEs.

2.2. **Nonlinear theory.** The assumptions on the nonlinearity $F : \mathbb{R} \times X \to X$ in (E) will be denoted by (**N**): Let us suppose there exists a $L \geq 0$ such that

($\mathbf{N_1}$) $F(t, 0) \equiv 0$ on $\mathbb{R}$ and $F(\cdot, u) : \mathbb{R} \to X$ is continuous for all $u \in X$

($\mathbf{N_2}$) $\|F(t, u) - F(t, \bar{u})\| \leq L \|u - \bar{u}\|$ for all $t \in \mathbb{R}$, $u, \bar{u} \in X$.

These conditions particularly imply continuity of $F : \mathbb{R} \times X \to X$. Clearly, ($E$) has the trivial solution. Using standard arguments one derives that (E) is well-posed in the following sense (cf., for instance [23, pp. 224ff]): For any pairs $(\tau, u_0) \in \mathbb{R} \times X$ there exists a unique continuous function $u(\cdot; \tau, u_0) : [\tau, \infty) \to X$ satisfying

$$u(t) = U(t, \tau)u_0 + \int_\tau^t U(t, s)F(s, u(s))\,\mathrm{d}s \quad \text{for all } \tau \leq t;$$

one speaks of a *mild solution*. The function $u(\cdot; \tau, u_0)$ defines a continuous 2-parameter semiflow on X, i.e. it fulfills

$$u(\tau; \tau, u_0) = u_0, \quad u(t; s, u(s; \tau, u_0)) = u(t; \tau, u_0) \quad \text{for all } \tau \leq s \leq t,\ u_0 \in X \tag{2.4}$$

and is denoted as *general solution* to (E). An *entire solution* of (E) exists on the real line. A nonautonomous set $\mathcal{A}$ is called *forward invariant* w.r.t. (E) under the inclusion $u(t; \tau, \mathcal{A}(\tau)) \subseteq \mathcal{A}(t)$ for all $\tau \leq t$ and *invariant*, provided equality holds.

Sometimes a further assumption on the nonlinearity is necessary, namely its global boundedness: This means there exists a $C \geq 0$ such that

(**B**) $\|F(t, u)\| \leq C$ for all $t \in \mathbb{R}$, $u \in X$.

3. **Integral manifolds and invariant foliations.** Throughout the section, we tacitly suppose that the assumptions (**L**) and (**N**) are fulfilled, choose reals

$$\sigma_n \in \left(0, \tfrac{\beta_n - \alpha_n}{2}\right], \qquad \gamma \in \Gamma_n := [\alpha_n + \sigma_n, \beta_n - \sigma_n]$$

for some fixed $n \in \mathbb{N}$ and define the constants

$$\ell_n := \frac{K_n L}{\sigma_n - 2K_n L} < \frac{2K_n L}{\sigma_n - 4K_n L} =: \ell_n^\star.$$

3.1. **Integral manifolds.** Our analysis is based on the *Lyapunov-Perron operator*

$$\begin{aligned} T_\tau^-(\phi, x_0) := \bar{U}(\cdot, \tau)P_n^-(\tau)x_0 - &\int_\cdot^\tau \bar{U}(\cdot, s)P_n^-(s)F(s, \phi(s))\,\mathrm{d}s \\ &+ \int_{-\infty}^\cdot U(\cdot, s)P_n^+(s)F(s, \phi(s))\,\mathrm{d}s, \end{aligned} \tag{3.1}$$

which formally defines a function between $(-\infty, \tau]$ and X, depending on $\tau \in \mathbb{R}$, a continuous $\phi : (-\infty, \tau] \to X$ and $x_0 \in X$.

Lemma 3.1 (the operator T_τ^-). *For $\tau \in \mathbb{R}$ the operator $T_\tau^- : B_{\tau,\gamma}^- \times X \to B_{\tau,\gamma}^-$ is well-defined with*

$$\operatorname{lip}_1 T_\tau^- \leq \frac{2K_n L}{\sigma_n}, \qquad \operatorname{lip}_2 T_\tau^- \leq K_n. \tag{3.2}$$

In the following, we conveniently abbreviate $T_\tau^-(t, \phi, x_0) := T_\tau^-(\phi, x_0)(t) \in X$ and proceed similarly with our further notation.

Proof. The well-definedness of T_τ^- will be tackled at the end of the proof. We thus suppose that this property is given for the moment. Then the second inequality in (3.2) is an immediate consequence of the dichotomy estimate (2.2) and $\gamma \in \Gamma_n$. Concerning the first Lipschitz estimate in (3.2) we obtain using ($\mathbf{N_2}$) that

$$\begin{aligned}
&\left\|T_\tau^-(t,\phi,x_0) - T_\tau^-(t,\bar\phi,x_0)\right\| e^{\gamma(\tau-t)}\\
\stackrel{(3.1)}{\le}\ &\left\|\int_t^\tau \bar U(t,s)P_n^-(s)[F(s,\phi(s)) - F(s,\bar\phi(s))]\,\mathrm{d}s\right\| e^{\gamma(\tau-t)}\\
&+\left\|\int_{-\infty}^t U(t,s)P_n^+(s)[F(s,\phi(s)) - F(s,\bar\phi(s))]\,\mathrm{d}s\right\| e^{\gamma(\tau-t)}\\
\stackrel{(2.2)}{\le}\ &K_nL\left(\frac{1}{\gamma-\alpha_n}+\frac{1}{\beta_n-\gamma}\right)\left\|\phi-\bar\phi\right\|_{\tau,\gamma}^- \quad \text{for all } t\le\tau
\end{aligned}$$

and $\phi, \bar\phi \in B_{\tau,\gamma}^-$, $x_0 \in X$. Passing to the least upper bound over $t \le \tau$ allows to infer the first estimate (3.2). In order to finally prove the well-definedness of T_τ^-, we observe that ($\mathbf{N_1}$) and the above estimates guarantee

$$\begin{aligned}
\left\|T_\tau^-(t,\phi,x_0)\right\| e^{\gamma(\tau-t)} &\le \left\|T_\tau^-(\phi,x_0) - T_\tau^-(0,x_0)\right\|_{\tau,\gamma}^- + \left\|T_\tau^-(0,x_0) - T_\tau^-(0,0)\right\|_{\tau,\gamma}^-\\
&\stackrel{(3.2)}{\le} \frac{2K_nL}{\sigma_n}\left\|\phi\right\|_{\tau,\gamma}^- + K_n\left\|x_0\right\| \quad \text{for all } t \le \tau.
\end{aligned}$$

Since $T_\tau^-(\phi,x_0) : (-\infty,\tau] \to X$ is continuous the lemma is established. □

Lemma 3.2 (the fixed point ϕ_τ^-). *Let $L < \frac{\beta_n-\alpha_n}{4K_n}$ and choose $\sigma_n \in \left(2K_nL, \frac{\beta_n-\alpha_n}{2}\right]$. For $\tau \in \mathbb{R}$ the operator $T_\tau^- : B_{\tau,\gamma}^- \times X \to B_{\tau,\gamma}^-$ has a unique fixed point function $\phi_\tau^- : X \to B_{\tau,\gamma}^-$; it satisfies*

$$\phi_\tau^-(0) = 0, \qquad \operatorname{lip}\phi_\tau^- \le \frac{\sigma_n}{\sigma_n - 2K_nL}. \tag{3.3}$$

Proof. In the above Lemma 3.1 we showed that $T_\tau^-(\cdot,x_0)$, $x_0 \in X$, maps the Banach space $B_{\tau,\gamma}^-$ into itself. Moreover, due to $\operatorname{lip}_1 T_\tau^- \le \frac{2K_nL}{\sigma_n} < 1$ (see (3.2)) the mapping T_τ^- is a uniform contraction in the first argument. Thus, the uniform contraction principle (cf., for example [17, p. 352, Thm. B.1.1]) yields a unique fixed point function $\phi_\tau^- : X \to B_{\tau,\gamma}^-$. Due to (3.2) it fulfills a global Lipschitz condition as claimed in (3.3). Finally, the relation $T_\tau^-(0,0) = 0$ implies that $\phi_\tau^-(0) = 0$. □

The following abstraction of the classical Hadamard-Perron theorem assures that both vector bundles $\mathcal{X}_n^-$ and $\mathcal{X}_n^+$ persist under nonlinear perturbations fulfilling ($\mathbf{N}$), provided the Lipschitz constant L is sufficiently small. In other words, for every gap in the dichotomy spectrum $\Sigma(A)$ there exist two integral manifolds intersecting along the trivial solution to (E):

Theorem 3.3 (pseudo-stable and -unstable manifolds). *Let $n \in \mathbb{N}$. If*

$$L < \tfrac{\beta_n-\alpha_n}{6K_n}, \qquad \sigma_n \in \left(3K_nL, \tfrac{\beta_n-\alpha_n}{2}\right], \tag{3.4}$$

then the following holds true for the evolution eqn. (E):

(a) The infinite-dimensional n-stable integral manifold

$$\mathcal{W}_n^+ := \left\{(\tau,u_0) \in \mathbb{R}\times X : u(\cdot;\tau,u_0) \text{ is } \gamma^+\text{-bounded}\right\} \tag{3.5}$$

is independent of $\gamma \in \Gamma_n$ and a forward invariant fiber bundle with

$$\mathcal{W}_n^+ = \left\{(\tau,\zeta + w_n^+(\tau,\zeta)) \in \mathbb{R}\times X : (\tau,\zeta) \in \mathcal{X}_n^+\right\}.$$

(b) The $d_1 + \ldots + d_n$-dimensional n-unstable integral manifold

$$\mathcal{W}_n^- := \left\{ (\tau, u_0) \in \mathbb{R} \times X : \begin{array}{l} (E) \text{ has a } \gamma^-\text{-bounded solution} \\ \phi : (-\infty, \tau] \to X \text{ with } \phi(\tau) = u_0 \end{array} \right\} \tag{3.6}$$

is independent of $\gamma \in \Gamma_n$ and an invariant fiber bundle with

$$\mathcal{W}_n^- = \left\{ (\tau, \xi + w_n^-(\tau, \xi)) \in \mathbb{R} \times X : (\tau, \xi) \in \mathcal{X}_n^- \right\}. \tag{3.7}$$

(c) The continuous functions $w_n^\pm : \mathbb{R} \times X \to X$ satisfy $w_n^\pm(\tau, 0) \equiv 0$, the inclusions $w_n^\pm(\tau, u_1) \in \mathcal{X}_n^\mp(\tau)$ for all $(\tau, u_1) \in \mathbb{R} \times X$ and the Lipschitz estimates

$$\operatorname{lip}_2 w_n^\pm \le \ell_n < 1. \tag{3.8}$$

(d) $\mathcal{W}_n^+ \cap \mathcal{W}_n^- = \mathbb{R} \times \{0\}$.

(e) For θ-periodic evolution eqns. (E) the functions $w_n^\pm$ are θ-periodic in, and for autonomous (E) even independent of the first variable.

Proof. Constructing integral manifolds by the Lyapunov-Perron method is a fairly well-established matter both in an autonomous (cf. [12, 3, 23]), as well as a non-autonomous context (cf. [4, 5, 6, 16, 24]). We thus only give a sketch focussing on preparations for our following considerations and differences in the present situation of an unbounded linear part in (E). Thereto, let $(\tau, x_0) \in \mathbb{R} \times X$ and $\gamma \in \Gamma_n$.

(a) allows a dual proof to the subsequent assertion

(b) The γ^--bounded mild solutions ϕ to (E) satisfying $P_n^-(\tau)\phi(\tau) = P_n^-(\tau)x_0$ can be characterized as fixed points of the operator $T_\tau^-(\cdot, x_0) : B_{\tau,\gamma}^- \to B_{\tau,\gamma}^-$ given in (3.1); this can be shown as in [23, p. 467ff, Proof of Lemma 71.2]. Thanks to Lemma 3.2 there exists a unique fixed point $\phi_\tau^-(x_0) \in B_{\tau,\gamma}^-$. If we define the function w_n^- as $w_n^-(\tau, x_0) := P_n^+(\tau)\phi_\tau^-(\tau, x_0)$, then

$$w_n^-(\tau, \xi) \overset{(3.1)}{=} \int_{-\infty}^{\tau} U(\tau, s) P_n^+(s) F(s, \phi_\tau^-(s, \xi))\, \mathrm{d}s \quad \text{for all } (\tau, \xi) \in \mathcal{X}_n^-. \tag{3.9}$$

(c) It results from Lemma 3.2 that $w_n^- : \mathbb{R} \times X \to X$ has the claimed properties. In particular, the Lipschitz estimates follow with (3.3).

(d) Suppose that $\phi^* \in B_\gamma$ is an entire solution of (E). Because of (**N**) the function $t \mapsto F(t, \phi^*(t))$ is γ-bounded. Hence, the unique γ-bounded mild solution of the linearly inhomogeneous equation

$$\dot{u} = A(t)u + F(t, \phi^*(t)) \tag{3.10}$$

can be characterized by means of the fixed point relation

$$\phi^* = \int_{-\infty}^{\cdot} U(\cdot, s) P_n^+(s) F(s, \phi^*(s))\, \mathrm{d}s - \int_{\cdot}^{\infty} \bar{U}(\cdot, s) P_n^-(s) F(s, \phi^*(s))\, \mathrm{d}s$$

in B_γ (see, e.g. [23, pp. 205–206, Thm. 45.7(3)]). Our assumptions yield that its right-hand side is globally Lipschitz in ϕ on the Banach space B_γ with constant

$$\frac{K_n L}{\gamma - \alpha_n} + \frac{K_n L}{\beta_n - \gamma} \le \frac{2K_n L}{\sigma_n} < 1.$$

Thus, the unique γ-bounded solution to (3.10) must be the trivial one, i.e. (d) holds.

(e) Given $\xi \in \mathcal{X}_n^-(\tau)$ let us consider the entire solution $\phi : \mathbb{R} \to X$ satisfying $\phi(\tau) = \xi + w_n^-(\tau, \xi)$. Thanks to the characterization (3.7) we know $\phi \in B_{\tau,\gamma}^-$. Then also the shifted function $\phi_\theta(t) := \phi(t - \theta)$ is γ^--bounded and, since (E) is θ-periodic, ϕ_θ is furthermore a mild solution to (E). We can conclude

$$w_n^-(\tau + \theta, \xi) \quad = \quad w_n^-(\tau + \theta, P_n^-(\tau)\phi(\tau - \theta + \theta))$$

$$\begin{aligned} &= w_n^-(\tau+\theta, P_n^-(\tau+\theta)\phi_\theta(\tau+\theta)) \\ &= P_n^+(\tau+\theta)\phi_\theta(\tau+\theta) = P_n^+(\tau)\phi(\tau) = w_n^-(\tau,\xi) \quad \text{for all } \tau \in \mathbb{R}, \end{aligned}$$

since the dichotomy projector P_n^- inherits θ-periodicity from (L). If (E) is even autonomous, then the above equation holds for all $\theta > 0$ and therefore w_n^- does not depend on the first variable. This completes the proof of Thm. 3.3. □

Corollary 3.4. *Under the additional assumption* (**B**) *one has the implication*

$$\alpha_n < 0 \quad \Rightarrow \quad \left\|w_n^-(\tau,\xi)\right\| \le \frac{K_n}{|\alpha_n|}C \quad \textit{for all } (\tau,\xi) \in \mathcal{X}_n^-.$$

Proof. Our argument is based on the relation (3.9) yielding

$$\left\|w_n^-(\tau,\xi)\right\| \overset{(2.2)}{\le} K_n C \int_{-\infty}^{\tau} e^{\alpha_n(\tau-s)}\,\mathrm{d}s \quad \text{for all } (\tau,\xi) \in \mathcal{X}_n^-.$$

The improper integral converges for $\alpha_n < 0$ and the claimed estimate follows. □

Besides the pseudo-stable and -unstable vector bundles $\mathcal{X}_n^+, \mathcal{X}_n^-$, also the spectral bundles $\mathcal{X}_n$ of (L) persist under small Lipschitzian perturbations. As demonstrated in the next proof, this follows by a geometric argument.

Theorem 3.5 (pseudo-center manifolds)**.** *Let $n > 1$. If*

$$L < \tfrac{\beta_j-\alpha_j}{6K_j}, \qquad \sigma_j \in \left(3K_jL, \tfrac{\beta_j-\alpha_j}{2}\right] \quad \textit{for all } j \in \{n, n-1\}, \tag{3.11}$$

then the d_n-dimensional n-center integral manifold

$$\mathcal{W}_n := \mathcal{W}_n^- \cap \mathcal{W}_{n-1}^+ = \left\{ (\tau,u_0) \in \mathbb{R}\times X : \begin{array}{l} u(\cdot;\tau,u_0) \textit{ is } \gamma_{n-1}^+\textit{-bounded and} \\ (E) \textit{ has a } \gamma_n^-\textit{-bounded solution} \\ \phi : (-\infty,\tau] \to X \textit{ with } \phi(\tau) = u_0 \end{array} \right\}$$

is independent of $\gamma_j \in \Gamma_j$, $j \in \{n, n-1\}$, and an invariant fiber bundle of the evolution eqn. (E) *allowing the representation*

$$\mathcal{W}_n = \{(\tau, \eta + w_n(\tau,\eta)) \in \mathbb{R}\times X : (\tau,\eta) \in \mathcal{X}_n\}$$

as graph of a continuous function $w_n : \mathbb{R}\times X \to X$ satisfying $w_n(\tau,0) \equiv 0$, the inclusion $w_n(\tau,u_1) \in \mathcal{X}_n^\perp(\tau)$ for all $(\tau,u_1) \in \mathbb{R}\times X$ and

$$\operatorname{lip}_2 w_n \le \frac{2\max_{j\in\{n,n-1\}} \ell_j}{1-\max_{j\in\{n,n-1\}} \ell_j}. \tag{3.12}$$

For θ-periodic evolution eqns. (E) *the function w_n is θ-periodic in, and for autonomous* (E) *even independent of the first variable.*

Remark 3.6 (hierarchies of integral manifolds)*.* The dynamical characterizations (3.5) and (3.6) yield the inclusions (cf. (1.1))

$$\begin{array}{ccccccccc}
\mathbb{R}\times\{0\} \subset \ldots \subset & \mathcal{W}_n^+ & \subset \mathcal{W}_{n-1}^+ & \subset \ldots \subset & \mathcal{W}_1^+ & \subset \mathcal{W}_0^+ := \mathbb{R}\times X \\
& \cup & \cup & & \cup & \cup \\
\ldots & \mathcal{W}_{n+1} & \mathcal{W}_n & \ldots & \mathcal{W}_2 & \mathcal{W}_1 \\
& \cap & \cap & & \cap & \| \\
\mathbb{R}\times X \supset \ldots \supset & \mathcal{W}_{n+1}^- \supset & \mathcal{W}_n^- & \supset \ldots \supset & \mathcal{W}_2^- \supset & \mathcal{W}_1^- \supset \mathcal{W}_0^- := \mathbb{R}\times\{0\},
\end{array} \tag{3.13}$$

where we supplemented Thm. 3.5 with the convention $\mathcal{W}_1 := \mathcal{W}_1^-$.

Proof. Let $\tau \in \mathbb{R}$. Thanks to our assumption (3.11) we obtain from Thm. 3.3 that the integral manifolds $\mathcal{W}^+_{n-1}$ and $\mathcal{W}^-_n$ can be represented as graphs using functions $w^+_{n-1}, w^-_n : \mathbb{R} \times X \to X$. In particular, the Lipschitz estimates (3.8) readily imply the inequalities $\operatorname{lip}_2 w^+_{n-1}, \operatorname{lip}_2 w^-_n \le \max_{j\in\{n,n-1\}} \ell_j < 1$.

(I) Given this, let us define the mapping

$$S : X^2 \times \mathbb{R} \times X \to X^2, \qquad S(x, z; \tau, y) := \begin{pmatrix} w^+_{n-1}(\tau, y + z) \\ w^-_n(\tau, x + y) \end{pmatrix},$$

which allows us to deduce the Lipschitz estimate

$$\begin{aligned} &\|S(x, z; \tau, y) - S(\bar{x}, \bar{z}; \tau, y)\| \\ &= \max\left\{\left\|w^+_{n-1}(\tau, y + z) - w^+_{n-1}(\tau, y + \bar{z})\right\|, \left\|w^-_n(\tau, x + y) - w^-_n(\tau, \bar{x} + y)\right\|\right\} \\ &\le \max_{j\in\{n,n-1\}} \ell_j \left\|\begin{pmatrix} x - \bar{x} \\ z - \bar{z} \end{pmatrix}\right\| \quad \text{for all } x, \bar{x}, y, z, \bar{z} \in X \end{aligned}$$

from (3.8). Therefore, S is a uniform contraction in the first two arguments. With the uniform contraction principle in e.g. [17, p. 352, Thm. B.1.1] we obtain a unique continuous fixed point function $(w^1_n, w^2_n) : \mathbb{R} \times X \to X^2$. From the definition of S the inclusions $w^1_n(\tau, y) \in \mathcal{X}^-_{n-1}(\tau)$ and $w^2_n(\tau, y) \in \mathcal{X}^+_n(\tau)$ follow for all $y \in X$. Using the Lipschitz estimates (3.8) again, given $x, z \in X$, $y, \bar{y} \in X$ one deduces

$$\|S(x, z; \tau, y) - S(x, z; \tau, \bar{y})\| \le \max_{j\in\{n,n-1\}} \ell_j \|y - \bar{y}\|. \tag{3.14}$$

(II) Let us show the representation of $\mathcal{W}_n$ as graph of a function w_n over $\mathcal{X}_n$. Thereto, suppose $u_0 \in X$. With Thm. 3.3(a) one has the inclusion $(\tau, u_0) \in \mathcal{W}^+_{n-1}$ if and only if there exists a $\zeta_0 \in \mathcal{X}^+_{n-1}(\tau)$ such that $u_0 = \zeta_0 + w^+_{n-1}(\tau, \zeta_0)$, which in turn implies $P^+_{n-1}(\tau)u_0 = \zeta_0 + P^+_{n-1}(\tau)w^+_{n-1}(\tau, \zeta_0) = \zeta_0$. This yields the equivalence $(\tau, u_0) \in \mathcal{W}^+_{n-1} \Leftrightarrow u_0 = P^+_{n-1}(\tau)u_0 + w^+_{n-1}(\tau, P^+_{n-1}(\tau)u_0)$ and analogously we deduce from Thm. 3.3(b) that $(\tau, u_0) \in \mathcal{W}^-_n \Leftrightarrow u_0 = P^-_n(\tau)u_0 + w^-_n(\tau, P^-_n(\tau)u_0)$. Now we can represent $u_0 = \zeta + \eta + \xi$ uniquely with components $\zeta \in \mathcal{X}^+_n(\tau)$, $\eta \in \mathcal{X}_n(\tau)$ and $\xi \in \mathcal{X}^-_{n-1}(\tau)$. Hence, thanks to the equivalences

$$\begin{aligned} (\tau, u_0) \in \mathcal{W}_n \Leftrightarrow u_0 &= P^+_{n-1}(\tau)u_0 + w^+_{n-1}(\tau, P^+_{n-1}(\tau)u_0) \text{ and} \\ u_0 &= P^-_n(\tau)u_0 + w^-_n(\tau, P^-_n(\tau)u_0) \\ \Leftrightarrow \xi &= w^+_{n-1}(\tau, \eta + \zeta) \text{ and } \zeta = w^-_n(\tau, \xi + \eta) \Leftrightarrow (\xi, \zeta) = S(\xi, \zeta; \tau, \eta) \end{aligned}$$

the pair $(\xi, \zeta) \in (\mathcal{X}^-_{n-1} \times \mathcal{X}^+_n)(\tau)$ is a fixed point of $S(\cdot; \tau, \eta)$. Referring to step (I) this fixed point is uniquely given by $(w^1_n, w^2_n)(\tau, \eta)$. Consequently, $w_n : \mathbb{R} \times X \to X$, $w_n(\tau, \eta) := (w^1_n + w^2_n)(\tau, \eta)$ is a continuous function with $w_n(\tau, \eta) \in \mathcal{X}^\perp_n(\tau)$. In addition, $(0, 0)$ is the unique fixed point of $S(\cdot; \tau, 0)$ and thus $w_n(\tau, 0) \equiv 0$ on $\mathbb{R}$.

(III) We next establish the claimed Lipschitz condition (3.12): In step (I) it was shown that $\operatorname{lip}_{(1,2)} S < 1$ and (3.14) means that S also fulfills a Lipschitz estimate in the parameter y. Accordingly, [17, p. 352, Thm. B.1.1(b)] implies that

$$\operatorname{lip}_2 \begin{pmatrix} w^1_n \\ w^2_n \end{pmatrix} \le \frac{\operatorname{lip}_4 S}{1 - \operatorname{lip}_{(1,2)} S} \le \frac{\max_{j\in\{n,n-1\}} \ell_j}{1 - \max_{j\in\{n,n-1\}} \ell_j}.$$

Thus, the relation $w_n = w^1_n + w^2_n$ leads to the desired estimate (3.12).

(IV) To complete the proof it remains to justify two assertions: First, as intersection of forward invariant sets, $\mathcal{W}_n = \mathcal{W}^+_{n-1} \cap \mathcal{W}^-_n$ itself is forward invariant and as a finite-dimensional set even invariant. Second, from Thm. 3.3(d) we deduce that

$S(x,z;\tau,y) = S(x,z;\tau+\theta,y)$ for $x,y,z \in X$. Hence, the respective unique fixed points $(w_n^1,w_n^2)(\tau,y)$, $(w_n^1,w_n^2)(\tau+\theta,y)$ coincide yielding a θ-periodic $w_n(\cdot,y)$. $\square$

Since the pseudo-unstable integral manifolds $\mathcal{W}_n^-$ are invariant, the 2-parameter semiflow of (E) restricted to each $\mathcal{W}_n^-$ fulfills the semilinear $d_1+\ldots+d_n$-dimensional nonautonomous ODEs

$$\boxed{\dot{x} = A_n^-(t)x + F_n^-(t,x)} \tag{E_n^-}$$

in the vector bundles $\mathcal{X}_n^-$; we have abbreviated

$$A_n^-(t) := A(t)P_n^-(t), \qquad F_n^-(t,x) := P_n^-(t)F(t,x+w_n^-(t,x))$$

and write x_n^- for the general solution of (E_n^-). Note that every solution $\phi_n : \mathbb{R} \to X$ to (E_n^-) yields an entire solution $u : \mathbb{R} \to X$ for (E) via $u(t) := \phi_n(t) + w_n^-(t,\phi_n(t))$.

For an insight into the dynamics of (E) on the finite-dimensional integral manifold $\mathcal{W}_n^-$ we next perform a similar analysis as above for every ODE (E_n^-). Keeping

$$\mathcal{X}_n^- = \mathcal{X}_1^k \oplus \mathcal{X}_{k+1}^n \quad \text{for all } 1 \le k < n$$

in mind, a counterpart to Thm. 3.3 reads as

Proposition 3.7 (reduced pseudo-stable and -unstable manifolds)**.** *Let $1 \le k < n$. If*

$$L < \tfrac{\beta_k-\alpha_k}{12K_k}, \qquad \sigma_k \in \left(6K_kL, \tfrac{\beta_k-\alpha_k}{2}\right]$$

and (3.4) hold, then the ODE (E_n^-) fulfills:

(a) The $d_{k+1}+\ldots+d_n$-dimensional (n,k)-stable integral manifold

$$\mathcal{W}_{n,k}^+ := \left\{(\tau,x_0) \in \mathcal{X}_n^- : x_n^-(\cdot;\tau,x_0) \text{ is } \gamma^+\text{-bounded}\right\}$$

is independent of $\gamma \in \Gamma_k$ and an invariant fiber bundle with

$$\mathcal{W}_{n,k}^+ = \left\{(\tau,\zeta + w_{n,k}^+(\tau,\zeta)) \in \mathcal{X}_n^- : (\tau,\zeta) \in \mathcal{X}_{k+1}^n\right\} \tag{3.15}$$

and the inclusion $w_{n,k}^+(\tau,x_1) \in \mathcal{X}_1^k(\tau)$ for all $(\tau,x_1) \in \mathcal{X}_n^-$.

(b) The $d_1+\ldots+d_k$-dimensional (n,k)-unstable integral manifold

$$\mathcal{W}_{n,k}^- := \left\{(\tau,x_0) \in \mathcal{X}_n^- : x_n^-(\cdot;\tau,x_0) \text{ is } \gamma^-\text{-bounded}\right\}$$

is independent of $\gamma \in \Gamma_k$ and an invariant fiber bundle with

$$\mathcal{W}_{n,k}^- = \left\{(\tau,\xi + w_{n,k}^-(\tau,\xi)) \in \mathcal{X}_n^- : (\tau,\xi) \in \mathcal{X}_1^k\right\}$$

and the inclusion $w_{n,k}^-(\tau,x_1) \in \mathcal{X}_{k+1}^n(\tau)$ for all $(\tau,x_1) \in \mathcal{X}_n^-$.

(c) The continuous functions $w_{n,k}^\pm : \mathcal{X}_n^\pm \to X$ satisfy $w_{n,k}^\pm(\tau,0) \equiv 0$ and

$$\operatorname{lip}_2 w_{n,k}^\pm \le \ell_k^\star < 1. \tag{3.16}$$

(d) $\mathcal{W}_{n,k}^+ \cap \mathcal{W}_{n,k}^- = \mathbb{R} \times \{0\}$.

(e) For θ-periodic evolution eqns. (E) the functions $w_{n,k}^\pm$ are θ-periodic in, and for autonomous (E) even independent of the first variable.

Proof. The assumption (3.4) ensures that $\mathcal{W}_n^-$ can be represented as graph over a function w_n^- as in Thm. 3.3(b). Since (E_n^-) is an ODE in the vector bundle $\mathcal{X}_n^-$, it offers itself to work with an adapted norm, namely the family

$$\|x\|_t := \left\|P_n^-(t)x\right\| \quad \text{for all } (t,x) \in \mathcal{X}_n^-. \tag{3.17}$$

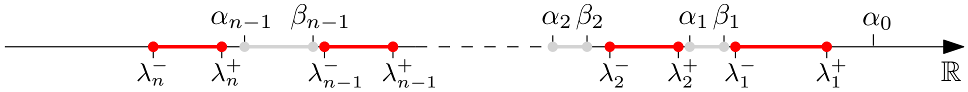

FIGURE 3. Bounded dichotomy spectrum $\Sigma(A_n^-)$ of $\dot{x} = A_n^-(t)x$ (in red) and the gap intervals $[\alpha_k, \beta_k]$, $1 \le k < n$

Then $\|\cdot\|_t$ is a norm on the fiber $\mathcal{X}_n^-(t)$ being equivalent to $\|\cdot\|$ uniformly in $t \in \mathbb{R}$. This allows us to show that (E_n^-) fulfills almost the same assumptions as (E):

ad (**L**): The dichotomy estimates (2.2) remain unchanged when using the operator norm induced by $\|\cdot\|_t$. In addition, the spectrum $\Sigma(A_n^-)$ of the linear part in (E_n^-) consists of the first n spectral intervals of $\Sigma(A)$ and is illustrated in Fig. 3.

ad (**N**): For all $t \in \mathbb{R}$ and $x, \bar{x} \in \mathcal{X}_n^-(t)$ one obtains the Lipschitz estimate

$$\begin{aligned} \left\|F_n^-(t,x) - F_n^-(t,\bar{x})\right\|_t &\overset{(3.17)}{=} \left\|F(t, x + w_n^-(t,x)) - F(t, \bar{x} + w_n^-(t,\bar{x}))\right\| \\ &\le L\left(\|x - \bar{x}\| + \left\|w_n^-(t,x) - w_n^-(t,\bar{x})\right\|\right) \\ &\overset{(3.8)}{\le} 2L\left\|P_n^-(t)(x - \bar{x})\right\| = 2L\left\|x - \bar{x}\right\|_t. \end{aligned}$$

Consequently, the remaining assertions can be deduced as in the proof of Thm. 3.3 with the constant L replaced by $2L$. Here, the estimates (3.8) are to be understood w.r.t. $\|\cdot\|_\tau$. However, since both functions $w_{n,k}^\pm$, as well as their second arguments have values in $\mathcal{X}_n^-(\tau)$, the estimates remain unchanged when using the norm $\|\cdot\|$. □

Corollary 3.8. *Under the additional assumption* (**B**) *one has the implication*

$$\alpha_k < 0 \quad \Rightarrow \quad \left\|w_{n,k}^-(\tau,\xi)\right\| \le \frac{K_k}{|\alpha_k|}C \quad \text{for all } (\tau,\xi) \in \mathcal{X}_1^k. \tag{3.18}$$

Proof. The proof of Prop. 3.7 is based on the adapted norms $\|\cdot\|_t$, $t \in \mathbb{R}$, defined in (3.17). Then using the estimate

$$\left\|F_n^-(t,x)\right\|_t = \left\|F(t, x + w_n^-(t,x))\right\| \le C \quad \text{for all } (t,x) \in \mathcal{X}_n^-,$$

the claim follows as in Cor. 3.4. □

Proposition 3.9 (reduced pseudo-center manifolds)**.** *Let* $1 < k < n$. *If*

$$L < \tfrac{\beta_j - \alpha_j}{12K_j}, \qquad \sigma_j \in \left(6K_jL, \tfrac{\beta_j - \alpha_j}{2}\right] \quad \text{for all } j \in \{k, k-1\} \tag{3.19}$$

and (3.4) *hold, then the* d_k*-dimensional* (n,k)-center integral manifold

$$\mathcal{W}_{n,k} := \mathcal{W}_{n,k}^- \cap \mathcal{W}_{n,k-1}^+ = \left\{(\tau, x_0) \in \mathcal{X}_n^- : x_n^-(\cdot;\tau,x_0) \text{ is } \gamma_{k-1}^+\text{- and } \gamma_k^-\text{-bounded}\right\}$$

does not depend on $\gamma_j \in \Gamma_j$, $j \in \{k, k-1\}$, *and is an invariant fiber bundle of the ODE* (E_n^-) *allowing the representation*

$$\mathcal{W}_{n,k} = \left\{(\tau, \eta + w_{n,k}(\tau,\eta)) \in \mathcal{X}_n^- : (\tau,\eta) \in \mathcal{X}_k\right\}$$

as graph of a continuous function $w_{n,k} : \mathcal{X}_n^- \to X$ *satisfying* $w_{n,k}(\tau, 0) \equiv 0$, *the inclusion* $w_{n,k}(\tau, x_1) \in (\mathcal{X}_k \oplus \mathcal{X}_{k+1}^n)(\tau)$ *for all* $(\tau, x_1) \in \mathcal{X}_n^-$ *and*

$$\operatorname{lip}_2 w_{n,k} \le \frac{2\max_{j\in\{k,k-1\}} \ell_j^\star}{1 - \max_{j\in\{k,k-1\}} \ell_j^\star}.$$

For θ*-periodic evolution eqns.* (E) *the function* $w_{n,k}$ *is* θ*-periodic in, and for autonomous* (E) *even independent of the first variable.*

Remark 3.10 (hierarchies of reduced integral manifolds). In analogy to Rem. 3.6 the reduced integral manifolds of (E_n^-) fulfill the inclusions

$$\begin{array}{rcccccl}
\mathbb{R}\times\{0\} =: \mathcal{W}^+_{n,n} \subset & \mathcal{W}^+_{n,n-1} & \subset \dots \subset & \mathcal{W}^+_{n,1} & \subset & \mathcal{W}^+_{n,0} & := \ \mathcal{X}^-_n \\
 & || & & \cup & & \cup & \\
 & \mathcal{W}_{n,n} & \dots & \mathcal{W}_{n,2} & & \mathcal{W}_{n,1} & \\
 & \cap & & \cap & & || & \\
\mathcal{X}^-_n \ =: & \mathcal{W}^-_{n,n} & \supset \dots \supset & \mathcal{W}^-_{n,2} & \supset & \mathcal{W}^-_{n,1} & \supset \mathcal{W}^-_{n,0} := \mathbb{R}\times\{0\}
\end{array}$$

Proof. First of all, the assumptions (3.4) ensure that $\mathcal{W}_n^-$ is graph of a function w_n^- with $\operatorname{lip}_2 w_n^- < 1$ (cf. Thm. 3.3(b)). Moreover, due to (3.19) the integral manifolds $\mathcal{W}^+_{n,k-1}$, $\mathcal{W}^-_{n,k}$ from Prop. 3.7 can be characterized by continuous functions $w^+_{n,k-1}$, $w^-_{n,k}$ with Lipschitz constant < 1. Under these conditions the proof of Thm. 3.5 directly applies to (E_n^-) having a nonlinearity fulfilling $\operatorname{lip}_2 F_n^- \leq 2L$ w.r.t. the adapted norms (3.17). This yields the assertions. □

Our following result aims to describe the dynamics of (E) restricted to the pseudo-center manifolds $\mathcal{W}_k$. It is determined by the d_k-dimensional ODE

$$\boxed{\dot{y} = A_k(t)y + F_k(t, y)} \tag{E_k}$$

in the spectral bundles $\mathcal{X}_k$, where we have abbreviated

$$A_k(t) := A(t)P_k(t), \qquad\qquad F_k(t) := P_k(t)F(t, y + w_k(t, y)).$$

If y_k denotes the general solution to (E_k), then $\hat{u} : \mathbb{R} \to X$,

$$\hat{u}(t) := y_k(t; \tau, \eta) + w_k(t, y_k(t; \tau, \eta)) \tag{3.20}$$

defines an entire solution of (E) in $\mathcal{W}_k$ (by the invariance properties from Thm. 3.5).

Corollary 3.11. *For $1 \leq k \leq n$ the following holds:*

(a) The general solutions x_k^- to (E_k^-) and $x^-_{n,k}$ to

$$\dot{x} = A_k^-(t)x + P_k^-(t)F\big(t, x + w^-_{n,k}(t, x) + w_n^-(t, x + w^-_{n,k}(t, x))\big) \tag{$E^-_{n,k}$}$$

coincide and for all $(\tau, \xi) \in \mathcal{X}_k^-$ we have

$$\xi + w_k^-(\tau, \xi) = \xi + w^-_{n,k}(\tau, \xi) + w_n^-(\tau, \xi + w^-_{n,k}(\tau, \xi)). \tag{3.21}$$

(b) The general solutions y_k to (E_k) and $y_{n,k}$ to

$$\dot{y} = A_k(t)y + P_k(t)F\big(t, y + w_{n,k}(t, y) + w_n^-(t, y + w_{n,k}(t, y))\big) \tag{$E_{n,k}$}$$

coincide and for all $(\tau, \eta) \in \mathcal{X}_k$ we have

$$\eta + w_k(\tau, \eta) = \eta + w_{n,k}(\tau, \eta) + w_n^-(\tau, \eta + w_{n,k}(\tau, \eta)). \tag{3.22}$$

Proof. Since assertion (a) can be shown analogously, we provide only a proof of (b). Given a pair $(\tau, \eta) \in \mathcal{X}_k$, the function $\hat{u} : \mathbb{R} \to X$ from (3.20) defines an entire solution to (E) in $\mathcal{W}_k$ and hence also in $\mathcal{W}_k^-$ (cf. Thm. 3.5), i.e.

$$\hat{u}(t) = y_k(t; \tau, \eta) + w_k^-(t, y_k(t; \tau, \eta)) \quad \text{for all } t \in \mathbb{R}.$$

Thanks to the hierarchy (3.13) this implies $\hat{u}(\tau) \in \mathcal{W}_k^-(\tau) \subseteq \mathcal{W}_n^-(\tau)$ and consequently also the representation

$$\hat{u}(\tau) = \xi + w_k^-(\tau, \xi) \quad \text{for some } (\tau, \xi) \in \mathcal{X}_k^-$$

holds. With the general solution x_n^- to (E_n^-) we then obtain that $u : \mathbb{R} \to X$,

$$u(t) := x_n^-(t; \tau, \xi) + w_n^-(t, x_n^-(t; \tau, \xi)) \quad \text{for all } t \in \mathbb{R}$$

is an entire solution of (E) in $\mathcal{W}_n^-$. Accordingly, the uniqueness of (entire) solutions in $\mathcal{W}_n^-$ implies $\hat{u} = u$. Our assumption (3.4) ensures $\operatorname{lip}_2 w_n^- < 1$ and therefore

$$\left\|x_n^-(t;\tau,\xi)\right\| \stackrel{(3.8)}{\leq} \tfrac{1}{1-\ell_n}\left\|u(t)\right\| \quad \text{for all } t \in \mathbb{R}.$$

Due to $u(\tau) \in \mathcal{W}_k$ this allows us to conclude that $x_n^-(\cdot;\tau,\xi)$ is γ_k^-- and γ_{k-1}^+-bounded from the corresponding properties of u. Hence, $(\tau,\xi) \in \mathcal{W}_{n,k}$ by Prop. 3.9 and one has the representation $\xi = \eta_k + w_{n,k}(\tau,\eta_k)$ for some $(\tau,\eta_k) \in \mathcal{X}_k$. The invariance of $\mathcal{W}_{n,k}$ yields $x_n(t;\tau,\xi) = y_{n,k}(t;\tau,\eta_k) + w_{n,k}(t, y_{n,k}(t;\tau,\eta_k))$ for all $t \in \mathbb{R}$ and thus

$$\begin{aligned} &y_k(t;\tau,\eta_k) + w_k(t, y_k(t;\tau,\eta_k)) = u(t) = x_n^-(t;\tau,\xi) + w_n^-(t, x_n^-(t;\tau,\xi)) \\ &= y_{n,k}(t;\tau,\eta_k) + w_{n,k}(t, y_{n,k}(t;\tau,\eta_k)) + w_n^-\big(t, y_{n,k}(t;\tau,\eta_k) + w_{n,k}(t, y_{n,k}(t;\tau,\eta_k))\big). \end{aligned}$$

Setting $t = \tau$ yields the claimed identity (3.22). The converse direction can be established using similar arguments. □

3.2. **Invariant foliations and asymptotic phase.**

In the previous Subsect. 3.1 we provided a geometrical description of the solution entities to (E) and (E_n^-) having a particular exponential growth behavior in relation to the trivial solution.

Rather than working with the zero solution, our present goal is to characterize solutions to (E) whose distance to an arbitrary forward solution allows a specific exponential estimate. Thus, given the solution $u(\cdot;\tau,u_0) : [\tau,\infty) \to X$ for initial pairs $(\tau,u_0) \in \mathbb{R} \times X$, let us investigate the evolution equation

$$\boxed{\dot{u} = A(t)u + G(t,u;\tau,u_0)}$$

with the nonlinearity $G : \{(t,u;\tau,u_0) \in \mathbb{R} \times X \times \mathbb{R} \times X : \tau \leq t\} \to X$,

$$G(t,u;\tau,u_0) := F(t, u + u(t;\tau,u_0)) - F(t, u(t;\tau,u_0)), \tag{3.23}$$

and (τ,u_0) understood as a parameter. The *Lyapunov-Perron operator*

$$\begin{aligned} S_\tau^+(\psi,y_0,u_0) := U(\cdot,\tau)[y_0 - P_n^+(\tau)u_0] + \int_\tau^\cdot U(\cdot,s)P_n^+(s)G(s,\psi(s);\tau,u_0)\,\mathrm{d}s \\ - \int_\cdot^\infty \bar{U}(\cdot,s)P_n^-(s)G(s,\psi(s);\tau,u_0)\,\mathrm{d}s \end{aligned} \tag{3.24}$$

formally introduces a continuous function between $[\tau,\infty)$ and X, depending on the pair $(\tau,y_0) \in \mathcal{X}_n^+$, a continuous $\psi : [\tau,\infty) \to X$ and $u_0 \in X$.

The analysis of the operators T_τ^- from (3.1) and S_τ^+ is largely dual. Therefore, our present approach complements the one from Subsect. 3.1. We actually focus on the dynamical meaning of $S_\tau^+ : B_{\tau,\gamma}^+ \times X^2 \to B_{\tau,\gamma}^+$:

Lemma 3.12. *Given $(\tau,y_0) \in \mathcal{X}_n^+$, $u_0 \in X$ the following statements are equivalent:*

(a) There exists a $u_1 \in X$ such that $\psi := u(\cdot;\tau,u_1) - u(\cdot;\tau,u_0) \in B_{\tau,\gamma}^+$ satisfies

$$P_n^+(\tau)\psi(\tau) = y_0 - P_n^+(\tau)u_0. \tag{3.25}$$

(b) $\psi \in B_{\tau,\gamma}^+$ fulfills $\psi = S_\tau^+(\psi,y_0,u_0)$.

Proof. Let $(\tau,y_0) \in \mathcal{X}_n^+$ and $u_0 \in X$.

(a) ⇒ (b) Assume there exists a $u_1 \in X$ so that $\psi := u(\cdot;\tau,u_1) - u(\cdot;\tau,u_0) \in B_{\tau,\gamma}^+$ and (3.25) hold. Then ψ is a γ^+-bounded solution to the inhomogeneous equation

$$\dot{u} = A(t)u + G(t,\psi(t);\tau,u_0)$$

and [23, pp. 205–207, Thm. 45.7(3)] yields the fixed point relation $\psi = S_\tau^+(\psi;\tau,u_0)$.

(*b*) $\Rightarrow$ (*a*) Suppose that $\psi \in B^+_{\tau,\gamma}$ is a fixed point of the operator $S^+_\tau(\cdot; y_0, u_0)$. With $u_1 := P^-_n(\tau)[u_0 + \psi(\tau)] + y_0$ and $\nu := \psi + u(\cdot; \tau, u_0)$ one obtains

$$\begin{aligned}\nu(\tau) &\overset{(2.4)}{=} \psi(\tau) + u_0 = P^-_n(\tau)\psi(\tau) + P^+_n(\tau)S^+_\tau(\tau, \psi; y_0, u_0) + u_0\\ &\overset{(3.24)}{=} P^-_n(\tau)\psi(\tau) + y_0 - P^+_n(\tau)u_0 + u_0 = P^-_n(\tau)[\psi(\tau) + u_0] + y_0 = u_1\end{aligned}$$

and moreover ν solves (E). Since forward solutions to (E) are unique, we conclude $\nu = u(\cdot; \tau, u_1)$, that is, $\psi = u(\cdot; \tau, u_1) - u(\cdot; \tau, u_0)$. Finally, it results

$$P^+_n(\tau)\psi(\tau) = P^+_n(\tau)[u_1 - u_0] = P^+_n(\tau)[y_0 - u_0] = y_0 - P^+_n(\tau)u_0$$

and the proof is finished. □

Proposition 3.13 (pseudo-stable leafs)**.** *Let $n \in \mathbb{N}$ and suppose that*

$$L < \tfrac{\beta_n - \alpha_n}{2K_n(K_n+2)}, \qquad \sigma_n \in \left(K_n(K_n+2)L, \tfrac{\beta_n - \alpha_n}{2}\right] \tag{3.26}$$

*is fulfilled. For every $u_0 \in X$ the infinite-dimensional n-*stable leaf

$$\mathcal{V}^+_n(u_0) := \left\{(\tau, u_1) \in \mathbb{R} \times X : \ u(\cdot; \tau, u_1) - u(\cdot; \tau, u_0) \text{ is } \gamma^+\text{-bounded}\right\}$$

does not depend on $\gamma \in \Gamma_n$ and is a forward invariant fiber bundle of (E) fulfilling:

(*a*) *It allows the representation*

$$\mathcal{V}^+_n(u_0) = \left\{(\tau, \zeta + v^+_n(\tau, \zeta; u_0)) \in \mathbb{R} \times X : \ (\tau, \zeta) \in \mathcal{X}^+_n\right\} \tag{3.27}$$

as graph of a continuous function $v^+_n : \mathbb{R} \times X^2 \to X$ satisfying the inclusion $v^+_n(\tau, u_1, u_0) \in \mathcal{X}^-_n(\tau)$ for all $(\tau, u_1) \in \mathbb{R} \times X$.

(*b*) *$v^+_n(\tau, u_1, u_0) = P^-_n(\tau)u_0 + V^+_n(\tau, u_1, u_0)$ for all $(\tau, u_1) \in \mathbb{R} \times X$ with a continuous function $V^+_n : \mathbb{R} \times X^2 \to X$ satisfying $V^+_n(\tau, u_1, u_0) \in \mathcal{X}^-_n(\tau)$.*

(*c*) *One has the Lipschitz estimates*

$$\operatorname{lip}_2 v^+_n \le K_n \ell_n < 1, \qquad \operatorname{lip}_2 V^+_n \le K_n \ell_n < 1. \tag{3.28}$$

Because (E) has the trivial solution it is clear that every $\mathcal{V}^+_n(0)$ defines a pseudo-stable integral manifold, i.e. $\mathcal{V}^+_n(0) = \mathcal{W}^+_n$.

Proof. Let $(\tau, u_0) \in \mathbb{R} \times X$ and choose $\gamma \in \Gamma_n$. Above all, the nonlinearity G given in (3.23) satisfies $G(t, 0; \tau, u_0) \equiv 0$ on $[\tau, \infty)$, inherits the Lipschitz condition $\operatorname{lip}_2 G \le L$ from ($\mathbf{N_2}$) and is finally continuous, since u has this property.

(I) As in the proof of Lemma 3.1 one shows that $S^+_\tau : B^+_{\tau,\gamma} \times \mathcal{X}^+_n(\tau) \times X \to B^+_{\tau,\gamma}$ is well-defined with

$$\operatorname{lip}_1 S^+_\tau \le \frac{2K_n L}{\sigma_n}, \qquad \operatorname{lip}_2 S^+_\tau \le K_n.$$

The condition (3.26) yields that $\frac{2K_n L}{\sigma_n} < 1$ and thus S^+_τ is a uniform contraction in the first argument. Again the uniform contraction principle guarantees a unique and continuous fixed point function $\psi^+_\tau : \mathcal{X}^+_n(\tau) \times X \to B^+_{\tau,\gamma}$ which satisfies

$$\operatorname{lip}_1 \psi^+_\tau \le \frac{\sigma_n}{\sigma_n - 2K_n L}.$$

(II) Let us show that the nonautonomous set $\mathcal{V}^+_n(u_0)$ is forward invariant. For $t \ge \tau$ choose $\hat{u}_0 \in u(t; \tau, \mathcal{V}^+_n(\tau, u_0))$. According to Lemma 3.12 this means there exists a $u_1 \in X$ such that $\hat{u}_0 = u(t; \tau, u_1)$ and $u(\cdot; \tau, u_1) - u(\cdot; \tau, u_0) \in B^+_{\tau,\gamma}$, hence

$$\begin{aligned}u(\cdot; t, \hat{u}_0) - u(\cdot; t, u(t; \tau, u_0)) &= u(\cdot; t, u(t; \tau, u_1)) - u(\cdot; t, u(t; \tau, u_0))\\ &\overset{(2.4)}{=} u(\cdot; \tau, u_1) - u(\cdot; \tau, u_0).\end{aligned}$$

Consequently, $\hat{u}_0 \in \mathcal{V}_n^+(t, u(t;\tau,u_0))$ and we verify the further assertions:

(a) Given the fixed point function ψ_τ^+ from (I) we define

$$v_n^+(\tau, y_0; u_0) := P_n^-(\tau)\left(u_0 + \psi_\tau^+(\tau, P_n^+(\tau)y_0, u_0)\right)$$

and obtain a continuous function $v_n^+ : \mathbb{R} \times X^2 \to X$.

(b) Also the function $V_n^+ : \mathbb{R} \times X^2 \to X$ is continuous, if we set

$$V_n^+(\tau, y_0, u_0) := P_n^-(\tau)\psi_n^+(\tau, P_n^+(\tau)y_0, u_0). \tag{3.29}$$

(c) Finally, the definition of v_n^+ implies the relation

$$v_n^+(\tau, y_0, u_0) \overset{(3.24)}{=} P_n^-(\tau)u_0 - \int_\tau^\infty \bar{U}(\tau,s)P_n^-(s)G(s, \psi_\tau^+(s, P_n^+(\tau)y_0, u_0); \tau, u_0)\,\mathrm{d}s,$$

from which the Lipschitz estimates (3.28) follow. □

Corollary 3.14. *Under the additional assumption* (**B**) *one has the implication*

$$\beta_n < 0 \quad \Rightarrow \quad \left\|V_n^+(\tau,\zeta,u_0)\right\| \le \frac{2K_n}{|\beta_n|}C \quad \text{for all } (\tau,\zeta) \in \mathcal{X}_n^+,\ u_0 \in X.$$

Proof. The function $V_n^+ : \mathbb{R} \times X^2 \to X$ defined in (3.29) allows the representation

$$V_n^+(\tau,\zeta,u_0) \overset{(3.24)}{=} -\int_\tau^\infty \bar{U}(\tau,s)P_n^-(s)G(s,\psi_n^+(s,P_n^+(\tau)\zeta,u_0);\tau,u_0)\,\mathrm{d}s$$

and the claim follows from (2.2) due to $\|G(t,u;\tau,u_0)\| \le 2C$. □

Pseudo-stable leafs allow to establish the following geometric property of pseudo-unstable integral manifolds:

Theorem 3.15 (asymptotic forward phase)**.** *Let $n \in \mathbb{N}$. If* (3.26) *holds, then the n-unstable integral manifold $\mathcal{W}_n^-$ has an* asymptotic forward phase*, i.e. there exists a continuous function $\pi_n^+ : \mathbb{R} \times X \to X$ and a bounded $C_n : \mathbb{R} \times [0,\infty) \to [0,\infty)$ such that the following holds for every $(\tau,u_0) \in \mathbb{R} \times X$:*

$$\left\|u(t;\tau,u_0) - u(t;\tau,\pi_n^+(\tau,u_0))\right\| \le C_n(\tau,\|u_0\|)e^{\gamma(t-\tau)} \quad \text{for all } \tau \le t \tag{3.30}$$

and $\gamma \in \Gamma_n$. Geometrically, $\pi_n^+(\tau,u_0)$ is given as unique intersection

$$\mathcal{W}_n^- \cap \mathcal{V}_n^+(u_0) = \left\{(t,\pi_n^+(t,u_0)) : t \in \mathbb{R}\right\} \quad \text{for all } u_0 \in X \tag{3.31}$$

and each $\pi_n^+(\tau,\cdot) : X \to \mathcal{W}_n^-(\tau)$ is a retraction onto $\mathcal{W}_n^-(\tau)$. In particular, there exists a unique continuous function $\xi_n^+ : \mathbb{R} \times X \to X$ with $\xi_n^+(\tau,u_0) \in \mathcal{X}_n^-(\tau)$ and

$$\pi_n^+(\tau,u_0) = \xi_n^+(\tau,u_0) + w_n^-(\tau,\xi_n^+(\tau,u_0)).$$

The mapping $\pi_n^+ : \mathbb{R} \times X \to X$ is linearly bounded, i.e.

$$\left\|\pi_n^+(\tau,u_0)\right\| \le K_n \tfrac{1+\ell_n^\star}{1-\ell_n^\star}\|u_0\| \quad \text{for all } (\tau,u_0) \in \mathbb{R} \times X. \tag{3.32}$$

For θ-periodic evolution eqns. (E) *the functions π_n^+ and ξ_n^+ are θ-periodic in, and for autonomous* (E) *even independent of the first variable.*

The boundedness of the real-valued function C_n means that for every $R > 0$ one has $\sup_{(t,x)\in\mathbb{R}\times[0,R)} C_n(t,x) < \infty$.

Proof. Let $(\tau, u_0) \in \mathbb{R} \times X$. Our assumption (3.26) implies (3.4). Hence, Thm. 3.3 yields the properties of $\mathcal{W}_n^-$ and Prop. 3.13 contains the necessary facts on $\mathcal{V}_n^+(u_0)$. We first show that the intersection (3.31) is a singleton. Thereto, every element $\hat{u} \in \mathcal{V}_n^+(\tau, u_0) \cap \mathcal{W}_n^-(\tau)$ allows the representation

$$\hat{u} = \zeta + v_n^+(\tau, \zeta, u_0), \qquad \hat{u} = \xi + w_n^-(\tau, \xi)$$

with $\zeta \in \mathcal{X}_n^+(\tau)$, $\xi \in \mathcal{X}_n^-(\tau)$. This is equivalent to

$$\xi = v_n^+(\tau, \zeta; u_0), \qquad \zeta = w_n^-(\tau, \xi)$$

and consequently $\binom{\xi}{\zeta}$ is a fixed point of the mapping $S : X^2 \times \mathbb{R} \times X \to X$,

$$S(x, z; \tau, u_0) := \begin{pmatrix} v_n^+(\tau, z; u_0) \\ w_n^-(\tau, x) \end{pmatrix}.$$

Since both $\operatorname{lip}_2 v_n^+ < 1$ and $\operatorname{lip}_2 w_n^- < 1$ hold due to (3.8) resp. (3.28), the uniform contraction principle yields unique continuous fixed point functions $\xi_n^+ : \mathbb{R} \times X \to X$, $\zeta_n^+ : \mathbb{R} \times X \to X$. In particular, by definition of S the inclusions $\xi_n^+(\tau, u_0) \in \mathcal{X}_n^-(\tau)$ and $\zeta_n^+(\tau, u_0) \in \mathcal{X}_n^+(\tau)$ hold true.

We will not show (3.30) and (3.32) since the argument is as in [16, Thm. 3.7]. $\square$

The sets $\mathcal{V}_n^+(u_0)$, $u_0 \in X$, allow us to foliate the extended state space $\mathbb{R} \times X$:

Corollary 3.16 (pseudo-stable foliation)**.** *The nonautonomous sets $\mathcal{V}_n^+(u_0)$ are leafs of a foliation over every fiber $\mathcal{W}_n^-(\tau)$, $\tau \in \mathbb{R}$.*

Proof. Let $(\tau, u_0) \in \mathbb{R} \times X$. Due to (3.30) it is $u(\cdot; \tau, u_0) - u(\cdot; \tau, \pi_n^+(\tau, u_0)) \in B_{\tau,\gamma}^+$ and Prop. 3.13 shows $u_0 \in \mathcal{V}_n^+(\tau, \pi_n^+(\tau, u_0))$. Because $u_0 \in X$ was arbitrary, one has $X = \bigcup_{\xi \in \mathcal{W}_n^-(\tau)} \mathcal{V}_n^+(\tau, \xi)$. The fibers $\mathcal{V}_n^+(\tau, u_1)$, $\mathcal{V}_n^+(\tau, u_2)$ are pairwise disjoint, $\emptyset = \{u_1\} \cap \{u_2\} = \mathcal{V}_n^+(\tau, u_1) \cap \mathcal{V}_n^+(\tau, u_2)$ for $u_1 \neq u_2$ and $u_1, u_2 \in \mathcal{W}_n^-(\tau)$. $\square$

The concepts of pseudo-stable foliations and asymptotic forward phases also apply to the nonautonomous ODEs (E_n^-) in the finite-dimensional vector bundles $\mathcal{X}_n^-$. Its solutions exist on $\mathbb{R}$ and in particular the unique existence of backward solutions is always given. This enables us to introduce the dual concepts of pseudo-unstable foliations and asymptotic backward phases:

Proposition 3.17 (reduced pseudo-stable and -unstable leafs)**.** *Let $1 \leq k < n$. If*

$$L < \tfrac{\beta_k - \alpha_k}{4K_k(K_k+2)}, \qquad \sigma_k \in \Big(2K_k(K_k+2)L, \tfrac{\beta_k - \alpha_k}{2}\Big] \tag{3.33}$$

and (3.4) hold, then the ODE (E_n^-) satisfies for all $(\tau, x_0) \in \mathcal{X}_n^-$:

(a) The $d_1 + \ldots + d_k$-dimensional (n, k)-stable leafs*, fiberwise given as*

$$\mathcal{V}_{n,k}^+(\tau, x_0) := \left\{x_1 \in \mathcal{X}_n^-(\tau) : x_n^-(\cdot; \tau, x_1) - x_n^-(\cdot; \tau, x_0) \text{ is } \gamma^+\text{-bounded}\right\}$$

do not depend on $\gamma \in \Gamma_k$ and are invariant fiber bundles of (E_n^-) satisfying:

(a_1) They allow the representation

$$\mathcal{V}_{n,k}^+(x_0) = \Big\{(\tau, \zeta + v_{n,k}^+(\tau, \zeta, x_0)) \in \mathcal{X}_n^- : (\tau, \zeta) \in \mathcal{X}_{k+1}^n\Big\} \tag{3.34}$$

as graph of a continuous function $v_{n,k}^+ : \mathbb{R} \times X^2 \to X$ fulfilling the inclusion $v_{n,k}^+(\tau, x_1, x_0) \in \mathcal{X}_1^k(\tau)$ for all $(\tau, x_1) \in \mathcal{X}_n^-$.

(a_2) $v_{n,k}^+(\tau, x_1, x_0) = P_1^k(\tau)x_0 + V_{n,k}^+(\tau, x_1, x_0)$ for all $(\tau, x_1) \in \mathcal{X}_n^-$ with a continuous $V_{n,k}^+ : \mathbb{R} \times X^2 \to X$ satisfying $V_{n,k}^+(\tau, x_1, x_0) \in \mathcal{X}_1^k(\tau)$.

(b) *The $d_{k+1}+\ldots+d_n$-dimensional (n,k)*-unstable leafs*, fiberwise given as*

$$\mathcal{V}^-_{n,k}(\tau,x_0) := \left\{x_1 \in \mathcal{X}^-_n(\tau) : x^-_n(\cdot;\tau,x_1) - x^-_n(\cdot;\tau,x_0) \text{ is } \gamma^-\text{-bounded}\right\}$$

do not depend on $\gamma \in \Gamma_k$ and are invariant fiber bundles of (E^-_n) satisfying:

(b_1) *They allow the representation*

$$\mathcal{V}^-_{n,k}(x_0) = \left\{(\tau, \xi + v^-_{n,k}(\tau,\xi,x_0)) \in \mathcal{X}^-_n : (\tau,\xi) \in \mathcal{X}^k_1\right\} \tag{3.35}$$

as graph of a continuous function $v^-_{n,k} : \mathbb{R}\times X^2 \to X$ fulfilling the inclusion $v^-_{n,k}(\tau,x_1,x_0) \in \mathcal{X}^n_{k+1}(\tau)$ for all $(\tau,x_1) \in \mathcal{X}^-_n$.

(b_2) *$v^-_{n,k}(\tau,x_1,x_0) = P^n_{k+1}(\tau)x_0 + V^-_{n,k}(\tau,x_1,x_0)$ for all $(\tau,x_1) \in \mathcal{X}^-_n$ with a continuous $V^-_{n,k} : \mathbb{R}\times X^2 \to X$ satisfying $V^-_{n,k}(\tau,x_1,x_0) \in \mathcal{X}^n_{k+1}(\tau)$.*

(c) *One has the Lipschitz estimates*

$$\operatorname{lip}_2 v^\pm_{n,k} \le K_k\ell^\star_k < 1, \qquad \operatorname{lip}_2 V^\pm_{n,k} \le K_k\ell^\star_k < 1. \tag{3.36}$$

(d) *There exists a unique continuous mapping $\Pi^n_k : \mathcal{X}^-_n \times \mathcal{X}^-_n \to X$ with*

$$\mathcal{V}^+_{n,k}(\tau,x_1) \cap \mathcal{V}^-_{n,k}(\tau,x_2) = \{\Pi^n_k(\tau,x_1,x_2)\} \quad \text{for all } (\tau,x_1,x_2) \in \mathcal{X}^-_n \times \mathcal{X}^-_n,$$

which, moreover, is also linearly bounded

$$\|\Pi^n_k(\tau,x_1,x_2)\| \le \frac{(1+2\ell^\star_k)(K_k+\ell^\star_k)}{1-\ell^{\star 2}_k}\left(\|x_1\|+\|x_2\|\right). \tag{3.37}$$

Since the ODE (E^-_n) shares the trivial solution with (E) one immediately obtains the relations $\mathcal{V}^+_{n,k}(0) = \mathcal{W}^+_{n,k}$ and $\mathcal{V}^-_{n,k}(0) = \mathcal{W}^-_{n,k}$.

Proof. With $\operatorname{lip}_2 F^-_n < 2L$ in mind, one mimics the proof of Prop. 3.13 using the adapted norm (3.17) for (a)–(c). The assertion (d) is shown in [16, Prop. 3.1(c)]. □

Corollary 3.18. *Let $(\tau,x_0) \in \mathcal{X}^-_n$. Under the additional assumption* (**B**) *it holds*

$$\beta_k < 0 \quad \Rightarrow \quad \left\|V^+_{n,k}(\tau,\zeta,x_0)\right\| \le \tfrac{2K_k}{|\beta_k|}C \quad \text{for all } \zeta \in \mathcal{X}^n_{k+1}(\tau), \tag{3.38}$$

$$\alpha_k < 0 \quad \Rightarrow \quad \left\|V^-_{n,k}(\tau,\xi,x_0)\right\| \le \tfrac{2K_k}{|\alpha_k|}C \quad \text{for all } \xi \in \mathcal{X}^k_1(\tau). \tag{3.39}$$

Proof. Proceed as in the proofs of Cor. 3.8 and Cor. 3.14. □

Proposition 3.19 (reduced asymptotic phases)**.** *Let $1 \le k < n$. If (3.4), (3.33) are satisfied, then there exists a bounded function $C_{n,k} : \mathbb{R}\times[0,\infty) \to [0,\infty)$ such that the following holds true for every $(\tau,x_0) \in \mathcal{X}^-_n$:*

(a) *The (n,k)-unstable integral manifold $\mathcal{W}^-_{n,k}$ of (E^-_n) has an* asymptotic forward phase*, i.e. there exists a continuous function $\pi^+_{k,n} : \mathcal{X}^-_n \to X$ such that*

$$\left\|x^-_n(t;\tau,x_0) - x^-_n(t;\tau,\pi^+_{n,k}(\tau,x_0))\right\| \le C_{n,k}(\tau,\|x_0\|)e^{\gamma(t-\tau)} \quad \text{for all } \tau \le t$$

and $\gamma \in \Gamma_k$. Geometrically, $\pi^+_{n,k}(\tau,x_0)$ is given as unique intersection

$$\mathcal{W}^-_{n,k} \cap \mathcal{V}^+_{n,k}(x_0) = \left\{(t,\pi^+_{n,k}(t,x_0)) \in \mathcal{X}^-_n : t \in \mathbb{R}\right\}$$

and each $\pi^+_{n,k}(\tau,\cdot) : \mathcal{X}^-_n(\tau) \to \mathcal{W}^-_{n,k}(\tau)$ is a retraction onto $\mathcal{W}^-_{n,k}(\tau)$. In particular, there exists a unique continuous function $\xi^+_{n,k} : \mathcal{X}^-_n \to X$ satisfying the inclusion $\xi^+_{n,k}(\tau,x_0) \in \mathcal{X}^k_1(\tau)$ and

$$\pi^+_{n,k}(\tau,x_0) = \xi^+_{n,k}(\tau,x_0) + w^-_{n,k}(\tau,\xi^+_{n,k}(\tau,x_0)). \tag{3.40}$$

(b) *The (n,k)-stable integral manifold $\mathcal{W}^+_{n,k}$ of (E^-_n) has an* asymptotic backward phase, *i.e. there exists a continuous function $\pi^-_{k,n} : \mathcal{X}^-_n \to X$ such that*

$$\left\| x^-_n(t;\tau,x_0) - x^-_n(t;\tau,\pi^-_{n,k}(\tau,x_0)) \right\| \le C_{n,k}(\tau,\|x_0\|)e^{\gamma(t-\tau)} \quad \text{for all } t \le \tau$$

and $\gamma \in \Gamma_k$. Geometrically, $\pi^-_{n,k}(\tau,x_0)$ is given as unique intersection

$$\mathcal{W}^+_{n,k} \cap \mathcal{V}^-_{n,k}(x_0) = \left\{ (t,\pi^-_{n,k}(t,x_0)) \in \mathcal{X}^-_n : t \in \mathbb{R} \right\}$$

and each $\pi^-_{n,k}(\tau,\cdot) : \mathcal{X}^-_n(\tau) \to \mathcal{W}^+_{n,k}(\tau)$ is a retraction onto $\mathcal{W}^+_{n,k}(\tau)$. In particular, there exists a unique continuous function $\zeta^-_{n,k} : \mathcal{X}^-_n \to X$ satisfying the inclusion $\zeta^-_{n,k}(\tau,x_0) \in \mathcal{X}^n_{k+1}(\tau)$ and

$$\pi^-_{n,k}(\tau,x_0) = \zeta^-_{n,k}(\tau,x_0) + w^+_{n,k}(\tau,\zeta^-_{n,k}(\tau,x_0)). \tag{3.41}$$

(c) *The mappings $\pi^\pm_{n,k} : \mathcal{X}^-_n \to X$ are linearly bounded, i.e.*

$$\left\| \pi^\pm_{n,k}(\tau,x_0) \right\| \le K_k \tfrac{1+\ell^\star_k}{1-\ell^\star_k} \|x_0\| \quad \text{for all } (\tau,x_0) \in \mathcal{X}^-_n \tag{3.42}$$

(d) *For θ-periodic evolution eqns. (E) the functions $\pi^\pm_{n,k}$ and $\xi^\pm_{n,k}$ are θ-periodic in, and for autonomous (E) even independent of the first variable.*

Proof. Using $\operatorname{lip}_2 F^-_n \le 2L$ the proof parallels that of Thm. 3.15. □

Corollary 3.20 (reduced pseudo-stable and -unstable foliation)**.** *The nonautonomous sets $\mathcal{V}^\pm_{n,k}(x_0)$ are leafs of a foliation of $\mathcal{X}^-_n$ over the fiber $\mathcal{W}^\mp_{n,k}(\tau)$, $\tau \in \mathbb{R}$.*

Proof. Referring to Prop. 3.19 this follows as in the proof of Cor. 3.16. □

4. **Topological decoupling.** For nonautonomous evolution eqns. (E), the concept of topological conjugation is not as straight forward as in the classical autonomous situation. Clearly, it is natural to allow time-dependent transformations here, but this alone offers too much flexibility: Then, as already demonstrated in [24, p. 72] one could actually conjugate arbitrary equations. Indeed further assumptions are due and we suggest the following notion (cf. [16, 24]):

Suppose also $\bar{F} : \mathbb{R} \times X \to X$ fulfills $(\mathbf{N})$. A continuous function $T : \mathbb{R} \times X \to X$ is called *topological conjugation* between the semi-linear evolution eqn. (E) and

$$\boxed{\dot{u} = A(t)u + \bar{F}(t,u),} \tag{$\bar{E}$}$$

provided $T_\tau : X \to X$, $T_\tau(x) := T(\tau,x)$ is a homeomorphism for every $\tau \in \mathbb{R}$, its inverse $\bar{T} : \mathbb{R} \times X \to X$, $\bar{T}(\tau,x) := T^{-1}_\tau(x)$ is continuous, and one has the properties:

(i) $\lim_{x\to 0} T(\tau,x) = \lim_{x\to 0} \bar{T}(\tau,x) = 0$ uniformly in $\tau \in \mathbb{R}$
(ii) for every solution ϕ of (E) the function $\bar{\phi}(t) := T(t,\phi(t))$ solves $(\bar{E})$
(iii) for every solution $\bar{\phi}$ of $(\bar{E})$ the function $\phi(t) := \bar{T}(t,\phi(t))$ solves (E).

In this case the differential eqns. (E) and $(\bar{E})$ are called *topologically conjugated.*

The condition (i) yields the canonical requirement that stability properties of the trivial solution to (E) are preserved under topological conjugation.

Proposition 4.1 (topological decoupling of (E^-_n))**.** *If $(\mathbf{L})$, $(\mathbf{N})$ and the inequality*

$$L < \frac{1}{4} \min_{k=1}^{n} \frac{\beta_k - \alpha_k}{K_k(K_k+2)} \tag{4.1}$$

hold for some $n > 1$, there exists a topological conjugation $T^n : \mathcal{X}_n^- \to X$ between (E_n^-) and the decoupled ODE

$$\boxed{\dot{x} = A_n^-(t)x + \sum_{k=1}^{n} P_k(t)F(t, P_k(t)x + w_k(t,x)).} \tag{D_n^-}$$

This result could be established solely as an inductive application of [16, Prop. 4.3] (see also [24, p. 219, 5.4 Satz]), which decouples the finite-dimensional ODEs (E_n^-) into two subsystems for every gap in $\Sigma(A_n^-)$. Yet, we combine this argument with [12, Thm. 5.1] in order to prepare an upcoming infinite-dimensional version.

Proof. Let $1 \le k \le n$ and $(\tau, x_0) \in \mathcal{X}_n^-$ with a fixed $n > 1$. We remind the reader that x_n^- denotes the general solution to the ODE (E_n^-) and y_k is the general solution for (E_k). Let us subdivide the proof into four steps:

(I) We argue on basis of Prop. 3.17 and Prop. 3.19, which apply because of (4.1): First, for $k = 1$ the intersection $\mathcal{W}_{n,1}^-(\tau) \cap \mathcal{V}_{n,1}^+(\tau, x_0)$ contains a unique element $\pi_{n,1}^+(\tau, x_0) \in \mathcal{X}_n^-(\tau)$. For some $\zeta \in \mathcal{X}_2^n(\tau)$ it allows the representation

$$\xi_{n,1}^+(\tau, x_0) + w_{n,1}^-(\tau, \xi_{n,1}^+(\tau, x_0)) \overset{(3.40)}{=} \pi_{n,1}^+(\tau, x_0) \overset{(3.34)}{=} \zeta + P_1(\tau)x_0 + V_{n,1}^+(\tau, \zeta, x_0).$$

Due to Prop. 3.7(c) we obtain from Lemma A.1 applied to the first part of this equation that $\eta_1^n := \xi_{n,1}^+ : \mathcal{X}_n^- \to X$ defines a continuous function satisfying

$$\|\eta_1^n(\tau, x_0)\| \overset{(3.16)}{\le} \tfrac{1}{1-\ell_1^\star} \left\|\pi_{n,1}^+(\tau, x_0)\right\| \overset{(3.42)}{\le} \tfrac{K_k(1+\ell_1^\star)}{(1-\ell_1^\star)^2} \|x_0\| \xrightarrow[x_0 \to 0]{} 0$$

uniformly in $\tau \in \mathbb{R}$ and $\eta_1^n(\tau, x_0) \in \mathcal{X}_1(\tau)$. Moreover, it holds

$$\eta_1^n(\tau, x_0) = P_1(\tau)x_0 + V_{n,1}^+(\tau, w_{n,1}^-(\tau, \eta_1^n(\tau, x_0)), x_0) \tag{4.2}$$

and by the invariance of leafs and manifolds this implies for all $t \in \mathbb{R}$ that

$$y_1(t; \tau, \eta_1^n(\tau, x_0)) = v_{n,1}^+\big(t, w_{n,1}^-\big(t, y_1(t; \tau, \eta_1^n(\tau, x_0))\big), x_n^-(t; \tau, x_0)\big).$$

Second, let $1 < k < n$: On the one hand, $\mathcal{W}_{n,k}^-(\tau)$ and $\mathcal{V}_{n,k}^+(\tau, x_0)$ intersect at a unique point $\pi_{n,k}^+(\tau, x_0) = \xi_{n,k}^+(\tau, x_0) + w_{n,k}^-(\tau, \xi_{n,k}^+(\tau, x_0))$. As above one shows that the continuous function $\xi_{n,k}^+$ fulfills $\xi_{n,k}^+(\tau, x_0) \in \mathcal{X}_1^k(\tau)$, $\lim_{x_0 \to 0} \xi_{n,k}^+(\tau, x_0) = 0$ uniformly in $\tau \in \mathbb{R}$ and

$$\xi_{n,k}^+(\tau, x_0) = P_1^k(\tau)x_0 + V_{n,k}^+(\tau, w_{n,k}^-(\tau, \xi_{n,k}^+(\tau, x_0)), x_0). \tag{4.3}$$

On the other hand, also $\mathcal{W}_{k,k-1}^+(\tau) \cap \mathcal{V}_{k,k-1}^-(\tau, \xi_{n,k}^+(\tau, x_0))$ consists of a unique element, which is of the form

$$\pi_{k,k-1}^-(\tau, \xi_{n,k}^+(\tau, x_0)) = \zeta_{k,k-1}^-(\tau, \xi_{n,k}^+(\tau, x_0)) + w_{k,k-1}^+\big(\tau, \zeta_{k,k-1}^-(\tau, \xi_{n,k}^+(\tau, x_0))\big)$$

with a continuous $\zeta_{k,k-1}^- : \mathcal{X}_n^- \to X$. As composition of continuous functions also $\eta_k^n(\tau, x_0) := \zeta_{k,k-1}^-(\tau, \xi_{n,k}^+(\tau, x_0)) \in \mathcal{X}_k(\tau)$ is continuous. Hence,

$$\begin{aligned}\eta_k^n(\tau, x_0) + w_{k,k-1}^+(\tau, \eta_k^n(\tau, x_0)) &\overset{(3.41)}{=} \pi_{k,k-1}^-(\tau, \xi_{n,k}^+(\tau, x_0)) \\ &\overset{(3.35)}{=} \xi + P_k(\tau)\xi_{n,k}^+(\tau, x_0) + V_{k,k-1}^-(\tau, \xi, \xi_{n,k}^+(\tau, x_0))\end{aligned}$$

holds for a $\xi \in \mathcal{X}_1^{k-1}(\tau)$, where the first equation implies $\lim_{x_0 \to 0} \eta_k^n(\tau, x_0) = 0$ uniformly in $\tau \in \mathbb{R}$. We furthermore deduce

$$\eta_k^n(\tau, x_0) = P_k(\tau)\xi_{n,k}^+(\tau, x_0) + V_{k,k-1}^-(\tau, w_{k,k-1}^+(\tau, \eta_k^n(\tau, x_0)), \xi_{n,k}^+(\tau, x_0))$$

and from (4.3) one has

$$\begin{aligned}\eta_k^n(\tau,x_0) = P_k(\tau)x_0 + P_k(\tau)V_{n,k}^+(\tau, w_{n,k}^-(\tau,\xi_{n,k}^+(\tau,x_0)), x_0)\\ + V_{k,k-1}^-(\tau, w_{k,k-1}^+(\tau,\eta_k^n(\tau,x_0)), \xi_{n,k}^+(\tau,x_0)).\end{aligned} \tag{4.4}$$

Since the involved integral manifolds and leafs are invariant, we conclude that

$$y_k(t;\tau,\eta_k^n(\tau,x_0)) = v_{k,k-1}^-\big(t, w_{k,k-1}^+\big(t, y_k(t;\tau,\eta_k^n(\tau,x_0))\big), \xi_{n,k}^+(t, x_n^-(t;\tau,x_0))\big)$$

holds for all $t \in \mathbb{R}$.

Third, for $k = n$, also the intersection $\mathcal{W}_{n,n-1}^+(\tau) \cap \mathcal{V}_{n,n-1}^-(\tau,x_0)$ is a singleton $\pi_{n,n-1}^-(\tau,x_0) = \zeta_{n,n-1}^-(\tau,x_0) + w_{n,n-1}^+(\tau,\zeta_{n,n-1}^-(\tau,x_0))$. Now $\eta_n^n := \zeta_{n,n-1}^- : \mathcal{X}_n^- \to X$ defines a continuous function with $\eta_n^n(\tau,x_0) \in \mathcal{X}_n(\tau)$, $\lim_{x_0\to 0}\eta_n^n(\tau,x_0) = 0$ uniformly in $\tau \in \mathbb{R}$ and

$$\eta_n^n(\tau,x_0) = P_n(\tau)x_0 + V_{n,n-1}^-(\tau, w_{n,n-1}^+(\tau,\eta_n^n(\tau,x_0)), x_0). \tag{4.5}$$

Moreover, the invariance of leafs and integral manifolds guarantees

$$y_n(t;\tau,\eta_n^n(\tau,x_0)) = v_{n,n-1}^-\big(t, w_{n,n-1}^+\big(t, y_n(t;\tau,\eta_n^n(\tau,x_0))\big), x_n^-(t;\tau,x_0)\big)$$

for all $t \in \mathbb{R}$. After these preparations we are in the position to introduce

$$T^n : \mathcal{X}_n^- \to X, \qquad\qquad T^n(\tau,x_0) := \sum_{k=1}^n \eta_k^n(\tau,x_0).$$

Thanks to the properties of its summands, $T_\tau^n : \mathcal{X}_n^-(\tau) \to \mathcal{X}_n^-(\tau)$ is well-defined and T^n is continuous with $\lim_{x_0\to 0} T^n(\tau,x_0) = 0$ uniformly in $\tau \in \mathbb{R}$. In the following, whenever confusion is absent, it is convenient to neglect the dependence of $\xi_{n,k}^+, \eta_k^n$ (and further quantities) on (τ,x_0). Given this, by means of (4.2), (4.4) and (4.5) one finally obtains the alternative representation

$$\begin{aligned}T_\tau^n(x_0) = x_0 + V_{n,1}^+(\tau, w_{n,1}^-(\tau,\eta_1^n), x_0) + V_{n,n-1}^-(\tau, w_{n,n-1}^+(\tau,\eta_n^n), x_0)\\ + \sum_{k=2}^{n-1}\Big[P_k(\tau)V_{n,k}^+(\tau, w_{n,k}^-(\tau,\xi_{n,k}^+), x_0) + V_{k,k-1}^-(\tau, w_{k,k-1}^+(\tau,\eta_k^n), \xi_{n,k}^+)\Big].\end{aligned} \tag{4.6}$$

(II) *Claim: $P_1^n(\tau)T_\tau^m(x_0 + w_{m,n}^-(\tau,x_0)) = T_\tau^n(x_0)$ for all $n < m$.* We merely give a sketch of the argument, since it is analogous to step (II) in the proof of the subsequent Thm. 4.2. Let us briefly write $\bar\eta_k^n(\tau,x_0) := \eta_k^n(\tau, x_0 + w_{m,n}^-(\tau,x_0))$.

- $k = 1$: The definition of T^m and T^n implies

$$\begin{aligned}\bar\eta_1^n + w_{m,1}^-(\tau,\bar\eta_1^n) &\in \mathcal{V}_{m,1}^+(\tau, x_0 + w_{m,n}^-(\tau,x_0)) \cap \mathcal{W}_{m,1}^-(\tau),\\ \eta_1^n + w_{n,1}^-(\tau,\eta_1^n) &\in \mathcal{V}_{n,1}^+(\tau,x_0) \cap \mathcal{W}_{n,1}^-(\tau)\end{aligned}$$

  and thanks to Cor. 3.11 one obtains

$$\begin{aligned}&\eta_1^n + w_{n,1}^-(\tau,\eta_1^n) + w_{m,n}^-(\tau, \eta_1^n + w_{n,1}^-(\tau,\eta_1^n)) \in \mathcal{W}_{m,1}^-(\tau),\\ &\quad \eta_1^n + w_{n,1}^-(\tau,\eta_1^n) + w_{m,n}^-\big(\tau, \eta_1^n + w_{n,1}^-(\tau,\eta_1^n)\big) \in \mathcal{V}_{m,1}^+(\tau, x_0 + w_{m,n}^-(\tau,x_0)).\end{aligned}$$

  Since the nonautonomous sets $\mathcal{W}_{m,1}^-$ and $\mathcal{V}_{m,1}^+(x_0 + w_{m,n}^-(\tau,x_0))$ possess fibers intersecting at a unique point, this implies $\eta_1^n = \bar\eta_1^m$.
- $1 < k < n$: One has $\xi_{n,k}^+(\tau, x_0 + w_{m,n}^-(\tau,x_0)) = \xi_{n,k}^+(\tau,x_0)$ due to the definition of T^n, T^m and consequently $\eta_k^n = \bar\eta_k^m$.
- $k = n$ is evident.

This finally gives us the assertion (II) because of

$$P_1^n(\tau)T_\tau^m(x_0+w_{m,n}^-(\tau,x_0)) = \sum_{k=1}^n \eta_k^m(\tau,x_0+w_{m,n}^-(\tau,x_0)) = \sum_{k=1}^n \eta_k^n(\tau,x_0) = T_\tau^n(x_0).$$

(III) *Claim: T^n is a topological conjugation between (E_n^-) and (D_n^-).* We use mathematical induction over $n \geq 2$ in order to prove that $T_\tau^n : \mathcal{X}_n^-(\tau) \to \mathcal{X}_n^-(\tau)$ possesses an inverse function $\bar{T}_\tau^n$ such that $\bar{T}^n(\tau,y) = \bar{T}_\tau^n(y)$ is continuous and satisfies the limit relation $\lim_{y\to 0} \bar{T}_\tau^n(y) = 0$ uniformly in $\tau \in \mathbb{R}$. For $n = 2$ the proof comes along [16, Prop. 3.1]: Given $(\tau,y) \in \mathcal{X}_2^-$ define

$$\bar{T}^2(\tau,y) := \Pi_1^2(\tau, P_1(\tau)y + w_{2,1}^-(\tau,y), P_2(\tau)y + w_{2,1}^+(\tau,y))$$

with Π_1^2 given in Prop. 3.17(d). Thanks to Prop. 3.7(c) the mapping $\bar{T}^2$ is continuous and (3.37) guarantees that the desired uniform limit relation holds. Due to

$$T_\tau^2(x) = P_1(\tau)\pi_{2,1}^+(\tau,x) + P_2(\tau)\pi_{2,1}^-(\tau,x)$$

we obtain for all $(\tau,x) \in \mathcal{X}_2^-$ that

$$\bar{T}_\tau^2(T_\tau(x)) = \Pi_1^2\big(\tau, \underbrace{\eta_1^2(\tau,x) + w_{2,1}^-(\tau,\eta_1^2(\tau,x))}_{\in \mathcal{V}_{2,1}^+(\tau,x)}, \underbrace{\eta_2^2(\tau,x) + w_{2,1}^+(\tau,\eta_2^2(\tau,x))}_{\in \mathcal{V}_{2,1}^-(\tau,x)}\big) = x$$

and it follows analogously that $T_\tau^2(\bar{T}_\tau^2(y)) = y$ for all $(\tau,y) \in \mathcal{X}_2^-$ holds. Hence, the mappings $T_\tau^2, \bar{T}_\tau^2$ are inverse to each other. As induction hypothesis we know that for every $(\tau,y_n) \in \mathcal{X}_n^-$ there exists a unique $\bar{T}_\tau^n(y_n) \in \mathcal{X}_n^-(\tau)$ with

$$y_n = T_\tau^n(\bar{T}_\tau^n(y_n)) = \sum_{k=1}^n \eta_k^n(\tau, \bar{T}_\tau^n(y_n)), \qquad \lim_{y\to 0} \bar{T}_\tau^n(y) = 0 \quad \text{uniformly in } \tau \in \mathbb{R}$$

and $\bar{T}^n : \mathcal{X}_n^- \to X$ being continuous. We define the continuous mapping $\bar{T}^{n+1}$ via

$$\bar{T}_\tau^{n+1}(y) := \Pi_n^{n+1}(\tau, \bar{T}_\tau^n(P_1^n(\tau)y) + w_{n+1,n}^-(\tau, \bar{T}_\tau^n(P_1^n(\tau)y)), P_{n+1}(\tau)y + w_{n+1,n}^+(\tau,y))$$

with the function Π_n^{n+1} from Prop. 3.17(d). Thanks to step (II) and the construction of T^{n+1} it holds that

$$T_\tau^{n+1}(\bar{T}_\tau^{n+1}(y)) = y, \qquad \bar{T}_\tau^{n+1}(T_\tau^{n+1}(x)) = x \quad \text{for all } (\tau,x),(\tau,y) \in \mathcal{X}_{n+1}^-$$

and thus T^{n+1} is shown to be bijective. The limit relation $\lim_{y\to 0} \bar{T}_\tau^{n+1}(y) = 0$ uniformly in $\tau \in \mathbb{R}$ results from (3.37).

(IV) In summary, the function T^n transforms solutions of (E_n^-) to solutions of the decoupled eqn. (D_n^-). Moreover, by uniqueness of solutions, the inverse $\bar{T}^n$ maps solutions of (D_n^-) to solutions of (E_n^-). Both $T^n, \bar{T}^n$ are continuous and fulfill uniform limit relations, i.e. T^n is the desired topological conjugation. □

Due to ($\mathbf{L_3}$) there exists an index $\kappa \in \mathbb{N}$ such that $\beta_k < 0$ for all $k \geq \kappa$ (cf. Fig. 1). This enables us to formulate a crucial decay condition for the remainder of the paper:

Theorem 4.2 (topological decoupling of (E))**.** *Suppose that* ($\mathbf{L}$)*,* ($\mathbf{N}$)*,* ($\mathbf{B}$) *and*

$$L < \frac{1}{5} \inf_{k\in\mathbb{N}} \frac{\beta_k - \alpha_k}{K_k(K_k+2)} \tag{4.7}$$

hold. If $(K_k)_{k\in\mathbb{N}}$ is bounded and the decay condition

$$\beta_k < 0 \quad \textit{for all } k \geq \kappa \quad \Rightarrow \quad \sum_{k\geq\kappa} \frac{K_k}{|\beta_k|} < \infty \tag{4.8}$$

is fulfilled, then (E) is topologically conjugated to the fully decoupled equation

$$\boxed{\dot{u} = A(t)u + \sum_{k\in\mathbb{N}} P_k(t)F(t, P_k(t)u + w_k(t,u)).} \tag{D}$$

The inequality (4.7) is not only a smallness assumption on the Lipschitz constant of our nonlinearity F. It also guarantees that the lengths of the intervals $[\alpha_k, \beta_k]$ and therefore the spectral gaps $(\lambda_{k+1}^+, \lambda_k^-)$ is bounded away from 0 (see Fig. 1).

Proof. Note that u and x_n^- stand for the respective general solution to (E) or (E_n^-). Let us construct a candidate for a topological conjugation $T : \mathbb{R} \times X \to X$ between (E) and (D): Thereto, choose $(\tau, u_0) \in \mathbb{R} \times X$ arbitrarily. Because of Prop. 3.13 and Thm. 3.15, which apply thanks to (4.7), for every $n \in \mathbb{N}$ the intersection

$$\mathcal{W}_n^- \cap \mathcal{V}_n^+(u_0) \overset{(3.31)}{=} \left\{(t, \pi_n^+(t,u_0)) \in \mathbb{R} \times X : t \in \mathbb{R}\right\}$$

allows the representations

$$\begin{aligned} P_n^+(\tau)\pi_n^+(\tau,u_0) + v_n^+(\tau, P_n^+(\tau)\pi_n^+(\tau,u_0), u_0) &\overset{(3.27)}{=} \pi_n^+(\tau,u_0) \\ &= \xi_n^+(\tau,u_0) + w_n^-(\tau, \xi_n^+(\tau,u_0)) \end{aligned}$$

with a unique $\xi_n^+(\tau,u_0) \in \mathcal{X}_n^-(\tau)$ defining a continuous function $\xi_n^+ : \mathbb{R} \times X \to X$. Thanks to Prop. 3.13(b) the point $\xi_n^+(\tau,u_0)$ moreover fulfills the fixed point equation

$$\xi = v_n^+(\tau, w_n^-(\tau,\xi), u_0) = P_n^-(\tau)u_0 + V_n^+(\tau, w_n^-(\tau,\xi), u_0). \tag{4.9}$$

First, for $n = 1$ let us define the mapping $\eta_1(\tau,u_0) := \xi_1^+(\tau,u_0) \in \mathcal{X}_1(\tau)$. Second, for $n > 1$ our Prop. 3.19(b) guarantees

$$\mathcal{W}_{n,n-1}^+(\tau) \cap \mathcal{V}_{n,n-1}^-(\tau, \xi_n^+(\tau,u_0)) = \left\{\left(t, \pi_{n,n-1}^-(t, \xi_n^+(\tau,u_0))\right) \in \mathcal{X}_n^- : t \in \mathbb{R}\right\}$$

with

$$\pi_{n,n-1}^-(t, \xi_n^+(\tau,u_0)) \overset{(3.41)}{=} \zeta_{n,n-1}^-(\tau, \xi_n^+(\tau,u_0)) + w_{n,n-1}^+\left(\tau, \zeta_{n,n-1}^-(\tau, \xi_n^+(\tau,u_0))\right) \tag{4.10}$$

and a unique continuous $\zeta_{n,n-1}^- : \mathcal{X}_n^- \to X$. Hence, as composition of continuous functions $\eta_n : \mathbb{R} \times X \to X$, $\eta_n(\tau,u_0) := \zeta_{n,n-1}^-(\tau, \xi_n^+(\tau,u_0)) \in \mathcal{X}_n(\tau)$ is continuous.

Whenever convenient and unambiguous, we neglect the dependence of ξ_n^+, η_n on the argument (τ,u_0). Due to the alternative representation in Prop. 3.17(b) we have

$$\eta_n + w_{n,n-1}^+(\tau,\eta_n) \overset{(4.10)}{=} \xi + P_n(\tau)\xi_n^+ + V_{n,n-1}^-(\tau, \xi, \xi_n^+)$$

with some $\xi \in \mathcal{X}_1^{n-1}(\tau)$. Consequently, the element η_n fulfills the fixed point relation $\eta_n = P_n(\tau)\xi_n^+ + V_{n,n-1}^-(\tau, w_{n,n-1}^+(\tau,\eta_n), \xi_n^+)$, which in turn implies

$$\begin{aligned} \eta_n \overset{(4.9)}{=} P_n(\tau)u_0 &+ P_n(\tau)V_n^+(\tau, w_n^-(\tau,\xi_n^+), u_0) \\ &+ V_{n,n-1}^-(\tau, w_{n,n-1}^+(\tau,\eta_n), \xi_n^+). \end{aligned} \tag{4.11}$$

After these preparations we now formally define the mapping $T : \mathbb{R} \times X \to X$ by

$$T(\tau,u_0) := \sum_{n\in\mathbb{N}} \eta_n(\tau,u_0) \tag{4.12}$$

and proceed in seven steps:

(I) *Claim: $T : \mathbb{R} \times X \to X$ is well-defined and continuous.* We initially establish that the infinite series (4.12) converges. Our assumption ($\mathbf{L_3}$) shows $\frac{1}{|\alpha_k|} < \frac{1}{|\beta_k|}$ for all $k \geq \kappa$. Hence, due to Cors. 3.14 and 3.18 the series

$$\sum_{n\geq 1} P_n(\tau)V_n^+(\tau, w_n^-(\tau, \xi_n^+), u_0), \sum_{n\geq 2} V_{n,n-1}^-\big(\tau, w_{n,n-1}^+(\tau, \eta_n), \xi_n^+\big)$$

have $\sum_{k=\kappa}^{\infty} \frac{K_k}{|\beta_k|}$ as convergent majorant, uniformly in $(\tau, u_0) \in \mathbb{R} \times X$. Referring to (4.9), (4.11) and complete spectral projectors due to ($\mathbf{L_3}$), T can be written as

$$\begin{aligned} T_\tau(u_0) = u_0 &+ \sum_{n\geq 1} P_n(\tau)V_n^+(\tau, w_n^-(\tau, \xi_n^+), u_0) \\ &+ \sum_{n\geq 2} V_{n,n-1}^-\big(\tau, w_{n,n-1}^+(\tau, \eta_n), \xi_n^+\big). \end{aligned}$$

Moreover, thanks to Lemma A.1 first we obtain

$$\left\|\xi_n^+\right\| \overset{(3.8)}{\leq} \tfrac{1}{1-\ell_n} \left\|\pi_n^+(\tau, u_0)\right\| \overset{(3.32)}{\leq} \tfrac{K_n(1+\ell_n^\star)}{(1-\ell_n^\star)^2} \left\|u_0\right\| \xrightarrow[u_0\to 0]{} 0 \quad \text{uniformly in } \tau \in \mathbb{R}$$

for all $n \in \mathbb{N}$ and second, (4.10) implies for every $n > 1$ that

$$\begin{aligned} \left\|\eta_n(\tau, u_0)\right\| &\overset{(3.16)}{\leq} \frac{1}{1-\ell_{n-1}^\star} \left\|\pi_{n,n-1}^-(\tau, \xi_n^+)\right\| \\ &\overset{(3.42)}{\leq} \frac{K_{n-1}(1+\ell_{n-1}^\star)}{(1-\ell_{n-1}^\star)^2} \left\|\xi_n^+\right\| \xrightarrow[u_0\to 0]{} 0 \quad \text{uniformly in } \tau \in \mathbb{R}. \end{aligned}$$

Hence, all summands in (4.12) are continuous and Lemma A.2 allows to conclude that the limit T is continuous fulfilling $\lim_{u\to 0} T_\tau(u) = 0$ uniformly in $\tau \in \mathbb{R}$.

(II) *Claim: $P_1^n(\tau)T_\tau(u_0) = T_\tau^n(\xi_n^+(\tau, u_0))$ for all $n > 1$.* Here, $T^n : \mathcal{X}_n^- \to X$ denotes the topological conjugation from Prop. 4.1 between the finite-dimensional eqns. (E_n^-) and (D_n^-). Based on the representations (cf. the proof of Prop. 4.1)

$$T_\tau^n(\xi_n^+) = \sum_{k=1}^{n} \bar{\eta}_k^n(\tau, u_0), \qquad P_1^n(\tau)T_\tau(u_0) = \sum_{k=1}^{n} \eta_k(\tau, u_0)$$

with $\bar{\eta}_k^n(\tau, u_0) := \eta_k^n(\tau, \xi_n^+(\tau, u_0))$ we establish that $\eta_k = \bar{\eta}_k^n$ for every $1 \leq k \leq n$:

- $k = 1$: Using Thm. 3.15 and the proof of Prop. 4.1 it is

$$\begin{aligned} \mathcal{W}_1^-(\tau) \cap \mathcal{V}_1^+(\tau, u_0) &= \left\{\eta_1 + w_1^-(\tau, \eta_1)\right\} \subseteq X, \\ \mathcal{W}_{n,1}^-(\tau) \cap \mathcal{V}_{n,1}^+(\tau, \xi_n^+) &= \left\{\bar{\eta}_1^n + w_{n,1}^-(\tau, \bar{\eta}_1^n))\right\} \subseteq \mathcal{X}_n^-(\tau). \end{aligned}$$

  On the one hand, due to (3.21) in Cor. 3.11 and Rem. 3.10 one has

$$\bar{\eta}_1^n + w_{n,1}^-(\tau, \bar{\eta}_1^n) + w_n^-(\tau, \bar{\eta}_1^n + w_{n,1}^-(\tau, \bar{\eta}_1^n)) = \bar{\eta}_1^n + w_1^-(\tau, \bar{\eta}_1^n) \in \mathcal{W}_1^-(\tau).$$

  On the other hand, the invariance of $\mathcal{W}_n^-$ implies that

$$\begin{aligned} u(t; \tau, \bar{\eta}_1^n + w_1^-(\tau, \bar{\eta}_1^n)) &= x_n^-(t; \tau, \bar{\eta}_1^n + w_{n,1}^-(\tau, \bar{\eta}_1^n)) + w_n^-\big(t, x_n^-(t; \tau, \bar{\eta}_1^n + w_{n,1}^-(\tau, \bar{\eta}_1^n))\big), \\ u(t; \tau, \pi_n^+(\tau, u_0)) &= x_n^-(t; \tau, \xi_n^+) + w_n^-(t, x_n^-(t; \tau, \xi_n^+)) \end{aligned}$$

  for all $t \geq \tau$ and consequently the triangle inequality leads to

$$\begin{aligned} &\left\|u(t; \tau, u_0) - u(t; \tau, \bar{\eta}_1^n + w_1^-(\tau, \bar{\eta}_1^n))\right\| \\ &\leq \left\|u(t; \tau, u_0) - u(t; \tau, \pi_n^+(\tau, u_0))\right\| + \left\|u(t; \tau, \pi_n^+(\tau, u_0)) - u(t; \tau, \bar{\eta}_1^n + w_1^-(\tau, \bar{\eta}_1^n))\right\| \\ &\overset{(3.8)}{\leq} \left\|u(t; \tau, u_0) - u(t; \tau, \pi_n^+(\tau, u_0))\right\| + 2\left\|x_n^-(t; \tau, \bar{\eta}_1^n + w_{n,1}^-(\tau, \bar{\eta}_1^n)) - x_n^-(t; \tau, \xi_n^+)\right\| \end{aligned}$$

for all $t \geq \tau$. Combining (3.30) and $\bar{\eta}_1^n + w_{n,1}^-(t, \bar{\eta}_1^n) \in \mathcal{V}_{n,1}^+(t, \xi_n^+)$ yields

$$u(\cdot; \tau, \bar{\eta}_1^n + w_n^-(\tau, \bar{\eta}_1^n)) - u(\cdot; \tau, u_0) \in B_{\tau,\gamma}^+ \quad \text{for all } \gamma \in \Gamma_n,$$

i.e. $\bar{\eta}_1^n + w_1^-(\tau, \bar{\eta}_1^n) \in \mathcal{V}_1^+(\tau, u_0)$ (cf. Prop. 3.13) and hence

$$\bar{\eta}_1^n + w_n^-(\tau, \bar{\eta}_1^n) = \pi_n^+(\tau, u_0) = \eta_1 + w_1^-(\tau, \eta_1).$$

Multiplication with $P_1^n(\tau)$ implies $\eta_1 = \bar{\eta}_1^n$.

- $1 < k < n$: Again from Thm. 3.15 and Prop. 3.19(a) it follows that

$$\mathcal{W}_k^-(\tau) \cap \mathcal{V}_k^+(\tau, u_0) = \left\{\xi_k^+ + w_k^-(\tau, \xi_k^+)\right\} \subseteq X,$$
$$\mathcal{W}_{n,k}^-(\tau) \cap \mathcal{V}_{n,k}^+(\tau, \xi_n^+) = \left\{\xi_{n,k}^+(\tau, \xi_n^+) + w_{n,k}^-(\tau, \xi_{n,k}^+(\tau, \xi_n^+))\right\} \subseteq \mathcal{X}_n^-(\tau).$$

Using the same arguments as above it results $\xi_k^+ = \xi_{n,k}^+(\tau, \xi_n^+)$ and therefore due to the definition of T one has $\eta_k = \bar{\eta}_k^n$.

- $k = n$: Finally, $\eta_k = \bar{\eta}_k^n$ results from the construction of T.

(III) *Claim: $T_\tau : X \to X$ is injective.* We assume $u_1, u_2 \in X$, $u_1 \neq u_2$. Then the function $\phi := u(\cdot; \tau, u_1) - u(\cdot; \tau, u_2)$ is a mild solution of the semilinear equation

$$\dot{u} = A(t)u + G(t, u) \tag{4.13}$$

with the nonlinearity $G(t, u) := F(t, u(t; \tau, u_1) + u) - F(t, u(t; \tau, u_1))$. It is clear that $G : [\tau, \infty) \times X \to X$ fulfills (**N**) with the Lipschitz constant L. Following the convention that the growth rates γ_n are always contained in Γ_n, one obtains:

- If $\phi \notin B_{\tau,\gamma_n}^+$ for all $n \in \mathbb{N}$, then $\mathcal{V}_1^+(u_1) \cap \mathcal{V}_1^+(u_2) = \emptyset$ (see Cor. 3.16) and hence the construction of T guarantees $P_1(\tau)T_\tau(u_1) \neq P_1(\tau)T_\tau(u_2)$.
- If there exists a $n \in \mathbb{N}$ with $\phi \in B_{\tau,\gamma_n}^+ \setminus B_{\tau,\gamma_{n+1}}^+$, then the definition of T establishes $\xi_{n+1}^+(\tau, u_1) \neq \xi_{n+1}^+(\tau, u_2)$, but $\xi_n^+(\tau, u_1) = \xi_n^+(\tau, u_2)$. Due to step (II) and Prop. 4.1 this implies that $P_{n+1}(\tau)T_\tau(u_1) \neq P_{n+1}(\tau)T_\tau(u_2)$.
- $\phi \in B_{\tau,\gamma_n}^+$ for all $n \in \mathbb{N}$ means that ϕ is a small solution to (4.13). Since (4.7) allows to apply [18, Thm. 4.1], we deduce $\phi = 0$ i.e. the contradiction $u_1 = u_2$.

(IV) *Claim: $T_\tau : X \to X$ is onto.* Given an arbitrary pair $(\tau, v_0) \in \mathbb{R} \times X$ we write $v_0 = \sum_{n\in\mathbb{N}} P_n(\tau)v_0$ and abbreviate $v_0^n := P_1^n(\tau)v_0$ for the partial sums. From Prop. 4.1 it is known that for every $n > 1$ and there exists a unique preimage $x_n \in \mathcal{X}_n^-(\tau)$ under T_τ^n. It is $x_n = \bar{T}_\tau^n(v_0^n)$ with a continuous $\bar{T}_n : \mathcal{X}_n^- \to X$ and

$$\lim_{y\to 0} \bar{T}^n(\tau, y) = 0 \quad \text{uniformly in } \tau \in \mathbb{R}. \tag{4.14}$$

We briefly write $\upsilon^n := \bar{T}_\tau^n(v_0^n)$ and aim to show that $(\upsilon^n + w_n^-(\tau, \upsilon^n))_{n\in\mathbb{N}}$ defines a Cauchy sequence in the Banach space X. Thereto, let $m > n$. On the one hand, the inclusion $\upsilon^m \in \mathcal{V}_{m,n}^+(\tau, \upsilon^n + w_{m,n}^+(\tau, \upsilon^m))$ leads to

$$\upsilon^n = v_{m,n}^+(\tau, w_{m,n}^-(\tau, \upsilon^n), \upsilon^m). \tag{4.15}$$

Having said this, $\upsilon^m \in \mathcal{V}_{m,n}^+(\tau, \upsilon^m) \cap \mathcal{V}_{m,n}^-(\tau, \upsilon^m)$ and due to our construction and Prop. 3.17(d) we arrive at the representation

$$\zeta^m + v_{m,n}^+(\tau, \zeta^m, \upsilon^m) \overset{(3.34)}{=} \upsilon^m \overset{(3.35)}{=} \xi^m + v_{m,n}^-(\tau, \xi^m, \upsilon^m),$$

which immediately implies

$$\zeta^m = v_{m,n}^-(\tau, \xi^m, \upsilon^m) \in \mathcal{X}_{n+1}^m(\tau), \qquad \xi^m = v_{m,n}^+(\tau, \zeta^m, \upsilon^m) \in \mathcal{X}_1^n(\tau) \tag{4.16}$$

as well as the decomposition $\upsilon^m = \xi^m + \zeta^m$. Thanks to (4.6) one obtains

$$y_m = \upsilon^m + V_{m,1}^+(\tau, w_{m,1}^-(\tau, \eta_1^m), \upsilon^m) + V_{m,m-1}^-(\tau, w_{m,m-1}^+(\tau, \eta_m^m), \upsilon^m)$$

$$+\sum_{k=2}^{m-1}\Big[P_k(\tau)V^+_{m,k}(\tau,w^-_{m,k}(\tau,\xi^+_{m,k}),\upsilon^m)+V^-_{k,k-1}(\tau,w^+_{k,k-1}(\tau,\eta^m_k),\upsilon^k)\Big]$$

and multiplication with $P^m_{n+1}(\tau)$ yields

$$\begin{aligned}\sum_{k=n+1}^{m}P_k(\tau)y_m=\zeta^m+\sum_{k=n+1}^{m-1}&\big[P_k(\tau)V^+_{m,k}(\tau,w^-_{m,k}(\tau,\xi^-_{m,k}),\upsilon^m)\\&+V^-_{k,k-1}(\tau,w^+_{k,k-1}(\tau,\eta^m_k),\upsilon^k)\big]+V^-_{m,m-1}(\tau,w^+_{m,m-1}(\tau,\eta^m_m),\upsilon^m).\end{aligned}\tag{4.17}$$

After these preparations we can verify the Cauchy property of the sequence $(\upsilon^n)_{n\in\mathbb{N}}$: Given $\varepsilon>0$, it follows from Cors. 3.4 and 3.18 combined with (4.8) that there exists a $N_1\in\mathbb{N}$ such that

$$\left\|w^-_n(\tau,\upsilon^n)\right\|<\tfrac{\varepsilon}{9},\quad\left\|V^-_{n,n-1}(\tau,w^+_{n,n-1}(\tau,\eta^n_n),\upsilon^n)\right\|<\tfrac{\varepsilon}{9}\quad\text{for all }n\geq N_1.\tag{4.18}$$

Because the limits

$$\sum_{k\geq1}P_k(\tau)y_m,\ \sum_{k\geq1}\Big(P_k(\tau)V^+_{m,k}(\tau,w^-_{m,k}(\tau,\xi^-_{m,k}),\upsilon^m)+V^-_{k,k-1}(\tau,w^+_{k,k-1}(\tau,\eta^m_k),\upsilon^k)\Big)$$

exist due to Cor. 3.18 and (4.8) again, there is a $N_2\geq N_1$ such that

$$\begin{aligned}\|\zeta^m\|&\overset{(4.17)}{\leq}\left\|\sum_{k=n+1}^{m-1}P_k(\tau)V^+_{m,k}(\tau,w^-_{m,k}(\tau,\xi^-_{m,k}),\upsilon^m)+V^-_{k,k-1}(\tau,w^+_{k,k}(\tau,\eta^m_k),\upsilon^k)\right\|\\&\quad+\left\|\sum_{k=n+1}^{m-1}P_k(\tau)y_m\right\|+\left\|V^-_{m,m-1}(\tau,w^+_{m,m-1}(\tau,\upsilon^m),\upsilon^m)\right\|\\&\overset{(4.18)}{<}\tfrac{\varepsilon}{9}+\tfrac{\varepsilon}{9}+\tfrac{\varepsilon}{9}=\tfrac{\varepsilon}{3}\quad\text{for all }m>n\geq N_2\end{aligned}$$

by the Cauchy criterion for infinite series. With (4.15), (4.16) one obtains

$$\begin{aligned}\|\upsilon^m-\upsilon^n\|&\leq\|\zeta^m\|+\|\xi^m-\upsilon^n\|\\&=\|\zeta^m\|+\left\|\upsilon^+_{m,n}(\tau,\zeta^m,\upsilon^m)-\upsilon^+_{m,n}(\tau,w^-_n(\tau,\upsilon^n),\upsilon^m)\right\|\\&\overset{(3.36)}{\leq}\|\zeta^m\|+\left\|\zeta^m-w^-_n(\tau,\upsilon^n)\right\|\leq2\|\zeta^m\|+\left\|w^-_n(\tau,\upsilon^n)\right\|\\&\overset{(4.18)}{<}\tfrac{2}{3}\varepsilon+\tfrac{1}{9}\varepsilon\quad\text{for all }m\geq n\geq N_2\end{aligned}\tag{4.19}$$

from the triangle inequality and therefore

$$\begin{aligned}&\left\|\upsilon^m+w^-_m(\tau,\upsilon^m)-(\upsilon^n+w^-_n(\tau,\upsilon^n))\right\|\\&\quad\leq\|\upsilon^m-\upsilon^n\|+\left\|w^-_m(\tau,\upsilon^m)\right\|+\left\|w^-_n(\tau,\upsilon^n)\right\|\overset{(4.18)}{<}\varepsilon\quad\text{for all }m\geq n\geq N_2.\end{aligned}$$

Hence, the Cauchy property is fulfilled in the Banach space X and

$$\bar{T}(\tau,v_0):=\lim_{n\to\infty}\left(\upsilon^n+w^-_n(\tau,\upsilon^n)\right)$$

is the unique T_τ-preimage of v_0. Indeed, setting $u_0:=\bar{T}(\tau,v_0)$ one obtains

$$\lim_{m\to\infty}P^n_1(\tau)\left[T_\tau(u_0)-T_\tau(\upsilon^m+w^-_m(\tau,\upsilon^m))\right]=0$$

from the continuity of T_τ and using step (II) we derive the relation

$$P^n_1(\tau)\left[T_\tau(u_0)-T_\tau(\upsilon^m+w^-_m(\tau,\upsilon^m))\right]=P^n_1(\tau)\left(T_\tau(u_0)-\sum_{k=1}^{n}\eta_k\right)\ \text{for all} n<m.$$

Consequently, $P_1^n(\tau)T_\tau(u_0) = \sum_{k=1}^n \eta_k$ and $T_\tau(u_0) = v_0$ results.

(V) *Claim:* $\bar{T} : \mathbb{R} \times X \to X$ *is continuous.* Let $(\tau_0, v_0) \in \mathbb{R} \times X$ be arbitrary, choose $\varepsilon > 0$ and abbreviate $v^n := P_1^n(\tau)v$, $v_0^n := P_1^n(\tau)v_0$, $n \in \mathbb{N}$.

- As in (4.19) there exist $N_1, N_2 \in \mathbb{N}$ such that both $\left\|\bar{T}_\tau^m(v^m) - \bar{T}_\tau^n(v^n)\right\| < \frac{\varepsilon}{4}$ for $m \geq n \geq N_1$ and $\left\|\bar{T}_{\tau_0}^m(v^m) - \bar{T}_{\tau_0}^n(v^n)\right\| < \frac{\varepsilon}{4}$ for $m \geq n \geq N_2$ hold. Hence, passing to the limit $m \to \infty$ and setting $N := \max\{N_1, N_2\}$ results in

$$\left\|\bar{T}_\tau(v) - \bar{T}_\tau^n(v^n)\right\| < \tfrac{\varepsilon}{3}, \qquad \left\|\bar{T}_{\tau_0}(v_0) - \bar{T}_{\tau_0}^n(v_0^n)\right\| < \tfrac{\varepsilon}{3} \quad \text{for all } n \geq N. \tag{4.20}$$

- Because $\bar{T}^N$ is continuous due to Prop. 4.1, there exists a $\delta_1 > 0$ such that the inclusion $(\tau, v^N) \in B_{\delta_1}(\tau_0, v_0^N)$ implies $\left\|\bar{T}_\tau^N(v^N) - \bar{T}_{\tau_0}^N(v_0^N)\right\| < \frac{\varepsilon}{3}$. Now we choose $\delta \in (0, \delta_1)$ so small that

$$\left\|v^N - v_0^N\right\| = \left\|P_1^N(\tau)[v - v_0]\right\| \overset{(2.2)}{\leq} K_N \left\|v - v_0\right\| < \delta_1 \quad \text{for all } v \in B_\delta(v_0)$$

and arrive at

$$\left\|\bar{T}_\tau^N(v^N) - \bar{T}_{\tau_0}^N(v_0^N)\right\| < \tfrac{\varepsilon}{3} \quad \text{for all } (\tau, v) \in B_\delta(\tau_0, v_0).$$

Given this, the triangle inequality yields the estimate

$$\begin{aligned}
\left\|\bar{T}_\tau(v) - \bar{T}_{\tau_0}(v_0)\right\| &\leq \left\|\bar{T}_\tau(v) - \bar{T}_\tau^N(v^N)\right\| + \left\|\bar{T}_\tau^N(v^N) - \bar{T}_{\tau_0}^N(v_0^N)\right\| \\
&\qquad + \left\|\bar{T}_{\tau_0}^N(v_0^N) - \bar{T}_{\tau_0}(v_0)\right\| \\
&\leq \left\|\bar{T}_\tau(v) - \bar{T}_\tau^N(v^N)\right\| + \tfrac{\varepsilon}{3} + \left\|\bar{T}_{\tau_0}^N(v_0^N) - \bar{T}_{\tau_0}(v_0)\right\| \overset{(4.20)}{<} \varepsilon
\end{aligned}$$

for all pairs $(\tau, v) \in B_\delta(\tau_0, v_0)$. This implies that $\bar{T}$ is continuous.

(VI) *Claim:* $\lim_{u \to 0} \bar{T}_\tau(u) = 0$ *uniformly in* $\tau \in \mathbb{R}$. According to their definition in step (IV) the mappings $v^n : \mathbb{R} \times X \to X$ fulfill $v^n(\tau, v) = \bar{T}_\tau^n(P_1^n(\tau)v)$ for all $n \in \mathbb{N}$, $v \in X$. Since the sequence $(K_k)_{k \in \mathbb{N}}$ of dichotomy constants is bounded, we can guarantee by means of (4.7) that the constants ℓ_k, ℓ_k^* stay below 1 uniformly in $k \in \mathbb{N}$; the factor $\frac{1}{5}$ in (4.7) was introduced to ensure this. Thus, the estimates (3.16), (3.37) can be realized to hold uniformly in $k \in \mathbb{N}$ as well. Due to the construction of $\bar{T}_\tau^n$ in step (III) of the proof to Prop. 4.1, this guarantees that the limit $\lim_{v \to 0} v^n(\tau, v) = 0$ holds uniformly in both $\tau \in \mathbb{R}$ and $n \in \mathbb{N}$. Since also (3.8) holds uniformly, we can conclude the claim from (4.14).

(VII) The invariance properties of the integral manifolds and leafs involved guarantee that T maps solutions of (E) to solutions of (D). It also follows that $\bar{T}$ maps solutions of the decoupled eqn. (D) to solutions of our initial evolution eqns. (E). □

5. **Topological linearization.** Having the above technical preparations at hand, we are finally in the position to formulate our main results. The sole missing ingredient is a linearization result for d_k-dimensional ODEs (E_k) with hyperbolic linear part. It is based on the premise that w.l.o.g. one can always choose the sequences $(\alpha_n)_{n \in \mathbb{N}_0}$, $(\beta_n)_{n \in \mathbb{N}}$ to have values different from 0 and thus also

$$\nu_n := \begin{cases} \beta_n, & \lambda_n^- > 0, \\ \alpha_{n-1}, & \lambda_n^+ < 0 \end{cases} \neq 0 \quad \text{for all } n \in \mathbb{N}.$$

Proposition 5.1. *Suppose that the assumptions* (**L**), (**N**), (**B**) *hold. If*

$$0 \notin \Sigma(A_k), \qquad\qquad 8K_k L < |\nu_k| \quad \textit{for some } k \in \mathbb{N},$$

then there is a topological conjugation $S^k : \mathcal{X}_k \to X$ between the ODEs (E_k) and

$$\boxed{\dot{y} = A_k(t)y} \tag{L_k}$$

in $\mathcal{X}_k$ with the following properties:

(a) $S^k(\tau, y) \in \mathcal{X}_k(\tau)$ and

$$\max\left\{\left\|S^k(\tau,y) - y\right\|, \left\|\bar{S}^k(\tau,y) - y\right\|\right\} \le \frac{4K_kC}{|\nu_k|} \quad \text{for all } (\tau, y) \in \mathcal{X}_k. \tag{5.1}$$

(b) For θ-periodic ODEs (E_k) the functions S and $\bar{S}$ are θ-periodic in, and for autonomous (E_k) even independent of the first variable.

Proof. First of all, we have $\Sigma(A_k) = \left[\lambda_k^-, \lambda_k^+\right]$ and we equip $\mathcal{X}_k(t)$ with the adapted norm $\|x\|_t := \|P_k(t)x\|$. Then the boundedness assumption (**B**) implies

$$\|F_k(t,y)\|_t = \|F(t, y + w_k(t,y))\| \le C \quad \text{for all } (t,y) \in \mathcal{X}_k$$

and similarly $\operatorname{lip}_2 F_k \le 2L$. These conditions allow us to apply [16, Prop. 5.2] to

- the ODEs (E_k) and (L_k) yielding a continuous mapping $S^k : \mathcal{X}_k \to X$ with the claimed properties
- the ODEs (L_k) and (E_k) guaranteeing a continuous inverse $\bar{S}^k$. In detail, the assumption $8K_kL < |\nu_k|$ combined with [16, (5.7)] guarantees that

$$\left\|\bar{S}^k(t,y) - y\right\| \le \frac{2K_kC}{|\nu_k| - 4K_kL} \le \frac{4K_kC}{|\nu_k|} \quad \text{for all } (t,y) \in \mathcal{X}_k^-.$$

Notice that the above estimates hold w.r.t. the norm $\|\cdot\|$, since the considered maps have images and arguments in $\mathcal{X}_k$. The periodicity assertion (b) can be shown as in [24, p. 222, 5.5. Lemma]. □

This finally brings us to our following main result:

Theorem 5.2 (Palmer-Šošitaĭšvili linearization of (E))**.** *Suppose that the assumptions* (**L**), (**N**), (**B**), *as well as the decay condition* (4.8) *hold with*

$$L < \frac{1}{5} \inf_{k \in \mathbb{N}} \min\left\{\frac{\beta_k - \alpha_k}{K_k(K_k+2)}, \frac{|\nu_k|}{2K_k}\right\} \tag{5.2}$$

and a bounded sequence $(K_k)_{k\in\mathbb{N}}$. If $\Sigma(A)$ contains a spectral interval satisfying

$$0 \in [\lambda_{k^*}^-, \lambda_{k^*}^+] \quad \text{for some } k^* \in \mathbb{N}, \tag{5.3}$$

then (E) is topologically conjugated to the decoupled equation

$$\boxed{\dot{u} = A(t)u + P_{k^*}(t)F(t, P_{k^*}(t)u + w_{k^*}(t,u)).} \tag{D^*}$$

Proof. In Thm. 4.2 we established a topological conjugation $T : \mathbb{R} \times X \to X$ between the semilinear eqn. (E) and the fully decoupled problem (D). Each component is of the form (E_k), i.e. a d_k-dimensional ODE whose linear part (L_k) has the dichotomy spectrum $[\lambda_k^-, \lambda_k^+]$, $k \in \mathbb{N}$. Moreover, for $k \ne k^*$ the assumptions of Prop. 5.1 hold. It yields a topological conjugation $S^k : \mathcal{X}_k \to X$ satisfying both $S^k(\tau, x) \in \mathcal{X}_k(t)$ and (5.1), which transforms each (E_k) into (L_k) for every $k \ne k^*$.

Due to our decay condition (4.8) it results from (5.1) and Lemma A.2 that

$$(\tau, x) \mapsto \sum_{k \ne k^*} \left(S^k(\tau, P_k(\tau)x) - P_k(\tau)x\right)$$

is continuous. Hence, the series $\sum_{k\ne k^*} S^k(\tau, P_k(\tau)x)$ exists as a function continuous in (τ, x) and we define $S : \mathbb{R} \times X \to X$, $S(\tau, x) := \sum_{k\ne k^*} S^k(\tau, P_k(\tau)x)$. With

Prop. 5.1 its inverse is $\bar{S}(\tau,x) := \sum_{k\neq k^*} \bar{S}^k(\tau,P_k(\tau)x)$, whose convergence and continuity is established as above. In conclusion, the composition $(\tau,x)\mapsto S_\tau(T_\tau(x))$ is the desired topological conjugation between (E) and (D^*). □

As an immediate consequence let us address the hyperbolic situation:

Corollary 5.3 (Hartman-Grobman linearization of (E))**.** *If rather than* (5.3) *one has the hyperbolicity condition*

$$0 \notin \Sigma(A),$$

then (E) *and its linear part* (L) *are topologically conjugated.*

Proof. Prop. 5.1 applies for all $k\in\mathbb{N}$ in the proof of Thm. 5.2. □

6. **Applications and perspectives.** At first glance the applicability of our above results seems to be somewhat limited due to the global assumptions ($\mathbf{N_2}$) and ($\mathbf{B}$) on the nonlinearity, as well as the specific spectrum required in ($\mathbf{L_3}$) combined with the summability assumption (4.8). Such objections are easy to debilitate:

For autonomous evolution equations the spectral intervals $[\lambda_n^-,\lambda_n^+]$ in ($\mathbf{L_3}$) degenerate to eigenvalue real parts $\Re\lambda_n$. Thus, in order to fulfill the summability condition (4.8), their asymptotic behavior must be of the form $\Re\lambda_n \sim Cn^\alpha$ in the limit $n\to\infty$ with some $\alpha>1$.

- When dealing with semilinear PDEs, under the standard boundary conditions this holds for the Laplacian Δ in one spatial dimension, or the poly-Laplacian $-(-\Delta)^m$ in $d<2m$ spatial dimensions. In [18, Sect. 3] we provide several examples of nonautonomous parabolic PDEs formulated as abstract evolution equations in $X=L^2(\Omega)$, where the assumptions ($\mathbf{L}$) can be justified.
- In the area of FDEs the decay condition (4.8) is more problematic. As shown in the classical paper [27, Thm. 5] already the simple delay differential equation $x'(t) = -\alpha x(t-1)$, $\alpha>0$, has a spectrum $\{\lambda_n : n\in\mathbb{N}\}$ with
$$\Re\lambda_n = -\ln\left(\tfrac{(4n+1)\pi}{2\alpha}\right) + O\left((\tfrac{\ln n}{n})^2\right) \quad \text{as } n\to\infty$$
and thus merely logarithmic decay.

Our main Thm. 5.2 and its Cor. 5.3 apply when interested in the local behavior near fixed reference solutions. By passing to the equation of perturbed motion one establishes ($\mathbf{N_1}$) (trivial solution). The global Lipschitz condition ($\mathbf{N_2}$), the boundedness assumption ($\mathbf{B}$) and (4.7), (5.2) hold after applying a standard cut-off procedure (cf., for instance [17, pp. 364ff, Sect. C.2]) yielding local results.

We illustrate these remarks by means of two classes of parabolic PDEs allowing a formulation as abstract eqn. (E) in the Hilbert space $H_0^1 := H_0^1(0,\ell)$, $0<\ell$. The open ball with radius r and center 0 in this space will be denoted by B_r.

6.1. **Nonautonomous reaction-diffusion equations in 1*d*.** Consider a nonautonomous reaction-diffusion equation

$$\boxed{u_t = a(t)\partial_x^2 u + g(t,u)} \tag{6.1}$$

equipped with homogeneous Dirichlet boundary conditions $u(t,0)=0=u(t,\ell)$ for some $\ell>0$. Here, $a:\mathbb{R}\to(0,\infty)$ is bounded and continuous, while $g:\mathbb{R}^2\to\mathbb{R}$ is a continuous function whose partial derivatives $D_2^i g$ exist as continuous functions such that for every bounded $B\subseteq\mathbb{R}$ there is a $C\geq 0$ fulfilling

$$\sup_{t\in\mathbb{R}} \left|D_2^i g(t,u)\right| \leq C \quad \text{for all } u\in B,\ i=1,2,3. \tag{6.2}$$

Let us moreover suppose that (6.1) has a bounded reference solution $u^*:\mathbb{R}\to\mathbb{R}$ which is independent of the spatial variable (e.g. a solution of $\dot u = g(t,u)$) with

$$\lim_{u\to 0} D_2 g(t,u^*(t)+u) = D_2 g(t,u^*(t)) \quad \text{uniformly in } t\in\mathbb{R}. \tag{6.3}$$

Linearizing (6.1) along u^* therefore yields

$$\partial_t u = [a(t)\partial_x^2 + b(t)]u + f(t,u), \tag{6.4}$$

where (relying on the mean value theorem)

$$b(t) := D_2 g(t,u^*(t)), \quad f(t,u) := \int_0^1 [D_2 g(t,u^*(t)+hu) - D_2 g(t,u^*(t))]\, u \,\mathrm{d}h.$$

Let us formulate (6.4) as abstract evolution eqn. (E) on $X = H_0^1$ with

$$(A(t)u)(x) := a(t)u_{xx}(x) + b(t)u(x), \quad F(t,u)(x) := f(t,u(x)) \quad \text{for all } x\in(0,\ell).$$

The linear part (L) generates an evolution family as required in ($\mathbf{L_1}$) and because $a,b:\mathbb{R}\to\mathbb{R}$ are bounded functions, also the growth bound ($\mathbf{L_2}$) holds. Furthermore, [18, Thm. 3.8 and Lemma 2.3] yields the dichotomy spectrum

$$\Sigma(A) = \bigcup_{n\in\mathbb{N}} \left[\underline{\beta}(b - (\tfrac{\pi n}{\ell})^2 a), \overline{\beta}(b - (\tfrac{\pi n}{\ell})^2 a)\right],$$

where $\underline{\beta},\overline{\beta}$ denote the lower resp. upper Bohl exponent.[1] Yet, to what extend ($\mathbf{L_3}$) holds, crucially depends on the diffusion coefficient a and deserves further remarks:

- For constant $a(t)\equiv\alpha>0$ the spectrum is a sequence of identical intervals $\Sigma(A) = \bigcup_{n\in\mathbb{N}}\{-\alpha(\frac{\pi n}{\ell})^2\} + [\underline{\beta}(b),\overline{\beta}(b)]$ decaying to $-\infty$ quadratically.
- For constant $b(t)\equiv\beta$ it is $\Sigma(A) = \{\beta\} + \bigcup_{n\in\mathbb{N}}\left[\underline{\beta}(-(\frac{\pi n}{\ell})^2 a), \overline{\beta}(-(\frac{\pi n}{\ell})^2 a)\right]$ and $\Sigma(A)$ could consist of only finitely many intervals violating ($\mathbf{L_3}$). For instance, this occurs when $\lim_{t\to\pm\infty} a(t) = \alpha^\pm$ with $\alpha^-\neq\alpha^+$ (cf. [18, Ex. 3.7]).
- Having θ-periodic functions a,b implies a discrete spectrum

  $$\Sigma(A) = \frac{1}{\theta}\int_0^\theta a(s)\,\mathrm{d}s \bigcup_{n\in\mathbb{N}}\{-(\tfrac{\pi n}{\ell})^2\} + \left\{\frac{1}{\theta}\int_0^\theta b(s)\,\mathrm{d}s\right\}$$

  and therefore the required behavior is fulfilled, unless $\int_0^\theta a(s)\,\mathrm{d}s = 0$.

If one has infinitely many spectral intervals, then $K_n = 1$, $\dim\mathcal{X}_n = 1$ holds eventually. The compactness property ($\mathbf{L_3}$) can be verified according to [23, pp. 244ff].

As in [23, p. 271] one shows that $F:\mathbb{R}\times H_0^1\to H_0^1$ is well-defined with $F(t,0)\equiv 0$ on $\mathbb{R}$ and locally Lipschitz, i.e. for every $r\geq 0$ there exists a $\ell(r)\geq 0$ such that

$$\|F(t,u)-F(t,\bar u)\|_{H_0^1} \leq \ell(r)\,\|u-\bar u\|_{H_0^1} \quad \text{for all } t\in\mathbb{R},\ u,\bar u\in B_r.$$

Thanks to (6.2), (6.3) the limit relation $\lim_{r\searrow 0}\ell(r) = 0$ holds. The radial retraction on the Hilbert space H_0^1 has Lipschitz constant 1 (cf. [17, p. 364, Lemma C.2.1])

[1] given a bounded continuous function $a:\mathbb{R}\to\mathbb{R}$ their *Bohl exponents* are defined as

$$\underline{\beta}(a) := \sup\left\{\omega\in\mathbb{R}|\,\exists K_\omega>0:\ K_\omega \leq \exp\left(\int_\tau^t a(s)-\omega\,\mathrm{d}s\right) \quad \text{for all } \tau\leq t\right\},$$

$$\overline{\beta}(a) := \inf\left\{\omega\in\mathbb{R}|\,\exists K_\omega>0:\ \exp\left(\int_\tau^t a(s)-\omega\,\mathrm{d}s\right)\leq K_\omega \quad \text{for all } \tau\leq t\right\}$$

and with [17, p. 365, Prop. C.2.5] there exists a globally Lipschitzian modification $F^\rho : \mathbb{R} \times H_0^1 \to H_0^1$ of F satisfying $F^\rho(t,u) \equiv F(t,u)$ on $\mathbb{R} \times B_\rho$ and

$$\operatorname{lip}_2 F^\rho \le \ell(\rho), \qquad \|F^\rho(t,u)\|_{H_0^1} \le \ell(\rho)\rho \quad \text{for all } (t,u) \in \mathbb{R} \times H_0^1.$$

Hence, both assumptions (**N**) and (**B**) can be satisfied with arbitrarily small constants $L, C \ge 0$, by choosing $\rho > 0$ sufficiently close to 0. In summary, provided solutions do not escape in forward time, within a ρ-neighborhood of a reference solution u^*, the reaction-diffusion eqn. (6.1) is locally topologically conjugated to its linearization

$$\boxed{\partial_t u = [a(t)\partial_x^2 + D_2 g(t, u^*(t))]u.}$$

6.2. **Nonautonomous convection equations in** $1d$**.** A further ad hoc application are convective reaction-diffusion equations

$$\boxed{\partial_t u = a(t)\partial_x^2 u + \partial_x(f(t,u)) + g(t,u)} \tag{6.5}$$

equipped with Dirichlet boundary conditions as in Subsect. 6.1 and the same assumption on the diffusion coefficient $a : \mathbb{R} \to (0,\infty)$. For simplicity, let the continuous functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ be polynomials in the second argument,

$$f(t,u) = \sum_{j=2}^{2k+1} f_j(t)u^j, \qquad g(t,u) = \sum_{j=2}^{2k+1} g_j(t)u^j \quad \text{for all } t,u \in \mathbb{R}$$

satisfying $g_{2k+1}(t) < 0$ for some $k \in \mathbb{N}$. Moreover, we assume there exist constants $C \ge 1$, $r_0 > 0$, $m \in \mathbb{N}$ so that

$$\left|D_2^{1+i} f(t,u)\right| \le C(1+|u|^{m-i}), \quad |D_2 g(t,u)| \le C(1+|u|^m) \quad \text{for all } t,u \in \mathbb{R},\ i = 0,1$$

and $g(t,u)u \le 0$ hold for all $t \in \mathbb{R}$, $|u| \ge r_0$.

Following [23, pp. 313ff] one writes (6.5) as abstract evolution equation

$$\dot u = A(t)u + B(t,u) + G(t,u) \tag{6.6}$$

in the Hilbert space $X = H_0^1$ with the operators $(A(t)u)(x) := a(t)\partial_x^2 u(x)$,

$$B(t,u)(x) := D_2 f(t,u(x))u'(x), \qquad G(t,u)(x) := g(t,u(x)).$$

In this setting, the spectrum of the linear part (L) reads as (cf. [18, Thm. 3.5])

$$\Sigma(A) = \bigcup_{n\in\mathbb{N}} \left[\underline{\beta}\big(-\big(\tfrac{\pi n}{\ell}\big)^2 a\big), \overline{\beta}\big(-\big(\tfrac{\pi n}{\ell}\big)^2 a\big)\right]$$

and the discussion whether (**L**) can be fulfilled is pursuant to Subsect. 6.1. The mappings $B, G : \mathbb{R} \times H_0^1 \to H_0^1$ are well-defined and differentiable in the second argument having the continuous derivatives

$$(D_2 B(t,u)v)(x) = D_2^2 f(t,u(x))u'(x)v(x) + D_2 f(t,u(x))v'(x),$$
$$(D_2 G(t,u)v)(x) = D_2 g(t,u(x))v(x).$$

Due to [23, p. 317, Lemma 53.3] the mild solutions to (6.6) generate a 2-parameter semiflow. It is clear that (6.5) and in turn (6.6) possess the trivial solution. Since it is $D_2 B(t,0) \equiv 0$, $D_2 G(t,0) \equiv 0$ on $\mathbb{R}$, the same cut-off technique as applied in Subsect. 6.1 allows to modify B, G outside a sufficiently small neighborhood of 0 in order to establish (**N**), (**B**). Hence, locally near the trivial solution the convection eqn. (6.5) is topologically conjugated to its linearization

$$\boxed{\partial_t u = a(t)\partial_x^2 u.}$$

6.3. **Outlook.** The decay condition (4.8) itself is clearly required for our construction to work. We do not know if it is of purely technical nature, i.e. whether there are for instance reaction-diffusion equations in spatial domains with $d > 1$ failing to allow a topological linearization á la Cor. 5.3?

Nevertheless, the above examples extend to corresponding systems of reaction-diffusion eqns. (6.1) and (6.5), but (due to the asymptotics of their eigenvalues) only on spatial domains $\Omega \subseteq \mathbb{R}$. This deficit is attenuated when dealing with higher-order parabolic equations, if they can be formulated abstractly as (E). Another possible application is the phase-field model discussed in [3, pp. 107ff].

With regard to further PDE examples, an obvious limitation of our approach is assumption ($\mathbf{N_2}$) requiring a Lipschitzian nonlinearity from an interpolation space X into itself, rather than between a pair of continuously (and densely) embedded spaces. In the autonomous case such an extension was given in [3] and yields an accordingly modified decay condition (4.8). Although being technically more involved than [12] (and hence the present paper), on a conceptional level the arguments remain quite similar yet.

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Appendix A. Let X, Y be Banach spaces and P be a metric space.

Lemma A.1 (Lipschitz inverse function theorem)**.** *Suppose $f : X \times P \to X$ satisfies $\operatorname{lip}_1 f < 1$. If $f(x, \cdot) : P \to X$ is continuous for all $x \in X$, then for every $y \in X$, $p \in P$ there exists a unique solution $x^*(y, p) \in X$ of the equation $x + f(x, p) = y$, where the function $x^* : X \times P \to X$ is continuous and fulfills $\operatorname{lip}_1 x^* \leq \frac{1}{1-\operatorname{lip}_1 f}$.*

Proof. Since $T : X \times P \times X \to X$, $T(x, y, p) := y - f(x, p)$ is a uniform contraction in the first argument, the assertion follows from, e.g. [17, p. 352, Thm. B.1.1]. □

Lemma A.2. *Suppose the real series $(\sum_{k=1}^n a_k)_{n\in\mathbb{N}}$ converges. If the continuous functions $T_k : X \times P \to X$ fulfill for all $k \in \mathbb{N}$ that*

(a) $\|T_k(x, p)\| \leq a_k$

(b) $\lim_{x\to 0} T_k(x, p) = 0$ *uniformly in* $p \in P$,

then $T : X \times P \to X$, $T(x, p) := \sum_{k=1}^\infty T_k(x, p)$ exists as a continuous function and satisfies the limit relation $\lim_{x\to 0} T(x, p) = 0$ uniformly in $p \in P$.

Proof. Since $(\sum_{k=1}^n a_k)_{n\in\mathbb{N}}$ converges, the Weierstraß M-test yields uniform convergence of $(\sum_{k=1}^n T_k(x, p))_{n\in\mathbb{N}}$ and thus T is well-defined and continuous. In order to establish the claimed limit relation, let $\varepsilon > 0$. By the uniform convergence we obtain that there exists a $N \in \mathbb{N}$ such that

$$\left\|\sum_{k=1}^n T_k(x, p) - T(x, p)\right\| < \frac{\varepsilon}{2} \quad \text{for all } n \geq N,\ x \in X \text{ and } p \in P.$$

The limit relation $\lim_{x\to 0} \sum_{k=1}^{N+1} T_k(x, p) = 0$ holds uniformly in $p \in P$ and so there exists a $\delta > 0$ satisfying $\left\|\sum_{k=1}^{N+1} T_k(x, p)\right\| < \frac{\varepsilon}{2}$ for all $x \in B_\delta(0)$, $p \in P$. Hence,

$$\|T(x, p)\| \leq \left\|T(x, p) - \sum_{k=1}^{N+1} T_k(x, p)\right\| + \left\|\sum_{k=1}^{N+1} T_k(x, p)\right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $x \in B_\delta(0)$, $p \in P$ and thus the claim. □

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