INTEGRAL MANIFOLDS UNDER EXPLICIT VARIABLE TIME-STEP DISCRETIZATION

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Abstract. We study the behavior of the "full hierarchy" of integral manifolds, i.e., in particular those of stable, center-stable, center, center-unstable and unstable type, for nonautonomous ordinary differential equations in Banach spaces under explicit one-step discretization with varying step-sizes. Our main results on C^{m-1}-closeness under such discretizations are formulated in a quantitative fashion and turn out to be an easy consequence of a general theorem on the existence of invariant fiber bundles within the "calculus on time scales."

This work is dedicated to the memory of our teacher Prof. Dr. Bernd Aulbach. As an excellent teacher he introduced us to the theory of dynamical systems, advised us throughout our studies and has always been an inspiration. As a mathematician he was full of ideas, visions and plans. And even beyond mathematics we were benefitting from his humanity.

1. Introduction

This paper is concerned with analytical discretization theory of ordinary differential equations (ODEs), i.e. the problem, which qualitative features of an ODE persist under discretization with a numerical scheme. The research in this area essentially never leaves the framework of classical continuous and discrete dynamical systems in the sense that one considers autonomous differential equations and numerical methods with constant step-sizes (cf., e.g., [SH98]). This approach has two drawbacks: On the one side, the study of differential equations near nonconstant solutions canonically leads to nonautonomous problems via the equation of perturbed motion, and on the other side, in real-world applications one usually works with a step-size control for the numerical scheme leading to variable time-steps.

Accordingly, we investigate the qualitative behavior of nonautonomous ODEs near not necessarily hyperbolic equilibria under explicit one-step discretizations and with "qualitative behavior" we mean the persistence of integral manifolds. More detailed, we consider the pseudo-stable and -unstable integral manifolds corresponding to equilibria, as well as their intersections leading to center-like manifolds.

The study of invariant manifolds under numerical discretization goes back to Wolf-Jürgen Beyn (cf. [Bey87]), who investigated the behavior near hyperbolic fixed points of autonomous finite-dimensional ODEs. He shows that the corresponding stable and unstable manifolds of the given continuous system and of its discretization are $O(h^p)$-close in the $\mathcal{C}^0$-topology, where p denotes the order of the method and $h > 0$ is the constant step-size. In a similar spirit, [Feč91] treats only Euler-discretizations, but obtains $\mathcal{C}^1$-closeness in this situation. The case of center-unstable manifolds under general one-step discretizations is considered in [BL87], where the authors obtain even $\mathcal{C}^k$-closeness; related results on center manifolds have been

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given by [Fum88] and another interesting approach to center manifolds under one-step schemes can be found in [ZB98]. The most general situation is contained in [Gar93], where pseudo-stable and -unstable invariant manifolds of ODEs in Banach spaces and their behavior under discretization are considered. Finally, the initiating results of [Bey87] are generalized to semi-linear parabolic equations in [AD91]. All the references given above, do not leave the mentioned autonomous constant step-size framework, though. However, nonautonomous ODEs and their stable manifolds are considered in [AG94], while we additionally consider the "full hierarchy" of integral manifolds, their differentiability, as well as varying step-sizes. For that purpose, the paper at hand is subdivided into several sections:

- Above all, we introduce our notation and try to convince the reader that the so-called "calculus on time scales" (cf. [Hil90, BP01]) is a useful and convenient tool to tackle problems in discretization theory, when it is important to formulate discrete and continuous systems simultaneously. This leads to the concept of so-called "dynamic equations." Nevertheless, we point out that no previous knowledge of this calculus is needed to fully understand the paper including its proofs.
- We work with nonautonomous ODEs in a weakly nonlinear form. Considering such systems has the advantage that one obtains strong global results, which can be verified in a technically elegant way. Using a cut-off technique, these results then easily carry over "locally" to more realistic applications — at least for stable and unstable integral manifolds; see Remark 4.1(2) for the more subtle case of center manifolds. In fact, we study the behavior of weakly nonlinear equations under explicit one-step discretizations and show in Proposition 3.4 that the discretized system is nothing else but a small perturbation of the original system.
- Afterwards we state a general result on the existence and persistence of invariant fiber bundles, which are the pendant of integral manifolds within the theory of dynamic equations on time scales. Actually, the mentioned Proposition 4.2 applies to nonautonomous, noninvertible dynamic equations on nearly arbitrary time scales, with a pseudo-hyperbolic linear part where the growth rates are not assumed to be constant. It will incorporate most of the technical work leading to our main Theorem 4.3 on the $O(H^p)$-closeness of the integral manifolds of the original and the discretized system in the C^{k-1}-topology; $H > 0$ denotes the maximal step-size and $k \geq 1$ the smoothness order of the invariant fiber bundles.

It is our intention to provide quantitative results and explicit estimates as far as possible, e.g., concerning the smallness of the step-sizes involved.

2. Preliminaries

First, $\mathbb{Z}$ is the set of integers, $\mathbb{N}$ is the set of positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}$ is the real and $\mathbb{C}$ the complex field.

Within this paper, Banach spaces $\mathcal{X}$ or $\mathcal{Y}$ are all real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$) and their norm is denoted by $\|\cdot\|_{\mathcal{X}}$, resp., $\|\cdot\|_{\mathcal{Y}}$ or simply by $\|\cdot\|$. $\mathcal{L}(\mathcal{X})$ is the Banach space of linear continuous endomorphism on $\mathcal{X}$ and $I_{\mathcal{X}}$ the identity map on $\mathcal{X}$. The open ball in $\mathcal{X}$ with center $x \in \mathcal{X}$ and radius $\varepsilon > 0$ is denoted by $B_\varepsilon(x)$ and $\bar{B}_\varepsilon(x)$ stands for the closed ball.

We write Df for the Fréchet derivative of a mapping f and if f depends differentiable on two or more variables, then the partial derivatives are denoted by $D_1 f$ and $D_2 f$, and so on. In addition, $C^m(\mathcal{X}, \mathcal{Y})$, $m \in \mathbb{N}$, is the linear space of m-times continuously differentiable mappings between $\mathcal{X}$ and $\mathcal{Y}$, and $C_b^m(\mathcal{X}, \mathcal{Y})$ is the subset of all such mappings with bounded derivatives of order $\leq m$. For some mapping $F : \mathcal{X} \times \mathcal{P} \to \mathcal{X}$, where $\mathcal{P}$ denotes a nonempty set, we define the Lipschitz constants

$$\begin{aligned}\operatorname{Lip} F(\cdot,p) &:= \inf\left\{L \ge 0 : \ \|F(x,p)-F(\bar{x},p)\| \le L\,\|x-\bar{x}\| \ \text{ for all } x,\bar{x}\in\mathcal{X}\right\},\\ \operatorname{Lip}_1 F &:= \sup_{p\in\mathcal{P}} \operatorname{Lip} F(\cdot,p),\end{aligned}$$

provided they exist. If the set $\mathcal{P}$ has a metric structure, one defines $\operatorname{Lip}_2 F$ analogously, and proceeds correspondingly, if F depends on more than two variables.

For a convenient notation of discrete and continuous results simultaneously, we use the calculus on time scales (cf. [Hil90, BP01]). In general, a *time scale* $\mathbb{T}$ is a canonically ordered closed subset of the reals, and the mapping $\sigma : \mathbb{T} \to \mathbb{T}$, $\sigma(t) := \inf\{s \in \mathbb{T} : t < s\}$ is the *jump operator*, while $\mu : \mathbb{T} \to \mathbb{R}$, $\mu(t) := \sigma(t) - t$ is denoted as *graininess*. For $\tau \in \mathbb{T}$ we abbreviate $\mathbb{T}_\tau^+ := \{s \in \mathbb{T} : \tau \le s\}$ and $\mathbb{T}_\tau^- := \{s \in \mathbb{T} : s \le \tau\}$. However, to broaden the audience of this paper, we restrict our considerations to the cases $\mathbb{T} = \mathbb{R}$ and to so-called *discrete time scales*, which are of the form $\mathbb{T} = \{t_k\}_{k\in\mathbb{I}}$ with a real sequence $(t_k)_{k\in\mathbb{I}}$, where $\mathbb{I} = \mathbb{N}_0$ or $\mathbb{I} = \mathbb{Z}$ and we always assume $t_k < t_{k+1}$ for $k \in \mathbb{I}$. Hence, discrete time scales are unbounded above, i.e., we have $\lim_{k\to\infty} t_k = \infty$ and in case $\mathbb{I} = \mathbb{Z}$ also unbounded below, because otherwise $\mathbb{T}$ would not be closed. The examples $t_k = hk$, $k \in \mathbb{I}$, for some $h > 0$ or the *harmonic numbers* $t_k = \sum_{n=0}^{k} \frac{1}{n+1}$, $k \in \mathbb{N}_0$, fit into our setting. For the time scales under consideration, we have $\sigma(t) = t$, $\mu(t) \equiv 0$ ($\mathbb{T} = \mathbb{R}$) and $\sigma(t_k) = t_{k+1}$, $\mu(t_k) = t_{k+1} - t_k$ ($\mathbb{T} = \{t_k\}_{k\in\mathbb{I}}$). So it is reasonable to interpret μ as *step-size* in numerical schemes.

$C(\mathbb{T},\mathcal{X})$ denotes the linear space of continuous functions from $\mathbb{T}$ to $\mathcal{X}$. *Growth rates* are functions $a \in C(\mathbb{T},\mathbb{R})$ such that $-1 < \inf_{t\in\mathbb{T}} \mu(t)a(t)$ and $\sup_{t\in\mathbb{T}} \mu(t)a(t) < \infty$ holds. Moreover, for $a, b \in C(\mathbb{T},\mathbb{R})$ we introduce the relations $\lfloor b-a \rfloor := \inf_{t\in\mathbb{T}}(b(t) - a(t))$,

$$a \lhd b \quad :\Leftrightarrow \quad 0 < \lfloor b-a \rfloor, \qquad\qquad a \unlhd b \quad :\Leftrightarrow \quad 0 \le \lfloor b-a \rfloor.$$

On the set $\mathcal{R} := \{a \in C(\mathbb{T},\mathbb{R}) : a \text{ is a growth rate and } 1 + \mu(t)a(t) > 0 \text{ for } t \in \mathbb{T}\}$ we define the product $(m \odot a)(t) := \lim_{h\searrow\mu(t)} \frac{(1+ha(t))^m - 1}{h}$ with $m \in \mathbb{N}$.

Now for an appropriate mapping $\phi : \mathbb{T} \to \mathcal{X}$, its *derivative* (cf. [Hil90, Section 2.4]) is denoted by $\phi^\Delta : \mathbb{T} \to \mathcal{X}$, and for $\mathbb{T} = \mathbb{R}$ given by the usual differential quotient $\phi^\Delta(t) = \lim_{s\to t} \frac{\phi(s)-\phi(t)}{s-t}$ in case it exists, whereas for discrete time scales $\mathbb{T} = \{t_k\}_{k\in\mathbb{I}}$ we obtain the forward difference quotient $\phi^\Delta(t_k) = \frac{\phi(t_{k+1})-\phi(t_k)}{t_{k-1}-t_k}$ for $k \in \mathbb{I}$.

For a nonautonomous, parameter-dependent dynamic equation

$$x^\Delta = f(t, x; p) \tag{2.1}$$

with a right-hand side $f : \mathbb{T} \times \mathcal{X} \times \mathcal{P} \to \mathcal{X}$, we say a mapping $\nu : I \to \mathcal{X}$ is a solution of (2.1), if $\nu^\Delta(t) \in \mathcal{X}$ exists and the identity $\nu^\Delta(t) = f(t, \nu(t); p)$ holds on a subset $I \subseteq \mathbb{T}$. For right-hand sides f guaranteeing existence and uniqueness of solutions in forward time (cf., e.g., [Pöt02, p. 38, Satz 1.2.17(a)]), let $\varphi_{(2.1)}(t; \tau, \xi; p)$ denote the *general solution* of (2.1), i.e. $\varphi_{(2.1)}(\cdot; \tau, \xi; p)$ solves (2.1) on $\mathbb{T}_\tau^+$ and satisfies $\varphi_{(2.1)}(\tau; \tau, \xi; p) = \xi$ for $\tau \in \mathbb{T}$, $\xi \in \mathcal{X}$, $p \in \mathcal{P}$. It fulfills the *cocycle property*

$$\varphi_{(2.1)}(t; s, \varphi_{(2.1)}(s; \tau, \xi; p); p) = \varphi_{(2.1)}(t; \tau, \xi; p) \quad \text{for } \tau, s, t \in \mathbb{T}_\tau^+,\ \tau \le s \le t \tag{2.2}$$

and $\xi \in \mathcal{X}$, $p \in \mathcal{P}$. As mentioned in the introduction, invariant fiber bundles are generalizations of invariant manifolds to nonautonomous equations. In order to be more precise, for fixed parameters $p \in \mathcal{P}$, we call a subset $S(p)$ of the extended state space $\mathbb{T} \times \mathcal{X}$ an *invariant fiber bundle* of (2.1), if for any pair $(\tau, \xi) \in S(p)$ one has $(t, \varphi_{(2.1)}(t; \tau, \xi; p)) \in S(p)$ for all $t \in \mathbb{T}_\tau^+$, and if $S(p)$ is graph of a mapping over a subset of $\mathbb{T} \times \mathcal{X}$. In case $\mathbb{T} = \mathbb{R}$ we speak of an *integral manifold* instead of an invariant fiber bundle.

Given a coefficient mapping $A \in C(\mathbb{T}, \mathcal{L}(\mathcal{X}))$, the *transition operator* $\Psi_{(2.3)}(t,\tau) \in \mathcal{L}(\mathcal{X})$, $\tau \leq t$, of a linear dynamic equation

$$x^\Delta = A(t)x \tag{2.3}$$

is the solution of the operator-valued initial value problem $X^\Delta = A(t)X$, $X(\tau) = I_\mathcal{X}$ in $\mathcal{L}(\mathcal{X})$. Besides, for $a \in \mathcal{R}$ the *exponential function* on $\mathbb{T}$ is denoted by $e_a(t,s) \in \mathbb{R}$, $s,t \in \mathbb{T}$, and we have $e_a(t,s) = \exp\left(\int_s^t a(\tau)d\tau\right)$ for $\mathbb{T} = \mathbb{R}$, while the representation

$$e_a(t_k, t_l) = \begin{cases} \prod_{n=\min\{k,l\}}^{\max\{k,l\}-1}(1 + \mu(t_n)a(t_n)) & \text{if } k \neq l \\ 1 & \text{if } k = l \end{cases}$$

holds on discrete time scales for $k,l \in \mathbb{I}$.

A projection-valued mapping $P : \mathbb{T} \to \mathcal{L}(\mathcal{X})$ is called a *projector*, we speak of an *invariant projector* of (2.3), if $P(t)\Psi_{(2.3)}(t,s) = \Psi_{(2.3)}(t,s)P(s)$ for $s \leq t$ holds, and finally, an invariant projector P is denoted as *regular*, if

$$I_\mathcal{X} + \mu(t)A(t)|_{\mathcal{R}(P(t))} : \mathcal{R}(P(t)) \to \mathcal{R}\big(P(\sigma(t))\big) \text{ is bijective for all } t \in \mathbb{T}.$$

Then the restriction $\bar{\Psi}_{(2.3)}(t,s) := \Psi_{(2.3)}(t,s)|_{\mathcal{R}(P(s))} : \mathcal{R}(P(s)) \to \mathcal{R}(P(t))$, $s \leq t$, is a well-defined isomorphism, and we denote its inverse by $\bar{\Psi}_{(2.3)}(s,t)$ (cf. [Pöt02, p. 86]). With an integer $N \geq 2$, the system (2.3) is said to possess an *exponential N-splitting*, if there exist invariant projectors $P_1, \ldots, P_N : \mathbb{T} \to \mathcal{L}(\mathcal{X})$ with

$$P_1(t) + \ldots + P_N(t) \equiv I_\mathcal{X}, \qquad P_i(t)P_j(t) \equiv 0 \quad \text{for } i \neq j,\ t \in \mathbb{T} \tag{2.4}$$

such that $P_2, \ldots, P_N$ are regular and for $1 \leq j < N$ we have the estimates

$$\left\|\Psi_{(2.3)}(t,s)P_j(s)\right\| \leq K_j^+ e_{a_j}(t,s), \qquad \left\|\bar{\Psi}_{(2.3)}(s,t)P_{j+1}(t)\right\| \leq K_j^- e_{b_j}(s,t) \quad \text{for } s \leq t \tag{2.5}$$

with reals $K_1^\pm, \ldots, K_{N-1}^\pm \geq 1$ and growth rates $a_j, b_j \in \mathcal{R}$, where $a_j \lhd b_j$ for $1 \leq j < N$ and $b_j \unlhd a_{j+1}$ for $1 \leq j < N-1$. In case $N = 2$ we speak of an *exponential dichotomy* and for $N = 3$ of an *exponential trichotomy.* In this situation, for illustrative reasons, we could call P_1, P_2 and P_3 the stable, center and unstable projector, since in many applications $b_1 \unlhd 0 \unlhd a_2$. This, however, is nowhere assumed here. Explicit examples for the occurrence of exponential N-splittings will be given later on.

To close this section, we introduce the so-called *quasiboundedness*, which is a convenient notion describing exponential growth of functions.

2.1. **Definition.** For $c \in \mathcal{R}$, $\tau \in \mathbb{T}$ we say that $\phi \in C(\mathbb{T}, \mathcal{X})$ is

(a) c^+*-quasibounded*, if $\|\phi\|_{\tau,c}^+ := \sup_{t \in \mathbb{T}_\tau^+} \|\phi(t)\| \, e_c(\tau, t) < \infty$,

(b) c^-*-quasibounded*, if $\|\phi\|_{\tau,c}^- := \sup_{t \in \mathbb{T}_\tau^-} \|\phi(t)\| \, e_c(\tau, t) < \infty$.

By $\mathcal{X}_{\tau,c}^+(\mathbb{T})$ and $\mathcal{X}_{\tau,c}^-(\mathbb{T})$ we denote the sets of c^+- and c^--quasibounded functions defined (at least) on $\mathbb{T}_\tau^+$ and $\mathbb{T}_\tau^-$, respectively.

Obviously $\mathcal{X}_{\tau,c}^+(\mathbb{T})$ and $\mathcal{X}_{\tau,c}^-(\mathbb{T})$ are nonempty and it is immediate that for any $c \in \mathcal{R}$, $\tau \in \mathbb{T}$, the sets $\mathcal{X}_{\tau,c}^+(\mathbb{T})$ and $\mathcal{X}_{\tau,c}^-(\mathbb{T})$ are Banach spaces with the norms $\|\cdot\|_{\tau,c}^+$ and $\|\cdot\|_{\tau,c}^-$, respectively. Finally, the spaces $\mathcal{X}_{\tau,c}^+(\mathbb{T})$ define a scale of Banach spaces, i.e., for $c, d \in \mathcal{R}$ we have the implication $c \unlhd d \Rightarrow \mathcal{X}_{\tau,c}^+(\mathbb{T}) \subseteq \mathcal{X}_{\tau,d}^+(\mathbb{T})$.

3. Discretization of weakly nonlinear equations

In the first part of this section we are interested in linear ordinary differential and dynamic equations only. Precisely we assume

Hypothesis (linear part)**.** *For given $p \in \mathbb{N}$ and $N \geq 2$ consider the linear ODE*

$$\dot{x} = A(t)x \tag{3.1}$$

under the following assumption:

$(H)_1$ *$A \in C_b^p(\mathbb{R}, \mathcal{L}(\mathcal{X}))$, set $|A|_n := \sup_{t\in\mathbb{R}} \|D^n A(t)\|$ for $0 \leq n \leq p$, and (3.1) is supposed to possess an exponential N-splitting, i.e., we have the estimates*

$$\left\|\Psi_{(3.1)}(t,s)P_j(s)\right\| \leq K_j^+ e^{\alpha_j(t-s)}, \qquad \left\|\Psi_{(3.1)}(s,t)P_{j+1}(t)\right\| \leq K_j^- e^{\beta_j(s-t)} \quad \text{for } s \leq t$$

with reals $K_j^\pm \geq 1$, $\alpha_j, \beta_j \in \mathbb{R}$, where $\alpha_j < \beta_j$ for $1 \leq j < N$, and $\beta_j \leq \alpha_{j+1}$ for $1 \leq j < N-1$ and invariant projectors P_i satisfying (2.4).

In the autonomous situation $A(t) \equiv A_0$ on $\mathbb{R}$, the linear ODE (3.1) possesses an exponential N-splitting, if the spectral points λ of the operator $A_0 \in \mathcal{L}(\mathcal{X})$ satisfy $\Re\lambda \in \bigcup_{i=1}^{N-1}(\beta_{i-1}, \alpha_i)$, where $\beta_0 = -\infty$. As another example, if (3.1) is a variational equation corresponding with an n-parameter family, $n \in \mathbb{N}_0$, of periodic solutions, then under certain nondegeneracy conditions, the equation (3.1) has an exponential trichotomy with $\dim \mathcal{R}(P_2(\tau)) = n$ for $\tau \in \mathbb{R}$ (cf. [Aul81]).

We start with an elementary preparatory result:

3.1. **Lemma.** *Assume that $\alpha < \beta$ are reals and define $\varepsilon_h(\alpha, \beta) := \frac{e^{\beta h} - e^{\alpha h}}{h}$. Then there exists a $H > 0$ such that*

$$\frac{\beta - \alpha}{2} < \lim_{t \searrow h} \varepsilon_t(\alpha, \beta) < \frac{3(\beta - \alpha)}{2} \quad \text{for } h \in [0, H]\,.$$

Proof. The assertion immediately follows from $\lim_{t\searrow 0} \frac{e^{\beta t} - e^{\alpha t}}{t} = \beta - \alpha$. □

For our further considerations it is convenient to introduce the abbreviation

$$E_\alpha^1(h) := \begin{cases} \frac{e^{\alpha h} - 1}{\alpha h} & \text{if } \alpha h \neq 0 \\ 1 & \text{if } \alpha h = 0 \end{cases}, \qquad E_\alpha^2(h) := \begin{cases} \frac{e^{\alpha h} - \alpha h - 1}{(\alpha h)^2} & \text{if } \alpha h \neq 0 \\ \frac{1}{2} & \text{if } \alpha h = 0 \end{cases}$$

with real numbers α, h. It is easy to see that $E_\alpha^i(h) \in \mathbb{R}$, $i = 1, 2$, are continuous in (α, h) and increasing (decreasing) in h, provided that $\alpha \geq 0$ ($\alpha \leq 0$).

3.2. **Lemma.** *Assume that Hypothesis $(H)_1$ is satisfied. Then*

$$\left\|A(t)\Psi_{(3.1)}(t,s) - A(s)\right\| \leq \left(|A|_0^2 \, E_{|A|_0}^1(t-s) + |A|_1\right)(t-s) \quad \text{for } s \leq t. \tag{3.2}$$

Proof. Using the mean value theorem (cf. [Lan93, p. 341, Theorem 4.2]) and $\Psi_{(3.1)}(\tau, \tau) = I_\mathcal{X}$, we obtain the identity

$$\begin{aligned} \Psi_{(3.1)}(t,s) &= I_\mathcal{X} + \int_0^1 D_1\Psi_{(3.1)}(s + \tau(t-s), s)\,d\tau(t-s) \\ &\overset{(3.1)}{=} I_\mathcal{X} + \int_0^1 A(s + \tau(t-s))\Psi_{(3.1)}(s + \tau(t-s), s)\,d\tau(t-s) \quad \text{for } s \leq t, \end{aligned} \tag{3.3}$$

and the mean value inequality (cf. [Lan93, p. 342, Corollary 4.3]), as well as an estimate derived in [AW96, Lemma 2.9], yields

$$\left\|A(t)\Psi_{(3.1)}(t,s) - A(s)\right\| \leq \left\|A(t)\Psi_{(3.1)}(t,s) - A(t)\right\| + \|A(t) - A(s)\|$$

$$
\begin{aligned}
&\leq |A|_0 \left\|\Psi_{(3.1)}(t,s) - I_{\mathcal{X}}\right\| + |A|_1 (t-s) \\
&\overset{(3.3)}{=} |A|_0 \left\| \int_0^1 A(s+\tau(t-s))\Psi_{(3.1)}(s+\tau(t-s),s)d\tau \right\| (t-s) + |A|_1 (t-s) \\
&\leq |A|_0^2 \int_0^1 \left\|\Psi_{(3.1)}(s+\tau(t-s),s)\right\| d\tau(t-s) + |A|_1 (t-s) \\
&\overset{(3.1)}{\leq} |A|_0^2 \int_0^1 e^{|A|_0(t-s)\tau} d\tau(t-s) + |A|_1 (t-s) \quad \text{for } s \leq t.
\end{aligned}
$$

An elementary integration finally gives us (3.2). □

Remark 3.1. It is possible to improve the above estimate (3.2) by using the *logarithmic norm* $\chi_0(B) := \lim_{s\searrow 0} \frac{\|I_{\mathcal{X}}+sB\|-1}{s}$ of an operator $B \in \mathcal{L}(\mathcal{X})$ (cf. [DV84, pp. 27–34, Section 1.5]), which can be used to estimate $\left\|\Psi_{(3.1)}(t,s)\right\|$. Note that one has $\chi_0(A(t)) \leq |A|_0$ for all $t \in \mathbb{R}$.

It is reasonable, to restrict our considerations to numerical schemes where the step-size is bounded above.

Hypothesis (time scale)**.** *Let $H > 0$ be real and $\mathbb{T} = \{t_k\}_{k\in\mathbb{I}}$ be a discrete time scale under the following assumption:*

$(H)_2$ $0 < \mu(t_k) \leq H$ *for all* $k \in \mathbb{I}$.

Differing from many other investigations in analytical discretization theory (see e.g. the monograph [SH98] or [Bey87, BL87, Fum88, Feč91, AD91, Gar93]), we do consider nonautonomous equations and varying step-sizes. Therefore, discretization theory of linear systems is not only part of a general perturbation theory of linear operators, and we have to employ different techniques.

3.3. **Lemma.** *Assume that Hypotheses $(H)_1$–$(H)_2$ are satisfied, $\bar\alpha_j, \bar\beta_j \in (\alpha_j, \beta_j)$ with $\bar\alpha_j < \bar\beta_j$ for $1 \leq j < N$, and $H > 0$ is chosen so small that*

$$
\max_{j=1}^{N-1} \left(\frac{K_1(j)}{\bar\alpha_j - \alpha_j} + \frac{K_2(j)}{\beta_j - \bar\beta_j} \right) \left(|A|_0^2 E^2_{|A|_0}(H) + \tfrac{1}{2} |A|_1 \right) H < \frac{1}{2} \tag{3.4}
$$

with $K_1(j) := \sum_{k=1}^{j} K_k^+$ and $K_2(j) := \sum_{k=j}^{N-1} K_k^-$. Then also the linear dynamic equation (2.3) *on the discrete time scale $\mathbb{T}$ possesses an exponential N-splitting with $\bar a_j, \bar b_j$, $\bar K_j^\pm$ for $1 \leq j < N$ and invariant projectors $Q_j : \mathbb{T} \to \mathcal{L}(\mathcal{X})$, where $\bar a_j(t) := \frac{e^{\bar\alpha_j \mu(t)}-1}{\mu(t)}$, $\bar b_j(t) := \frac{e^{\bar\beta_j \mu(t)}-1}{\mu(t)}$ for $t \in \mathbb{T}$,*

$$
\bar K_j^+ := \frac{K_1(j)}{1 - 2\max_{i=1}^{N-1}\left(\frac{K_1(i)}{\bar\alpha_j-\alpha_j} + \frac{K_2(i)}{\beta_j-\bar\beta_j}\right)}, \quad \bar K_j^- := \frac{K_2(j)}{1 - 2\max_{i=1}^{N-1}\left(\frac{K_1(i)}{\bar\alpha_j-\alpha_j} + \frac{K_2(i)}{\beta_j-\bar\beta_j}\right)}
$$

and Q_j satisfies $\|Q_j(t) - P_j(t)\|_{\mathcal{L}(\mathcal{X})} = O(H)$ for $1 \leq j < N$, uniformly in $t \in \mathbb{T}$.

Proof. Obviously $\Psi_{(3.1)}|_{\mathbb{T}\times\mathbb{T}}$ is the transition operator of the dynamic equation

$$
x^\Delta = \hat A(t)x, \qquad \hat A(t_k) := \tfrac{1}{\mu(t_k)}\left(\Psi_{(3.1)}(t_{k+1},t_k) - I_{\mathcal{X}}\right) \quad \text{for } k \in \mathbb{I} \tag{3.5}
$$

on the discrete time scale $\mathbb{T}$ and, thus, it is easy to see that (3.5) possesses an exponential N-splitting with $\hat a_j(t) := \frac{e^{\alpha_j\mu(t)}-1}{\mu(t)}$, $\hat b_j(t) := \frac{e^{\beta_j\mu(t)}-1}{\mu(t)}$, $1 \leq j < N$, for $t \in \mathbb{T}$ and invariant projectors $P_j|_{\mathbb{T}}$, $1 \leq j \leq N$. Using (3.3) one obtains

$$
\left\|A(t_k) - \hat A(t_k)\right\| \leq \left\| \int_0^1 A(t_k) - A(t_{k+1} + \tau\mu(t_k))\Psi_{(3.1)}(t_k + \tau\mu(t_k), t_k)d\tau \right\|
$$

$$\overset{(3.2)}{\leq} \int_0^1 \left(|A|_0^2 \, E^1_{|A|_0}(\tau\mu(t_k)) + |A|_1\right) \tau d\tau \mu(t_k) \quad \text{for } k \in \mathbb{I}.$$

and distinguishing the cases $A = 0$ and $A \neq 0$ we get by an easy integration

$$\left\| A(t_k) - \hat{A}(t_k) \right\| \leq \left(|A|_0^2 \, E^2_{|A|_0}(H) + \tfrac{1}{2} |A|_1\right) H \quad \text{for } k \in \mathbb{I}.$$

Consequently, under the assumption (3.4), a roughness argument for exponential N-splittings yields the assertion, which is essentially a consequence of (cf. [Pöt05]. □

From now on, and similarly to [AG94], we work with semi-linear systems, where the non-linear perturbations are globally small Lipschitzian. Furthermore, they are assumed to have polynomially decaying derivatives in the state space, here.

Hypothesis (nonlinear part). *For given $p \in \mathbb{N}$, $q \in \mathbb{N}_0$ consider the ODE*

$$\dot{x} = A(t)x + F(t,x) \tag{3.6}$$

under the following assumptions:

$(H)_3$ $F : \mathbb{R}\times\mathcal{X} \to \mathcal{X}$ *such that the derivatives D^nF exist and are continuous for $0 \leq n \leq p+1$, $F(t,0) \equiv 0$ on $\mathbb{R}$, and we impose the growth conditions*

$$\sup_{(t,x)\in\mathbb{R}\times\mathcal{X}} \|D^nF(t,x)\| < \infty \quad \textit{for } 1 \leq n \leq p+1,$$

$$\sup_{(t,x)\in\mathbb{R}\times\mathcal{X}} \left\|D_2^n D_1^k F(t,x)\right\| \|x\|^{n-1} < \infty \quad \textit{for } k \geq 0,\ n \geq 2,\ k+n \leq p+1.$$

$(H)_3'$ D_2^nF *exist and are continuous for $p+1 \leq n \leq p+q+2$.*

Remark 3.2. (1) By standard theorems on ODEs (cf., e.g., [Lan93, p. 377, Theorem 5.2]), the assertions $(H)_1$, $(H)_3$ imply that the general solution $\varphi_{(3.6)}$ has $\mathbb{R}\times\mathbb{R}\times\mathcal{X}$ as its domain of definition, is of class C^p, and the partial derivative $D_1\varphi_{(3.6)}$ is of class C^p, i.e., $D_1^{p+1}\varphi_{(3.6)}$ exists and is continuous.

(2) Using a cut-off technique one can weaken $(H)_3$ to:

($*$) F is of class C^{p+1} on a set of the form $\mathbb{R}\times B_R(0)$, $R > 0$, $F(t,0) \equiv 0$ on $\mathbb{R}$ and suppose that there exists a $\rho_0 \in \left(0, \frac{R}{2}\right)$ such that $D^nF(\mathbb{R}\times B_{2\rho_0}(0))$ is bounded for $1 \leq n \leq p+1$.

Nevertheless, an important assumption here is that $\mathcal{X}$ is a C^{p+1}-Banach space, i.e., the norm $\|\cdot\|_{\mathcal{X}} : \mathcal{X} \to \mathbb{R}$ is of class C^{p+1} away from 0. Finite dimensional or Hilbert spaces are C^∞-Banach spaces, while the general situation is considered in [KM97, pp. 127–152]. Then there exists a C^{p+1}-bump function $\Theta_{\rho_0} : \mathcal{X} \to [0,1]$ with $\Theta_{\rho_0}(x) \equiv 1$ on $\bar{B}_{\rho_0}(0)$ and $\Theta_{\rho_0}(x) \equiv 0$ outside of $\bar{B}_{2\rho_0}(0)$. We define the mapping $F_\rho : \mathbb{R}\times\mathcal{X} \to \mathcal{X}$, $F_\rho(t,x) := \Theta_\rho(x)F(t,x)$ and in the light of ($*$), the modified mapping F_ρ satisfies $(H)_3$ for $\rho \in (0,\rho_0]$. Additionally, (3.6) coincides with

$$\dot{x} = A(t)x + F_\rho(t,x) \tag{3.7}$$

on $\mathbb{R}\times\bar{B}_\rho(0)$. Finally, under the assumption $\lim_{x\to 0} D_2F(t,x) = 0$ uniformly in $t \in \mathbb{T}$, one obtains the limit relation $\lim_{\rho\searrow 0} \operatorname{Lip}_2 F_\rho = 0$, which is essential to apply Proposition 4.2 or Theorem 4.3 to the dynamic equation (3.7). A similar construction in a C^{p+q+2}-Banach space can be used to fulfill Hypothesis $(H)_3'$.

Now we investigate a dynamic equation on a discrete time scale $\mathbb{T}$, which has the restriction $\varphi_{(3.6)}(\cdot;\tau,\xi)|_{\mathbb{T}}$, $\tau \in \mathbb{T}$, $\xi \in \mathcal{X}$, as its general solution. This will be called the $\mathbb{T}$-*equation* of (3.6) and is our substitute for the time-h-map in a nonautonomous setting.

Thereto, with given $k, n \in \mathbb{N}$ we write $\#N$ for the cardinality of a finite set $N \subset \mathbb{N}$ and

$$P_k^<(n) := \left\{ (N_1, \ldots, N_k) \,\middle|\, \begin{array}{l} N_i \subseteq \{1, \ldots, n\} \text{ and } N_i \neq \emptyset \text{ for } i \in \{1, \ldots, k\}, \\ N_1 \cup \ldots \cup N_k = \{1, \ldots, n\}, \\ N_i \cap N_j = \emptyset \text{ for } i \neq j,\, i, j \in \{1, \ldots, k\}, \\ \max N_i < \max N_{i+1} \text{ for } i \in \{1, \ldots, k-1\} \end{array} \right\}$$

for the set of *ordered partitions* of $\{1, \ldots, n\}$ with length k.

3.4. **Proposition** ($\mathbb{T}$-equation)**.** *Assume that Hypotheses $(H)_1$–$(H)_3$ are satisfied and $\tau \in \mathbb{T}$, $\xi \in \mathcal{X}$. Then $\varphi_{(3.6)}(\cdot; \tau, \xi)|_{\mathbb{T}}$ solves the dynamic equation*

$$x^\Delta = A(t)x + F(t, x) + \phi(t, x, \mu(t)) \tag{3.8}$$

on $\mathbb{T}$ with mappings

$$\phi(t, x, h) := \sum_{n=1}^{p-1} h^n \phi_n(t, x) + h^p \phi_p(t, x, h),$$

$$\phi_n(t, x) := \frac{1}{(n+1)!} \sum_{k=1}^{n} \sum_{(N_1, \ldots, N_k) \in P_k^<(n)} D^k f(t, x) \hat{f}_{\#N_1}(t; t, x) \cdot \ldots \cdot \hat{f}_{\#N_k}(t; t, x),$$

$$\phi_p(t, x, h) := \int_0^1 \frac{(1-s)^p}{p!} \sum_{k=1}^{p} \sum_{(N_1, \ldots, N_k) \in P_k^<(p)} D^k f(t, \varphi_{(3.6)}(t + sh; t, x)) \cdot$$

$$\cdot \hat{f}_{\#N_1}(t + sh; t, x) \cdot \ldots \cdot \hat{f}_{\#N_k}(t + sh; t, x) ds,$$

for $1 \leq n < p$ with $f(t, x) := A(t)x + F(t, x)$, and

$$\hat{f}_1(s; t, x) := \begin{pmatrix} 1 \\ f(s, \varphi_{(3.6)}(s; t, x)) \end{pmatrix}, \qquad \hat{f}_k(s; t, x) := \begin{pmatrix} 0 \\ D_1^k \varphi_{(3.6)}(s; t, x) \end{pmatrix} \quad \text{for } 2 \leq k \leq p.$$

Moreover, we have:

(a) *$\phi_n(t, 0) \equiv 0$, $\phi_p(t, 0, h) \equiv 0$ on $\mathbb{T}$ for $1 \leq n < p$ and $h \in [0, H]$.*

(b) *The mappings $\phi_1, \ldots, \phi_p$ are continuous and globally Lipschitzian in $x \in \mathcal{X}$ (uniformly in $t \in \mathbb{T}$, $h \in [0, H]$). Under Hypothesis $(H)_3'$, $\phi_1, \ldots, \phi_p$ are $(q+1)$-times continuously differentiable in the second argument with globally bounded derivatives $D_2^k \phi_n$, $1 \leq k \leq q+1$, for $1 \leq n \leq p$.*

(c) *The mapping $\phi : \mathbb{T} \times \mathcal{X} \times [0, H] \to \mathcal{X}$ is globally Lipschitzian in $x \in \mathcal{X}$ (uniformly in $t \in \mathbb{T}$, $h \in [0, H]$) with*

$$\operatorname{Lip}_2 \phi \leq \sum_{n=1}^{p} h^n \operatorname{Lip}_2 \phi_n. \tag{3.9}$$

Under Hypothesis $(H)_3'$, ϕ is $(q+1)$-times continuously differentiable in the second argument with globally bounded derivatives $D_2^k \phi$, $1 \leq k \leq q+1$.

Proof. Let $h \in (0, H]$, $\tau, t \in \mathbb{T}$ and $x \in \mathcal{X}$ be arbitrary. Keeping in mind Remark 3.2(1), the partial derivatives $D_1^n \varphi_{(3.6)}$ exist and are continuous up to order $n = p+1$. Hence, differentiating the identity $D_1 \varphi_{(3.6)}(t; \tau, \xi) = f(t, \varphi_{(3.6)}(t; \tau, \xi))$ with the higher order chain rule (cf. [But87, p. 156, Lemma 302A]) leads to

$$D_1^{n+1} \varphi_{(3.6)}(t; \tau, x) = \sum_{k=1}^{n} \sum_{(N_1, \ldots, N_k) \in P_k^<(n)} D^k f(t, \varphi_{(3.6)}(t; \tau, x)) \hat{f}_{\#N_1}(t; \tau, x) \cdot \ldots \cdot \hat{f}_{\#N_k}(t; \tau, x)$$

for $1 \le n \le p$, with the above abbreviations for $\hat{f}_1, \dots, \hat{f}_p$. In particular, we have $\phi_n(t,x) = \frac{1}{(n+1)!} D_1^{n+1} \varphi_{(3.6)}(t;t,x)$, $1 \le n < p$, since $\varphi_{(3.6)}(t;t,x) = x$. Using Taylor's formula (cf. [Lan93, p. 349]) we obtain the relation

$$\begin{aligned} (3.10)\quad & \frac{\varphi_{(3.6)}(t+h;t,x) - x}{h} \\ &= \sum_{n=0}^{p-1} \frac{h^n}{(n+1)!} D_1^{n+1} \varphi_{(3.6)}(t;t,x) + h^p \int_0^1 \frac{(1-s)^p}{p!} D_1^{p+1} \varphi_{(3.6)}(t+sh;t,x)\, ds \\ &= A(t)x + F(t,x) + \sum_{n=1}^{p-1} h^n \phi_n(t,x) + h^p \phi_p(t,x,h)\, ds = A(t)x + F(t,x) + \phi(t,x,h), \end{aligned}$$

and $\varphi_{(3.6)}(\cdot;\tau,x)^\Delta(t) = \frac{\varphi_{(3.6)}(\sigma(t);\tau,x) - \varphi_{(3.6)}(t;\tau,x)}{\mu(t)}$ for $\tau, t \in \mathbb{T}$, as well as (2.2), yields that $\varphi_{(3.6)}(\cdot;\tau,\xi)$ is a solution of (3.8).

(a) We show the identity $\phi_n(t,0) \equiv 0$ for $1 \le n \le k$ by induction over $k \in \{1,\dots,p-1\}$. From Hypothesis $(H)_3$ we have $\varphi_{(3.6)}(t;\tau,0) = 0$ and consequently (3.10) implies $\phi(t,0,h) \equiv 0$ for $t \in \mathbb{T}$, $h \in (0,H]$. Therefore, one has the identity $0 = \phi(t,0,h) = h\phi_1(t,0) + \sum_{n=2}^{p-1} h^n \phi_n(t,0) + h^p \phi_p(t,0,h)$ and thus $\|\phi_1(t,0)\| \le \sum_{n=2}^{p-1} h^{n-1} \|\phi_n(t,0)\| + h^{p-1} \|\phi_p(t,0,h)\|$, which in the limit $h \searrow 0$ yields $\phi_1(t,0) = 0$, since $\lim_{h \searrow 0} \phi_p(t,0,h)$ exists by definition of ϕ_p. Now assume $\phi_n(t,0) = 0$ for $1 \le n \le k$ and $k \le p-2$. Then similarly $0 = \phi(t,0,h) = h^{k+1}\phi_{k+1}(t,0) + \sum_{n=k+2}^{p-1} h^n \phi_n(t,0) + h^p \phi_p(t,0,h)$, and so $\|\phi_{k+1}(t,0)\| \le \sum_{n=k+2}^{p-1} h^{n-k-1} \|\phi_n(t,0)\| + h^{p-k-1} \|\phi_p(t,0,h)\|$. Reasoning as above, passing over to the limit $h \searrow 0$ gives us necessarily $\phi_n(t,0) \equiv 0$, $t \in \mathbb{T}$. Hence, we have shown $\phi_n(t,0) \equiv 0$ for $1 \le n < p$ and this, in turn, implies $\phi_p(t,0,h) = \phi(t,0,h) - \sum_{n=0}^{p-1} h^n \phi_n(t,0) = 0$.

(b) The proof of assertion (b) is quite technical and we refer to [Kel99, p. 92, Satz 4.1.9] for the details.

(c) The estimate (3.9) is a consequence of (b) and properties of globally Lipschitzian mappings from, e.g., [AMR88, p. 138, Exercise 2.5K(i)]. □

An *explicit one-step method* to solve (3.6) is a recursion of the form

$$x_{k+1} := x_k + \mu(t_k)\Phi(t_k, x_k, \mu(t_k)),$$

where $\mathbb{T} = \{t_k\}_{k \in \mathbb{I}}$ is a discrete time scale and the mapping Φ satisfies some, e.g., consistency assumptions (cf., for instance, [SB83]). One can easily embed such one-step methods into the calculus on time scales. Actually, for a given initial value $x_0 = \xi$, the sequence $(x_k)_{k \in \mathbb{N}_0}$ is just the solution of the dynamic equation

$$(3.11)\qquad x^\Delta = \Phi(t, x, \mu(t))$$

on the time scale $\mathbb{T} = \{t_k\}_{k \in \mathbb{I}}$, i.e., one has $\varphi_{(3.11)}(t_k; t_0, \xi) = x_k$ for all $k \in \mathbb{N}_0$.

From now on, we denote (3.11) as *explicit one-step discretization* of (3.6) and assume

Hypothesis (discretization)**.** *Let $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, $\mathbb{T}$ be a discrete time scale like in $(H)_2$. Consider a mapping $\Phi : \mathbb{T} \times \mathcal{X} \times [0,H] \to \mathcal{X}$ under the following assumptions:*

$(H)_4$ $\Phi(t,0,h) \equiv 0$ *for $t \in \mathbb{T}$, $h \in [0,H]$, Φ generates an explicit one-step method of order p, i.e., for each $t \in \mathbb{T}$, $x \in \mathcal{X}$ there exists a $K(t,x) \ge 0$ depending only on A and F, but not on $h \in [0,H]$, with*

$$(3.12)\qquad \|A(t)x + F(t,x) + \phi(t,x,h) - \Phi(t,x,h)\| \le K(t,x)h^p \quad \textit{for } h \in [0,H],$$

where ϕ is given in Proposition 3.4, and the partial derivatives $D_3^n\Phi(t,x,\cdot)$, $0 \le n \le p$, $t \in \mathbb{T}$, $x \in \mathcal{X}$, exist and are continuous, and $\operatorname{Lip}_2 D_3^p\Phi < \infty$.

$(H)_4'$ *$D_3^p\Phi$ is $(q+1)$-times continuously differentiable in the second variable with globally bounded partial derivatives $D_2^n D_3^p\Phi$ for $1 \le n \le q+1$.*

Remark 3.3. (1) Instead of (3.12), it is more common in numerical analysis to work with the local discretization error

$$e_h(t,x) := \varphi_{(3.6)}(t+h;t,x) - x - h\Phi(t,x,h)$$

for $h \in [0,H]$, $t \in \mathbb{T}$ and $x \in \mathcal{X}$, but obviously the inequality (3.12) is equivalent to $\|e_h(t,x)\| \le K(t,x)h^{p+1}$ (cf. (3.10)).

(2) If Φ is given by an explicit r-stage Runge-Kutta method of order $p \in \mathbb{N}$ to solve (3.6) under the assumptions of $(H)_3$, then $(H)_4$ is fulfilled. Similarly, $(H)_3'$ implies $(H)_4'$ (cf. [Kel99, pp. 95–102, Abschnitt 4.2]).

3.5. **Proposition.** *Under Hypothesis $(H)_4$ the mapping Φ has the representation*

$$\Phi(t,x,h) = A(t)x + F(t,x) + \sum_{n=1}^{p-1} h^n\Phi_n(t,x) + h^p\Phi_p(t,x,h) \tag{3.13}$$

for $t \in \mathbb{T}$, $x \in \mathcal{X}$ and $h \in [0,H]$, with mappings

$$\Phi_n(t,x) := \tfrac{1}{n!}D_3^n\Phi(t,x,0) \quad \text{for } 0 \le n < p,$$

$$\Phi_p(t,x,h) := \frac{1}{(p-1)!}\int_0^1 (1-s)^{p-1} D_3^p\Phi(t,x,sh)\,ds.$$

Moreover, under the Hypotheses $(H)_1$–$(H)_2$ we have:

(a) *$\Phi_n = \phi_n$ for $1 \le n < p$, $\Phi_p(t,0,h) \equiv 0$ for all $t \in \mathbb{T}$ and $h \in [0,H]$.*

(b) *The mappings $\Phi_1, \dots, \Phi_p$ are continuous and globally Lipschitzian in $x \in \mathcal{X}$ (uniformly in $t \in \mathbb{T}$, $h \in [0,H]$), and in particular we have*

$$\operatorname{Lip}_2 \Phi_p \le \tfrac{1}{p!}\operatorname{Lip}_2 D_3^p\Phi. \tag{3.14}$$

Under Hypothesis $(H)_3'$, $\Phi_1, \dots, \Phi_p$ are $(q+1)$-times continuously differentiable in the second argument with globally bounded derivatives $D_2^k\Phi_n$, $1 \le k \le q+1$, for $1 \le n \le p$.

Proof. Let $h \in (0,H]$, $t \in \mathbb{T}$, $x \in \mathcal{X}$ be arbitrary. The representation (3.13) is just a Taylor expansion of $\Phi(t,x,h)$ of order p w.r.t. h around 0 (cf. [Lan93, p. 349]).

(a) Using Proposition 3.4 and (3.13), as well as the abbreviation $f(t,x) = A(t)x + F(t,x)$, we obtain the representations

$$\phi(t,x,h) = f(t,x) + \sum_{n=1}^{k} h^n\phi_n(t,x) + \sum_{n=k+1}^{p-1} h^n\phi(t,x) + h^p\phi_p(t,x,h),$$

$$\Phi(t,x,h) = \sum_{n=0}^{k} h^n\Phi_n(t,x) + \sum_{n=k+1}^{p-1} h^n\Phi(t,x) + h^p\Phi_p(t,x,h)$$

for any $1 \le k < p$, and proceed inductively now. For $k = 0$ we get from the triangle inequality

$$K(t,x)h^p \overset{(3.12)}{\ge} \|f(t,x) + \phi(t,x,h) - \Phi(t,x,h)\| = \Bigg\| f(t,x) - \Phi_0(t,x) + h\left[\sum_{n=1}^{p-1} h^{n-1}\left(\phi_n(t,x) - \Phi_n(t,x)\right) + h^{p-1}\left(\phi_p(t,x,h) - \Phi_p(t,x,h)\right)\right]\Bigg\|$$

$$\geq \|f(t,x) - \Phi_0(t,x)\| \\ - h \left\| \sum_{n=1}^{p-1} h^{n-1} \left(\phi_n(t,x) - \Phi_n(t,x)\right) + h^{p-1} \left(\phi_p(t,x,h) - \Phi_p(t,x,h)\right) \right\|$$

and in the limit $h \searrow 0$, therefore, $\Phi_0(t,x) = f(t,x)$ for all $t \in \mathbb{T}$, $x \in \mathcal{X}$. Now assume that $\Phi_n(t,x) = \phi_n(t,x)$ for $1 \leq n \leq k$ and $k \leq p-1$. Then similarly, by the induction hypothesis

$$\begin{aligned} K(t,x)h^{p-k-1} &\overset{(3.12)}{\geq} \Bigg\| \phi_{k+1}(t,x) - \Phi_{k+1}(t,x) \\ &+ h \left[\sum_{n=k+2}^{p-1} h^{n-k-2} \left(\phi_n(t,x) - \Phi_n(t,x)\right) + h^{p-k-2} \left(\phi_p(t,x,h) - \Phi_p(t,x,h)\right) \right] \Bigg\| \\ &\geq \|\phi_{k+1}(t,x) - \Phi_{k+1}(t,x)\| \\ &- h \left\| \sum_{n=k+2}^{p-1} h^{n-k-2} \left(\phi_n(t,x) - \Phi_n(t,x)\right) + h^{p-k-2} \left(\phi_p(t,x,h) - \Phi_p(t,x,h)\right) \right\|, \end{aligned}$$

which yields $\phi_{k+1}(t,x) = \Phi_{k+1}(t,x)$ for $h \searrow 0$. This, in turn, together with $(H)_4$ and Proposition 3.4(a) gives us

$$\Phi_p(t,0,h) \overset{(3.13)}{=} \frac{1}{h^p} \left(\Phi(t,0,h) - \sum_{n=0}^{p-1} h^n \phi_n(t,0) \right) = 0.$$

(b) The continuity and the Lipschitz property of $\Phi_0, \dots, \Phi_{p-1}$ is an easy consequence of assertion (a) and Proposition 3.4(b). On the other hand, the mapping Φ_p satisfies the Lipschitz condition

$$\begin{aligned} \|\Phi_p(t,x,h) - \Phi_p(t,\bar{x},h)\| &\leq \frac{1}{(p-1)!} \int_0^1 (1-s)^{p-1} \|D_3^p \Phi(t,x,sh) - D_3^p \Phi(t,\bar{x},sh)\| \, ds \\ &\leq \frac{1}{(p-1)!} \int_0^1 (1-s)^{p-1} \operatorname{Lip}_2 D_3^p \Phi \, ds \, \|x - \bar{x}\| = \frac{\operatorname{Lip}_2 D_3^p \Phi}{p!} \|x - \bar{x}\| \end{aligned}$$

for $t \in \mathbb{T}$, $x, \bar{x} \in \mathcal{X}$, $h \in [0, H_0]$ and due to the discrete topology of $\mathbb{T}$ and [AW96, Lemma B.4] we obtain the continuity of $\Phi_p : \mathbb{T} \times \mathcal{X} \times [0,H] \to \mathcal{X}$. Now assume, $(H)'_3$ holds. Then assertion (a) and Proposition 3.4(b) yield that $\Phi_1, \dots, \Phi_{p-1}$ are $(q+1)$-times continuously differentiable in the second variable with globally bounded partial derivatives. Moreover, $(H)'_4$ implies the assertions for Φ_p. □

4. Asymptotics of Invariant Fiber Bundles

In the first part of the present section, $\mathbb{T}$ can be an arbitrary time scale which is unbounded above and has bounded graininess, like, e.g., $\mathbb{R}$ or a discrete time scale. We begin with a useful tool about invariant fiber bundles for dynamic equations of the form

$$x^\Delta = A(t)x + F_1(t,x) + \theta F_2(t,x), \tag{4.1}$$

with continuous mappings $F_1, F_2 : \mathbb{T} \times \mathcal{X} \to \mathcal{X}$. It incorporates all the technical machinery to prove our main result, and generalizes several earlier approaches for $F_2 = 0$, which can be traced back to [Aul87], who considers finite-dimensional autonomous ODEs, while [Aul95] works with nonautonomous difference equations in general Banach spaces, and the case of infinite-dimensional Carathéodory differential equations can be found in [AW96].

We precisely assume the following

4.1. **Hypothesis.** *Let $\theta \in \mathbb{F}$ and $N \geq 2$. We consider semi-linear perturbations* (4.1) *of the dynamic equation* (2.3) *and assume:*

(i) *The linear dynamic equation* (2.3) *has an exponential N-splitting, i.e., the estimates* (2.5) *hold.*

(ii) *For $i = 1, 2$ one has the identities $F_i(t, 0) \equiv 0$ on $\mathbb{T}$ and there exist $L_i \in [0, \infty)$ such that the mappings F_i satisfy the following global Lipschitz estimates*

$$\operatorname{Lip}_2 F_i \leq L_i.$$

Moreover, we define $K_1(j) := \sum_{k=1}^{j} K_k^+$, $K_2(j) := \sum_{k=j}^{N-1} K_k^-$ for $1 \leq j < N$, for some real $\delta_{\max} > 0$ we require

$$L_1 < \frac{\delta_{\max}}{2K_{\max}}, \qquad K_{\max} := \max_{i=1}^{N-1} \left(K_1(i) + K_2(i) + K_1(i)K_2(i)\right),$$

choose a fixed $\delta \in (2K_{\max}L_0, \delta_{\max})$ and abbreviate $\Theta := \{\theta \in \mathbb{F} : L_2 |\theta| \leq L_1\}$,

$$\Gamma_j := \{c \in \mathcal{R} : a_j + \delta \lhd c \lhd b_j - \delta\} \quad \text{for } 1 \leq j < N.$$

(iii) *Assume the partial derivatives $D_2^n F_i(t, \cdot)$, $t \in \mathbb{T}$, exist, are continuous on $\mathcal{X}$ up to order $m \in \mathbb{N}$, and suppose they are globally bounded, i.e. for $2 \leq n \leq m$ we have*

$$\sup_{(t,x) \in \mathbb{T} \times \mathcal{X}} \|D_2^n F_i(t, x)\| < \infty \quad \text{for } i = 1, 2.$$

For the sake of a convenient notation and with invariant projectors P_i like in (2.4), we define $P_i^j := P_i + \ldots + P_j$, $1 \leq i \leq j \leq N$. Using (2.4), we see that P_i^j is an invariant projector of (2.3); moreover, P_i^j is regular for $i \geq 2$ and we define the complementary subspaces

$$\mathcal{X}_i^j(\tau) := \mathcal{R}(P_i^j(\tau)) = \bigoplus_{k=i}^{j} \mathcal{R}(P_k(\tau)), \qquad \bar{\mathcal{X}}_i^j(\tau) := \mathcal{N}(P_i^j(\tau)) = \bigcap_{k=i}^{j} \mathcal{N}(P_k(\tau))$$

for any $\tau \in \mathbb{T}$. Before proceeding, for growth rates $a, b \in \mathcal{R}(\mathbb{T}, \mathbb{R})$ and $m, n \in \mathbb{N}$ satisfying $m \odot a \lhd b$, $a \lhd n \odot b$, we get from [PS04, Lemma 4.1] that

$$\rho_s^m[a, b] := \inf_{t \in \mathbb{T}} \lim_{h \searrow \mu(t)} \frac{1 + ha(t)}{h} \left(\sqrt[m]{\frac{1 + ha(t) + 1 + hb(t)}{1 + ha(t) + (1 + ha(t))^m}} - 1 \right) > 0,$$

$$\rho_r^n[a, b] := \inf_{t \in \mathbb{T}} \lim_{h \searrow \mu(t)} \frac{1 + hb(t)}{h} \left(1 - \sqrt[n]{\frac{1 + ha(t) + 1 + hb(t)}{1 + hb(t) + (1 + hb(t))^m}} \right) > 0.$$

4.2. **Proposition** (hierarchies of invariant fiber bundles)**.** *Assume that Hypothesis 4.1(i)–(ii) holds with $\delta_{\max} = \frac{\min_{i=1}^{N-1} \lfloor b_i - a_i \rfloor}{2}$, let $1 \leq j \leq i \leq N$, $(j, i) \neq (1, N)$, where in case $j > 1$ we additionally suppose that $\mathbb{T}$ is also unbounded below. Then for all $\theta \in \Theta$ the sets*

$$C_{i,j}(\theta) := \begin{cases} \{(\tau, x_0) \in \mathbb{T} \times \mathcal{X} : \varphi_{(4.1)}(\cdot; \tau, x_0; \theta) \in \mathcal{X}_{\tau,c}^+(\mathbb{T}) \text{ for all } c \in \Gamma_i\} & \text{for } j = 1 \\ \left\{(\tau, x_0) \in \mathbb{T} \times \mathcal{X} : \begin{array}{l} \text{there exists a solution } \nu : \mathbb{T} \to \mathcal{X} \text{ of } (2.1) \\ \text{with } \nu(\tau) = x_0 \text{ and } \nu \in \mathcal{X}_{\tau,c}^-(\mathbb{T}) \text{ for all } c \in \Gamma_{j-1} \end{array} \right\} & \text{for } i = N \\ C_{i,1}(\theta) \cap C_{N,j}(\theta) & \text{else} \end{cases}$$

are invariant fiber bundles of (4.1) *admitting the so-called* extended hierarchy

$$
\begin{array}{ccccccccc}
C_{1,1}(\theta) & \subset & C_{2,1}(\theta) & \subset & \dots & \subset & C_{N-1,1}(\theta) & \subset & \mathbb{T}\times\mathcal{X} \\
 & & \cup & & & & \cup & & \cup \\
 & & C_{2,2}(\theta) & \subset & \dots & \subset & C_{N-1,2}(\theta) & \subset & C_{N,2}(\theta) \\
 & & & & & & \cup & & \cup \\
 & & & & & \ddots & \vdots & & \vdots \\
 & & & & & & \cup & & \cup \\
 & & & & & & C_{N-1,N-1}(\theta) & \subset & C_{N,N-1}(\theta) \\
 & & & & & & & & \cup \\
 & & & & & & & & C_{N,N}(\theta).
\end{array} \tag{4.2}
$$

Each $C_{i,j}(\theta)$ *possesses the representation as graph*

$$C_{i,j}(\theta) = \left\{(\tau, \eta + c_{i,j}(\tau, \eta; \theta)) \in \mathbb{T} \times \mathcal{X} : \tau \in \mathbb{T}, \eta \in \mathcal{X}_j^i(\tau)\right\}$$

with a uniquely determined continuous mapping $c_{i,j} : \mathbb{T} \times \mathcal{X} \times \Theta \to \mathcal{X}$ *satisfying*

$$c_{i,j}(\tau, x_0; \theta) = c_{i,j}(\tau, P_j^i(\tau)x_0; \theta) \in \bar{\mathcal{X}}_j^i(\tau) \quad \textit{for } \tau \in \mathbb{T},\ x_0 \in \mathcal{X}.$$

Furthermore, it holds:

(a) $c_{i,j}(\tau, 0; \theta) \equiv 0$ *on* $\mathbb{T} \times \Theta$,

(b) $c_{i,j} : \mathbb{T} \times \mathcal{X} \times \Theta \to \mathcal{X}$ *satisfies the Lipschitz estimates*

$$\operatorname{Lip} c_{i,j}(\tau, \cdot; \theta) \le \begin{cases} \frac{K_1(i)K_2(i)(L_1+|\theta|L_2)}{\delta-(K_1(i)+K_2(i))(L_1+|\theta|L_2)} & \textit{for } j = 1 \\ \frac{K_1(j-1)K_2(j-1)(L_1+|\theta|L_2)}{\delta-(K_1(j-1)+K_2(j-1))(L_1+|\theta|L_2)} & \textit{for } i = N \\ \max_{k\in\{i,j-1\}} \frac{2K_1(k)K_2(k)(L_1+|\theta|L_2)}{\delta-(K_1(k)+K_2(k)+K_1(k)K_2(k))(L_1+|\theta|L_2)} & \textit{else} \end{cases},$$

$$\operatorname{Lip} c_{i,j}(\tau, x_0; \cdot) \le \begin{cases} \frac{\delta K_1(i)K_2(i)(K_1(i)+K_2(i))L_2}{[\delta-2(K_1(i)+K_2(i))L_1]^2} \|x_0\| & \textit{for } j = 1 \\ \frac{\delta K_1(j-1)(K_1(j-1)+K_2(j-1))L_2}{[\delta-2(K_1(j-1)+K_2(j-1))L_1]^2} \|x_0\| & \textit{for } i = N \\ \frac{2L_{i,j} \max_{k\in\{i,j-1\}} \frac{\delta K_1(k)K_2(k)(K_1(k)+K_2(k))L_2}{[\delta-2(K_1(k)+K_2(k))L_1]^2}}{1-\max_{k\in\{i,j-1\}} \frac{2K_1(k)K_2(k)L_1}{\delta-2(K_1(k)+K_2(k))L_1}} \|x_0\| & \textit{else} \end{cases} \tag{4.3}$$

for all $\tau \in \mathbb{T}$, $x_0 \in \mathcal{X}$ *and* $\theta \in \Theta$, *with*

$$L_{i,j} := 1 + \max_{k\in\{i,j-1\}} \frac{2K_1(k)K_2(k)L_1}{\delta - 2(K_1(k) + K_2(k) + K_1(k)K_2(k))L_1},$$

(c) *if additionally Hypothesis 4.1(iii) and the gap conditions*

$$\begin{cases} m_{i,j} \odot a_i \lhd b_i & \textit{for } j = 1 \\ a_{j-1} \lhd m_{i,j} \odot b_{j-1} & \textit{for } i = N \\ m_{i,j} \odot a_i \lhd b_i,\ a_{j-1} \lhd m_{i,j} \odot b_{j-1} & \textit{else} \end{cases}$$

hold for some $m_{i,j} \in \{1, \dots, m\}$, *and if we set*

$$\delta_{\max} := \begin{cases} \min\left\{\frac{\lfloor b_i - a_i\rfloor}{2}, \rho_s^{m_{i,j}}[a_i, b_i]\right\} & \textit{for } j = 1 \\ \min\left\{\frac{\lfloor b_{j-1} - a_{j-1}\rfloor}{2}, \rho_r^{m_{i,j}}[a_{j-1}, b_{j-1}]\right\} & \textit{for } i = N \\ \min\left\{\frac{\lfloor b_i - a_i\rfloor}{2}, \frac{\lfloor b_{j-1} - a_{j-1}\rfloor}{2}, \rho_s^{m_{i,j}}[a_i, b_i], \rho_r^{m_{i,j}}[a_{j-1}, b_{j-1}]\right\} & \textit{else} \end{cases},$$

then the partial derivatives $D^n_{(2,3)}c_{i,j}$ *exist, are continuous up to order* $m_{i,j}$*, and there exist reals* $M^n_{i,j}, N^n_{i,j} > 0$*, such that*

$$\begin{aligned} \|D_2^n c_{i,j}(\tau, x_0; \theta)\| &\leq M^n_{i,j} \quad \text{for } 1 \leq n \leq m_{i,j}, \\ \|D_3 D_2^n c_{i,j}(\tau, x_0; \theta)\| &\leq N^n_{i,j}\,\|x_0\| \quad \text{for } 0 \leq n < m_{i,j} \end{aligned} \tag{4.4}$$

for all $\tau \in \mathbb{T}$*,* $x_0 \in \mathcal{X}$ *and* $\theta \in \Theta$*.*

Proof. The assertion is a special case of [Pöt05, Theorem 4.2]. □

At this point it is important to illuminate the lengthy Proposition 4.2 in the light of classical (continuous) dynamical systems and an exponential trichotomy. Thereto consider the autonomous ODE

$$\dot{x} = Ax + F(x),$$

such that the spectrum of A allows a decomposition into three spectral sets $\sigma_1 \dot\cup \sigma_2 \dot\cup \sigma_3$, where the points in σ_i have reals parts in an open interval (β_{i-1}, α_i) for $i = 1, 2, 3$, with real numbers $\beta_0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 := \infty$, and $\operatorname{Lip} F$ is assumed to be "small" w.r.t. the spectral gaps of A. This guarantees an exponential 3-splitting like in $(H)_1$ for the linear part. Then in case $0 \in (\beta_1, \alpha_2)$, the above ODE possesses five "classical" invariant manifolds (cf. [Kel67]) $C_{1,1}, C_{2,1}, C_{2,2}, C_{3,2}, C_{3,3} \subseteq \mathcal{X}$, namely the stable, the center-stable, the center, the center-unstable and the unstable manifold, respectively.

Now, for $\mathbb{T} = \mathbb{R}$, the above ODE is obviously a special case of (4.1), and if we suppose $N = 3$ and $b_1 \trianglelefteq 0 \trianglelefteq a_2$, Proposition 4.2 allows the following interpretation. The dynamic equation (4.1) has five nontrivial invariant fiber bundles, denoted as

- *Stable fiber bundle* $C_{1,1}$: Because of $c_1 \triangleleft b_1$ and the dynamical characterization in Theorem 4.2(a) all solutions of (4.1) on $C_{1,1}$ converge to 0 exponentially for $t \to \infty$.
- *Center-stable fiber bundle* $C_{2,1}$: All solutions of (4.1) which are not growing too fast as $t \to \infty$ (in the sense that they are c_2^+-quasibounded with $c_2 \trianglelefteq b_2 - \delta$) are contained in $C_{2,1}$, like e.g., solutions bounded in forward time.
- *Center-unstable fiber bundle* $C_{3,2}$: All solutions of (4.1) which exist and are not growing too fast as $t \to -\infty$ (in the sense of c_1^--quasiboundedness with $a_1 + \delta \trianglelefteq c_1$) lie on $C_{3,2}$, like e.g., solutions bounded in backward time.
- *Unstable fiber bundle* $C_{3,3}$: All solutions on the unstable fiber bundle exist in backward time and converge exponentially to 0 as $t \to -\infty$.
- *Center fiber bundle* $C_{2,2}$: The center fiber bundle consists of those solutions which are contained both in the center-stable and the center-unstable fiber bundle. Particularly, all bounded solutions lie on this fiber bundle.

Here we have suppressed the dependence on the parameter $\theta \in \Theta$.

For the remaining section we target the question, whether the invariant fiber bundles from Proposition 4.2 in the special case of ODEs ($\mathbb{T} = \mathbb{R}$) persist under explicit one-step discretization of (3.6). In fact, under our assumptions the ODE (3.6) possesses an extended hierarchy of integral manifolds $C^{i,j}_{\mathbb{R}}$ and we prove that they essentially coincide with the inertial fiber bundles $C^{i,j}_{\mathbb{T}}$ of the $\mathbb{T}$-equation (3.8). Furthermore, also the one-step discretization (3.11) has an extended hierarchy of invariant fiber bundles $\hat{C}^{i,j}_{\mathbb{T}}$ and we are able to estimate their distance to $C^{i,j}_{\mathbb{T}}$. Precisely, we have

4.3. **Theorem** (discretization of integral manifolds)**.** *Let* $\bar\alpha_j, \bar\beta_j \in (\alpha_j, \beta_j)$ *be reals satisfying* $\bar\alpha_j < \bar\beta_j$ *for* $1 \leq j < N$*. Assume that the Hypotheses* $(H)_1$*–*$(H)_4$ *are fulfilled and let the global*

Lipschitz constant of F satisfy

$$0 \le \operatorname{Lip}_2 F < \frac{\delta_{\max}}{4K_{\max}}, \qquad K_{\max} := \max_{i=1}^{N-1} \left(\bar K_1(i) + \bar K_2(i) + \bar K_1(i)\bar K_2(i)\right) \tag{4.5}$$

with $\delta_{\max} := \frac{\min_{j=1}^{N-1}\{\bar\beta_j - \bar\alpha_j\}}{4}$, *choose a fixed* $\delta \in (4K_{\max}\operatorname{Lip}_2 F, \delta_{\max})$ *and assume that the maximal step-size* $H > 0$ *is so small that beyond* (3.4) *we have*

$$\frac{\bar\beta_j - \bar\alpha_j}{4} < \varepsilon_H\left(\bar\alpha_j, \frac{\bar\alpha_j + \bar\beta_j}{2}\right), \qquad \frac{\bar\beta_j - \bar\alpha_j}{4} < \varepsilon_H\left(\frac{\bar\alpha_j + \bar\beta_j}{2}, \bar\beta_j\right),$$
$$\frac{\bar\beta_j - \bar\alpha_j}{2} < \varepsilon_H(\bar\alpha_j, \bar\beta_j) \tag{4.6}$$

for $1 \le j < N$ *and*

$$\sum_{n=1}^{p} H^n \operatorname{Lip}_2 \phi_n < \frac{\min_{j=1}^{2}\left\{\bar\beta_j - \bar\alpha_j\right\}}{8K_{\max}} - \operatorname{Lip}_2 F, \tag{4.7}$$

$$\frac{H^p}{p!}\operatorname{Lip}_2 D_3^p \Phi \le \operatorname{Lip}_2 F + \sum_{n=1}^{p-1} H^n \operatorname{Lip}_2 \phi_n. \tag{4.8}$$

Then the semi-linear ODE (3.6) *satisfies the assumptions of Proposition 4.2 with*

- $\mathbb{T} = \mathbb{R}$, $\theta = 0$, $F_1 = F$, $F_2 = 0$, α_j, β_j *instead of* a_j, b_j, $L_0 = \operatorname{Lip}_2 F$, $L_1 = 0$,

and the $\mathbb{T}$*-equation* (3.8) *satisfies the assumptions of Proposition 4.2 with*

- $\mathbb{T} = \{t_k\}_{k\in\mathbb{I}}$, $\theta = 0$, $F_1 = F + \phi$, $F_2 = 0$, $\bar a_j, \bar b_j$ *instead of* a_j, b_j, $\bar K_j^+, \bar K_j^-$ *instead of* K_j^+, K_j^-, *and the projectors* Q_j *instead of* P_j *(cf. Lemma 3.3)*, $L_0 = \operatorname{Lip}_2 F + \sum_{n=1}^{p} H^n \operatorname{Lip}_2 \phi_n$, $L_1 = 0$.

If we define the complementary subspaces

$$\mathcal{Y}_j^i(\tau) := \mathcal{R}(Q_j^i(\tau)) = \bigoplus_{k=j}^{i} \mathcal{R}(Q_k(\tau)), \qquad \bar{\mathcal{Y}}_j^i(\tau) := \mathcal{N}(Q_j^i(\tau)) = \bigcap_{k=j}^{i} \mathcal{N}(Q_k(\tau)) \quad \text{for } \tau \in \mathbb{T}$$

and the sets

$$\Gamma_{\mathbb{R}}^i := (\alpha_i + \delta, \beta_i - \delta), \qquad \bar\Gamma_{\mathbb{T}}^i := \{c \in \mathcal{R} : a_i + \delta \lhd c \lhd b_i - \delta\},$$

then in addition, for any $1 \le j \le i \le N$, $(j,i) \ne (1,N)$, *where in case* $j > 1$ *we additionally suppose* $\mathbb{I} = \mathbb{Z}$, *the following holds:*

(a) *The set*

$$C_{\mathbb{R}}^{i,j} := \begin{cases} \left\{(\tau, x_0) \in \mathbb{R}\times\mathcal{X} : \varphi_{(3.6)}(\cdot;\tau,x_0;\theta) \in \mathcal{X}_{\tau,\gamma}^{+}(\mathbb{R}) \text{ for all } \gamma \in \Gamma_{\mathbb{R}}^i\right\} & \text{for } j = 1 \\ \left\{(\tau, x_0) \in \mathbb{R}\times\mathcal{X} : \varphi_{(3.6)}(\cdot;\tau,x_0;\theta) \in \mathcal{X}_{\tau,\gamma}^{-}(\mathbb{R}) \text{ for all } \gamma \in \Gamma_{\mathbb{R}}^{j-1}\right\} & \text{for } i = N \\ C_{\mathbb{R}}^{i,1} \cap C_{\mathbb{R}}^{N,j} & \text{else} \end{cases}$$

is an invariant fiber bundle of the ODE (3.6) *possessing the representation*

$$C_{\mathbb{R}}^{i,j} = \left\{(\tau, \eta + c_{\mathbb{R}}^{i,j}(\tau,\eta)) \in \mathbb{R}\times\mathcal{X} : \tau \in \mathbb{R}, \eta \in \mathcal{X}_i^j(\tau)\right\}$$

with a uniquely determined continuous mapping $c_{\mathbb{R}}^{i,j} : \mathbb{R}\times\mathcal{X} \to \mathcal{X}$,

(b) *the set*

$$C_{\mathbb{T}}^{i,j} := \begin{cases} \left\{(\tau, x_0) \in \mathbb{T} \times \mathcal{X} : \varphi_{(3.8)}(\cdot; \tau, x_0; \theta) \in \mathcal{X}_{\tau,c}^{+}(\mathbb{T}) \text{ for all } c \in \bar{\Gamma}_{\mathbb{T}}^{i}\right\} & \text{for } j = 1 \\ \left\{(\tau, x_0) \in \mathbb{T} \times \mathcal{X} : \begin{array}{l} \text{there exists a solution } \nu : \mathbb{T} \to \mathcal{X} \text{ of } (3.8) \\ \text{with } \nu(\tau) = x_0 \text{ and } \nu \in \mathcal{X}_{\tau,c}^{-}(\mathbb{T}) \text{ for all } c \in \bar{\Gamma}_{\mathbb{T}}^{j-1} \end{array}\right\} & \text{for } i = N \\ C_{\mathbb{T}}^{i,1} \cap C_{\mathbb{T}}^{N,j} & \text{else} \end{cases}$$

is an invariant fiber bundle of the $\mathbb{T}$*-equation* (3.8) *possessing the representation*

$$C_{\mathbb{T}}^{i,j} = \left\{(\tau, \eta + c_{\mathbb{T}}^{i,j}(\tau, \eta)) \in \mathbb{T} \times \mathcal{X} : \tau \in \mathbb{T}, \eta \in \mathcal{Y}_i^j(\tau)\right\}$$

with a uniquely determined continuous mapping $c_{\mathbb{T}}^{i,j} : \mathbb{T} \times \mathcal{X} \to \mathcal{X}$,

(c) $C_{\mathbb{T}}^{i,j} = C_{\mathbb{R}}^{i,j} \cap (\mathbb{T} \times \mathcal{X})$.

Moreover, the set

$$\hat{C}_{\mathbb{T}}^{i,j} = \begin{cases} \left\{(\tau, x_0) \in \mathbb{T} \times \mathcal{X} : \varphi_{(3.11)}(\cdot; \tau, x_0; \theta) \in \mathcal{X}_{\tau,c}^{+}(\mathbb{T}) \text{ for all } c \in \bar{\Gamma}_{\mathbb{T}}^{i}\right\} & \text{for } j = 1 \\ \left\{(\tau, x_0) \in \mathbb{T} \times \mathcal{X} : \begin{array}{l} \text{there exists a solution } \nu : \mathbb{T} \to \mathcal{X} \text{ of } (3.11) \\ \text{with } \nu(\tau) = x_0 \text{ and } \nu \in \mathcal{X}_{\tau,c}^{-}(\mathbb{T}) \text{ for all } c \in \bar{\Gamma}_{\mathbb{T}}^{j-1} \end{array}\right\} & \text{for } i = N \\ C_{\mathbb{T}}^{i,1} \cap C_{\mathbb{T}}^{N,j} & \text{else} \end{cases}$$

is an invariant fiber bundle of the one-step discretization (3.11) *possessing the representation*

$$\hat{C}_{\mathbb{T}}^{i,j} = \left\{(\tau, \eta + \hat{c}_{\mathbb{T}}^{i,j}(\tau, \eta)) : \tau \in \mathbb{T}, \eta \in \mathcal{Y}_i^j(\tau)\right\}$$

with a unique continuous mapping $\hat{c}_{\mathbb{T}}^{i,j} : \mathbb{T} \times \mathcal{X} \to \mathcal{X}$ *satisfying* $\hat{c}_{\mathbb{T}}^{i,j}(\tau, x_0) = \hat{c}_{\mathbb{T}}^{i,j}(\tau, Q_j^i(\tau)x_0; \theta) \in \mathcal{Y}_i^j(\tau)$ *for* $\tau \in \mathbb{T}$, $x_0 \in \mathcal{X}$. *Furthermore, we have:*

(d) $\hat{c}_{\mathbb{T}}^{i,j}(\tau, 0) \equiv 0$ *on* $\mathbb{T}$,

(e) $\hat{c}_{\mathbb{T}}^{i,j} : \mathbb{T} \times \mathcal{X} \to \mathcal{X}$ *satisfies the estimates*

$$\operatorname{Lip} \hat{c}_{\mathbb{T}}^{i,j}(\tau, \cdot) \le \begin{cases} \frac{\bar{K}_1(i)\bar{K}_2(i)\left(\bar{L}_0(H)+H^p\bar{L}_1\right)}{\delta-\left(\bar{K}_1(i)+\bar{K}_2(i)\right)\left(\bar{L}_0(H)+H^p\bar{L}_1\right)} & \text{for } j = 1 \\ \frac{\bar{K}_1(j-1)\bar{K}_2(j-1)\left(\bar{L}_0(H)+H^p\bar{L}_1\right)}{\delta-\left(\bar{K}_1(j-1)+\bar{K}_2(j-1)\right)\left(\bar{L}_0(H)+H^p\bar{L}_1\right)} & \text{for } i = N \\ \max_{k\in\{i,j-1\}} \frac{2\bar{K}_1(k)\bar{K}_2(k)\left(\bar{L}_0(H)+H^p\bar{L}_1\right)}{\delta-(\bar{K}_1(k)+\bar{K}_2(k)+\bar{K}_1(k)\bar{K}_2(k))\left(\bar{L}_0(H)+H^p\bar{L}_1\right)} & \text{else} \end{cases},$$

$$\left\|\hat{c}_{\mathbb{T}}^{i,j}(\tau, x_0) - c_{\mathbb{T}}^{i,j}(\tau, x_0)\right\| \le M_{i,j}(H)H^p \|x_0\| \tag{4.9}$$

for all $\tau \in \mathbb{T}$ *and* $x_0 \in \mathcal{X}$, *with*

$$M_{i,j}(H) := \begin{cases} \frac{\bar{K}_1(i)\bar{K}_2(i)\left(\bar{L}_0(H)+H^p\bar{L}_1\right)}{\left[\delta-2\left(\bar{K}_1(i)+\bar{K}_2(i)\right)\bar{L}_0(H)\right]^2} & \text{for } j = 1 \\ \frac{\bar{K}_1(j-1)\bar{K}_2(j-1)\left(\bar{L}_0(H)+H^p\bar{L}_1\right)}{\left[\delta-2\left(\bar{K}_1(j-1)+\bar{K}_2(j-1)\right)\bar{L}_0(H)\right]^2} & \text{for } i = N \\ \frac{2\bar{L}_{i,j}(H)\max_{k\in\{i,j-1\}} \frac{\delta\bar{K}_1(k)(\bar{K}_1(k)+\bar{K}_2(k))\bar{L}_1}{\left[\delta-2(\bar{K}_1(k)+\bar{K}_2(k))\bar{L}_0(H)\right]^2}}{1-\max_{k\in\{i,j-1\}} \frac{2\bar{K}_1(k)\bar{K}_2(k)\bar{L}_0(H)}{\delta-2(\bar{K}_1(k)+\bar{K}_2(k))\bar{L}_0(H)}} & \text{else} \end{cases},$$

(f) *if additionally* $(H_3)'$, $(H_4)'$ *and the gap conditions*

$$\begin{cases} m_{i,j}\alpha_i < \beta_i & \text{for } j = 1 \\ \alpha_{j-1} < m_{i,j}\beta_{j-1} & \text{for } i = N \\ m_{i,j}\alpha_i < \beta_i,\ \alpha_{j-1} < m_{i,j}\beta_{j-1} & \text{else} \end{cases} \tag{4.10}$$

hold for some $m_{i,j} \in \{1, \ldots, q+1\}$*, and if we set*

$$\delta_{\max} := \begin{cases} \min\left\{\frac{\lfloor \bar{b}_i - \bar{a}_i \rfloor}{2}, \rho_s^{m_{i,j}}[\bar{a}_i, \bar{b}_i]\right\} & \text{for } j=1 \\ \min\left\{\frac{\lfloor \bar{b}_{j-1} - \bar{a}_{j-1} \rfloor}{2}, \rho_r^{m_{i,j}}[\bar{a}_{j-1}, \bar{b}_{j-1}]\right\} & \text{for } i=N \\ \min\left\{\frac{\lfloor \bar{b}_i - \bar{a}_i \rfloor}{2}, \frac{\lfloor \bar{b}_{j-1} - \bar{a}_{j-1} \rfloor}{2}, \rho_s^{m_{i,j}}[\bar{a}_i, \bar{b}_i], \rho_r^{m_{i,j}}[\bar{a}_{j-1}, \bar{b}_{j-1}]\right\} & \text{else} \end{cases},$$

then the partial derivatives $D^n_{(2,3)} c^{i,j}_{\mathbb{T}}, D^n_{(2,3)} \hat{c}^{i,j}_{\mathbb{T}}$ *exist, are continuous up to order* $m_{i,j}$*, and there exist reals* $M^n_{i,j} > 0$ *such that*

$$\left\| D^n_2 \hat{c}^{i,j}_{\mathbb{T}}(\tau, x_0) - D^n_2 c^{i,j}_{\mathbb{T}}(\tau, x_0) \right\| \le M^n_{i,j} \|x_0\| H^p \tag{4.11}$$

for all $\tau \in \mathbb{T}$*,* $x_0 \in \mathcal{X}$ *and* $0 \le n < m_{i,j}$*,*

with the abbreviations

$$\bar{L}_0(H) := \operatorname{Lip}_2 F + \sum_{n=1}^{p} H^n \operatorname{Lip}_2 \phi_n, \quad \bar{L}_1 := \tfrac{1}{p!} \operatorname{Lip}_2 D_3^p \Phi + \operatorname{Lip}_2 \phi_p,$$

$$\bar{L}_{i,j}(H) := 1 + \max_{k \in \{i, j-1\}} \frac{2\bar{K}_1(k)\bar{K}_2(k)\bar{L}_0(H)}{\delta - 2(\bar{K}_1(k) + \bar{K}_2(k) + \bar{K}_1(k)\bar{K}_2(k))\bar{L}_0(H)}.$$

Remark 4.1. (1) Note that the individual fibers of the graphs $C^{i,j}_{\mathbb{T}}$ and $\hat{C}^{i,j}_{\mathbb{T}}$ are only $O(H)$-close due to the dichotomy Lemma 3.3, although one has the better estimates (4.9), (4.11) concerning the mappings $c^{i,j}_{\mathbb{T}}$ and $\hat{c}^{i,j}_{\mathbb{T}}$.

(2) It is crucial for assertion (c) that we work under global assumptions on the nonlinearities and consequently have a dynamical characterization of $C^{i,j}_{\mathbb{R}}$ and $C^{i,j}_{\mathbb{T}}$ available. In a local framework, such assertions need not to hold, since for example local center manifolds of time-h-maps need not to be local center manifolds of the basic flow (cf. [Kri05]).

Proof. We subdivide the proof into three steps:

(I) In terms of the inequalities

$$\operatorname{Lip}_2 F < \frac{\min_{j=1}^{N-1}\{\beta_j - \alpha_j\}}{4K_{\max}}, \qquad \max_{i=1}^{N-1}\{K_1(i) + K_2(i) + K_1(i)K_2(i)\} \le K_{\max},$$

the assumption (4.5) guarantees that one can apply Proposition 4.2 in case of the time scale $\mathbb{T} = \mathbb{R}$, $F_1 = F$, $\theta = 0$ and $a_j(t) \equiv \alpha_j$, $b_j(t) \equiv \beta_j$ for $1 \le j < N$ to the semi-linear ODE (3.6). Thus, for $1 \le j \le i \le N$, $(j,i) \ne (1,N)$, there exist mappings $c^{i,j}_{\mathbb{R}} : \mathbb{R} \times \mathcal{X} \to \mathcal{X}$ such that the assertion (a) holds.

(II) We introduce a "homotopy" between the $\mathbb{T}$-equation (3.8) and the one-step discretization (3.11), namely the system (4.1) on the discrete time scale $\mathbb{T}$, with mappings $F_1, F_2 : \mathbb{T} \times \mathcal{X} \to \mathcal{X}$,

$$F_1(t,x) := F(t,x) + \phi(t,x,\mu(t))$$
$$F_2(t,x) := \left(\frac{\mu(t)}{H}\right)^p (\Phi_p(t,x,\mu(t)) - \phi_p(t,x,\mu(t))).$$

Obviously, for $\theta = 0$, the equation (4.1) coincides with (3.8) (cf. Proposition 3.4), and for $\theta = H^p$, equation (4.1) coincides with (3.11) (cf. Proposition 3.5). From Lemma 3.3 and the assumption (3.4) we obtain that the linear part (2.3) possesses an exponential N-splitting with

$\bar{a}_j, \bar{b}_j, \bar{K}_j^-, \bar{K}_j^+$ given in Lemma 3.3 and the complementary invariant projectors Q_j. Furthermore, due to Proposition 3.4(c) and Proposition 3.5(b) we get the Lipschitz estimates

$$L_0 := \sup_{t\in\mathbb{T}} \operatorname{Lip} F_1(t,\cdot) \overset{(3.9)}{\leq} \operatorname{Lip}_2 F + \sum_{n=1}^{p} H^n \operatorname{Lip}_2 \phi_n,$$

$$L_1 := \sup_{t\in\mathbb{T}} \operatorname{Lip} F_2(t,\cdot) \overset{(3.14)}{\leq} \tfrac{1}{p!} \operatorname{Lip}_2 D_3^p\Phi + \operatorname{Lip}_2 \phi_p$$

from, e.g., [AMR88, p. 138, Exercise 2.5K(i)]. Because of the assumption (4.6), one obtains the inequality $\frac{\bar{\beta}_j-\bar{\alpha}_j}{2} < \frac{e^{\bar{\beta}_j\mu(t)}-e^{\bar{\alpha}_j\mu(t)}}{\mu(t)} = \bar{b}_j(t) - \bar{a}_j(t)$ for $1 \leq j < N$, $t \in \mathbb{T}$ and, therefore, (4.7) enables us to apply Proposition 4.2 to the above dynamic equation (4.1). Here, in particular, the assumption (4.8) guarantees $H^p \in \Theta$, and trivially we have $0 \in \Theta$. In any case, for $1 \leq j \leq i \leq N$, $(j,i) \neq (1,N)$, Proposition 4.2 yields the existence of uniquely determined continuous mappings $c_{i,j} : \mathbb{T} \times \mathcal{X} \times \Theta \to \mathcal{X}$, such that the graph

$$C_{i,j}(\theta) = \left\{(\tau, \eta + c_{i,j}(\tau,\eta;\theta)) : \tau \in \mathbb{T}, \eta \in \mathcal{Y}_i^j(\tau)\right\} \quad \text{for } \theta \in \Theta$$

is an invariant fiber bundle of (4.1) possessing the dynamical characterization

$$C_{i,j}(\theta) = \begin{cases} \left\{(\tau,x_0) \in \mathbb{T}\times\mathcal{X} : \varphi_{(4.1)}(\cdot;\tau,x_0;\theta) \in \mathcal{X}^+_{\tau,c}(\mathbb{T}) \text{ for all } c \in \Gamma_i\right\} & \text{for } j=1 \\ \left\{(\tau,x_0) \in \mathbb{T}\times\mathcal{X} : \begin{array}{l}\text{there exists a solution } \nu : \mathbb{T}\to\mathcal{X} \text{ of (2.1)} \\ \text{with } \nu(\tau)=x_0 \text{ and } \nu \in \mathcal{X}^-_{\tau,c}(\mathbb{T}) \text{ for all } c \in \Gamma_{j-1}\end{array}\right\} & \text{for } i=N \\ C_{i,1}(\theta) \cap C_{N,j}(\theta) & \text{else} \end{cases}$$

for all $\theta \in \Theta$, and we define

$$c_{\mathbb{T}}^{i,j}(\tau,x_0) := c_{i,j}(\tau,x_0;0), \qquad \hat{c}_{\mathbb{T}}^{i,j}(\tau,x_0) := c_{i,j}(\tau,x_0;H^p),$$
$$C_{\mathbb{T}}^{i,j} := C_{i,j}(0), \qquad \hat{C}_{\mathbb{T}}^{i,j} := C_{i,j}(H^p), \tag{4.12}$$

for $\tau \in \mathbb{T}$ and $x_0 \in \mathcal{X}$. These mappings and graphs satisfy the assertions (b), (d), as well as the first Lipschitz estimate in (e). Furthermore, we have

$$\left\| c_{\mathbb{T}}^{i,j}(\tau,x_0) - \hat{c}_{\mathbb{T}}^{i,j}(\tau,x_0) \right\| \overset{(4.12)}{=} \| c_{i,j}(\tau,x_0;0) - c_{i,j}(\tau,x_0;H^p)\| \overset{(4.3)}{\leq} \operatorname{Lip} c_{i,j}(\tau,x_0;\cdot)H^p$$

for all $\tau \in \mathbb{T}$, $x_0 \in \mathcal{X}$, which gives us the second inequality in (e). To verify (f), we note that the gap conditions (4.10) guarantee

$$\begin{cases} m_{i,j} \odot \bar{a}_i \lhd \bar{b}_i & \text{for } j=1 \\ \bar{a}_{j-1} \lhd m_{i,j} \odot \bar{b}_{j-1} & \text{for } i=N \\ m_{i,j} \odot \bar{a}_i \lhd \bar{b}_i,\ \bar{a}_{j-1} \lhd m_{i,j} \odot \bar{b}_{j-1} & \text{else} \end{cases}.$$

Therefore, Proposition 4.2(c) implies that $c_{i,j}(\tau,\cdot) : \mathcal{X}\times\Theta \to \mathcal{X}$ is of class $C^{m_{i,j}}$ and the mean value inequality (cf. [Lan93, p. 342, Corollary 4.3]) yields

$$\begin{aligned} \left\| D_2^n c_{\mathbb{T}}^{i,j}(\tau,x_0) - D_2^n \hat{c}_{\mathbb{T}}^{i,j}(\tau,x_0) \right\| &\overset{(4.12)}{=} \| D_2^n c_{i,j}(\tau,x_0;0) - D_2^n c_{i,j}(\tau,x_0;H^p)\| \\ &\overset{(4.4)}{\leq} N_{i,j}^n \|x_0\| H^p \quad \text{for } 0 \leq n < m_{s_i}, \end{aligned}$$

and $\tau \in \mathbb{T}$, $x_0 \in \mathcal{X}$. This gives us (f).

(III) It remains to show the identity in (c). Hereto, for $1 \leq i < N$, we define $\gamma_i := \frac{\bar{\alpha}_i+\bar{\beta}_i}{2} \in \Gamma_{\mathbb{R}}^i$ and for the growth rate $c_i(t) := \frac{e^{\gamma_i\mu(t)}-1}{\mu(t)}$ one obtains

$$\delta < \frac{\bar{\beta}_i - \bar{\alpha}_i}{2} = \frac{\gamma_i - \bar{\alpha}_i}{2} < \frac{e^{\gamma_i\mu(t)} - e^{\bar{\alpha}_i\mu(t)}}{\mu(t)} = c_i(t) - a_i(t) \quad \text{for } t \in \mathbb{T}.$$

In order to derive (c) we have to verify two inclusions. Here we restrict ourselves to the situation $j = 1$, since the other cases yield analogously:
$(\subseteq)$ Let $(\tau, x_0) \in \mathbb{T} \times \mathcal{X}$ such that $(\tau, x_0) \in C_{\mathbb{R}}^{i,1}$ holds. Then the mapping $\varphi_{(3.6)}(\cdot; \tau, x_0)$ is γ_i^+-quasibounded and since the general solutions of the ODE (3.6) and of the $\mathbb{T}$-equation (3.8) coincide on $\mathbb{T}$ (cf. Lemma 3.4), we get

$$\left\|\varphi_{(3.8)}(t; \tau, x_0)\right\| e_{c_i}(\tau, t) \leq \left\|\varphi_{(3.6)}(t; \tau, x_0)\right\| e^{\gamma_i(\tau - t)} \leq \left\|\varphi_{(3.6)}(\cdot; \tau, x_0)\right\|_{\tau, \gamma_i}^{+}$$

for all $t \in \mathbb{T}_\tau^+$; this implies $\varphi_{(3.8)}(\cdot; \tau, x_0) \in \mathcal{X}_{\tau, c_i}^+(\mathbb{T})$, i.e., $(\tau, x_0) \in C_{\mathbb{T}}^{i,1}$.
$(\supseteq)$ For $(\tau, x_0) \in C_{\mathbb{T}}^{i,1}$ we have the inclusion $\varphi_{(3.8)}(\cdot; \tau, x_0) \in \mathcal{X}_{\tau, c_i}^+(\mathbb{T})$ and we are going to show $\varphi_{(3.6)}(\cdot; \tau, x_0) \in \mathcal{X}_{\tau, \gamma_i}^+(\mathbb{R})$. Thereto, an elementary Gronwall argument leads to the inequality $\left\|\varphi_{(3.6)}(t; t_k, x_0)\right\| \leq e^{(|A|_0 + L_1)H} \|x_0\|$ for $t \in [t_k, t_{k+1}]$, $k \in \mathbb{N}_0$ and consequently we arrive at

$$\begin{aligned}
\left\|\varphi_{(3.6)}(t; \tau, x_0)\right\| &\overset{(2.2)}{=} \left\|\varphi_{(3.6)}(t; t_k, \varphi_{(3.6)}(t_k; \tau, x_0))\right\| \\
&\leq e^{(|A|_0 + L_1)H} \left\|\varphi_{(3.6)}(t_k; \tau, x_0)\right\| \\
&\leq e^{(|A|_0 + L_1)H} \left\|\varphi_{(3.8)}(\cdot; \tau, x_0)\right\|_{\tau, c_i}^{+} e_{c_i}(t_k, \tau) \\
&\leq e^{(|A|_0 + L_1)H} \max\left\{1, e^{\gamma_i H}\right\} \left\|\varphi_{(3.8)}(\cdot; \tau, x_0)\right\|_{\tau, c_i}^{+} e^{\gamma_i(t - \tau)} \quad \text{for } t \in \mathbb{R}_\tau^+,
\end{aligned}$$

which implies that the solution $\varphi_{(3.6)}(\cdot; \tau, x_0)$ is γ_i^+-quasibounded, i.e., $\varphi_{(3.6)}(\cdot; \tau, x_0) \in C_{\mathbb{R}}^{i,j}$.
Therefore, the proof of Theorem 4.3 is finished. □

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