DISSIPATIVE DELAY ENDOMORPHISMS
AND ASYMPTOTIC EQUIVALENCE

CHRISTIAN PÖTZSCHE

ABSTRACT. Using an invariant manifold theorem we demonstrate that the dynamics of nonautonomous dissipative delayed difference equations (with delay M) is asymptotically equivalent to the long-term behavior of an N-dimensional first order difference equation (with $N \leq M$) – assumed the nonlinearity is small Lipschitzian on the absorbing set. As consequence we obtain a result of Kirchgraber that multi-step methods for the numerical solution of ordinary differential equations are essentially one-step methods, and generalize it to varying step-sizes.

1. MOTIVATION AND INTRODUCTION

Scalar real delay (or higher order) difference equations appear in a variety of applications, ranging from biosciences (Mackey-Glass, Wazewska-Czyzewska and Lasota equation, etc.) to numerical analysis (multi-step methods for the numerical solution of ordinary, or discretizations of delay differential equations). Provided their delay is M (or equivalently, their order is $M + 1$), they can be formulated as first order equations in $\mathbb{R}^{M+1}$ and become accessible to the theory of discrete dynamical systems. It is well-known that the possible complexity of their long-time behavior depends on the dimension of the state space, i.e., the size of M. Ideally, one wants to keep M as small as possible, but frequently the delay is dictated from the given model.

In this article the geometry of delay difference equations is studied using invariant manifold theory. The problems under consideration are assumed to be dissipative with contractive linear part and a locally Lipschitzian nonlinearity. Provided the Lipschitz constants on the absorbing set are small, we can associate an asymptotically equivalent lower dimensional equation to the original problem. Hence, we are able to reduce the dimension of the state space for a delay difference equation without loosing information on its long-time behavior.

Such a global reduction principle is well-known in the area of dissipative evolutionary partial differential equations. Here the existence of a so-called *inertial manifold* ensures that the corresponding infinite-dimensional semiflow asymptotically behaves as the solutions of a finite-dimensional ordinary differential equation (the *inertial form*; cf., e.g., [SY02, pp. 569ff, Chapter 8]). In order to study the persistence of inertial manifolds under (numerical) discretizations one needs flexible invariant manifold theorems for mappings. Such results have been derived for instance in [Pöt07a, Pöt07b] and can be simplified to Theorem C.1 of this paper, allowing an application to delay difference equations. Attractive invariant manifold theorems for mappings already date back to [Har64] and were also considered by [KS78] with generalizations in [NS92]. Differing from center-unstable manifolds they provide a reduction principle on the whole absorbing set and not only in a small neighborhood.

Our basic contribution is to discuss nonautonomous difference equations instead of discrete dynamical systems (mappings). Dissipativity is understood in the sense of pullback convergence (cf., e.g., [Klo00]) and invariant manifolds generalize to invariant fiber bundles.

After some necessary terminology is presented in Section 2, we provide a simple criterion for dissipativity in Section 3 and demonstrate its applicability to nonautonomous versions of various well-studied delay difference equations. The following Section 4 equips us with sufficient conditions that delay difference equations (with delay M) feature asymptotically the same dynamical behavior as first order difference equations in $\mathbb{R}^N$ with $N \leq M$. As an application we show in the final Section 5 that strongly stable multi-step methods for the numerical integration of ordinary differential equations are conjugated to one-step methods. In the autonomous constant step-size setting this is a result due to [Kir86] with generalizations to general linear methods by [Sto93]. Our approach is intended to demonstrate how such results can be lifted to schemes with varying step-sizes and nonautonomous ordinary differential equations. Due to the lack of space, our analysis is somehow crude neglecting convergence issues; for that we refer the interested reader to the nice and comprehensive discussion in [Sto93]. Finally, for the reader's convenience, the Appendix contains some notions important for our nonautonomous perspective, namely nonautonomous sets, exponential dichotomies and invariant fiber bundles.

Concerning terminology, $\mathbb{Z}_\kappa^+ := [\kappa, \infty) \cap \mathbb{Z}$ for integers $\kappa \in \mathbb{Z}$. We index real N-tupels $x \in \mathbb{R}^N$ according to $x = (x_{-N+1}, \ldots, x_0)$ and to provide largely explicit assumptions we use the norm $\|x\| := \max_{j=-N+1}^0 |x_j|$ throughout. For the open ball in $\mathbb{R}^N$ centered in 0 with radius $r > 0$ we simply write B_r. Given a matrix $A \in \mathbb{R}^{N \times M}$ we denote its transpose by $A^T A \in \mathbb{R}^{M \times N}$, the ith row, jth column element by $A_{i,j} \in \mathbb{R}$, and obtain the induced norm

$$\|A\| = \max_{i=-N+1}^0 \sum_{j=-M+1}^0 |A_{i,j}|.$$

Moreover, $\operatorname{im} A$ denotes the *range* (*image*) and $\ker A$ the *kernel* (*nullspace*) of A. The identity matrix on $\mathbb{R}^N$ is I_N and O_N the zero matrix.

2. Delay difference equations versus delay endomorphisms

This paper deals with scalar nonautonomous $M + 1$-th order difference equations

(2.1)
$$\boxed{x(k+1) = f(k, x(k-M), \ldots, x(k-1), x(k)),}$$

where $M \in \mathbb{Z}_0^+$ is interpreted as *delay*, $f : \mathbb{Z} \times I^{M+1} \to I$ is the *right hand side* and $I \subseteq \mathbb{R}$ an interval. Given an initial time $\kappa \in \mathbb{Z}$, a *solution* of the *delay difference equation* is a sequence $\phi : \mathbb{Z}_{\kappa-M}^+ \to I$ satisfying the identity (2.1) and initial value problems are well-posed, if beyond $\phi(\kappa) \in I$ also supplementary values $\phi(\kappa-M), \cdots, \phi(\kappa-1) \in I$ are known. The unique solution $\varphi(\cdot; \kappa, \xi_{-M}, \ldots, \xi_0) : \mathbb{Z}_{\kappa-M}^+ \to I$ starting at time $\kappa \in \mathbb{Z}$ with initial values $x(\kappa+j) = \xi_j$ for $-M \leq j \leq 0$ is the *general solution* of (2.1). It satisfies the *cocycle property*

$$\varphi(k; l, \varphi(l-M; \kappa, \xi), \ldots, \varphi(l; \kappa, \xi)) = \varphi(k; \kappa, \xi) \quad \text{for all } k \geq l-M, \, l \geq \kappa - M,$$

where we have abbreviated $\xi = (\xi_{-M}, \ldots, \xi_0) \in I^{M+1}$.

From the above it is clear that the natural state space for (2.1) is $M + 1$-dimensional. In order to make this more precise, we introduce the mapping $\hat{f} : \mathbb{Z} \times I^{M+1} \to I^{M+1}$,

$$\hat{f}(k, x_{-M}, \ldots, x_0) := \begin{pmatrix} x_{-M+1} \\ \vdots \\ x_0 \\ f(k, x_{-M}, \ldots, x_{-1}, x_0) \end{pmatrix},$$

denoted as *delay endomorphism* associated with (2.1). Having this notion at hand, the delay difference equation (2.1) is equivalent to the first order difference equation

$$(2.2) \qquad \boxed{x(k+1) = \hat{f}(k, x(k))}$$

in the higher dimensional set $I^{M+1} \subseteq \mathbb{R}^{M+1}$ in the following sense: The solutions $\phi : \mathbb{Z}^+_{\kappa-M} \to I$ of (2.1) and $\hat{\phi} : \mathbb{Z}^+_{\kappa} \to I^{M+1}$ of (2.2) are related by the identities

$$(2.3) \qquad \phi_j(k) = \hat{\phi}_0(k+j) \quad \text{for all } -M \leq j < 0, \qquad\qquad \phi_0(k) = \hat{\phi}_0(k)$$

and the general solution $\hat{\varphi}$ of (2.2) can be interpreted as discrete (2-parameter) semiflow of the delay difference equation (2.1).

In case the mapping f is linear, we can write (2.1) as

$$(2.4) \qquad \boxed{x(k+1) = \sum_{j=-M}^{0} \ell_j(k) x(k+j)}$$

with sequences $\ell_{-M}, \dots, \ell_0 : \mathbb{Z} \to \mathbb{R}$, and borrowing terminology from [HNW93, pp. 402ff] we denote the delay endomorphism associated with the linear equation (2.4) as *companion matrix*

$$(2.5) \qquad L(k) := \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ \ell_{-M}(k) & \ell_{-M+1}(k) & \dots & \ell_{-1}(k) & \ell_0(k) \end{pmatrix} \qquad \text{for all } k \in \mathbb{Z}.$$

The equivalence between (2.1) and (2.2) enables us to apply many results from the theory of nonautonomous dynamical systems to delay difference equations (see Appendix A for some terminology needed in the following). Here, due to the special algebraic structure of the delay endomorphism $\hat{f}$, certain assumptions can be simplified.

For the purpose of studying the dynamical behavior of (2.1) we restrict to a class of delay difference equations, whose solutions eventually enter a bounded set.

Definition 2.1. A delay difference equation (2.1) is said to be *pullback dissipative*, if I has a bounded subset $A \subseteq I$ such that for all bounded $B \subseteq I$ there exists a $N = N(B) \in \mathbb{Z}^+_0$ such that

$$\varphi(k; k-n, \xi) \in A \quad \text{for all } k \in \mathbb{Z}, \, n \geq N, \, \xi \in B^{M+1}.$$

One denotes A as *absorbing set* of (2.1).

Remark 2.2. (1) The existence of an absorbing set $A \subseteq I$ has far reaching consequences. For a continuous right hand side $f : \mathbb{Z} \times I^{M+1} \to I$ one can prove the existence of a *global pullback attractor* $\mathcal{A}$ for (2.1) (cf. [Klo00, Theorem 3.6]). This is a uniquely determined nonautonomous set $\mathcal{A} \subseteq \mathbb{Z} \times A^{M+1}$ with the following properties:

    (a) $\mathcal{A}(k)$ is compact,
    (b) $\mathcal{A}(k) = \hat{\varphi}(k; \kappa, \mathcal{A}(\kappa))$ for all $\kappa \leq k$ (*invariance*), and
    (c) $\lim_{n\to\infty} \text{dist}(\hat{\varphi}(\kappa, \kappa-n, \xi), \mathcal{A}(\kappa-n)) = 0$ for all $\kappa \in \mathbb{Z}, \xi \in I^{M+1}$ (*attractivity*).

Dynamically the pullback attractor $\mathcal{A}$ consists of all pairs $(\kappa, \xi_{-M}, \dots, \xi_0) \in \mathcal{A}$ such that there exists a bounded solution $\phi : \mathbb{Z} \to I$ of (2.1) with $\phi(\kappa-M) = \xi_{-M}, \dots, \phi(\kappa) = \xi_0$. In particular, $\mathcal{A}$ contains stationary, periodic, homoclinic and heteroclinic solutions of (2.1).

(2) For bounded intervals I the delay difference equation (2.1) is trivially pullback dissipative with absorbing set I.

3. Dissipative delay difference equations

For the specific scalar delay difference equation

$$(3.1) \qquad \boxed{x(k+1) = \lambda x(k) + g(k, x(k-M), \ldots, x(k-1), x(k)),}$$

we can deduce dissipativity criteria under the assumption $\lambda \in (0,1)$ and $g : \mathbb{Z} \times I^{M+1} \to I$. They are based on a simple lemma guaranteeing forward boundedness of solutions.

Lemma 3.1. *For each $\kappa \in \mathbb{Z}$ the following holds:*

(a) *If there exists a $K^+ \geq 0$ such that $g(k, x_{-M}, \ldots, x_0) \leq K^+$ for all $k \in \mathbb{Z}_\kappa^+$ and $x_{-M}, \ldots, x_0 \in I$, then the general solution φ of (3.1) satisfies*

$$\varphi(k; \kappa, \xi_{-M}, \ldots, \xi_0) \leq \lambda^{k-\kappa} \xi_0 + \frac{K^+}{1-\lambda} \quad \text{for all } k \in \mathbb{Z}_\kappa^+, \, \xi_{-M}, \ldots, \xi_0 \in I.$$

(b) *If $-I \subseteq I$ and there exists a $K^- \geq 0$ such that $-K^- \leq g(k, x_{-M}, \ldots, x_0)$ for all $k \in \mathbb{Z}_\kappa^+$ and $x_{-M}, \ldots, x_0 \in I$, then the general solution φ of (3.1) satisfies*

$$-\lambda^{k-\kappa} \xi_0 - \frac{K^-}{1-\lambda} \leq \varphi(k; \kappa, \xi_{-M}, \ldots, \xi_0) \quad \text{for all } k \in \mathbb{Z}_\kappa^+, \, \xi_0, \ldots, \xi_{-M} \in I.$$

Proof. Let $\kappa \in \mathbb{Z}$ and abbreviate the tupel $\xi = (\xi_{-M}, \ldots, \xi_0) \in I^{M+1}$.

(a) Define $\psi(k) := \lambda^{\kappa-k} \varphi(k, \kappa, \xi)$. Then one has

$$\begin{aligned}
\psi(j+1) - \psi(j) &= \lambda^{\kappa-j-1} \left(\varphi(j+1; \kappa, \xi) - \lambda \varphi(j; \kappa, \xi) \right) \\
&\overset{(3.1)}{=} \lambda^{\kappa-j-1} g(j, \varphi(j; \kappa, \xi), \ldots, \varphi(j-M; \kappa, \xi)) \leq K^+ \lambda^{\kappa-j-1}
\end{aligned}$$

for all $j \in \mathbb{Z}_\kappa^+$ and summation yields

$$\psi(k) - \psi(\kappa) = \sum_{j=\kappa}^{k-1} (\psi(j+1) - \psi(j)) \leq \frac{K^+}{\lambda} \sum_{j=0}^{k-\kappa-1} \lambda^{-j} = \frac{K^+}{\lambda-1} \left(1 - \lambda^{\kappa-k} \right)$$

for all $k \in \mathbb{Z}_\kappa^+$. The definition of ψ implies our claimed estimate.

(b) If $\varphi(\cdot; \kappa, \xi)$ is the general solution of (3.1), then $-\varphi(\cdot; \kappa, -\xi)$ is the general solution of the transformed delay difference equation

$$x(k+1) = \lambda x(k) - g(k, -x(k), -x(k-1), \ldots, -x(k-M)).$$

Utilizing this observation we obtain from (a) that $-\varphi(k; \kappa, -\xi) \leq -\lambda^{k-\kappa} \xi_0 + \frac{K^-}{1-\lambda}$ for all $k \in \mathbb{Z}_\kappa^+$ and this establishes the estimate in (b). $\qquad \square$

Proposition 3.2 (pullback dissipativity). (a) *If $I \subseteq [0, \infty)$ and if there exists a $K^+ \geq 0$ such that $0 \leq g(k, x_{-M}, \ldots, x_0) \leq K^+$ for all $k \in \mathbb{Z}$ and $x_{-M}, \ldots, x_0 \in I$, then (3.1) is pullback dissipative with an absorbing set $A = [0, R^+]$, where $R^+ > \frac{K^+}{1-\lambda}$.*

(b) *If $-I \subseteq I$ and if there exist $K^\pm \geq 0$ such that $-K^- \leq g(k, x_{-M}, \ldots, x_0) \leq K^+$ for all $k \in \mathbb{Z}$ and $x_{-M}, \ldots, x_0 \in I$, then (3.1) is pullback dissipative with an absorbing set $A = [R^-, R^+]$, where $R^+ > \frac{K^+}{1-\lambda}$ and $R^- < \frac{-K^-}{1-\lambda}$.*

Remark 3.3. (1) In both cases the delay difference equation (3.1) possesses a global pullback attractor, provided $g : \mathbb{Z} \times I^{M+1} \to I$ is continuous (cf. Remark 2.2).

(2) A further dissipativity criterion for (3.1) can be obtained by an application of [CM04, Theorem 5.2] to the associated delay endomorphism. In addition, this enables us to weaken the boundedness assumption on g.

Proof. Let $B \subseteq I$ be bounded, i.e., $B \subseteq [b_-, b_+]$ with reals $b_- < 0 < b_+$, and choose the initial state $\xi = (\xi_{-M}, \ldots, \xi_0) \in B^{M+1}$.

(a) The above Lemma 3.1(a) implies

$$0 \leq \varphi(k; k - n, \xi) \leq \lambda^n \xi_0 + \frac{K^+}{1 - \lambda} \leq \lambda^n b_+ + \frac{K^+}{1 - \lambda}$$

and due to $\lambda \in (0, 1)$ there exists a $N = N(B) \in \mathbb{Z}_0^+$ such that $\lambda^n b_+ \in \left[0, R^+ - \frac{K^+}{1-\lambda}\right]$ for $n \geq N$, thus $\varphi(k; k - n, B) \subseteq [0, R^+]$ for all $k \in \mathbb{Z}, n \geq N$.

(b) This can be shown analogously to (a) using Lemma 3.1(a) and (b). $\qquad \square$

Now we present some dissipative delay difference equations. The *Mackey-Glass equation* models dynamics of *haematopoiesis*, i.e., white blood cell production in the human body (cf. [MG77]).

Example 3.4 (Mackey-Glass equation). Let $I = [0, \infty)$ and $(\beta_k)_{k \in \mathbb{Z}}$ be a bounded sequence in I. The discrete nonautonomous *Mackey-Glass equation* is given by

$$x(k + 1) = \lambda x(k) + \frac{\beta_k}{1 + x(k - M)^p},$$

where $g : \mathbb{Z} \times I \to \mathbb{R}$, $g(k, y) := \frac{\beta_k}{1 + y^p}$ and the parameter $p > 0$. From Proposition 3.2(a) we obtain that $[0, R]$ is an absorbing set for $R > \frac{1}{1-\lambda} \sup_{k \in \mathbb{Z}} \beta_k$. A further delay difference equation suggested by Mackey and Glass reads as

$$x(k + 1) = \lambda x(k) + \frac{\beta_k x(k - M)}{1 + x(k - M)^p},$$

where $g : \mathbb{Z} \times I \to \mathbb{R}$, $g(k, y) := \frac{\beta_k y}{1 + y^p}$ and parameters $p > 1$. Since the function $y \mapsto \frac{y}{1+y^p}$ has the maximal value $\frac{1}{p}(p-1)^{1-1/p}$ for $y = (p-1)^{-1/p}$ we derive from Proposition 3.2(a) that the interval $[0, R]$ is an absorbing set for $R > \frac{1}{p(1-\lambda)}(p-1)^{1-1/p} \sup_{k \in \mathbb{Z}} \beta_k$.

Another important model fitting into our approach is the discrete *Wazewska-Czyzewska and Lasota equation* describing the erythropoietic (red blood-cell) system (cf. [LWC76]), i.e., the survival of red blood-cells in an animal.

Example 3.5 (Wazewska-Lasota equation). Let $I = [0, \infty)$ and $(\beta_k)_{k \in \mathbb{Z}}$, $(\gamma_k)_{k \in \mathbb{Z}}$ be two sequences in I, where $(\beta_k)_{k \in \mathbb{Z}}$ is assumed to be bounded. We investigate the equation

$$x(k + 1) = \lambda x(k) + \beta_k e^{-\gamma_k x(k - M)},$$

where $g : \mathbb{Z} \times I \to \mathbb{R}$, $g(k, y) := \beta_k e^{-\gamma_k y}$. From Proposition 3.2(a) we deduce that $[0, R]$ is an absorbing set, if one has $R > \frac{1}{1-\lambda} \sup_{k \in \mathbb{Z}} \beta_k$.

Our final example will be a delay difference equation defined on the reals. It is a discrete model for the behavior of a single, self-excitatory neuron with graded delayed response (cf. [Her94]).

Example 3.6. Let $I = \mathbb{R}$ and $(\beta_k)_{k \in \mathbb{Z}}$, $(\gamma_k)_{k \in \mathbb{Z}}$ be real sequences with bounded $(\beta_k)_{k \in \mathbb{Z}}$. Consider the discrete delay difference equation

$$x(k + 1) = \lambda x(k) + \beta_k \tanh(\gamma_k x(k - M)),$$

where $g : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$, $g(k, y) := \beta_k \tanh(\gamma_k y)$. Using Proposition 3.2(b) we see that the compact interval $[-R, R]$ is an absorbing set, if $R > \frac{1}{1-\lambda} \sup_{k \in \mathbb{Z}} |\beta_k|$.

Remark 3.7. For the autonomous version of the delay difference equations introduced in Example 3.4–3.6 one can use a criterion of [Iva94, Invariance property] to show that the respective minimal absorbing intervals are invariant.

4. ASYMPTOTIC EQUIVALENCE FOR DELAY DIFFERENCE EQUATIONS

In this section we consider delay difference equations

$$(4.1) \qquad x(k+1) = \sum_{j=-M}^{0} \ell_j(k)x(k+j) + g(k, x(k-M), \dots, x(k))$$

with $\ell_{-M}, \dots, \ell_0 : \mathbb{Z} \to \mathbb{R}$ and a function $g : \mathbb{Z} \times \mathbb{R}^{M+1} \to \mathbb{R}$. The linear part (2.4) is said to have an *exponential dichotomy*, if the associated companion matrix L (see (2.5)) satisfies the conditions given in Definition B.1. In particular for an autonomous linear part (2.4), an exponential dichotomy is determined by the roots of the characteristic polynomial

$$z^{M+1} - \sum_{j=-M}^{0} \ell_j z^{M+j} = 0$$

(cf. Appendix B), which are the eigenvalues of the constant companion matrix L.

Now we are in a position to state our main result. Its assumptions guarantee the existence of an exponentially attractive positively invariant submanifold of the state space for (4.1), which dominates the dynamical behavior. At first glance Theorem 4.1 seems technical, but an interpretation will be given below before we prove it.

Theorem 4.1 (asymptotic equivalence). *Assume the following holds for* (4.1):

(i) *The linear delay difference equation* (2.4) *admits an exponential dichotomy with growth rates $0 < \Lambda < \lambda \leq 1$, constants $K^{\pm} \geq 1$ and projectors $P_{\pm}$ with $N_{\pm} \equiv \dim \operatorname{im} P_{\pm}(k)$.*

(ii) *One has $\sup_{k \in \mathbb{Z}} |g(k, 0, \dots, 0)| < \infty$ and with functions $l^{\pm} : [0, \infty) \to [0, \infty)$ the following local Lipschitz conditions hold for all $k \in \mathbb{Z}$, $r \geq 0$:*

$$(4.2) \qquad \max_{j=-M}^{0} |P_{\pm}(k+1)_{j,0}| \, |g(k,x) - g(k, \bar{x})| \leq l^{\pm}(r) \, \|x - \bar{x}\| \quad \text{for all } x, \bar{x} \in B_r.$$

(iii) *The nonlinear delay equation* (4.1) *is pullback absorbing with absorbing set $A \subseteq \mathbb{R}$.*

Choose $\rho > 0$ so large that $\bar{A}^{M+1} \subseteq B_\rho$ and suppose the following spectral gap condition:

$$(4.3) \qquad K^- l^-(\rho) + K^+ l^+(\rho) + \max\left\{ K^- l^-(\rho), K^+ l^+(\rho) \right\} < \frac{\lambda - \Lambda}{4},$$

$$(4.4) \qquad 4K^- l^-(\rho) < \lambda.$$

Then there exists a Lyapunov transformation $T : \mathbb{Z} \to \mathbb{R}^{(M+1)\times(M+1)}$, a nonempty open non-autonomous set $\mathcal{O} \subseteq \mathcal{P}_-$ and a continuous function $w : \mathcal{O} \to \mathbb{R}^{M+1}$ with $w(k,x) \in \mathcal{P}_+(k)$, $\operatorname{Lip}_2 w < 1$ such that the following holds:

(a) *For every solution $\phi : \mathbb{Z}_{\kappa-M}^+ \to I$ of the delay difference equation* (4.1) *there exists a constant $C \geq 0$ and a further solution $\phi^* : \mathbb{Z}_{\kappa^*-M}^+ \to I$ of* (4.1) *satisfying*

$$(4.5) \qquad \|\phi(k) - \phi^*(k)\| \leq C\gamma^{k-\kappa} \quad \text{for all } k \in \mathbb{Z}_{\kappa^*}^+,$$

where $\gamma \in (\Lambda, 1)$, $\kappa \leq \kappa^*$ and $(\phi^*(\kappa^*-M), \dots, \phi^*(\kappa^*))^T = \eta + w(\kappa, \eta)$ with $\eta \in \mathcal{O}(\kappa^*)$.

(b) *The solution ϕ^* allows the representation*

$$\begin{pmatrix} \phi^*(k-M) \\ \vdots \\ \phi^*(k)) \end{pmatrix} = T(k)^{-1}\psi(k; \kappa^*, T(\kappa^*)\eta) + w(k, T(k)^{-1}\psi(k; \kappa^*, T(\kappa^*)\eta))$$

for all $k \geq \kappa^*$, where ψ is the general solution of the reduced equation

$$(4.6) \quad \begin{pmatrix} y_{-M}(k+1) \\ \vdots \\ y_{-N_-}(k+1) \\ y_{1-N_-}(k+1) \\ \vdots \\ y_0(k+1) \end{pmatrix} = T(k+1)L(k) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ y_{1-N_-}(k) \\ \vdots \\ y_0(k) \end{pmatrix} + G(k, y_{1-N_-}(k), \ldots, y_0(k))$$

and using $S(k) := T(k)^{-1}$ we have defined the nonlinearity

$$G(k, y_{1-N_-}, \ldots, y_0) := g\left(k, S(k)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ y_{1-N_-} \\ \vdots \\ y_0 \end{pmatrix} + w\left(k, S(k)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ y_{1-N_-} \\ \vdots \\ y_0 \end{pmatrix}\right)\right)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ S_-(k+1)_{1-N_-,0} \\ \vdots \\ S_-(k+1)_{0,0} \end{pmatrix}.$$

In order to provide an interpretation of Theorem 4.1 we remark that the canonical state space for the reduced equation (4.6) is $\mathbb{R}^{N_-}$, since its solutions are uniquely determined by N_- initial values. The relation (4.5) guarantees that every solution ϕ of (4.1) is asymptotically equivalent to another solution ϕ^* of (4.1), which is uniquely driven by the first order difference equation (4.6). Consequently, the long-term behavior of the delay difference equation (4.1) with state space $\mathbb{R}^{M+1}$ is ultimately determined by the N_--dimensional problem (4.6).

Remark 4.2. While it is a problem of linear algebra to construct the Lyapunov transformation $T : \mathbb{Z} \to \mathbb{R}^{(M+1)\times(M+1)}$ from the dichotomy data for L (see [Pöt98, p. 166, Lemma A.6.1]), it is practically impossible to derive an explicit expression for the mapping w, which satisfies the *invariance equation* (cf. (C.5)) for $\kappa \in \mathbb{Z}$, $\eta = (\eta_{-M}, \ldots, \eta_0) \in \mathcal{P}_-(\kappa)$ given by

$$w\left(\kappa+1, \eta_{-M+1}, \ldots, \eta_0, \sum_{j=-M}^{0} f_j(\kappa)\eta_j + g(\kappa, \eta + w(\kappa, \eta))\right)_i$$
$$= w(\kappa, \eta)_{i+1} \quad \text{for all } -M \leq i < 0,$$
$$w\left(\kappa+1, \eta_{-M+1}, \ldots, \eta_0, \sum_{j=-M}^{0} f_j(\kappa)\eta_j + g(\kappa, \eta + w(\kappa, \eta))\right)_0$$
$$= \sum_{j=-M}^{0} \ell_j(\kappa)w(\kappa, \eta)_j + g(\kappa, \eta + w(\kappa, \eta)) \sum_{j=-M}^{0} P_+(\kappa+1)_{j,0}.$$

Nonetheless, under differentiability assumptions on the nonlinearity g one can use this functional equation to compute Taylor approximations to w. It is worth to point out that is a dynamical and not an algebraic problem (see [PR05b, PR05b]). Such approximations seem of little use in the present more global setting. Therefore, Theorem 4.1 is a primarily theoretical result.

Proof of Theorem 4.1. We successively establish both assertions.

(a) Our main technical tool in the proof will be Theorem C.1. Thereto, in this first step we are going to show that the delay endomorphism associated with (4.1) satisfies its assumptions. We consider the first order difference equation (C.1), where $L(k)$ is the companion matrix of (2.4) and nonlinearity $F(k, x) := (0, \ldots, 0, g(k, x_{-M}, \ldots, x_0))^T$.

- ad $(H)_1$: This is just the present assumption (i). Note that we have $\lambda \leq 1$.
- ad $(H)_2$: Obviously, (C.2) is satisfied and, thanks to the particular algebraic structure of the mapping F, the local Lipschitz estimate (C.3) reduces to (4.2).
- ad $(H)_3$: From (iii) it is easy to see that the nonautonomous set $\mathbb{Z} \times A^{M+1}$ is uniformly pullback absorbing w.r.t. (C.1).

Consequently, Theorem C.1 guarantees the existence of an attractive nonautonomous set $\mathcal{W} \subseteq \mathbb{Z} \times \mathbb{R}^{M+1}$ being graph of a function w with properties stated in Theorem C.1(a). Let $\phi : \mathbb{Z}^+_{\kappa-M} \to I$ be a solution of (4.1) and $\hat{\phi}$ be the associated solution of (C.1) by virtue of the identities (2.3). Referring to Theorem C.1(b) there exist $\kappa^* \in \mathbb{Z}^+_\kappa$, $\eta^* \in \mathcal{P}_-(\kappa^*)$ such that (C.6) holds with $\xi^* = \eta^* + w(\kappa^*, \eta^*)$. Then the last component $\phi^* := \hat{\varphi}(\cdot; \kappa^*, \xi^*)_0$ is a solution of (4.1) and the asymptotic equivalence (4.5) holds true. Thus, we have shown assertion (a).

(b) Reflecting our above construction, the solution ϕ^* allows the representation (cf. (2.3))

$$(\phi^*(k-M), \ldots, \phi^*(k))^T = \tilde{\varphi}(k; \kappa^*, \eta^*) + w(k, \tilde{\varphi}(k; \kappa^*, \eta^*)) \quad \text{for all } k \geq \kappa^*,$$

where $\tilde{\varphi}$ is the general solution of the reduced equation (C.7) in the pseudo-unstable bundle $\mathcal{P}_-$. Referring to, for instance [Pöt98, p. 33, Satz 1.5.7], we know from the exponential dichotomy assumption (i) that the companion matrix (2.5) is kinematically similar to a block-diagonal matrix. In particular, there exists a bounded sequence $T : \mathbb{Z} \to \mathbb{R}^{(M+1)\times(M+1)}$ of invertible matrices, such that also the sequence of inverses $T(\cdot)^{-1} : \mathbb{Z} \to \mathbb{R}^{(M+1)\times(M+1)}$ is bounded, and one has

$$T(k)P_-(k)T(k)^{-1} \equiv \begin{pmatrix} 0_{N_+} & \\ & I_{N_-} \end{pmatrix} \quad \text{on } \mathbb{Z}.$$

Thus, the transformation $y = T(k)x$ brings (C.7) into the form (4.6) and the general solution ψ of this N_--dimensional equation satisfies $\psi(k; \kappa^*, T(\kappa)\eta^*) = T(k)\tilde{\varphi}(k; \kappa^*, \eta^*)$ for all $k \geq \kappa^*$. $\square$

As further illustration of Theorem 4.1 we discuss a class of examples sufficiently large to include the Examples 3.4–3.6. On the other hand, it is simple enough to illuminate our lengthy, but quantitative assumptions, in particular on the exponential dichotomy.

Example 4.3. Let $\lambda \in (0,1)$ and $g_0 : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ be a function such that the following holds:

- g_0 is bounded, i.e., $|g_0(k,x)| \leq K$ for all $k \in \mathbb{Z}$, $x \in \mathbb{R}$,
- the partial derivative $D_2 g_0$ exists and with some $R > \frac{K}{1-\lambda}$ one has

$$l := \sup_{k \in \mathbb{Z}} \sup_{|x| \leq R} |D_2 g_0(k,x)| < \infty.$$

These assumptions and Proposition 3.2(b) guarantee that the single delay difference equation

$$(4.7) \qquad\qquad x(k+1) = \lambda x(k) + g_0(k, x(k-M))$$

is pullback dissipative with absorbing set $[-R, R]$. The associated constant companion matrix[1]

$$L = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda \end{pmatrix} \in \mathbb{R}^{(M+1)\times(M+1)}$$

---

[1] Note that this particular splitting of (4.7) into linear part and nonlinearity is quite arbitrary and not required from Theorem 4.1. Another choice would be writing (4.7) as $x(k+1) = \lambda_0 x(k) + g(k, x(k-M), x(k))$ with nonlinearity $g(k, x_{-M}, x_0) = (\lambda - \lambda_0)x_0 + g_0(k, x_{-M})$, where $\lambda_0 \in (0,1)$. Then we obtain a different gap condition.

has the spectrum $\sigma(L) = \{0, \lambda\}$, which implies that L possesses a (Λ, λ)-decomposition for any real $\Lambda \in (0, \lambda)$. Moreover, L admits an exponential dichotomy with growth rates Λ, λ, constants $K^\pm = 2(1 + \lambda^{-M})\Lambda^{-M}$ and constant projectors

$$
P_+ = \begin{pmatrix} 1 & 0 & & \dots & -\lambda^{-M} \\ & 1 & 0 & \dots & -\lambda^{-M+1} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & -\lambda^{-1} \\ & & & & 0 \end{pmatrix}, \qquad
P_- = \begin{pmatrix} 0 & 0 & & \dots & \lambda^{-M} \\ & 0 & 0 & \dots & \lambda^{-M+1} \\ & & \ddots & \ddots & \vdots \\ & & & 0 & \lambda^{-1} \\ & & & & 1 \end{pmatrix}
$$

with $\dim \operatorname{im} P_- = 1$. Having this information available, it is not difficult to verify that our spectral gap conditions (4.3) and (4.4) reduce to

$$
48(1 + \lambda^{-M})(\lambda\Lambda)^{-M} l < \lambda - \Lambda, \qquad\qquad 8(1 + \lambda^{-M})(\lambda\Lambda)^{-M} l < \lambda,
$$

respectively. Provided these conditions hold, Theorem 4.1 ensures that the dynamical behavior of the delay difference equation (4.7) is asymptotically equivalent to a scalar first order difference equation (note $\dim \operatorname{im} P_- = 1$). However, this example clearly demonstrates that the assumptions (4.3)–(4.4) are rather restrictive and indicates that large delays require very weak nonlinearities, i.e., small local Lipschitz constants for g_0.

5. STRONGLY STABLE MULTI-STEP METHODS

Our second application of Theorem C.1 comes from the numerical integration of ordinary differential equations (see, for instance, [HNW93]). Thanks to a theoretically interesting result of Kirchgraber (cf. [Kir86], see also [Sto93]) we know that strongly stable multi-step methods are asymptotically equivalent to one-step methods, hence essentially one-step methods. Here we will indicate how to generalize this to multi-step methods with varying step-sizes.

To keep our presentation accessible and simple we restrict to scalar, yet nonautonomous ordinary differential equations

$$
(5.1) \qquad\qquad \boxed{\dot{x} = f(t, x),}
$$

equipped with an initial condition $x(\tau) = \xi$ with $\tau, \xi \in \mathbb{R}$. For further simplicity let us suppose $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous with

$$
\operatorname{Lip}_2 f < \infty, \qquad\qquad \sup_{t \in \mathbb{R}} |f(t, 0)| < \infty;
$$

then the general solution $\chi(\cdot; \tau, \xi) : \mathbb{R} \to \mathbb{R}$ of (5.1) is well-defined (cf., e.g., [Har64]).

In order to discretize the problem (5.1) we prescribe a discrete set of time steps $\{t_k\}_{k \in \mathbb{Z}}$ satisfying $t_0 = \tau$, $t_k < t_{k+1}$ for all $k \in \mathbb{Z}$ and define the ratios $\omega_k := \frac{t_{k+1} - t_k}{t_k - t_{k-1}}$. Note that for constant step-sizes one has the identity $\omega_k \equiv 1$ on $\mathbb{Z}$. Following [HNW93, p. 396, Section III.5], a *linear variable step-size multi-step method* to solve (5.1) is a difference equation

$$
x(k+1) = -\sum_{j=-M}^{0} \alpha_{M+j}(\omega_k, \dots, \omega_{k-M}) x(k+j)
$$

$$
(5.2) \qquad\qquad + (t_{k+1} - t_k) \sum_{j=-M}^{0} \beta_{M+j}(\omega_k, \dots, \omega_{k-M}) f(t_{k+j}, x(k+j)),
$$

which obviously fits in the framework of our delay difference equation (4.1) with

$$
\ell_j(k) = -\alpha_{M+j}(\omega_k, \dots, \omega_{k-M}) \quad \text{for all } -M \le j \le 0, \ k \in \mathbb{Z},
$$

$$g(k, x_{-M}, \ldots, x_0) = (t_{k+1} - t_k) \sum_{j=-M}^{0} \beta_{M+j}(\omega_k, \ldots, \omega_{k-M}) f(t_{k+j}, x_j).$$

Then the values $\varphi(k; 0, \xi_{-M}, \cdot, \xi_{-1}, \xi)$ obtained from (5.2) approximate the solution $\chi(\cdot; \tau, \xi)$ of the ordinary differential equation (5.1) at times $t = t_k$.

For explicit examples of variable step-size multi-step methods we again refer to [HNW93, p. 396, Section III.5] and remark that the following Hypothesis is typically satisfied.

Hypothesis 5.1. *Suppose the multi-step method (5.2) satisfies:*

  (o) *There exists a $H > 0$ and a neighborhood $\Omega \subseteq (0, \infty)$ of 1 such that*

$$0 < t_{k+1} - t_k \leq H, \qquad\qquad \omega_k \in \Omega \quad \text{for all } k \in \mathbb{Z},$$

  (i) *$\alpha_0, \ldots, \alpha_M : \Omega^{M+1} \to \mathbb{R}$ are continuous and $\beta_0, \ldots, \beta_M : \Omega^{M+1} \to \mathbb{R}$ are bounded,*
  (ii) *the method (5.2) is of* order $p \geq 0$, *i.e.,*

$$1 + \sum_{j=-M}^{0} \alpha_{M+j}(\omega_k, \ldots, \omega_{k-M}) \equiv 0 \quad \text{on } \mathbb{Z},$$

  (iii) *for $\omega_k \equiv 1$ the method (5.2) is* strongly stable, *i.e., all roots $z_{-M}, \ldots, z_0 \in \mathbb{C}$ of*

$$z^{M+1} + \sum_{j=-M}^{0} \alpha_{M+j}(1, \ldots, 1) z^{M+j} = 0$$

  lie inside the open unit disc $B_1 \subseteq \mathbb{C}$ except the simple root $z_0 = 1$.

Remark 5.2. Let $L_0 \in \mathbb{R}^{(M+1) \times (M+1)}$ be the companion matrix of the constant step-size formula (where $\omega_k \equiv 1$). By Hypothesis 5.1(iii) we can transform L_0 into block diagonal form

$$T^{-1} L_0 T = \begin{pmatrix} \tilde{L}_0 & \\ & 1 \end{pmatrix}, \qquad \tilde{L}_0 = \begin{pmatrix} z_{-M} & \varepsilon_{-M} & & \\ & \ddots & \ddots & \\ & & \ddots & \varepsilon_{-2} \\ & & & z_{-1} \end{pmatrix} \in \mathbb{R}^{M \times M}$$

with an invertible matrix $T \in \mathbb{R}^{(M+1) \times (M+1)}$, where $|\varepsilon_j| < 1 - |z_j|$ for all $-M \leq j < 0$ and

$$(5.3) \qquad\qquad\qquad\qquad \left\| \tilde{L}_0 \right\| < 1.$$

This is an easy consequence of the Jordan canonical form (yielding a transformation T_0 and values $\varepsilon_j \in \{0, 1\}$) and an appropriate multiplication of the columns of T_0 (yielding that (5.3) holds).

The following perturbation result ensures that the linear part of (5.2) admits an exponential dichotomy for weakly varying time steps.

Lemma 5.3. *Assume Hypothesis 5.1 holds and choose $\Lambda \in \left(\left\| \tilde{L}_0 \right\|, 1 \right)$. Then there exists a neighborhood $\Omega_0 \subseteq \Omega$ of 1 such that $\omega_k \in \Omega_0$ for all $k \in \mathbb{Z}$ implies that (2.4) admits an exponential dichotomy with growth rates $\Lambda, 1$, constants $K^\pm = \|T\| \left\| T^{-1} \right\|$ and constant projectors*

$$P_+ = T \begin{pmatrix} I_M & \\ & 0 \end{pmatrix} T^{-1}, \qquad\qquad P_- = T \begin{pmatrix} 0_M & \\ & 1 \end{pmatrix} T^{-1}$$

with $\dim \operatorname{im} P_- = 1$, where $T \in \mathbb{R}^{(M+1) \times (M+1)}$ is the transformation matrix from Remark 5.2.

Proof. Note that the last column of T, the eigenvector of L_0 corresponding to the eigenvalue 1, is given by $e_0 = (1, \ldots, 1)^T$. In addition, by Hypothesis 5.1(ii) this vector $e_0 \in \mathbb{R}^{M+1}$ is also an eigenvector of each companion matrix $L(k)$, $k \in \mathbb{Z}$. Therefore, T transforms every $L(k)$ into block diagonal form

$$T^{-1}L(k)T = \begin{pmatrix} \tilde{L}(k) & \\ & 1 \end{pmatrix} \quad \text{for all } k \in \mathbb{Z},$$

where the matrix sequence $\tilde{L} : \mathbb{Z} \to \mathbb{R}^{M \times M}$ satisfies

$$\left\| \tilde{L}(k) \right\| \leq \left\| \tilde{L}_0 \right\| + \left\| \tilde{L}(k) - \tilde{L}_0 \right\| \leq \left\| \tilde{L}_0 \right\| + \left\| T^{-1} \left[L(k) - L_0 \right] T \right\|$$

$$\leq \left\| \tilde{L}_0 \right\| + \|T\| \left\| T^{-1} \right\| \sum_{j=-M}^{0} |\alpha_j(\omega_k, \ldots, \omega_{k-M}) - \alpha_j(1, \ldots, 1)| \quad \text{for all } k \in \mathbb{Z}.$$

Thus, by continuity of the functions $\alpha_j : \Omega^{M+1} \to \mathbb{R}$ (cf. Hypothesis 5.1(i)) and (5.3) we know that for each $\Lambda \in \left(\left\| \tilde{L}_0 \right\|, 1 \right)$ there exists a neighborhood $\Omega_0 \subseteq \Omega$ of 1 such that $\left\| \tilde{L}(k) \right\| \leq \Lambda$, provided $\omega_k \in \Omega_0$ holds for all $k \in \mathbb{Z}$. Now it is straight forward to verify that the sequence L satisfies the dichotomy estimates (B.1). $\qquad \square$

For the initialization of a multi-step method (5.2), one needs a *starting procedure S*. This is a mapping $S : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^{M+1}$ providing starting values $\xi_{-M}, \ldots, \xi_0 \in \mathbb{R}$ for (5.2) according to

$$S(\kappa, \xi) = (\xi_{-M}, \ldots, \xi_0);$$

i.e., for each incomplete initial condition $x(\kappa) = \xi$ for (5.2) the starting procedure S delivers a full set of initial values $\xi_{-M}, \ldots, \xi_0$. Typically the value of a starting procedure will be the iterates of a one-step scheme to integrate (5.1).

Hypothesis 5.4. *Suppose the starting procedure $S : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^{M+1}$ satisfies for all $\kappa \in \mathbb{Z}$ that $P_- S(\kappa, \cdot) : \mathbb{R} \to \operatorname{im} P_-$ is a homeomorphism.*

Theorem 5.5. *Assume that Hypotheses 5.1, 5.4 hold, choose $\Lambda \in \left(\left\| \tilde{L}_0 \right\|, 1 \right)$ and that the discrete time steps are small and weakly varying in the sense of*

$$(5.4) \qquad \sup_{(\theta_0, \ldots, \theta_M) \in \Omega_0^{M+1}} \sum_{j=-M}^{0} |\beta_j(\theta_0, \ldots, \theta_M)| \operatorname{Lip}_2 f \, H < \frac{1-\Lambda}{6K^+ K^-}, \qquad \omega_k \in \Omega_0$$

with constants $K^{\pm} \geq 1$ and the neighborhood $\Omega_0 \subseteq \Omega$ of 1 from Lemma 5.3. Then there exists a continuous function $w : \mathbb{Z} \times \operatorname{im} P_- \to \operatorname{im} P^+$ with $\operatorname{Lip}_2 w < 1$ and graph

$$\mathcal{W} := \left\{ (\kappa, \eta + w(\kappa, \eta)) \in \mathbb{Z} \times \mathbb{R}^{M+1} : \eta \in \operatorname{im} P_- \right\},$$

as well as a scalar nonautonomous difference equation

$$(5.5) \qquad \boxed{x(k+1) = G(k, x(k))}$$

with continuous right hand side $G : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ such that the following holds for all $\kappa \in \mathbb{Z}$:

(a) *The multi-step method (5.2) (with starting procedure S) and the scalar difference equation (5.5) are conjugated on $\mathcal{W}$, i.e., their general solutions φ and ψ, respectively, satisfy*

$$S_k \circ \psi(k; \kappa, \cdot) = \begin{pmatrix} \varphi(k-M; \kappa, \cdot) \circ S_\kappa \\ \vdots \\ \varphi(k; \kappa, \cdot) \circ S_\kappa \end{pmatrix} \quad \text{for all } k \in \mathbb{Z}_\kappa^+$$

with a homeomorphism $S_k : \mathbb{R} \to \mathcal{W}(k)$,

(b) *there is a $\gamma \in (\Lambda, 1)$ such that for all $\xi_{-M}, \ldots, \xi_0 \in \mathbb{R}$ there exist a constant $C \geq 0$ and an initial value $\eta \in \mathbb{R}$ with*

$$\left\| \begin{pmatrix} \varphi(k - M; \kappa, \xi_{-M}, \ldots, \xi_0) \\ \vdots \\ \varphi(k; \kappa, \xi_{-M}, \ldots, \xi_0) \end{pmatrix} - S_k \psi(k; \kappa, \eta) \right\| \leq C \gamma^{k-\kappa} \quad \text{for all } k \in \mathbb{Z}_\kappa^+.$$

Proof. In this proof we can use the global version of Theorem C.1 described in Remark C.2. Thereto, let L be the companion matrix associated with the multi-step method (5.2), define the mapping $F : \mathbb{Z} \times \mathbb{R}^{M+1} \to \mathbb{R}^{M+1}$ by

$$F(k, x) := \left(0, \ldots, 0, (t_{k+1} - t_k) \sum_{j=-M}^{0} \beta_{M+j}(\omega_k, \ldots, \omega_{k-M}) f(t_{k+j}, x_j) \right)^T.$$

and $\hat{\varphi}$ is the general solution of (C.1). The assumptions of Theorem C.1 (or Remark C.2) hold:

- ad $(H)_1$: By Lemma 5.3 the companion matrix L admits an exponential dichotomy with $\Lambda, 1$ and $\dim \operatorname{im} P_- = 1$.
- ad $(H)_2$: An easy estimate shows that (5.4) implies (C.2), while the other assumptions are not necessary in the present global setting.

Thus, there exists a function w with graph $\mathcal{W}$ as claimed above. Let $\kappa \in \mathbb{Z}$ be given.

(a) With the starting procedure S from Hypothesis 5.4 we define the mapping $S_\kappa : \mathbb{R} \to \mathcal{W}(\kappa)$ by $S_\kappa(x) := P_- S(\kappa, x) + w(\kappa, P_- S(\kappa, x))$. Thanks to Hypothesis 5.4 and $\operatorname{Lip}_2 w < 1$ we know that S_κ is a homeomorphism. Thus, let us define the right hand side $G : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ of (5.5) as $G(\kappa, x) := S_{\kappa+1}^{-1}(L(\kappa) S_\kappa(x) + F(\kappa, S_\kappa(x)))$, which is obviously continuous. Mathematical induction yields $S_k \circ \psi(k; \kappa, \cdot) = \hat{\varphi}(k; \kappa, \cdot) \circ S_\kappa$ for $k \in \mathbb{Z}_\kappa^+$ and using (2.3) we get assertion (a).

(b) Let $\xi_{-M}, \ldots, \xi_0 \in \mathbb{R}$ and set $\xi = (\xi_{-M}, \ldots, \xi_0)$. Referring to Theorem C.1(b) there exists a unique initial value $\xi^* \in \mathcal{W}(\kappa)$ such that

$$\| \hat{\varphi}(k; \kappa, \xi) - \hat{\varphi}(k; \kappa, \xi^*) \| \leq C \gamma^{k-\kappa} \quad \text{for all } k \in \mathbb{Z}_\kappa^+$$

(due to our global assumptions we have $\kappa^* = \kappa$). Then (b) follows with $\eta := S_\kappa^{-1}(\xi^*)$. $\qquad\square$

APPENDIX A. NONAUTONOMOUS DYNAMICS

Let $D \subseteq \mathbb{R}^N$ be nonempty. We consider a first order nonautonomous difference equation

(A.1) $$\boxed{x(k+1) = f(k, x(k))}$$

with *right hand side* $f : \mathbb{Z} \times D \to D$. A sequence $\phi : \mathbb{Z}_\kappa^+ \to D$, $\kappa \in \mathbb{Z}$, satisfying the identity $\phi(k+1) \equiv f(k, \phi(k))$ on $\mathbb{Z}_\kappa^+$ is called *solution* of (A.1). We define the *general solution* $\varphi(\cdot; \kappa, \xi)$, $\xi \in D$, of (A.1) as unique solution satisfying the initial condition $x(\kappa) = \xi$. We remark that in general $\varphi(k; \kappa, \cdot)$ does not exist for $k < \kappa$.

A set $\mathcal{A} \subseteq \mathbb{Z} \times D$ is called *nonautonomous set* with *k-fiber* $\mathcal{A}(k) := \{x \in D : (k, x) \in \mathcal{A}\}$ for $k \in \mathbb{Z}$. Such a set $\mathcal{A}$ is said to be *positively invariant* w.r.t. (A.1), if $f(k, \mathcal{A}(k)) \subseteq \mathcal{A}(k+1)$ holds, and it is called *invariant*, if one has equality $f(k, \mathcal{A}(k)) = \mathcal{A}(k+1)$ for $\kappa \leq k$. Moreover, we denote (A.1) as difference equation *in $\mathcal{A}$*, if $\mathcal{A}$ is positively invariant.

APPENDIX B. DISCRETE DICHOTOMIES

We deal with matrix-valued sequences $L : \mathbb{Z} \to \mathbb{R}^{N \times N}$ with *evolution operator*

$$\Phi(k, \kappa) := \begin{cases} I_N & \text{for } k = \kappa \\ L(k-1) \cdots L(\kappa) & \text{for } k > \kappa \end{cases};$$

if the linear mapping $L(k) : \mathbb{R}^N \to \mathbb{R}^N$, $k \in \mathbb{Z}$, is invertible (possibly between appropriate subspaces of $\mathbb{R}^N$), then $\Phi(k, \kappa) := L(k)^{-1} \cdots L(\kappa - 1)^{-1}$ for $k < \kappa$.

Definition B.1 (exponential dichotomy). Let $0 < \Lambda < \lambda$ and $K^\pm \geq 1$ be given. Then a matrix sequence $L : \mathbb{Z} \to \mathbb{R}^{N \times N}$ is said to possess an *exponential dichotomy*, if there exist complementary projections $P_\pm : \mathbb{Z} \to \mathbb{R}^{N \times N}$ with $P_-(k+1)L(k) = L(k)P_-(k)$, the mappings

$$L(k)|_{\operatorname{im} P_-(k)} : \operatorname{im} P_-(k) \to \operatorname{im} P_-(k+1)$$

are invertible with associate evolution operator $\bar{\Phi}(k, \kappa)$, and the dichotomy estimates hold:

(B.1) $$\left\| \Phi(k,l)P_+(l) \right\| \leq K^+ \Lambda^{k-l}, \qquad \left\| \bar{\Phi}(l,k)P_-(k) \right\| \leq K^- \lambda^{l-k} \quad \text{for all } l \leq k.$$

Of particular importance are the two nonautonomous sets

$$\mathcal{P}_- := \left\{ (k, x) \in \mathbb{R}^N : x \in \operatorname{im} P_-(k) \right\}, \qquad \mathcal{P}_+ := \left\{ (k, x) \in \mathbb{R}^N : x \in \operatorname{im} P_+(k) \right\},$$

denoted as *pseudo-unstable* and *pseudo-stable bundle*, respectively.

In general it is difficult to determine wether a given matrix sequence admits an exponential dichotomy. Nonetheless, for the special cases of constant and periodic matrices a dichotomy is fully determined by spectral properties. Thereto, we need certain preliminaries from linear algebra (cf. [HS74, pp. 109–133]).

For given $0 < \Lambda < \lambda$ we say a matrix $T \in \mathbb{R}^{N \times N}$ possesses a (Λ, λ)-*decomposition*, if the disjoint sets $\sigma^+ := \{\nu \in \sigma(T) : |\nu| \leq \Lambda\}$, $\sigma^- := \{\nu \in \sigma(T) : \lambda \leq |\nu|\}$ are nonempty with $\sigma(T) = \sigma^+ \cup \sigma^-$, i.e., $\sigma(T)$ can be separated by an annulus with center 0 and radii $\Lambda < \lambda$. This at hand, we introduce the direct sums

$$V_T^\pm := \bigoplus_{\substack{\nu \in \sigma^\pm, \\ \Im\nu = 0}} \ker \left(T - \nu I_N\right)^N \oplus \bigoplus_{\substack{\nu \in \sigma^\pm, \\ \Im\nu > 0}} \ker \left(T^2 - 2\Re\nu T + |\nu|^2 I_N\right)^N$$

and integers $n_\pm := \dim V_T^\pm$. Let $\{x_1^\pm, \ldots, x_{n_\pm}^\pm\}$ be a basis of $V_T^\pm$. Using the invertible matrix $C := (x_1^+, \ldots, x_{n_+}^+, x_1^-, \ldots, x_{n_-}^-) \in \mathbb{R}^{N \times N}$ we introduce the complementary projections

$$Q_T^+ := C \begin{pmatrix} I_{n_+} & \\ & 0_{n_-} \end{pmatrix} C^{-1}, \qquad Q_T^- := C \begin{pmatrix} 0_{n_+} & \\ & I_{n_-} \end{pmatrix} C^{-1},$$

fulfilling $\ker Q_T^\pm = V_T^\mp$ and $\operatorname{im} Q_T^\pm = V_T^\pm$.

Constant matrix sequences: Suppose $L(k)$ is independent of $k \in \mathbb{Z}$, i.e., $L(k) \equiv L$. An eigenvalue ν of $L \in \mathbb{R}^{N \times N}$ is said to be *semisimple*, if its algebraic and geometric multiplicities coincide. If the matrix L possesses a (Λ, λ)-decomposition and eigenvalues of L with modulus Λ and λ are semisimple, then L possesses an exponential dichotomy with growth rates Λ, λ and constant invariant projectors $P_\pm = Q_A^\pm$ (cf. [PR05a, Proposition 2.1]).

Periodic matrix sequences: Suppose $L(k)$ is ω-periodic, i.e., $L(k) \equiv L(k + \omega)$ on $\mathbb{Z}$. Note that the matrix $M_\omega(k) := \Phi(k + \omega, k)$, $k \in \mathbb{Z}$, has the same eigenvalues as the so-called *monodromy matrix* $M_\omega(0)$. They are the *Floquet multipliers* of L. If the monodromy matrix $M_\omega(0)$ possesses an $(\Lambda^\omega, \lambda^\omega)$-decomposition and the Floquet multipliers with modulus Λ^ω and λ^ω are semisimple, then L possesses an exponential dichotomy with growth rates Λ, λ and ω-periodic invariant projectors $P_\pm(k) := Q_{M_\omega(k)}^\pm$ for all $k \in \mathbb{Z}$ (cf. [PR05a, Proposition 2.2]).

Appendix C. Attractive fiber bundles

Now we turn to nonlinear first order difference equations

$$(C.1) \qquad \boxed{x(k+1) = L(k)x(k) + F(k, x(k)),}$$

where $L : \mathbb{Z} \to \mathbb{R}^{N \times N}$ and $F : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}^N$ specifies the nonlinearity. For the general solution of (C.1) we write $\hat{\varphi}$. Then our invariant manifold theorem reads as follows:

Theorem C.1 (attractive fiber bundles)**.** *Assume the following holds for* (C.1)*:*

$(H)_1$ *The sequence $L : \mathbb{Z} \to \mathbb{R}^{N \times N}$ has an exponential dichotomy with projectors $P_{\pm}$, growth rates $0 < \Lambda < \lambda$ and constants $K^{\pm}$.*

$(H)_2$ *One has the growth condition*

$$(C.2) \qquad \sup_{k < \kappa} \|F(k, 0)\| \lambda^{\kappa - k} < \infty \quad \text{for all } \kappa \in \mathbb{Z}$$

and there exist functions $l^{\pm} : [0, \infty) \to [0, \infty)$ such that for all $r > 0$, $k \in \mathbb{Z}$ we have

$$(C.3) \qquad \|P_{\pm}(k+1)\left[F(k, x) - F(k, \bar{x})\right]\| \leq l^{\pm}(r) \|x - \bar{x}\| \quad \text{for all } x, \bar{x} \in B_r.$$

$(H)_3$ *The difference equation (C.1) possesses a uniformly pullback absorbing set $\mathcal{A} \subseteq \mathbb{Z} \times \mathbb{R}^N$, i.e., $\mathcal{A}$ is bounded and for every nonempty bounded subset $\mathcal{B} \subseteq \mathbb{Z} \times \mathbb{R}^N$ there exists $N = N(\mathcal{B}) \in \mathbb{Z}_0^+$ such that*

$$\hat{\varphi}(k; k-n, \mathcal{B}(k-n)) \subseteq \mathcal{A}(k) \quad \text{for all } k \in \mathbb{Z}, \, n \geq N.$$

Choose $\rho > 0$ so large that $\bar{\mathcal{A}} \subseteq \mathbb{Z} \times B_\rho$ and suppose the following spectral gap condition*:*

$$K^- l^-(\rho) + K^+ l^+(\rho) + \max\left\{K^- l^-(\rho), K^+ l^+(\rho)\right\} < \tfrac{\lambda - \Lambda}{4},$$

$$(C.4) \qquad \qquad 4K^- l^-(\rho) < \lambda.$$

Then there exists a nonautonomous set $\mathcal{W} \subseteq \mathbb{Z} \times \mathbb{R}^N$ (denoted as attractive invariant fiber bundle*), which is positively invariant w.r.t. (C.1), and possesses the following properties:*

(a) *$\mathcal{W}$ is graph of a function $w : \mathcal{O} \to \mathbb{R}^N$ over a nonempty open nonautonomous set $\mathcal{O} \subseteq \mathcal{P}_-$, i.e., $\mathcal{W} = \{(\kappa, \eta + w(\kappa, \eta)) : (\kappa, \eta) \in \mathcal{O}\}$, the functions $w(\kappa, \cdot) : \mathcal{O}(\kappa) \to \mathcal{P}_+(\kappa)$ are well-defined and satisfy:*

(a_1) *They are globally Lipschitzian with $\mathrm{Lip}_2 w < 1$,*

(a_2) *one has the functional equation (invariance equation)*

$$(C.5) \qquad w(\kappa+1, \eta_1) = L(\kappa)w(\kappa, \eta) + P_+(\kappa+1)F(\kappa, \eta + w(\kappa, \eta)),$$

for all $(\kappa, \eta) \in \mathcal{O}$ such that $\eta_1 := L(\kappa)\eta + F(\kappa, \eta + w(\kappa, \eta)) \in \mathcal{O}(\kappa+1)$,

(b) *$\mathcal{W}$ is asymptotically complete, i.e., for every pair $(\kappa, \xi) \in \mathbb{Z} \times \mathbb{R}^N$ there exists a point $(\kappa^*, \xi^*) \in \mathcal{W}$ with $\kappa \leq \kappa^*$ such that*

$$(C.6) \qquad \|\hat{\varphi}(k; \kappa, \xi) - \hat{\varphi}(k; \kappa^*, \xi^*)\| \leq C \gamma^{k - \kappa} \quad \text{for all } k \in \mathbb{Z}_{\kappa^*}^+,$$

and a $\gamma \in (\Lambda, \lambda)$, where the real $C \geq 0$ depends boundedly on κ, ξ.

(c) *The nonautonomous difference equation*

$$(C.7) \qquad x(k+1) = L(k)x(k) + P_-(k+1)F\big(k, x(k) + w(k, x(k))\big)$$

in the pseudo-unstable bundle $\mathcal{P}_-$ is denoted as reduced equation *of (C.1) and the general solution $\tilde{\varphi}$ of (4.6) is related to $\hat{\varphi}$ by*

$$\hat{\varphi}(k; \kappa, \eta + w(\kappa, \eta)) = \tilde{\varphi}(k; \kappa, \eta) + w(k, \tilde{\varphi}(k; \kappa, \eta)) \quad \text{for all } \eta \in \mathcal{O}(\kappa).$$

Remark C.2. If the Lipschitz constants $l^{\pm} : [0, \infty) \to [0, \infty)$ are bounded, it is possible to disclaim assumptions $(H)_3$, (C.4) and to deduce a stronger global version of Theorem C.1: Then the function w is defined on $\mathcal{P}_-$ and in Theorem C.1(b) the point $(\kappa^*, \xi^*) \in \mathcal{W}$ is uniquely determined with $\kappa^* = \kappa$ (cf. [Pöt07a, Proposition 2.1 and Theorem 2.5]).

Proof. The above theorem is a special case of [Pöt07b, Theorem 4.1]. $\qquad\square$

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CHRISTIAN PÖTZSCHE, SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS MN, 55455, USA

E-mail address: christian.poetzsche@ma.tum.de