DISCRETE INERTIAL MANIFOLDS

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ABSTRACT. This work is devoted to attractive invariant manifolds for nonautonomous difference equations, occurring in the discretization theory for evolution equations. Such invariant sets provide a discrete counterpart to inertial manifolds of dissipative FDEs and evolutionary PDEs. We discuss their essential properties, like smoothness, the existence of an asymptotic phase, normal hyperbolicity and attractivity in a nonautonomous framework of pullback attraction.

As application we show that inertial manifolds of the Allen-Cahn and complex Ginzburg-Landau equation persist under discretization. For the Ginzburg-Landau equation we can also determine the dimension of the inertial manifold.

1. INTRODUCTION

1.1. **Introduction and Motivation:** The study of the long-term behavior of evolutionary equations is a problem of interest in many areas of mathematical physics and other applied sciences. In fact, a large variety of evolutionary processes in mechanics, physics or biology can be described using nonlinear dissipative functional differential equations (FDEs) or partial differential equations (PDEs) generating infinite dimensional dynamical systems. The reduction of such a system in an infinite dimensional state space to a finite one preserving its long-time behavior, is a relevant and interesting problem in both pure and applied mathematics. Firstly, it simplifies theoretical considerations on the existence and properties of attractors for the underlying system. On the other side, the resolution and numerical simulation of these processes is perhaps one of the most challenging problems in applied mathematics and engineering science. The numerical approximation of these kind of problems and related phenomena such as turbulence essentially requires a reduction of the size of the problem. For that reason, one is interested in describing the behavior of the infinite dimensional problem by means of merely finitely many degrees of freedom.

A description of the long-term behavior of a dynamical system ultimately means to determine its attractor. It has been found out that in many cases the global attractor can be embedded into exponentially attractive finite dimensional manifolds. Consequently, it turned out that so-called *inertial manifolds* are often an appropriate tool for the studies of questions related to the long-term behavior of evolutionary equations. By definition, these inertial manifolds are finite dimensional, positively invariant, Lipschitzian and they attract all solutions at an exponential rate, which implies that they contain the global attractor. Furthermore, inertial manifolds allow a reduction of the dynamics to a finite dimensional ordinary differential equation (ODE). In a way, the long-term dynamics of a FDE or PDE with an inertial manifold is completely determined by the solutions of an ODE in finite dimensions, and one can use their well-established theory for the qualitative analysis in an infinite dimensional setting. During the last few years it has been shown that many dissipative infinite dimensional evolutionary equations, including small-delay equations, certain reaction diffusion equations, the Cahn-Hilliard, complex Ginzburg-Landau, or the Ginzburg-Landau equation actually possess inertial manifolds. We refer to [SY02, Chapter 8] for a comprehensive introduction and historical comments. Their construction is similar to center-unstable manifolds. However, center-unstable manifolds are a tool to describe the local behavior close to equilibria or more general invariant sets, whereas inertial manifolds define a more global approach to embed universal attractors into finite dimensional sets and to get a global reduction principle.

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The property of normal hyperbolicity is a key issue in the general theory of inertial manifolds, since it guarantees their robustness, which in turn, is essential for discretizations matters. Indeed, a normally hyperbolic invariant manifold is stable under small perturbations of the right-hand side in the problem (see [Fen71, PS01]); in case of inertial manifolds this is guaranteed by the spectral gap condition.

As part of their definition, inertial manifolds are exponentially attractive sets. Nevertheless, in many cases one can establish a stronger assertion concerning the way solutions approach inertial manifolds. One of them is the so-called *asymptotic completeness*, which roughly speaking means that every trajectory of the system is exponentially attracted by some trajectory lying on the manifold, so that both trajectories have the same ω-limit set. Here, however, the solution on the inertial manifold is allowed to start at a later time. The purpose of this time translate is to wait for the given solution to get close enough to the inertial manifold. If the solution on the inertial manifold can be continued in backward time, then the time translate can be dropped, and one speaks of an *asymptotic phase*.

The present paper deals with such questions of existence and exponential attraction to invariant manifolds in the framework of nonautonomous (ordinary) difference equations, instead of evolutionary differential equations. The theory of attractive invariant manifolds for discrete dynamical systems has a certain tradition, which can be traced back at least to [KS78], who consider finite dimensional Lipschitzian maps. Their results have been generalized and extended to general Banach spaces in [NS92], where also smoothness questions are considered; moreover, [KS78, NS92] obtain an asymptotic phase property.

Another source for attractive invariant manifolds of autonomous difference equations is the discretization theory for inertial manifolds: [JS95, vDL99] work in a Lipschitzian setting, and the C^1-case is considered in [JST98]. These references prove exponential attraction and utilize Hadamard's graph transform to construct discrete inertial manifolds. An approach using the Lyapunov-Perron method is also possible. Results concerning existence and exponential attraction are shown in [DG91, Theorem 2.1]. Without proofs, similar statements and a stronger asymptotic phase property can be found in [Kob94, Kob95, Theorem 2.1], [Kob99] (PDEs) or [Far02] (FDEs). To deduce an asymptotic phase for a given inertial manifold one typically uses invariant foliations over the manifold. In a setting of not necessarily invertible mappings, existence and C^1-smoothness results for invariant manifolds are obtained in [CHT97] by a Lyapunov-Perron approach.

In the present paper we generalize the above results to nonautonomous difference equations, where the right-hand side and the state spaces are allowed to depend explicitly on time. Moreover, compared to the above references, our assumptions on the linear part are more general and we state a higher order smoothness of the inertial manifolds and invariant foliations.

The main advantages of our nonautonomous approach read as follows:

- Our results can be applied to variable time-step numerical discretizations of evolutionary PDEs. Due to their "stiff" behavior, step-size control is commonly used and consequently non-constant step-sizes are a more realistic assumption. Beyond that, our variable state spaces allow to consider schemes being adaptive in the spatial variable as well.
- A larger class of equations fit into our setting, since we can handle discretizations of nonautonomous evolutionary equations, which do not generate an autonomous dynamical system.
- The nonlinearities of our systems are allowed to be unbounded in time, as long as the growth rate is dominated by their linear part.

In addition, our set-up is sufficiently flexible to obtain discrete versions of inertial manifolds, as well as of unstable manifolds. We construct these invariant sets using a Lyapunov-Perron technique. Compared to other methods, this functional-analytical approach has advantages when it comes to smoothness proofs and we can refer to earlier results in a similar setting. Related references concerning invariant manifolds of nonautonomous difference equations (so-called *invariant fiber bundles*) include [Aul98, PS04], where pseudo-stable and -unstable fiber bundles and their smoothness is addressed. A different construction of invariant foliations can be found in [AW03].

Since our nonautonomous inertial manifolds are designed to contain global attractors, it is important to work with an appropriate concept of attraction. Apparently canonical generalizations of concepts like limit, absorbing or attracting sets, based on forward convergence, are often too restrictive in the nonautonomous context, since for example limit sets need not to be invariant then. Instead, it is adequate to use families of sets, which are parametrized by time, leading to notions like nonautonomous sets and pullback convergence (cf., e.g., [Kl00]). The invariant sets constructed in this paper are pullback attracting and they contain the global pullback attractor.

At the end of this introduction we outline our approach: It is helpful to read Section 2 containing our basic hypotheses, as well as abstract results on the existence of invariant fiber bundles, their asymptotic phase and invariant foliations. After these preliminaries, we begin our investigations in Section 3 dealing with normal hyperbolicity of invariant fiber bundles. Having pointed out to aim at applications in discretization theory, from a theoretical view this normal hyperbolicity is a crucial prerequisite. So far, our results are global in nature and supposed to hold under Lipschitz conditions on the whole state space. This global assumption is weakened in Section 4 to obtain our discrete counterpart of an inertial manifold, where we basically assume the existence of a pullback absorbing set for the nonautonomous difference equation under consideration. Compared to discrete dynamical systems, our nonautonomous setting demands some additional technical considerations leading to uniform properties in the time parameter. The paper concludes with two examples in Section 5: An implicit Euler scheme applied to an Allen-Cahn equation, and a finite-difference method for the complex Ginzburg-Landau equation demonstrate that inertial manifolds persist under temporal, as well as full discretizations, respectively.

1.2. **Basic Notation and Nonautonomous Sets.** Dealing with discrete equations, our "time axis" is the set of integers denoted by $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$, a *discrete interval* is the intersection of a real interval with $\mathbb{Z}$, in particular we conveniently write $\mathbb{Z}_\kappa^+ := \{k \in \mathbb{Z} :\, \kappa \le k\}$, $\mathbb{Z}_\kappa^- := \{k \in \mathbb{Z} :\, k \le \kappa\}$ for $\kappa \in \mathbb{Z}$, and $\mathbb{N} := \mathbb{Z}_1^+$ are the natural numbers. The integer functions are defined by $\lfloor x \rfloor := \sup\{k \in \mathbb{Z} :\, k \le x\}$ and $\lceil x \rceil := \inf\{k \in \mathbb{Z} :\, x \le k\}$.

Banach spaces X, Y considered in this paper are real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$), and their norm is denoted by $\|\cdot\|_X, \|\cdot\|_Y$, respectively. The open ball in X with center 0 and radius $r > 0$ is $B_r(X)$ and $\bar{B}_r(X)$ the corresponding closed ball. We write $L(X, Y)$ for the bounded linear maps between X and Y, $L(X) := L(X, X)$, and I_X for the identity map on X. The space of bounded n-linear operators from X to Y is $L_n(X, Y)$, $n \in \mathbb{N}$.

We use the following definitions for subsets $A, B \subseteq X$. The *distance* of a point $a \in A$ from the set B is $\operatorname{dist}(a, B) := \inf_{b \in B} \|a - b\|_X$ and the *Hausdorff separation* of A and B is $h(A, B) := \sup_{a \in A} \operatorname{dist}(a, B)$. For a mapping $F : X \times Z \to Y$, where $Z \neq \emptyset$ is a set, we define the Lipschitz constants

$$\begin{aligned} \operatorname{Lip} F(\cdot, z) &:= \inf\left\{L \ge 0 :\, \|F(x, z) - F(\bar{x}, z)\|_Y \le L\, \|x - \bar{x}\|_X \text{ for all } x, \bar{x} \in X\right\}, \\ \operatorname{Lip}_1 F &:= \sup_{z \in Z} \operatorname{Lip} F(\cdot, z), \end{aligned}$$

provided they exist. If the set Z has a metric structure, one defines the Lipschitz constant w.r.t. the second variable analogously as $\operatorname{Lip}_2 F$, and proceeds correspondingly, if F depends on more than two variables. Moreover, a mapping $f : X \to Y$ is said to be of *class* C^m, if it is m-times continuously Fréchet differentiable.

Our primary interest are nonautonomous difference equations. Here we even allow time-dependent state spaces, and thereto let X_k, $k \in \mathbb{Z}$, be a sequence of Banach spaces and $\mathcal{X} := \bigcup_{k \in \mathbb{Z}} X_k$. For $\kappa \in \mathbb{Z}$ and reals $\gamma > 0$ we introduce the weighted sequence spaces

$$\mathcal{X}_{\kappa,\gamma}^- := \left\{ \phi : \mathbb{Z}_\kappa^- \to \mathcal{X} \;\middle|\; \begin{array}{l} \phi(k) \in X_k \text{ for all } k \in \mathbb{Z}_\kappa^- \text{ and} \\ \sup_{k \in \mathbb{Z}_\kappa^-} \|\phi(k)\|_{X_k} \gamma^{\kappa - k} < \infty \end{array} \right\}$$

and equip them with the norm $\|\phi\|_{\kappa,\gamma}^\pm := \sup_{k \in \mathbb{Z}_\kappa^\pm} \|\phi(k)\|_{X_k} \gamma^{\kappa - k}$. Moreover, a sequence $\phi : \mathbb{I} \to \bigcup_{k \in \mathbb{I}} X_k$ defined on a discrete interval $\mathbb{I}$ with $\mathbb{Z}_\kappa^- \subseteq \mathbb{I}$ is called γ^--*quasibounded*, if $\phi|_{\mathbb{Z}_\kappa^-} \in \mathcal{X}_{\kappa,\gamma}^-$ holds.

For (not necessarily invertible) linear operators $A(k) : X_k \to X_{k+1}$, $k \in \mathbb{Z}$, we define the associate *evolution operator* $\Phi(k, \kappa) : X_\kappa \to X_k$, $\kappa, k \in \mathbb{Z}$, $\kappa \le k$, as the linear mapping given by

$$\Phi(k, \kappa) := \begin{cases} I_{X_\kappa} & \text{for } k = \kappa \\ A(k-1) \cdots A(\kappa) & \text{for } k > \kappa \end{cases},$$

and if $A(k)$ is invertible for $k < \kappa$, then $\Phi(k, \kappa) := A(k)^{-1} \cdots A(\kappa - 1)^{-1}$ for $k < \kappa$.

Let $\mathbb{I}$ stand for a discrete interval. Given a sequence $\phi : \mathbb{I} \to \mathcal{X}$, we define $\phi'(k) := \phi(k+1)$ for all $k \in \mathbb{I}$ such that $k + 1 \in \mathbb{I}$, and use a similar notation for sequences with set- or operator-values.

For the applications in mind, it is reasonable to consider implicit problems. To denote such (ordinary) difference equations (the notions *recursion* or *iteration* are also frequently used) we prefer the notation

$$x' = f(k, x, x') \tag{1.1}$$

instead of the conventional ones

$$x(k+1) = f(k, x(k), x(k+1)) \qquad \text{or} \qquad x_{k+1} = f(k, x_k, x_{k+1}).$$

The *right-hand side* of (1.1) is a function $f(k,\cdot) : X_k \times X_{k+1} \to X_{k+1}$, $k \in \mathbb{Z}$. Then a sequence $\phi : \mathbb{I} \to \mathfrak{X}$ satisfying $\phi(k) \in X_k$ for $k \in \mathbb{I}$ and $\phi'(k) = f(k, \phi(k), \phi'(k))$ for $k \in \mathbb{I}$ with $k+1 \in \mathbb{I}$ is called *solution* of (1.1). We say (1.1) is *well-defined on* $\mathcal{X}$, if for all initial pairs $\kappa \in \mathbb{Z}$, $\xi \in X_\kappa$ there exists a unique solution $\varphi(\cdot; \kappa, \xi)$ on $\mathbb{Z}_\kappa^+$; we speak of the *general solution* to (1.1). The *cocycle property*

$$\varphi(k; \kappa, \xi) = \varphi(k; l, \varphi(l; \kappa, \xi)) \quad \text{for all } k \geq l \geq \kappa \tag{1.2}$$

holds for the mapping φ. If the nonautonomous difference equation (1.1) is *explicit*, i.e., if its right-hand side does not depend on x', then the general solution trivially exists and can be defined recursively

$$\varphi(k; \kappa, \xi) := \begin{cases} \xi & \text{for } k = \kappa \\ f(k-1, \varphi(k-1; \kappa, \xi)) & \text{for } k > \kappa \end{cases}.$$

Keep in mind, nevertheless, that $\varphi(k; \kappa, \cdot)$ does not exist in general for $k < \kappa$.

We write $\mathcal{X} := \{(k, x) : k \in \mathbb{Z}, x \in X_k\}$ for the *extended state space* of (1.1). A subset $\mathcal{A} \subseteq \mathcal{X}$ is called a *nonautonomous set* with *k-fiber* $\mathcal{A}(k) := \{x \in X_k : (k, x) \in \mathcal{A}\}$ for $k \in \mathbb{Z}$. Such a set $\mathcal{A}$ is called *positively invariant* w.r.t. (1.1), if the inclusion $\varphi(k; \kappa, \mathcal{A}(\kappa)) \subseteq \mathcal{A}(k)$ holds for $\kappa \leq k$, and it is called *invariant*, if one has equality $\varphi(k; \kappa, \mathcal{A}(\kappa)) = \mathcal{A}(k)$ for $\kappa \leq k$. Abbreviating $\mathcal{A}'(k) := \mathcal{A}(k+1)$, then positive invariance means $f(k, \mathcal{A}(k), \mathcal{A}'(k)) \subseteq \mathcal{A}'(k)$ and invariance is equivalent to $f(k, \mathcal{A}(k), \mathcal{A}'(k)) = \mathcal{A}'(k)$ for $k \in \mathbb{Z}$. Moreover, we denote (1.1) as difference equation *in* $\mathcal{A}$, if $\mathcal{A}$ is positively invariant. We call $\mathcal{A}$ an *invariant fiber bundle* (IFB for short) of (1.1), if it is *invariant* and each fiber $\mathcal{W}_\theta(k)$, $k \in \mathbb{Z}$, is a submanifold of X_k. A nonautonomous set $\mathcal{A}$ is called a *vector bundle*, if each fiber $\mathcal{A}(k)$ is a linear subspace of X_k, and for two vector bundles $\mathcal{A}, \mathcal{B} \subseteq \mathcal{X}$ we define the *Whitney sum* $\mathcal{A} \oplus \mathcal{B} := \{(k, x) \in \mathcal{X} : x \in \mathcal{A}(k) \oplus \mathcal{B}(k)\}$.

Let $\mathcal{B}$ be a further nonautonomous set in $\mathcal{X}$. We write $\mathcal{A} \subseteq \mathcal{B}$ if $\mathcal{A}(k) \subseteq \mathcal{B}(k)$ holds for $k \in \mathbb{Z}$ and introduce $\mathcal{A} \times \mathcal{B} := \{(k, a, b) : k \in \mathbb{Z}, a \in \mathcal{A}(k), b \in \mathcal{B}(k)\}$. With a real $\rho > 0$ we define the *closed ball* $\mathcal{U}_\rho := \left\{(k, x) \in \mathcal{X} : \|x\|_{X_k} \leq \rho\right\}$ and the set $\mathcal{A}$ is said to be *bounded*, in case there exists a $R > 0$ satisfying $\mathcal{A} \subseteq \mathcal{U}_R$; note that this notion of boundedness is uniform in $k \in \mathbb{Z}$. Moreover, $\mathcal{A}$ is said to possess a certain property (e.g., being nonempty, open, closed or compact), if all fibers $\mathcal{A}(k)$ possess this property. In particular, the *closure* of $\mathcal{A}$ is given by the $\bar{\mathcal{A}} := \{(k, x) \in \mathcal{X} : x \in \operatorname{cl}_{X_k} \mathcal{A}(k)\}$.

2. Preliminaries on Attractive Invariant Fiber Bundles

In this section we introduce the kind of difference equations under consideration and give the precise notational framework used throughout this paper. Moreover, we prepare some basic abstract results on attractive invariant fiber bundles, which are fundamental for our further investigations. Corresponding proofs, further remarks and possible applications can be found in our preparative paper [Pöt07].

2.1. **Standing Hypothesis.** Let $\Theta \subseteq \mathbb{F}$ denote a nonempty bounded set. The parameters $\theta \in \Theta$ can be interpreted as upper bound for step-sizes in numerical schemes. Moreover, let Y_k, $k \in \mathbb{Z}$, be a further sequence of Banach spaces. The paper deals with implicit θ-dependent nonautonomous recursions

$$y' = A(k)y + \theta K'(k) F(k, y, y'), \tag{2.1}$$

where $A(k) : Y_k \to Y_{k+1}$, $K(k) \in L(Y_k)$ are linear operators and $F(k, \cdot) : Y_k \times Y_{k+1} \to Y_{k+1}$ denotes the nonlinearity for $k \in \mathbb{Z}$. Hence, a priori (2.1) is a difference equation in $\mathcal{Y} := \{(k, y) : k \in \mathbb{Z}, y \in Y_k\}$. The recursion (2.1) can be implicit, but only in the nonlinearity F. In order to capture also linearly implicit schemes, we have introduced the function K, which will be a resolvent operator in applications.

We formulate our assumptions, which basically state that the linear part of (2.1), given by

$$y' = A(k)y, \tag{2.2}$$

possesses an exponential dichotomy and the nonlinearity F is Lipschitzian.

Hypothesis. *Let X_k, Y_k, $k \in \mathbb{Z}$, be Banach spaces with the embedding $X_k \subseteq Y_k$ for all $k \in \mathbb{Z}$, let $\nu \geq 0$, $0 < \Lambda < \lambda$, $K_1^-, K_2^-, K_1^+, K_2^+, K_3^+ \geq 0$, $L_2^-, L_2^+, L_3^-, L_3^+ \geq 0$ be reals and assume the following holds:*

- $(H)_0$ *The nonautonomous difference equation* (2.1) *is well-defined on $\mathcal{X}$ with continuous general solution $\varphi(k;\kappa,\cdot) : X_\kappa \to X_k$, $k, \kappa \in \mathbb{Z}$, $\kappa < k$.*
- $(H)_1$ *Let $A(k) \in L(X_k, X_{k+1})$ for all $k \in \mathbb{Z}$ and assume there exist complementary projections $P_-(k), P_+(k)$ on Y_k with $P_-(k) \in L(Y_k)$, $P_-(k)Y_k \subseteq X_k$, $P_+(k)X_k \subseteq X_k$,*

$$P_-'(k)A(k) = A(k)P_-(k), \qquad K(k)P_-(k) = P_-(k)K(k) \quad \textit{for all } k \in \mathbb{Z}, \tag{2.3}$$

one has the inclusions $A(k)P_+(k)X_k \subseteq X_{k+1}$, $P_+(k)K(k)Y_k \subseteq X_k$ for all $k \in \mathbb{Z}$, the mappings

$$A(k)|_{P_-(k)X_k} : P_-(k)X_k \to P_-'(k)X_{k+1} \tag{2.4}$$

are invertible with associate evolution operator $\bar{\Phi}(k,\kappa)$, we have

$$\bar{C} := \sup_{k\in\mathbb{Z}} \|K(k)\|_{L(Y_k)} < \infty \tag{2.5}$$

and for all $k, l \in \mathbb{Z}$ one finally has the dichotomy estimates

$$\|\Phi(k,l)P_+(l)\|_{L(X_l,X_k)} \leq K_1^+\Lambda^{k-l} \quad \textit{for all } l \leq k, \tag{2.6}$$

$$\|\Phi(k,l)P_+(l)K(l)\|_{L(Y_l,X_k)} \leq (K_2^+ + K_3^+ |\theta|^{-\nu} (k-l+1)^{-\nu})\Lambda^{k-l} \quad \textit{for all } l \leq k, \tag{2.7}$$

$$\left\|\bar{\Phi}(k,l)P_-(l)\right\|_{L(X_l,X_k)} \leq K_1^- \lambda^{k-l} \quad \textit{for all } k \leq l, \tag{2.8}$$

$$\left\|\bar{\Phi}(k,l)P_-(l)\right\|_{L(Y_l,X_k)} \leq K_2^- \lambda^{k-l} \quad \textit{for all } k < l, \tag{2.9}$$

where $\Phi(k,l)$ is the evolution operator for A.
- $(H)_2$ *Let $K'(k)F(k,\cdot) : X_k \times X_{k+1} \to X_{k+1}$, $k \in \mathbb{Z}$, be continuous and suppose that the constants*

$$C_\kappa^\pm := \sup_{k<\kappa} \left\|P_\pm'(k)F(k,0,0)\right\|_{Y_{k+1}} \lambda^{\kappa-k} \quad \textit{for one (and hence) every } \kappa \in \mathbb{Z} \tag{2.10}$$

are finite, we have the global Lipschitz estimates

$$\begin{aligned} \left\|P_\pm'(k)\left[F(k,x,y) - F(k,\bar{x},y)\right]\right\|_{Y_{k+1}} &\leq L_2^\pm \|x-\bar{x}\|_{X_k} \quad \textit{for all } k \in \mathbb{Z}, x, \bar{x} \in X_k, y \in X_{k+1}, \\ \left\|P_\pm'(k)\left[F(k,x,y) - F(k,x,\bar{y})\right]\right\|_{Y_{k+1}} &\leq L_3^\pm \|y-\bar{y}\|_{X_{k+1}} \quad \textit{for all } k \in \mathbb{Z}, x \in X_k, y, \bar{y} \in X_{k+1} \end{aligned} \tag{2.11}$$

and we require the spectral gap condition*: There exist reals $0 < \sigma < \sigma_{\max} \leq \frac{\lambda-\Lambda}{2}$ such that*

$$|\theta|\, \Sigma(\bar{\sigma}) < 1 \quad \textit{for all } \bar{\sigma} \in (\sigma, \sigma_{\max}) \tag{2.12}$$

holds with a function $\Sigma : (\sigma, \sigma_{\max}) \to \mathbb{R}^+$ to be specified later, but depending on the dichotomy data and $L_2^\pm, L_3^\pm$; we then define the nonempty interval $\bar{\Gamma} := [\Lambda + \sigma, \lambda - \sigma]$.
- $(H)_3$ *Let the Fréchet derivatives $D_{(2,3)}^n F(k,\cdot) : X_k \times X_{k+1} \to L_n(X_k \times X_{k+1}, X_{k+1})$, $k \in \mathbb{Z}$, exist and be continuous, and assume the Fréchet derivatives $D_{(2,3)}^n F(k,\cdot) : X_k \times X_{k+1} \to L_n(X_k \times X_{k+1}, Y_{k+1})$, $k \in \mathbb{Z}$, exist, are continuous and one has the global boundedness*

$$\sup_{k\in\mathbb{Z}} \sup_{(x,y)\in X_k\times X_{k+1}} \left\|D_{(2,3)}^n F(k,x,y)\right\|_{L_n(X_k\times X_{k+1}, Y_{k+1})} < \infty \quad \textit{for all } n \in \{1,\dots,m\}.$$

Remark 2.1. (1) The above Hypothesis $(H)_0$ circumvents a number of problems coming from the implicit nature of (2.1). However, instead of imposing restrictive condition implying $(H)_0$ we think it is advantageous to leave the verification to concrete individual examples.

(2) To provide a compact notation throughout the paper, we often write $P_\pm$ to denote either P_- or P_+ (for example in (2.10),(2.11), etc.) and proceed similarly with other objects.

(3) The left relation in (2.3) implies positive invariance of the sets $\mathcal{P}_\pm := \{(k,x) \in \mathcal{X} : x \in P_\pm(k)X_k\}$, and from the *regularity condition* (2.4) one gets the invariance of $\mathcal{P}_-$ w.r.t. (2.2). From a dynamical characterization of $\mathcal{P}_\pm$ it is reasonable to denote $\mathcal{P}_+$ as *pseudo-stable* and $\mathcal{P}_-$ as *pseudo-unstable vector bundle* of (2.2), respectively.

(4) For an explicit difference equation (2.1) the Hypotheses $(H)_1$–$(H)_2$ guarantee that (2.1) is a difference equation in $\mathcal{X}$, as well as the continuity of the general solution $\varphi(k;\kappa,\cdot) : X_\kappa \to X_k$ for $k \in \mathbb{Z}_\kappa^+$.

2.2. **Invariant Fiber Bundles.** In the beginning of this subsection we provide an invariant nonautonomous set $\mathcal{W}_\theta$ for (2.1), which generalizes the pseudo-unstable vector bundle $\mathcal{P}_-$ to a nonlinear setting. Thanks to our global assumption $(H)_2$, each of the fibers $\mathcal{W}_\theta(\kappa)$ will be a submanifold of X_κ given by the graph of a globally Lipschitzian mapping over $\mathcal{P}_-(\kappa)$, $\kappa \in \mathbb{Z}$.

Constructions of inertial manifolds for evolutionary differential equations using the Lyapunov-Perron technique frequently lead to spectral gap conditions involving the Γ-function (cf., e.g., [Kob94, Kob95]). In our setting we encounter a discrete counterpart of the Γ-function, namely the *polylogarithm* (cf. [Lew82, pp. 236–238]), which is the strictly increasing unbounded function $\operatorname{Li}_\nu : [0,1) \to \mathbb{R}$, $\nu \in [0,\infty)$,

$$\operatorname{Li}_\nu(x) := \sum_{n=1}^{\infty} n^{-\nu} x^n.$$

Proposition 2.1 (existence of IFBs)**.** *Let $\theta \in \Theta$ and assume Hypotheses $(H)_0$–$(H)_2$ with $\sigma_{\max} = \frac{\lambda-\Lambda}{2}$ and*

$$\Sigma(\sigma) := L^-(\lambda-\sigma)\frac{\bar{C}K_2^-}{\sigma} + L^+(\lambda-\sigma)\left(\frac{K_2^+}{\sigma} + |\theta|^{-\nu} K_3^+ \operatorname{Li}_\nu\left(\frac{\Lambda}{\Lambda+\sigma}\right)\right). \tag{2.13}$$

Then the nonautonomous set

$$\mathcal{W}_\theta := \left\{ (\kappa,\xi) \in \mathcal{X} \;\middle|\; \begin{array}{l} \text{there exists a solution } \phi : \mathbb{Z} \to \mathfrak{X} \text{ of} \\ (2.1) \text{ with } \phi(\kappa) = \xi \in X_\kappa \text{ and } \phi \in \mathfrak{X}_{\kappa,\gamma}^- \end{array} \right\}$$

is an invariant fiber bundle of equation (2.1)*, which is independent of $\gamma \in \bar{\Gamma}$ and possesses the representation $\mathcal{W}_\theta = \{(\kappa, \eta + w_\theta(\kappa,\eta)) : (\kappa,\eta) \in \mathcal{P}_-\}$ as graph of a unique mapping $w_\theta(\kappa,\cdot) : Y_\kappa \to X_\kappa$ satisfying*

$$w_\theta(\kappa,\xi) = w_\theta(\kappa, P_-(\kappa)\xi) \in \mathcal{P}_+(\kappa) \quad \text{for all } (\kappa,\xi) \in \mathcal{Y} \tag{2.14}$$

and the invariance equation

$$\begin{aligned} w_\theta(\kappa+1,\eta_1) &= A(\kappa)w_\theta(\kappa,\eta) + \theta P_+'(\kappa)K'(\kappa)F(\kappa,\eta + w_\theta(\kappa,\eta), \eta_1 + w_\theta(\kappa+1,\eta_1)) \\ \eta_1 &= A(\kappa)\eta + \theta K'(\kappa)F(\kappa,\eta + w_\theta(\kappa,\eta), \eta_1 + w_\theta(\kappa+1,\eta_1)) \end{aligned} \tag{2.15}$$

for all $(\kappa,\eta) \in \mathcal{P}_-$. Furthermore, for all $\gamma \in \bar{\Gamma}$ and $\theta \in \Theta$ it holds:

(a) *$w_\theta(\kappa,\cdot) : Y_\kappa \to X_\kappa$ is linearly bounded*

$$\|w_\theta(\kappa,\xi)\|_{X_\kappa} \le |\theta|\,\ell^+(\gamma)\left[C_\kappa^+ + \frac{L^+(\gamma)}{1-|\theta|\,\ell(\gamma)}\left(|\theta|\,\Gamma_\kappa(\gamma) + K_1^-\,\|P_-(\kappa)\xi\|_{X_\kappa}\right)\right] \quad \text{for all } (\kappa,\xi) \in \mathcal{Y},$$

(b) *$w_\theta(\kappa,\cdot)$ is globally Lipschitzian with*

$$\operatorname{Lip}_2 w_\theta \le |\theta|\,K_1^- \frac{L^+(\gamma)\ell^+(\gamma)}{1-|\theta|\,\ell(\gamma)}, \tag{2.16}$$

(c) *assume Hypothesis $(H)_3$ is satisfied with $\Lambda < \lambda^m$, $m \in \mathbb{N}$, and if the spectral gap condition* (2.12) *holds with $\sigma_{\max} = \min\left\{\frac{\lambda-\Lambda}{2}, \lambda\left(1 - \sqrt[m]{\frac{\lambda+\Lambda}{\lambda+\lambda^m}}\right)\right\}$, then the partial derivatives $D_2^n w_\theta(\kappa,\cdot) : Y_\kappa \to L_n(Y_\kappa, X_\kappa)$ exist, are continuous, and globally bounded for $n \in \{1,\dots,m\}$,* *where the constants* $L^{\pm}(\gamma), \Gamma_\kappa(\gamma), \ell(\gamma), \ell^{\pm}(\gamma)$ *are given by* $L^{\pm}(\gamma) := L_2^{\pm} + \gamma L_3^{\pm}$,

$$\Gamma_\kappa(\gamma) := C_\kappa^+ \ell^+(\gamma) + C_\kappa^- \ell^-(\gamma), \qquad \ell(\gamma) := L^+(\gamma)\ell^+(\gamma) + L^-(\gamma)\ell^-(\gamma),$$

$$\ell^+(\gamma) := \frac{K_2^+}{\gamma - \Lambda} + \frac{|\theta|^{-\nu} K_3^+}{\Lambda} \operatorname{Li}_\nu\left(\tfrac{\Lambda}{\gamma}\right), \qquad \ell^-(\gamma) := \frac{\bar{C} K_2^-}{\lambda - \gamma}.$$

Proof. We only present a very rough sketch of the proof and refer to [Pöt07, Theorem 3.5] for the details. Thereto, let $(\kappa, \xi) \in \mathcal{Y}$ and $\gamma > 0$. We define the *Lyapunov-Perron operator* $\mathcal{T}_\kappa : \mathfrak{X}_{\kappa,\gamma}^- \times X_\kappa \to \mathfrak{X}_{\kappa,\gamma}^-$,

$$\begin{aligned} \mathcal{T}_\kappa(\phi; \xi) := \bar{\Phi}(\cdot, \kappa) P_-(\kappa)\xi + \theta \sum_{n=-\infty}^{\cdot-1} \Phi(\cdot, n+1) P_+'(k) K'(n) F(n, \phi(n), \phi'(n)) \\ - \theta \sum_{n=\cdot}^{\kappa-1} \bar{\Phi}(\cdot, n+1) P_-'(k) K'(n) F(n, \phi(n), \phi'(n)), \end{aligned}$$

which is a uniform contraction in $\xi \in Y_\kappa$. Denoting its unique fixed point by $\phi_\kappa(\xi) \in \mathfrak{X}_{\kappa,\gamma}^-$, we define

$$w_\theta(\kappa, \xi) := P_+(\kappa)\big(\phi_\kappa(\xi)\big)(\kappa) \tag{2.17}$$

and this mapping satisfies the assertions of Proposition 2.1. □

In general, through a point $(\kappa, \xi) \in \mathcal{X}$ there may exist no or a multiple number of backward solutions. The following Corollary ensures that exactly one of them lies on $\mathcal{W}_\theta$. Moreover, it relates the dynamics of (2.1) to a reduced nonautonomous difference equation (the so-called *inertial form*), which is finite dimensional provided the projections $P_-(k)$ have finite dimensional range.

Corollary 2.2 (inertial form)**.** *The nonautonomous difference equation*

$$x' = A(k)x + \theta P_-'(k) K'(k) F(k, x + w_\theta(k, x), x' + w_\theta(k+1, x')) \tag{2.18}$$

in the pseudo-unstable vector bundle $\mathcal{P}_-$ *is denoted as* inertial form *of* (2.1)*. One has:*

(a) *The general solution* $\hat{\varphi}$ *of* (2.18) *is defined on* $\mathbb{Z} \times \mathcal{P}_-$*,*
(b) φ *is defined on* $\mathbb{Z} \times \mathcal{W}_\theta$*, related to* $\hat{\varphi}$ *by virtue of*

$$\hat{\varphi}(k; \kappa, \xi) = P_-(k)\varphi(k; \kappa, \xi + w_\theta(\kappa, \xi)) \quad \textit{for all } (k, \kappa, \xi) \in \mathbb{Z} \times \mathcal{P}_- \tag{2.19}$$

and in case $|\theta|\,(1 + \operatorname{Lip}_2 w_\theta)\, \bar{C} K_2^- L^-\left(\frac{1}{\lambda}\right) < \lambda$ *one has*

$$\operatorname{Lip} \varphi(k; \kappa, \cdot)|_{\mathcal{W}_\theta(\kappa)} \le K_1^- \left(K_1^- + \frac{K_2^- \bar{C} L_3^-}{\lambda^2} (1 + \operatorname{Lip}_2 w_\theta) \right) (1 + \operatorname{Lip}_2 w_\theta)\, \omega_-^{k-\kappa} \tag{2.20}$$

for all $k \in \mathbb{Z}_\kappa^-$ *with* $\omega_- := \lambda - |\theta|\,(1 + \operatorname{Lip}_2 w_\theta)\, \bar{C} K_2^- L^-\left(\frac{1}{\lambda}\right)$.

Proof. Let $\theta \in \Theta$ and $\kappa \in \mathbb{Z}$ be given. First of all, we show that the general solution φ of (2.1) is defined on $\mathbb{Z} \times \mathcal{W}_\theta$. Due to the invariance of $\mathcal{W}_\theta$ we have that $\varphi(\kappa + 1; \kappa, \cdot) : \mathcal{W}_\theta(\kappa) \to \mathcal{W}_\theta'(\kappa)$ is onto. Let us show now that the inverse of this mapping is given by $\xi \mapsto \phi_{\kappa+1}(\kappa, \xi)$, with the $\phi_{\kappa+1}(\xi)$ from the proof of Proposition 2.1. Indeed, for each $\xi \in \mathcal{W}_\theta'(\kappa)$ there exists a γ^--quasibounded solution of (2.1), given by $\phi_{\kappa+1}(\xi) : \mathbb{Z}_{\kappa+1}^- \to \mathfrak{X}$, and [Pöt07, Lemma 3.3]yields

$$\xi = P_-'(\kappa)\xi + w_\theta(\kappa + 1, P_-'(\kappa)\xi) \overset{(2.17)}{=} P_-'(\kappa)\xi + P_+'(\kappa)\phi_{\kappa+1}(\kappa + 1, \xi) = \phi_{\kappa+1}(\kappa + 1, \xi).$$

Since $\phi_{\kappa+1}(\xi)$ is a solution of (2.1), one therefore has $\varphi(\kappa + 1; \kappa, \phi_{\kappa+1}(\kappa, \xi)) = \phi_{\kappa+1}(\kappa + 1, \xi) = \xi$. It remains to show $\phi_{\kappa+1}(\kappa, \varphi(\kappa + 1, \kappa, \xi)) = \xi$ for $\xi \in \mathcal{W}_\theta(\kappa)$. Thereto, we define $\mu(\kappa + 1) := \varphi(\kappa + 1; \kappa, \xi)$ and $\mu(k) := \phi_{\kappa+1}(k + 1, \xi)$ for $k \le \kappa$. Then $\mu : \mathbb{Z}_{\kappa+1}^- \to \mathfrak{X}$ is a γ^--quasibounded solution of (2.1) and [Pöt07, Lemma 3.3] implies $\mu(\kappa) = \phi_{\kappa+1}(\kappa, \mu'(\kappa)) = \phi_{\kappa+1}(\kappa, \varphi(\kappa; \kappa + 1, \xi))$. Noting that $\mu(\kappa) = \phi_{\kappa+1}(\kappa + 1, \xi) = \xi$ we are done. Hence, each $\varphi(\kappa + 1; \kappa, \cdot) : \mathcal{W}_\theta(\kappa) \to \mathcal{W}_\theta'(\kappa)$ is bijective and therefore φ exists on $\mathbb{Z} \times \mathcal{W}_\theta$, i.e., we established the first assertion of (b).

By multiplying the solution identity for φ with $P'_-(k)$, using (2.3) and the invariance of $\mathcal{W}_\theta$, it is easily seen that (2.19) holds and that $\hat\varphi$ is defined on $\mathbb{Z}\times\mathcal{P}_-$, yielding the assertion (b).

It remains to show the Lipschitz estimate (2.20). Let $\xi_1,\xi_2\in\mathcal{W}_\theta(\kappa)$ and the invariance of $\mathcal{W}_\theta$ implies

$$\varphi(k;\kappa,\xi_i)=\varphi(k;\kappa,P_-(\kappa)\xi_i)+w_\theta(k,\varphi(k;\kappa,P_-(\kappa)\xi_i))\quad\text{for all } k\in\mathbb{Z},\ i=1,2,$$

which, in turn, yields the estimate

$$\|\varphi(k;\kappa,\xi_1)-\varphi(k;\kappa,\xi_2)\|_{X_k}\le(1+\operatorname{Lip}_2 w_\theta)\,\|\hat\varphi(k;\kappa,P_-(\kappa)\xi_1)-\hat\varphi(k;\kappa,P_-(\kappa)\xi_2)\|_{X_k}\tag{2.21}$$

for all $k\in\mathbb{Z}$. To obtain an estimate for the difference $\hat\varphi(k;\kappa,\bar\xi_1)-\hat\varphi(k;\kappa,\bar\xi_2)$, $\bar\xi_1,\bar\xi_2\in\mathcal{P}_-(\kappa)$, we remark that the variation of constants formula in backward time yields

$$\begin{aligned}\hat\varphi(k;\kappa,\bar\xi_i)=\bar\Phi(k;\kappa)\bar\xi_i-\theta\sum_{n=k}^{\kappa-1}\bar\Phi(k,n+1)P'_-(n)K'(n)&\\ \cdot F\big(n,\hat\varphi(n;\kappa,\bar\xi_i)+w_\theta(n,\hat\varphi(n;\kappa,\bar\xi_i)),\hat\varphi(n+1;\kappa,\bar\xi_i)+w_\theta(n+1,\hat\varphi(n+1;\kappa,\bar\xi_i))\big)&\end{aligned}$$

for all $k\in\mathbb{Z}_\kappa^-$ and $i=1,2$. Thus, (2.8), (2.9), (2.5), (2.11) and (2.16) imply the inequality

$$\begin{aligned}&\left\|\hat\varphi(k;\kappa,\bar\xi_1)-\hat\varphi(k;\kappa,\bar\xi_2)\right\|_{X_k}\lambda^{\kappa-k}\\ &\quad\le K_1^-\left\|\bar\xi_1-\bar\xi_2\right\|+|\theta|\,\frac{(1+\operatorname{Lip}_2 w_\theta)\,\bar CK_2^-L^-\left(\frac1\lambda\right)}{\lambda}\sum_{n=k}^{\kappa-1}\lambda^{\kappa-n}\left\|\hat\varphi(n;\kappa,\bar\xi_1)-\hat\varphi(n;\kappa,\bar\xi_2)\right\|_{X_n}\end{aligned}$$

for all $k\in\mathbb{Z}_\kappa^-$, so that the above assumptions allow us to apply Gronwall's lemma in backward time (cf. [Aul98, Lemma 2.1(b)]), which leads to

$$\begin{aligned}\left\|\hat\varphi(k;\kappa,\bar\xi_1)-\hat\varphi(k;\kappa,\bar\xi_2)\right\|_{X_k}&\le\left(K_1^-+\frac{K_2^-\bar CL_3^-}{\lambda^2}(1+\operatorname{Lip}_2 w_\theta)\right)\\ &\cdot\left[\lambda-|\theta|\,(1+\operatorname{Lip}_2 w_\theta)\,\bar CK_2^-L^-\left(\tfrac1\lambda\right)\right]^{k-\kappa}\left\|\bar\xi_1-\bar\xi_2\right\|\quad\text{for all } k\in\mathbb{Z}_\kappa^-.\end{aligned}$$

This, together with (2.21) and (2.8) implies (2.20). □

Before proceeding, we need a technical result for later purpose in Section 4. It states that the general solution φ of (2.1) satisfies a Lipschitz estimate, provided that the linear part of the inertial form (2.18) has bounded growth in forward time. Note that assumption (2.22) becomes void for explicit equations.

Corollary 2.3. *If* $\omega:=\sup_{k\in\mathbb{Z}}\max\left\{\|A(k)P_-(k)\|_{L(X_k,X_{k+1})},\|A(k)P_-(k)\|_{L(Y_k,X_{k+1})}\right\}<\infty$ *and*

$$|\theta|\,\bar CL_3^-\,(1+\operatorname{Lip}_2 w_\theta)<\omega^2\tag{2.22}$$

holds, then one has the Lipschitz estimate $\operatorname{Lip}\varphi(k;\kappa,\cdot)|_{\mathcal{W}_\theta(\kappa)}\le\frac{K_1^-\omega^2(1+\operatorname{Lip}_2 w_\theta)}{\omega^2-|\theta|\bar CL_3^-(1+\operatorname{Lip}_2 w_\theta)}\omega_+^{k-\kappa}$ *for all* $k\in\mathbb{Z}_\kappa^+$ *with* $\omega_+:=\omega+|\theta|\,\frac{\bar C(1+\operatorname{Lip}_2 w_\theta)}{\omega^2-|\theta|\bar CL_3^-(1+\operatorname{Lip}_2 w_\theta)}$.

Proof. Due to (2.3), as well as our bounded growth assumption we have $\|\Phi(k,l)P_-(l)\|_{L(X_l,X_k)}\le\omega^{k-l}$ and $\|\Phi(k,l)P_-(l)\|_{L(Y_l,X_k)}\le\omega^{k-l}$ for $l\le k$. Then the remaining argument follows along the lines to deduce the estimate (2.20): The difference is to apply the corresponding results in forward time $k\in\mathbb{Z}_\kappa^+$, namely the variation of constants formula for (2.18), as well as the Gronwall's lemma. □

2.3. **Invariant Fibers and Asymptotic Phase.** In the following we investigate the attraction properties of the IFB $\mathcal{W}_\theta$ from Proposition 2.1 using invariant fibers. These fibers serve as leaves for an invariant foliation of the extended state space $\mathcal{X}$ and enable us to construct an asymptotic phase for $\mathcal{W}_\theta$. This means that $\mathcal{W}_\theta$ is not only exponentially attracting, but solutions are also synchronized with corresponding solutions on the IFB $\mathcal{W}_\theta$.

Proposition 2.4 (invariant fibers)**.** *Let $\theta \in \Theta$ and assume Hypotheses $(H)_0$–$(H)_2$ with $\sigma_{\max} = \frac{\lambda-\Lambda}{2}$ and Σ given by* (2.13)*. Then for all $(\kappa, \xi) \in \mathcal{X}$ the so-called* fiber through (κ, ξ), *given by*

$$\mathcal{V}_{\xi,\theta}(\kappa) := \left\{ \zeta \in X_\kappa : \sup_{k \in \mathbb{Z}_\kappa^+} \gamma^{\kappa-k} \left\| \varphi(\cdot; \kappa, \zeta) - \varphi(\cdot; \kappa, \xi) \right\|_{X_k} < \infty \right\}$$

is independent of $\gamma \in \bar{\Gamma}$, positively invariant w.r.t. (2.1)*, i.e.,*

$$\varphi(k; \kappa, \mathcal{V}_{\xi,\theta}(\kappa)) \subseteq \mathcal{V}_{\varphi(k;\kappa,\xi),\theta}(k) \quad \text{for all } k \in \mathbb{Z}_\kappa^+ \tag{2.23}$$

and possesses the representation $\mathcal{V}_{\xi,\theta} = \{(\kappa, \eta + v_\theta(\kappa, \eta, \xi)) : (\kappa, \eta) \in \mathcal{P}_+\}$ as graph of a uniquely determined mapping $v_\theta(\kappa, \cdot) : X_\kappa \times X_\kappa \to X_\kappa$ satisfying

$$v_\theta(\kappa, \eta, \xi) \in \mathcal{P}_-(\kappa) \quad \text{for all } (\kappa, \eta, \xi) \in \mathcal{P}_+ \times \mathcal{X} \tag{2.24}$$

and the invariance equation

$$\begin{aligned} v_\theta(\kappa+1, \eta_1, \xi_1) &= A(\kappa) v_\theta(\kappa, \eta, \xi) + \theta P_-'(\kappa) K'(\kappa) F(\kappa, \eta + v_\theta(\kappa, \eta, \xi), \eta_1 + v_\theta(\kappa+1, \eta_1, \xi_1)), \\ \eta_1 &= A(\kappa)\eta + \theta P_+'(\kappa) K'(\kappa) F(\kappa, \eta + v_\theta(\kappa, \eta, \xi), \eta_1 + v_\theta(\kappa+1, \eta_1, \xi)), \\ \xi_1 &= A(\kappa)\xi + \theta K'(\kappa) F(\kappa, \xi, \xi_1) \end{aligned} \tag{2.25}$$

for all $(\kappa, \eta, \xi) \in \mathcal{P}_+ \times \mathcal{X}$. Furthermore, for all $\gamma \in \bar{\Gamma}$ and $\theta \in \Theta$ it holds:

(a) *$v_\theta(\kappa, \cdot) : \mathcal{P}_+(\kappa) \times X_\kappa \to X_\kappa$ is continuous and linearly bounded*

$$\|v_\theta(\kappa, \eta, \xi)\|_{X_\kappa} \le \|P_-(\kappa)\xi\|_{X_\kappa} + |\theta| K_1^+ \frac{L^-(\gamma)\ell^-(\gamma)}{1 - |\theta| \ell(\gamma)} \|\eta - P_+(\kappa)\xi\|_{X_\kappa} \quad \text{for all } (\kappa, \eta, \xi) \in \mathcal{P}_+ \times \mathcal{X},$$

(b) *$v_\theta(\kappa, \cdot, \xi)$ is globally Lipschitzian with*

$$\operatorname{Lip}_2 v_\theta \le |\theta| K_1^+ \frac{L^-(\gamma)\ell^-(\gamma)}{1 - |\theta| \ell(\gamma)}, \tag{2.26}$$

(c) *assume Hypothesis $(H)_3$ is satisfied with $\Lambda^m < \lambda$, $m \in \mathbb{N}$, and if the spectral gap condition* (2.12) *holds with $\sigma_{\max} := \min\left\{ \frac{\lambda-\Lambda}{2}, \lambda\left(\sqrt[m]{\frac{\Lambda+\lambda}{\Lambda+\Lambda^m}}\right) - 1 \right\}$, then $v_\theta(\kappa, \cdot) : \mathcal{P}_+(\kappa) \times X_\kappa \to X_\kappa$ is of class C^1 and the partial derivatives $D_2^n v_\theta(\kappa, \cdot) : \mathcal{P}_+(\kappa) \times X_\kappa \to L_n(\mathcal{P}_+(\kappa), X_\kappa)$ exist, are continuous, and globally bounded for $n \in \{1, \ldots, m\}$,*

where the constants $L^-(\gamma), \ell(\gamma), \ell^-(\gamma)$ are defined in Proposition 2.1.

Proof. See [Pöt07, Proposition 4.4]. □

In a more descriptive way, the subsequent asymptotic phase property is sometimes referred as "exponential tracking" of the IFB $\mathcal{W}_\theta$. It states that convergence to $\mathcal{W}_\theta$ is actually "in phase" with solutions on the IFB $\mathcal{W}_\theta$, and for that reason we speak of an *asymptotic phase*.

Theorem 2.5 (asymptotic phase)**.** *Let $\theta \in \Theta$, $\kappa \in \mathbb{Z}$ and assume Hypotheses $(H)_0$–$(H)_2$ with $\sigma_{\max} = \frac{\lambda-\Lambda}{2}$ and*

$$\begin{aligned} \Sigma(\sigma) := {} & L^-(\lambda - \sigma)\frac{\bar{C}K_2^-}{\sigma} + L^+(\lambda - \sigma)\left(\frac{K_2^+}{\sigma} + |\theta|^{-\nu} K_3^+ \operatorname{Li}_\nu\left(\frac{\Lambda}{\Lambda+\sigma}\right)\right) \\ & + \max\left\{ L^-(\lambda - \sigma)\frac{\bar{C}K_2^-}{\sigma}, L^+(\lambda - \sigma)\left(\frac{K_2^+}{\sigma} + |\theta|^{-\nu} K_3^+ \operatorname{Li}_\nu\left(\frac{\Lambda}{\Lambda+\sigma}\right)\right) \right\}. \end{aligned} \tag{2.27}$$

Then the invariant fiber bundle $\mathcal{W}_\theta$ from Proposition 2.1 possesses an asymptotic phase*, i.e., for every $\kappa \in \mathbb{Z}$ there exists a retraction $\pi(\kappa, \cdot) : X_\kappa \to \mathcal{W}_\theta(\kappa)$ onto $\mathcal{W}_\theta(\kappa) \subseteq X_\kappa$ with the property:*

$$\left\|\varphi(k;\kappa,\xi) - \varphi(k;\kappa,\pi(\kappa,\xi))\right\|_{X_\kappa} \le \frac{K_1^+}{1-|\theta|\,\ell(\gamma)} \left(\left\|P_+(\kappa)\xi\right\|_{X_\kappa} + \tilde{C}_\kappa^+(\xi,\gamma)\right)\gamma^{k-\kappa} \quad \textit{for all } k \in \mathbb{Z}_\kappa^+ \tag{2.28}$$

and all $\xi \in X_\kappa$ with $\gamma \in \bar{\Gamma}$. Geometrically, $\pi(\kappa,\xi)$ is the unique intersection $\mathcal{W}_\theta(\kappa) \cap \mathcal{V}_{\xi,\theta}(\kappa) = \{\pi(\kappa,\xi)\}$ for all $\xi \in X_\kappa$ and one has:

(a) *$\pi(\kappa,\cdot) : X_\kappa \to \mathcal{W}_\theta(\kappa)$ is continuous, linearly bounded $\left\|\pi(\kappa,\xi)\right\|_{X_\kappa} \le \tilde{C}_\kappa^+(\xi,\gamma) + \tilde{C}_\kappa^-(\xi,\gamma)$ for all $\xi \in X_\kappa$ and, therefore, it maps bounded subsets of X_κ on bounded subsets of $\mathcal{W}_\theta(\kappa)$,*

(b) *$\varphi(k;\kappa,\cdot) \circ \pi(\kappa,\cdot) = \pi(k,\cdot) \circ \varphi(k;\kappa,\cdot)$ for $k \in \mathbb{Z}_\kappa^+$,*

(c) *if Hypothesis $(H)_3$ is satisfied, then $\pi(\kappa,\cdot) : X_\kappa \to X_\kappa$ is of class C^1,*

where the constants $L^\pm(\gamma)$, $\ell(\gamma), \ell^\pm(\gamma)$ are defined in Proposition 2.1 and $\tilde{\ell}(\gamma) := \frac{L^+(\gamma)\ell^+(\gamma)}{1-|\theta|\ell(\gamma)} \frac{L^-(\gamma)\ell^-(\gamma)}{1-|\theta|\ell(\gamma)}$,

$$\tilde{C}_\kappa^+(\xi,\gamma) := |\theta| \frac{\ell^+(\gamma)C_\kappa^+ + \frac{L^+(\gamma)\ell^+(\gamma)}{1-|\theta|\ell(\gamma)}\left(|\theta|\,\Gamma_\kappa(\gamma) + K_1^-\left\|P_-(\kappa)\xi\right\|_{X_\kappa}\right) + |\theta|\,K_1^+K_1^-\tilde{\ell}(\gamma)\left\|P_+(\kappa)\xi\right\|_{X_\kappa}}{1-|\theta|^2 K_1^+K_1^-\tilde{\ell}(\gamma)},$$

$$\tilde{C}_\kappa^-(\xi,\gamma) := \frac{\left\|P_-(\kappa)\xi\right\|_{X_\kappa} + |\theta|\,K_1^+\frac{L^-(\gamma)\ell^-(\gamma)}{1-|\theta|\ell(\gamma)}\left(\left\|P_+(\kappa)\xi\right\|_{X_\kappa} + |\theta|\,\ell^+(\gamma)C_\kappa^+\right) + |\theta|^3 K_1^+\tilde{\ell}(\gamma)\Gamma_\kappa(\gamma)}{1-|\theta|^2 K_1^+K_1^-\tilde{\ell}(\gamma)}.$$

Remark 2.2. (1) The fact that the gap condition (2.12) holds with the more restrictive function Σ given in (2.27), implies that the mappings w_θ, v_θ are globally Lipschitzian in their second argument, i.e.,

$$\operatorname{Lip}_2 w_\theta < 1, \qquad \operatorname{Lip}_2 v_\theta < 1. \tag{2.29}$$

(2) As immediate consequence of Proposition 2.5 we obtain that for each $(\kappa,\xi) \in \mathcal{W}_\theta$ the fibers $\mathcal{V}_{\xi,\theta}(\kappa)$ are mutually disjoint and form a foliation of X_κ (cf. [Pöt07, Corollary 4.6]).

Proof. See [Pöt07, Theorem 4.5]. □

In case $\lambda + \sigma < 1$, the asymptotic phase from Theorem 2.5 implies forward convergence of every solution to the nonautonomous set $\mathcal{W}_\theta$, but it does not instantly imply the convergence to a specific fiber $\mathcal{W}_\theta(k)$. In order to do so, one needs to start "progressively earlier" leading to the concept of pullback attraction (cf., e.g., [Kl00]). Under two additional assumptions we can prove such an attraction property of the IFB $\mathcal{W}_\theta$. At first, we suppose a constant state space X, and for the second assumption we need the following notion: A real nonnegative sequence $(x_k)_{k\in\mathbb{Z}}$ is called *backward tempered*, if one has

$$\lim_{k\to-\infty} \varepsilon^k x_k = 0 \quad \text{for all } \varepsilon \in (1,\infty).$$

Backward tempered sequences are allowed to grow polynomially, but not exponentially for $k \to -\infty$.

Corollary 2.6 (pullback attraction)**.** *Assume $X_k = X$ for all $k \in \mathbb{Z}$, $\lambda - \sigma < 1$ and that the sequences $(C_\kappa^\pm)_{\kappa\in\mathbb{Z}}$ from* (2.10) *are backward tempered. Then the IFB $\mathcal{W}_\theta$ from Proposition 2.1 is* pullback attracting*, i.e., for every bounded set $\mathcal{B} \subseteq \mathcal{X}$ one has the exponential convergence*

$$\lim_{n\to\infty} h(\varphi(k;k-n,\mathcal{B}(k-n)),\mathcal{W}_\theta(k)) = 0 \quad \textit{for all } k \in \mathbb{Z}.$$

Proof. We denote the norm on the common state space $X = X_k$ by $\|\cdot\|$. Let $\theta \in \Theta$, $\gamma \in \bar{\Gamma}$ and $\mathcal{B} \subseteq \mathcal{X}$ be bounded. Choose $(\kappa,\xi) \in \mathcal{B}$ and w.l.o.g. assume $\mathcal{B} = \mathcal{U}_R$ for some $R > 0$. The dichotomy estimates (2.6), (2.10) imply that the sequences $(\|P_-(\kappa)\xi\|)_{\kappa\in\mathbb{Z}}$, $(\|P_+(\kappa)\xi\|)_{\kappa\in\mathbb{Z}}$ are bounded by K_1^-R respectively K_1^+R. Hence, they are backward tempered uniformly in $\xi \in B_R(X)$. Moreover, the assumption on $(C_\kappa^\pm)_{\kappa\in\mathbb{Z}}$ ensures that the sequence $(\tilde{C}_\kappa^\pm(\xi,\gamma))_{\kappa\in\mathbb{Z}}$, where $\tilde{C}_\kappa^\pm(\xi,\gamma)$ is given in Theorem 2.5, is backward tempered uniformly in $\xi \in B_R(X)$. Thus, if we choose $\varepsilon \in \left(1, \frac{1}{\gamma}\right)$ there exists an integer $K = K(\gamma,\varepsilon,R)$ such that

$$\tilde{C}_\kappa^\pm(\xi,\gamma) \le \varepsilon^{-\kappa} \quad \text{for all } \kappa \le K,\ \xi \in B_R(X). \tag{2.30}$$

For each $\xi \in B_R(X)$ the invariance of $\mathcal{W}_\theta$ implies

$$\begin{aligned}\operatorname{dist}(\varphi(k;k-n,\xi),\mathcal{W}_\theta(k)) &= \operatorname{dist}\big(\varphi(k;k-n,\xi),\varphi(k;k-n,\mathcal{W}_\theta(k-n))\big)\\ &\leq \|\varphi(k;k-n,\xi)-\varphi(k;k-n,\pi(k-n,\xi))\|\\ &\overset{(2.28)}{\leq} \frac{K_1^+}{1-|\theta|\,\ell(\gamma)}\Big(K_1^+R+\tilde{C}_{k-n}^+(\xi,\gamma)\Big)\gamma^n \quad \text{for all } k\in\mathbb{Z},\ n\in\mathbb{Z}_0^+\end{aligned}$$

and together with (2.30) this guarantees

$$\operatorname{dist}(\varphi(k;k-n,\xi),\mathcal{W}_\theta(k)) \leq \frac{K_1^+}{1-|\theta|\,\ell(\gamma)}\left(K_1^+R\gamma^n+\gamma^{-k}(\varepsilon\gamma)^n\right) \quad \text{for all } (k,\xi)\in\mathcal{U}_R$$

and $n \geq k-N$. Since the right-hand side of this estimate does not depend on ξ we get

$$\operatorname{dist}(\varphi(k;k-n,\xi),\mathcal{W}_\theta(k)) \xrightarrow[n\to\infty]{} 0 \quad \text{for all } k\in\mathbb{Z},$$

where the choice of γ implies convergence at an exponential rate. □

3. Normal Hyperbolicity

In order to motivate the present section, it is convenient to turn to the linear problem (2.2). As we have seen, the IFB $\mathcal{W}_\theta$ for equation (2.1), as formulated here, is a perturbation of the pseudo-unstable bundle $\mathcal{P}_-$, and the linear spectral gap condition $\lambda - \Lambda > 0$ implies that $\mathcal{P}_-$ is normally hyperbolic in the sense that (2.2) possesses an exponential dichotomy. Now we tackle the problem if this normal hyperbolicity persists under nonlinear perturbations.

As important result from the previous two sections we know that the IFB $\mathcal{W}_\theta$ and its invariant foliation $\mathcal{V}_{\xi,\theta}$ are of class C^1, if the difference equation (2.1) has this property. Consequently, for each triple $(\kappa,x,y)\in\mathcal{W}_\theta\times\mathcal{X}$ we can define the tangent spaces

$$\begin{aligned}T_x\mathcal{W}_\theta(\kappa) &:= \{\xi + D_2w_\theta(\kappa,x)\xi \in X_\kappa : \xi\in\mathcal{P}_-(\kappa)\},\\ T_y\mathcal{V}_{x,\theta}(\kappa) &:= \{\eta + D_2v_\theta(\kappa,P_+(\kappa)y,x)\eta \in X_\kappa : \eta\in\mathcal{P}_+(\kappa)\}.\end{aligned}$$

For simplicity we assume in this section that the difference equation (2.1) is explicit. As a consequence, our Hypothesis $(H)_0$ holds trivially and the invariance equations (2.15), (2.25) become easier to handle. The subsequent lemma roughly states that the two tangential spaces defined above provide a splitting of each fiber X_κ of the extended state space $\mathcal{X}$ for (2.1).

Lemma 3.1. *Let $\theta\in\Theta$ and assume Hypotheses $(H)_1$–$(H)_3$ with $\sigma_{\max} = \frac{\lambda-\Lambda}{2}$ and Σ given in* (2.27)*. Then for each $\kappa\in\mathbb{Z}$ we have the decomposition*

$$X_\kappa = T_x\mathcal{W}_\theta(\kappa)\oplus T_y\mathcal{V}_{x,\theta}(\kappa) \quad \text{for all } x\in\mathcal{W}_\theta(\kappa),\ y\in X_\kappa \tag{3.1}$$

and the splitting is continuous in $(x,y)\in\mathcal{W}_\theta(\kappa)\times X_\kappa$.

Proof. Let $\theta\in\Theta$, fix a triple $(\kappa,x,y)\in\mathcal{W}_\theta\times\mathcal{X}$. To prove that the tangent spaces $T_x\mathcal{W}_\theta(\kappa)$ and $T_y\mathcal{V}_{x,\theta}(\kappa)$ satisfy (3.1) we show that each $\zeta\in X_\kappa$ possesses the representation $\zeta = \bar{\xi}+\bar{\eta}$ with unique $\bar{\xi}\in T_x\mathcal{W}_\theta(\kappa)$ and $\bar{\eta}\in T_y\mathcal{V}_{x,\theta}(\kappa)$. This is equivalent to the unique existence of $\xi\in\mathcal{P}_-(\kappa)$, $\eta\in\mathcal{P}_+(\kappa)$ such that $\zeta = \xi + D_2w_\theta(\kappa,x)\xi+\eta+D_2v(\kappa,P_+(\kappa)y,x)\eta$, which holds if and only if (cf. (2.14) and (2.24))

$$P_-(\kappa)\zeta = \xi + D_2v_\theta(\kappa,P_+(\kappa)y,x)\eta, \qquad P_+(\kappa)\zeta = \eta + D_2w_\theta(\kappa,x)\xi$$

and this, in turn, is equivalent to

$$\begin{aligned}\xi &= P_-(\kappa)\zeta - D_2v_\theta(\kappa,P_+(\kappa)y,x)P_+(\kappa)\zeta + D_2v_\theta(\kappa,P_+(\kappa)y,x)D_2w_\theta(\kappa,x)\xi,\\ \eta &= P_+(\kappa)\zeta - D_2w_\theta(\kappa,x)P_-(\kappa)\zeta + D_2v_\theta(\kappa,x)D_2v_\theta(\kappa,P_+(\kappa)y,x)\eta.\end{aligned}$$

By Proposition 2.1(b) and Proposition 2.4(b) the Lipschitz constants $\operatorname{Lip}_2 w_\theta$, $\operatorname{Lip}_2 v_\theta$, respectively, exist and their product is less than 1 (cf. (2.29)), so that the operators $I_{X_\kappa} - D_2v_\theta(\kappa,P_+(\kappa)y,x)D_2w_\theta(\kappa,x)$ and I_{X_κ} −

$D_2 w_\theta(\kappa, x) v_\theta(\kappa, P_+(\kappa)y, x)$ are invertible in $L(X_\kappa)$. Therefore, one can represent $\zeta \in X_\kappa$ uniquely as $\zeta = \hat{P}_-(\kappa, x, y)\zeta + \hat{P}_+(\kappa, x, y)\zeta$, where $\hat{P}_-(\kappa, x, y) \in L(X_\kappa)$ is the projection of X_κ onto $T_x \mathcal{W}_\theta(\kappa)$ along $T_y \mathcal{V}_{x,\theta}(\kappa)$,

$$\hat{P}_+(\kappa, x, y) := \left[I_{X_\kappa} - D_2 v_\theta(\kappa, P_+(\kappa)y, x) D_2 w_\theta(\kappa, x)\right]^{-1} \left[P_-(\kappa) - D_2 v_\theta(\kappa, P_+(\kappa)y, x)\right],$$

and $\hat{P}_+(\kappa, x, y) \in L(X_\kappa)$ is the projection of X_κ onto $T_y \mathcal{V}_{x,\theta}(\kappa)$ along $T_x \mathcal{W}_\theta(\kappa)$ given by

$$\hat{P}_-(\kappa, x, y) := \left[I_{X_\kappa} - D_2 w_\theta(\kappa, x) D_2 v_\theta(\kappa, P_+(\kappa)y, x)\right]^{-1} \left[P_+(\kappa) - D_2 w_\theta(\kappa, x)\right].$$

Due to Propositions 2.1(c), 2.4(c) and the fact that the inversion $\cdot^{-1} : L(X_\kappa) \to L(X_\kappa)$ is C^∞, we see that $\hat{P}_-(\kappa, x, y)$, $\hat{P}_+(\kappa, x, y)$ depend continuously on $(x, y) \in \mathcal{W}_\theta(\kappa) \times X_\kappa$. Thus, the splitting (3.1) is continuous. $\square$

Consider the difference equation in $\mathcal{X} \times \mathcal{X}$ given by (2.1) and the corresponding variational equation

$$\text{(3.2)} \qquad \begin{cases} y' = A(k)y + \theta K'(k) F(k, y) \\ z' = \left[A(k) + \theta K'(k) D_2 F(k, y)\right] z \end{cases};$$

its general solution will be denoted by (φ, ϕ). In the following it is our aim to show that the IFB $\mathcal{W}_\theta$ is normally hyperbolic; that is to say that the tangential and normal bundle for $\mathcal{W}_\theta$ are invariant under (3.2), and that we have an exponential dichotomy w.r.t. these bundles. To be more precise, we have

Lemma 3.2 (tangent bundle)**.** *Let $\theta \in \Theta$, assume Hypotheses $(H)_1$–$(H)_3$ with $\sigma_{\max} = \frac{\lambda - \Lambda}{2}$, Σ given in* (2.27) *and $|\theta| (1 + \operatorname{Lip}_2 w_\theta) \bar{C} K_1^- L^-(0) < \lambda$. Then the* tangent bundle

$$T\mathcal{W}_\theta := \{(\kappa, \xi, \zeta) : (\kappa, \xi) \in \mathcal{W}_\theta, \zeta \in T_\xi \mathcal{W}_\theta(\kappa)\}$$

is invariant w.r.t. (3.2)*, the general solution (φ, ϕ) of* (3.2) *exists on $\mathbb{Z} \times T\mathcal{W}_\theta$ and one has the backward estimate*

$$\text{(3.3)} \qquad \|\phi(k; \kappa, \xi, \zeta)\|_{X_k} \le K_1^- (1 + \operatorname{Lip}_2 w_\theta) \left[\lambda - |\theta| \bar{C} K_2^- L^-(\lambda - \sigma)(1 + \operatorname{Lip}_2 w_\theta)\right]^{k-\kappa} \|P_-(\kappa)\zeta\|_{X_\kappa}$$

for $k \in \mathbb{Z}_\kappa^-$ and $(\kappa, \xi, \zeta) \in T\mathcal{W}_\theta$.

Proof. Let $\theta \in \Theta$, choose any triple $(\kappa, \xi, \zeta) \in T\mathcal{W}_\theta$ and thus we have representations $\xi = \xi_0 + w_\theta(\kappa, \xi_0)$, $\zeta = \zeta_0 + D_2 w_\theta(\kappa, \xi_0)\zeta_0$ for some $\xi_0, \zeta_0 \in \mathcal{P}_-(\kappa)$. Then Corollary 2.2(a) implies that the general solution $\hat{\varphi}$ of the inertial form (2.18) is defined on $\mathbb{Z} \times \mathcal{W}_\theta$. The further proof is subdivided into four steps:

(I) Claim: *The general solution $\tilde{\varphi}$ of the variational equation for* (2.18)*, i.e.,*

$$\text{(3.4)} \qquad x' = A(k)x + \theta P_-'(k) K'(k) D_2 F\big(k, \hat{\varphi}(k; \kappa, \xi_0) + w_\theta(k, \hat{\varphi}(k; \kappa, \xi_0))\big) \left[x + D_2 w_\theta(k, \hat{\varphi}(k; \kappa, \xi_0))x\right]$$

is defined on $\mathbb{Z} \times \mathcal{P}_-$.

A differentiation of the solution identity for $\hat{\varphi}$ w.r.t. ξ_0 yields that $D_2 \hat{\varphi}(\cdot; \kappa, \xi_0)$ is an operator-valued solution of (3.4) satisfying the initial condition $x(\kappa) = I_{X_\kappa}$. Then $\tilde{\varphi}(k; \kappa, \xi_0) := D_2 \hat{\varphi}(k; \kappa, \xi_0)\xi_0$ defines the general solution of (3.4) for $k \in \mathbb{Z}$, $(\kappa, \xi_0) \in \mathcal{P}_-$.

(II) Claim: *The tangent bundle $T\mathcal{W}_\theta$ is positively invariant w.r.t.* (3.2)*.*

Define the sequence $\psi_1 : \mathbb{Z}_\kappa^+ \to \mathfrak{X}$, $\psi_1(k) := \hat{\varphi}(k; \kappa, \xi_0) + w_\theta(k, \hat{\varphi}(k; \kappa, \xi_0))$ and due to the inclusion $\hat{\varphi}(k; \kappa, \xi_0) \in \mathcal{P}_-(k)$ one obviously has $\psi_1(k) \in \mathcal{W}_\theta(k)$ for all $k \in \mathbb{Z}_\kappa^+$. In addition, from the invariance equation (2.15) we see that ψ_1 is a solution of the first equation in (3.2) with $\psi_1(\kappa) = \xi$ and this yields $\varphi(k; \kappa, \xi) = \psi_1(k) \in \mathcal{W}_\theta(k)$ for all $k \in \mathbb{Z}_\kappa^+$. Next we define the sequence $\psi_2 : \mathbb{Z}_\kappa^+ \to \mathfrak{X}$, $\psi_2(k) = \tilde{\varphi}(k; \kappa, \zeta_0) + D_2 w_\theta(k, \varphi(k; \kappa, \xi))\tilde{\varphi}(k; \kappa, \zeta_0)$. Observing $\tilde{\varphi}(k; \kappa, \zeta_0) \in \mathcal{P}_-(k)$ one has $\psi_2(k) \in T_{\varphi(k;\kappa,\xi)} \mathcal{W}_\theta(k)$ for all $k \in \mathbb{Z}_\kappa^+$. Using an identity obtained by differentiating the invariance equation (2.15) w.r.t. the variable in $\mathcal{P}_-(k)$, one verifies that ψ_2 solves the second equation in (3.2) and satisfies $\psi_2(\kappa) = \zeta_0 + D_2 w_\theta(k, \xi)\zeta_0$. Hence, $\phi(k; \kappa, \xi, \zeta) = \psi_2(k) \in T_{\varphi(k;\kappa,\xi)} \mathcal{W}_\theta(k)$ and the tangent bundle $T\mathcal{W}_\theta$ is positively invariant.

(III) The fact that φ is defined on $\mathbb{Z} \times \mathcal{W}_\theta$ is given in Corollary 2.2(b) and we will show that the second component ϕ is defined on $\mathbb{Z} \times T\mathcal{W}_\theta$. Thereto, let $k \in \mathbb{Z}$. From Step (II) we know that $\phi(k+1; k, \cdot) : T_{\varphi(k;\kappa,\xi)} \mathcal{W}_\theta(k) \to T_{\varphi(k+1;\kappa,\xi)} \mathcal{W}_\theta'(k)$ is well-defined and it suffices to show that this mapping is bijective. Thereto, let $\eta \in T_{\varphi(k+1;\kappa,\xi)}\mathcal{W}'_\theta(k)$, i.e., $\eta = \eta_1 + D_2 w_\theta(k+1, \varphi(k+1; k, \xi))\eta_1$ for some $\eta_1 \in \mathcal{P}'_-(k)$; note that $\mathrm{Lip}_2\, w_\theta < 1$ and consequently η_1 uniquely determines the point η. We show that the mapping

$$A(k) + \theta P'_-(k)K'(k)D_2F(k, \varphi(k;\kappa,\xi))\,[I_{X_k} + D_2 w_\theta(k, \varphi(k;\kappa,\xi))] : \mathcal{P}_-(k) \to \mathcal{P}'_-(k)$$

is bijective. Let us abbreviate $\Phi_k := P'_-(k)K'(k)D_2F(k, \varphi(k;\kappa,\xi))\,[I_{X_k} + D_2 w_\theta(k, \varphi(k;\kappa,\xi))]$ and from (2.3), (2.5), (2.11) and (2.16) one derives $\|\Phi_k\|_{L(\mathcal{P}_-(k),\mathcal{P}'_-(k))} \le (1 + \mathrm{Lip}_2\, w_\theta)\,\bar{C}L^-(\lambda - \sigma)$ for all $k \in \mathbb{Z}$. On the other hand, from Hypothesis $(H)_1$ we know that the inverse $A(k)|^{-1}_{L(\mathcal{P}_-(k),\mathcal{P}'_-(k))}$ exists and (2.8) implies $\|A(k)\|_{L(\mathcal{P}_-(k),\mathcal{P}'_-(k))} \le K_1^- \lambda^{-1}$ for all $k \in \mathbb{Z}$. Then [Aul98, Corollary 6.2] guarantees the invertibility of the sum $A(k) + \theta\Phi_k \in L(\mathcal{P}_-(k), \mathcal{P}'_-(k))$, and $\eta_0 := [A(k) + \Phi_k]^{-1}\,\eta_1$ is the unique point in $\mathcal{P}_-(k)$ satisfying the relation $\phi(k+1; k, \xi, \eta_0 + D_2 w_\theta(k, \varphi(k;\kappa,\xi))\eta_0) = \eta$.

(IV) Referring to Step (I) we know that the general solution $\tilde{\varphi}(k;\kappa,\cdot)$ of the variational equation (3.4) exists for $k \in \mathbb{Z}^-_\kappa$. Consequently, the variation of constants formula in backward time implies the relation

$$\begin{aligned}\tilde{\varphi}(k;\kappa,\zeta_0) = \bar{\Phi}(k;\kappa)\zeta_0 - \theta\sum_{n=k}^{\kappa-1}\bar{\Phi}(k,n+1)P'_-(n)K'(n)D_2F\big(n, \hat{\varphi}(n;\kappa,\xi_0) + w_\theta(n, \hat{\varphi}(n;\kappa,\xi_0))\big)\\ \cdot [I_{X_n} + D_2 w_\theta(n, \hat{\varphi}(n;\kappa,\xi_0))]\,\tilde{\varphi}(n;\kappa,\zeta_0) \quad \text{for all } k \in \mathbb{Z}^-_\kappa\end{aligned}$$

and completely analogous to the proof of (2.20) in Corollary 2.2 one gets the Lipschitz estimate (3.3). □

Lemma 3.3 (normal bundle). *Let $\theta \in \Theta$, assume Hypotheses $(H)_1$–$(H)_3$ with $\sigma_{\max} = \frac{\lambda-\Lambda}{2}$ and Σ given in* (2.27). *Then the* normal bundle

$$N\mathcal{W}_\theta := \{(\kappa,\xi,\zeta) : (\kappa,\xi) \in \mathcal{W}_\theta, \zeta \in T_\xi\mathcal{V}_{\xi,\theta}(\kappa)\}$$

is positively invariant w.r.t. (3.2)*, and one has the forward estimate*

$$\|\phi(k;\kappa,\xi,\zeta)\|_{X_k} \le K_1^+\,(1 + \mathrm{Lip}_2\, v_\theta)\left[\Lambda + |\theta|\,(1 + \mathrm{Lip}_2\, v_\theta)\left(K_2^+ + |\theta|^{-\nu}\,K_3^+\right)L^+(\lambda - \sigma)\right]^{k-\kappa}\|P_+(\kappa)\zeta\|_{X_\kappa}$$

for $k \in \mathbb{Z}^+_\kappa$ and $(\kappa,\xi,\zeta) \in N\mathcal{W}_\theta$.

Proof. Let $\theta \in \Theta$ and $(\kappa,\xi,\zeta) \in N\mathcal{W}_\theta$. We proceed in two steps:

(I) To show the positive invariance of $N\mathcal{W}_\theta$ we choose an arbitrary $\eta \in \mathcal{W}_\theta(\kappa)$ and let $\eta_0 \in \mathcal{P}_+(\kappa)$ be such that $\eta = \eta_0 + v_\theta(\kappa, \eta_0, \xi)$. From the positive invariance of $\mathcal{V}_{\xi,\theta}$ guaranteed by Proposition 2.4 (cf. (2.23)) we know that there exists a sequence of points $\psi_1(k) \in \mathcal{P}_+(k)$ satisfying

$$\varphi(k;\kappa,\eta) = \psi_1(k) + v_\theta(k, \psi_1(k), \varphi(k)) \quad \text{for all } k \in \mathbb{Z}^+_\kappa, \tag{3.5}$$

where we abbreviate $\varphi(k) = \varphi(k;\kappa,\xi)$ from now on, since $\xi \in \mathcal{W}_\theta(\kappa)$ is fixed. If we multiply the solution identity for φ with $P'_+(k)$, we see that $\psi_1 = P_+(\cdot)\varphi(\cdot;\kappa,\eta) : \mathbb{Z}^+_\kappa \to \mathfrak{X}$ solves the difference equation

$$y' = A(k)y + \theta P'_+(k)K'(k)F\big(k, y + v_\theta(k, y, \varphi(k))\big). \tag{3.6}$$

Let ψ denote the general solution of (3.6). Then the partial derivative $D_3\psi$ exists and $D_3\psi(\cdot;\kappa,\eta_0)P_+(\kappa)\zeta$ is a solution of the variational equation (cf. (3.5))

$$\begin{aligned} y' =& A(k)y + \theta P'_+(k)K'(k)D_2F\big(k, \psi(k;\kappa,\eta_0) + v_\theta(k, \psi(k;\kappa,\eta_0), \varphi(k;\kappa,\eta))\big)\\ &\cdot [y + D_2 v_\theta(k, P_+(k)\varphi(k;\kappa,\eta), \varphi(k))y]\end{aligned} \tag{3.7}$$

satisfying the initial condition $y(\kappa) = P_+(\kappa)\zeta$. On the other hand, the invariance equation (2.25) yields

$$\begin{aligned}& v_\theta(k+1, \psi(k+1;\kappa,\eta_0), \varphi'(k))\\ \equiv\;& A(k)v_\theta(k; \psi(k;\kappa,\eta_0), \varphi(k)) + \theta P'_+(k)K'(k)F\big(k, \psi(k;\kappa,\eta_0) + v_\theta(k, \psi(k;\kappa,\eta_0), \varphi(k))\big) \quad \text{on } \mathbb{Z}^+_\kappa\end{aligned}$$

and if we differentiate this identity w.r.t. η_0 and apply $P_+(\kappa)\zeta$ one gets

$$D_2 v_\theta(k+1, \psi(k+1;\kappa,\eta_0), \varphi'(k))D_3\psi(k+1;\kappa,\eta_0)P_+(\kappa)\zeta$$

$$\begin{aligned}\equiv\quad & A(k)D_2v_\theta(k;\psi(k;\kappa,\eta_0),\varphi(k))D_3\psi(k;\kappa,\eta_0)P_+(\kappa)\zeta\\ & +\theta P_+'(k)K'(k)D_2F\big(k,\psi(k;\kappa,\eta_0)+v_\theta(k,\psi(k;\kappa,\eta_0),\varphi(k))\big)\\ & \quad\cdot[D_3\psi(k;\kappa,\eta_0)+D_2v_\theta(k,D_3\psi(k;\kappa,\eta_0),\varphi(k))D_3\psi(k;\kappa,\eta_0)]\,P_+(\kappa)\zeta \quad \text{on } \mathbb{Z}_\kappa^+.\end{aligned}$$

From this, and the solution identity for $D_3\psi(\cdot;\kappa,\eta_0)P_+(\kappa)\zeta$ (cf. (3.7)) we see that the sum

$$\begin{aligned}\sigma(k) &:= D_3\psi(k;\kappa,\eta_0)P_+(\kappa)\zeta+D_2v_\theta(k;\psi(k;\kappa,\eta_0),\varphi(k))D_3\psi(k;\kappa,\eta_0)P_+(\kappa)\zeta\\ &= D_3\psi(k;\kappa,\eta_0)P_+(\kappa)\zeta+D_2v_\theta(k;P_+(k)\varphi(k;\kappa,\eta_0),\varphi(k))D_3\psi(k;\kappa,\eta_0)P_+(\kappa)\zeta\\ &\in T_{\varphi(k;\kappa,\eta)}\mathcal{V}_{\varphi(k)}(k),\theta \quad \text{for all } k\in\mathbb{Z}_\kappa^+\end{aligned}$$

is a solution of the linear difference equation $y'=A(k)y+\theta K'(k)D_2F(k,\varphi(k;\kappa,\eta))y$ satisfying $\sigma(\kappa)=P_+(\kappa)\zeta+D_2v_\theta(\kappa,P_+(\kappa)\eta,\zeta)P_+(\kappa)\zeta$. Since $\eta\in\mathcal{V}_{\xi,\theta}(\kappa)$ was arbitrary, we can choose $\eta=\pi(\kappa,\xi)$ now, and $\xi\in\mathcal{W}_\theta(\kappa)$ yields $\eta=\pi(\kappa,\xi)=\xi$ (cf. Theorem 2.5). Hence, $\sigma(k)\in T_{\varphi(k)}\mathcal{V}_{\varphi(k),\theta}(k)$ for all $k\in\mathbb{Z}_\kappa^+$ and $\sigma(\kappa)=\zeta$. The uniqueness of forward solutions implies $\phi(k;\kappa,\xi,\zeta)=\sigma(k)$, i.e.,

$$\phi(k;\kappa,\xi,\zeta)=D_3\psi(k;\kappa,\eta_0)P_+(\kappa)\zeta+D_2v_\theta(k;P_+(k)\varphi(k;\kappa,\xi),\varphi(k))D_3\psi(k;\kappa,\eta_0)P_+(\kappa)\zeta \tag{3.8}$$

and due to the invariance of $\mathcal{W}_\theta$ we have $(\varphi,\phi)(k;\kappa,\xi,\zeta)\in N\mathcal{W}_\theta(k)$ for all $k\in\mathbb{Z}_\kappa^+$.

(II) It remains to deduce the claimed forward estimate. Thereto, (2.7) yields the crude estimate

$$\|\Phi(k,l)P_+(l)K(l)\|_{L(Y_l,X_k)}\le K^+\Lambda^{k-l} \quad \text{for all } l\le k \tag{3.9}$$

with $K^+:=K_2^++|\theta|^{-\nu}K_3^+$. Then the variation of constants formula, applied to (3.7), gives us

$$\begin{aligned}D_3\psi(k;\kappa,\eta_0)P_+(\kappa)\zeta &= \Phi(k,\kappa)P_+(\kappa)\zeta+\theta\sum_{n=\kappa}^{k-1}\Phi(k,n+1)P_+'(n)K'(n)\\ &\quad\cdot D_2F\big(n,\psi(n;\kappa,\eta_0)+v_\theta(n,\psi(k;\kappa,\eta_0),\varphi(n;\kappa,\eta))\big)\\ &\quad\cdot[D_3\psi(k;\kappa,\eta_0)+D_2v_\theta(k,P_+(k)\varphi(k;\kappa,\eta),\varphi(k))D_3\psi(k;\kappa,\eta_0)]\,P_+(\kappa)\zeta\end{aligned}$$

for all $k\in\mathbb{Z}_\kappa^+$, and from (2.6), (3.9), (2.3), (2.11) and (2.26) we get

$$\begin{aligned}&\|D_3\psi(k;\kappa,\eta_0)P_+(\kappa)\zeta\|_{X_k}\Lambda^{\kappa-k}\\ &\le K_1^+\|P_+(\kappa)\zeta\|_{X_\kappa}+|\theta|\left(1+\operatorname{Lip}_2 v_\theta\right)K^+L^+(\lambda-\sigma)\sum_{n=\kappa}^{k-1}\Lambda^{\kappa-n}\|\psi(n;\kappa,\eta_0)P_+(\kappa)\zeta\|_{X_n}\end{aligned}$$

for all $k\in\mathbb{Z}_\kappa^+$. The Gronwall lemma in forward time (cf. [Aul98, Lemma 2.1(a)]) implies

$$\|D_3\psi(k;\kappa,\eta_0)P_+(\kappa)\zeta\|_{X_k}\Lambda^{\kappa-k}\le K_1^+\left[\Lambda+|\theta|\left(1+\operatorname{Lip}_2 v_\theta\right)K^+L^+(\lambda-\sigma)\right]^{k-\kappa}\|P_+(\kappa)\zeta\|_{X_\kappa}$$

for all $k\in\mathbb{Z}_\kappa^+$ and the relation (3.8) together with (2.26) leads to our assertion. □

Theorem 3.4 (normal hyperbolicity). *Let $\theta\in\Theta$, assume Hypotheses $(H)_1$–$(H)_3$ with $\sigma_{\max}=\frac{\lambda-\Lambda}{2}$,*

$$|\theta|\left[\left(K_2+|\theta|^{-\nu}K_3^+\right)L^+(\lambda-\sigma)+\bar{C}K_2^-L^-(\lambda-\sigma)\right]<\frac{\lambda-\Lambda}{2}, \tag{3.10}$$

and Σ given in (2.27)*. Then the IFB $\mathcal{W}_\theta$ is* normally hyperbolic *as follows: One has the* Whitney sum $\mathcal{X}\times\mathcal{X}=T\mathcal{W}_\theta\oplus N\mathcal{W}_\theta$, *where the splitting is continuous in each fiber. Moreover, the nonautonomous sets $T\mathcal{W}_\theta$ and $N\mathcal{W}_\theta$ possess the properties stated in Lemma 3.2 and Lemma 3.3, respectively. In particular, the contraction in the normal direction of $\mathcal{W}_\theta$ is stronger than in the tangential direction.*

Proof. For every $\theta\in\Theta$ the claim follows readily from the above Lemma 3.1, Lemma 3.2 and Lemma 3.3. Here the assumption (3.10) together with (2.29) implies $\Lambda+|\theta|\left(1+\operatorname{Lip}_2 v_\theta\right)\left(K_2+|\theta|^{-\nu}K_3^+\right)L^+(\lambda-\sigma)<\lambda-|\theta|\left(1+\operatorname{Lip}_2 w_\theta\right)\bar{C}K_2^-L^-(\lambda-\sigma)$ and we are done. □

4. INERTIAL FIBER BUNDLES

In the first instance, the goal of this paper was to provide a discrete counterpart for the concept of an inertial manifold. Regarding this, our approach so far lacks certain features. Thus, let us reconsider the theory developed in this paper from an applied point of view. Here two aspects need to be addressed:

- At least in the autonomous or time-periodic situation, classical spectral or Floquet theory provides sufficient criteria that the linear part of (2.1) meets the exponential dichotomy assumption $(H)_1$. The global Lipschitz condition $(H)_2$ on the nonlinearity F, however, will hardly be satisfied in relevant applications. More often the nonlinear term F is only Lipschitzian on bounded sets.
- The existence of inertial manifolds relies on a certain kind of dissipativity. Hence, we need appropriate counterparts of notions like absorbing sets or attractors in our nonautonomous framework. Here the concept of pullback convergence will serve as the right tool.

To incorporate these two points into our theory we weaken $(H)_2$ by imposing the following

Hypothesis. *Assume that there exist functions $l_2^\pm, l_3^\pm : [0,\infty) \to [0,\infty)$ such that for all $k \in \mathbb{Z}$ one has*

$(H)_4$ $K'(k)F(k,\cdot,y) : X_k \to X_{k+1}$ *is continuous for all $y \in X_{k+1}$, the estimates* (2.10) *are fulfilled and for each $r > 0$ we have the local Lipschitz conditions*

$$\|K'(k)\,[F(k,x,y) - F(k,x,\bar y)]\|_{X_{k+1}} \le l_0(r)\,\|y-\bar y\|_{X_{k+1}} \quad \text{for all } x \in B_r(X_k), y,\bar y \in B_r(X_{k+1}), \tag{4.1}$$

$$\begin{aligned} \left\|P_\pm'(k)\,[F(k,x,y) - F(k,\bar x,y)]\right\|_{Y_{k+1}} &\le l_2^\pm(r)\,\|x-\bar x\|_{X_k} \quad \text{for all } x,\bar x \in B_r(X_k), y \in B_r(X_{k+1}),\\ \left\|P_\pm'(k)\,[F(k,x,y) - F(k,x,\bar y)]\right\|_{Y_{k+1}} &\le l_3^\pm(r)\,\|y-\bar y\|_{X_{k+1}} \quad \text{for all } x \in B_r(X_k), y,\bar y \in B_r(X_{k+1}). \end{aligned} \tag{4.2}$$

$(H)_5$ *The equation* (2.1) *possesses a* uniformly pullback absorbing set $\mathcal{A} \subseteq \mathcal{X}$, *i.e., $\mathcal{A}$ is bounded and for every nonempty bounded subset $\mathcal{B} \subseteq \mathcal{X}$ there exists $N = N(\mathcal{B}) \in \mathbb{Z}_0^+$ such that*

$$\varphi(k; k-n, \mathcal{B}(k-n)) \subseteq \mathcal{A}(k) \quad \text{for all } k \in \mathbb{Z},\ n \ge N.$$

To prove smoothness assertions, we impose the assumption, that each X_k is a C^m-Banach space; that is, the norm on X_k is of class C^m away from 0. A characterization of such spaces and concrete examples, can be found in [AMR88, pp. 332ff]; e.g., Hilbert spaces are C^∞-Banach spaces.

Theorem 4.1 (inertial fiber bundles). *Let $\theta \in \Theta$, assume Hypotheses $(H)_0$–$(H)_1$, $(H)_4$–$(H)_5$, choose $\rho > 0$ so large that $\bar{\mathcal{A}} \subseteq \mathcal{U}_\rho$ holds,*

$$2\,|\theta|\,l_0(\rho) < 1, \tag{4.3}$$

$$4\,|\theta|\,\bar C l_3^-(\rho) < \sup_{k\in\mathbb{Z}} \max\left\{\|A(k)P_-(k)\|_{L(X_k,X_{k+1})}, \|A(k)P_-(k)\|_{L(Y_k,X_{k+1})}\right\}^2 < \infty \tag{4.4}$$

and suppose the following spectral gap condition*: There exists a real $0 < \sigma < \frac{\lambda-\Lambda}{2}$ such that*

$$\begin{aligned} &|\theta|\,l^-(\rho)\tfrac{\bar C K_2^-}{\bar\sigma} + |\theta|\,l^+(\rho)\left(\tfrac{K_2^+}{\bar\sigma} + |\theta|^{-\nu}K_3^+\operatorname{Li}_\nu\left(\tfrac{\Lambda}{\Lambda+\bar\sigma}\right)\right)\\ &+ |\theta|\max\left\{l^-(\rho)\tfrac{\bar C K_2^-}{\bar\sigma}, l^+(\rho)\left(\tfrac{K_2^+}{\bar\sigma} + |\theta|^{-\nu}K_3^+\operatorname{Li}_\nu\left(\tfrac{\Lambda}{\Lambda+\bar\sigma}\right)\right)\right\} < \tfrac12 \quad \text{for all } \bar\sigma \in \left(\sigma, \tfrac{\lambda-\Lambda}{2}\right), \end{aligned} \tag{4.5}$$

$$4\,|\theta|\,\bar C K_2^- l^-(\rho) < \lambda \tag{4.6}$$

with the constants $l^\pm(\rho) := l_2^\pm(\rho) + (\lambda-\bar\sigma)l_3^\pm(\rho)$. Then there exists a nonautonomous set $\mathcal{W}_\theta \subseteq \mathcal{X}$, which is positively invariant w.r.t. (2.1)*, and possesses the following properties:*

(a) *$\mathcal{W}_\theta$ is graph of a function w_θ over a nonempty open set $\mathcal{O}_\theta \subseteq \mathcal{P}_-$, i.e., $\mathcal{W}_\theta = \{(\kappa, \eta + w_\theta(\kappa,\eta)) : (\kappa,\eta) \in \mathcal{O}_\theta\}$, the functions $w_\theta(\kappa,\cdot) : \mathcal{O}_\theta(\kappa) \to \mathcal{P}_+(\kappa)$ are well-defined and satisfy:*

  (a_1) *They are globally Lipschitzian with $\operatorname{Lip}_2 w_\theta < 1$,*

(a_2) *one has the functional equation* (invariance equation)

$$w_\theta(\kappa+1,\eta_1) = A(\kappa)w_\theta(\kappa,\eta) + \theta P_+'(\kappa)K'(\kappa)F(\kappa,\eta+w_\theta(\kappa,\eta),\eta_1+w_\theta(\kappa+1,\eta_1)),$$
$$\eta_1 = A(\kappa)\eta + \theta K'(\kappa)F(\kappa,\eta+w_\theta(\kappa,\eta),\eta_1+w_\theta(\kappa+1,\eta_1))$$

for all $(\kappa,\eta)\in\mathcal{O}_\theta$ *such that* $\eta_1\in\mathcal{O}_\theta'(\kappa)$,

(a_3) *assume additionally that each* X_k *is a* C^m*-Banach space, that Hypothesis* $(H)_3$ *is satisfied with* $\Lambda<\lambda^m$, $m\in\mathbb{N}$*, and that the following stronger spectral gap condition holds: There exists a real* $0<\sigma<\min\left\{\frac{\lambda-\Lambda}{2},\lambda\left(1-\sqrt[m]{\frac{\lambda+\Lambda}{\lambda+\lambda^m}}\right)\right\}$ *such that*

$$\begin{aligned}&|\theta|\,l^-(2\rho)\frac{\bar{C}K_2^-}{\bar\sigma}+|\theta|\,l^+(2\rho)\left(\frac{K_2^+}{\bar\sigma}+|\theta|^{-\nu}K_3^+\operatorname{Li}_\nu\left(\frac{\Lambda}{\Lambda+\bar\sigma}\right)\right)\\ &\qquad+|\theta|\max\left\{l^-(2\rho)\frac{\bar{C}K_2^-}{\bar\sigma},l^+(2\rho)\left(\frac{K_2^+}{\bar\sigma}+|\theta|^{-\nu}K_3^+\operatorname{Li}_\nu\left(\frac{\Lambda}{\Lambda+\bar\sigma}\right)\right)\right\}<\frac{1}{3},\\ &\qquad 4\,|\theta|\,\bar{C}K_2^-l^-(2\rho)<\lambda\end{aligned}\tag{4.7}$$

for $\bar\sigma\in\left(\sigma,\min\left\{\frac{\lambda-\Lambda}{2},\lambda\left(1-\sqrt[m]{\frac{\lambda+\Lambda}{\lambda+\lambda^m}}\right)\right\}\right)$. *Then* $w_\theta(\kappa,\cdot):\mathcal{O}_\theta(\kappa)\to\mathcal{P}_+(\kappa)$ *is of class* C^m,

(b) *the nonautonomous set* $\mathcal{W}_\theta$ *is* asymptotically complete*, i.e., for every* $(\kappa,\xi)\in\mathcal{X}$ *there exists a point* $(\kappa_0,\eta)\in\mathcal{W}_\theta$ *with* $\kappa\le\kappa_0$ *such that*

$$\|\varphi(k;\kappa,\xi)-\varphi(k;\kappa_0,\eta)\|_{X_k}\le C\gamma^{k-\kappa}\quad\textit{for all } k\in\mathbb{Z}_{\kappa_0}^+,$$

where the real constant $C\ge 0$ *depends boundedly on* κ, ξ *and* $\gamma\in\bar\Gamma$.

Under the assumption $\dim\mathcal{P}_-(\kappa)<\infty$ *for one* $\kappa\in\mathbb{Z}$ *we denote* $\mathcal{W}_\theta$ *as* inertial fiber bundle *of* (2.1).

Remark 4.1. (1) Note that (4.1), (4.3), (4.4) become void for explicit difference equations (2.1). Moreover, with Hilbert spaces X_k, $k\in\mathbb{Z}$, one can weaken (4.3) to $|\theta|\,l_0(\rho)<1$, the spectral gap condition (4.5) to

$$\begin{aligned}&|\theta|\,l^-(\rho)\frac{\bar{C}K_2^-}{\bar\sigma}+|\theta|\,l^+(\rho)\left(\frac{K_2^+}{\bar\sigma}+|\theta|^{-\nu}K_3^+\operatorname{Li}_\nu\left(\frac{\Lambda}{\Lambda+\bar\sigma}\right)\right)\\ &\qquad+|\theta|\max\left\{l^-(\rho)\frac{\bar{C}K_2^-}{\bar\sigma},l^+(\rho)\left(\frac{K_2^+}{\bar\sigma}+|\theta|^{-\nu}K_3^+\operatorname{Li}_\nu\left(\frac{\Lambda}{\Lambda+\bar\sigma}\right)\right)\right\}<1\quad\text{for all }\bar\sigma\in\left(\sigma,\tfrac{\lambda-\Lambda}{2}\right)\end{aligned}$$

and replace the factor 4 in (4.4), (4.6) by 2. However, the gap conditions (4.7) remain unchanged, whereas X_k are C^∞-Banach spaces, since their norm is induced by an inner product.

(2) From Theorem 4.1 we see that the inertial fiber bundle $\mathcal{W}_\theta$ can be interpreted as a nonautonomous discrete counterpart of an inertial manifold. If we have $\dim\mathcal{P}_-(\kappa)<\infty$ for one $\kappa\in\mathbb{Z}$, then (2.4) guarantees that all fibers $\mathcal{W}_\theta(k)$, $k\in\mathbb{Z}$, possess the same finite dimension, they are Lipschitzian, $\mathcal{W}_\theta$ is positively invariant w.r.t. (2.1) and in case $\lambda-\sigma<1$ also exponentially attractive.

(3) There are two possible approaches, in order to fulfill the spectral gap condition (4.5):

- If one is interested in discretization theory (cf., e.g., [DG91, JS95, JST98, vDL99]), then λ,Λ are a priori given by a continuous problem (e.g., λ,Λ are chosen such that a spectral gap condition for an evolutionary PDE is satisfied, yielding the existence of an inertial manifold), and one considers values (step-sizes) for $\theta\in\Theta$ sufficiently small that (4.5) and (4.6) hold.
- From a less applied point of view, for fixed $\theta\in\Theta$ satisfying (4.6), the condition (4.5) is fulfilled only, if the spectral gap $\lambda-\Lambda$ is sufficiently large. Hence, beyond condition (4.6), the existence of inertial fiber bundles for (2.1) primarily depends on a sufficiently large spectral gap $\lambda-\Lambda>0$.

The usual procedure to prove Theorem 4.1 is to replace (2.1) by an appropriately modified difference equation and to apply our previous results from Section 2 to the modified equation. It then remains to show that this modification does not affect the long term dynamics.

Proof. Let $\theta \in \Theta$ and $\mathcal{A} \subseteq \mathcal{X}$ be the uniformly pullback absorbing set for (2.1) from Hypothesis $(H)_5$. Since $\mathcal{A}$ is bounded, there exists a $\rho > 0$ with $\mathcal{A} \subseteq \mathcal{U}_\rho$. For this fixed $\rho > 0$ we define modified nonlinearities $F_\rho(k,x,y) := F(k, \rho r_k(x/\rho), \rho r_{k+1}(y/\rho))$, where $r_k : X_k \to \bar{B}_1(X_k)$ is the radial retraction in X_k,

$$r_k(x) := \begin{cases} x & \text{for } \|x\|_{X_k} \leq 1 \\ \frac{x}{\|x\|_{X_k}} & \text{for } \|x\|_{X_k} > 1 \end{cases}.$$

This gives us $F_\rho(k,x,y) = F(k,x,y)$ for $(k,x,y) \in \mathcal{U}_\rho \times \mathcal{U}'_\rho$. Due to the fact $\operatorname{Lip} r_k \leq 2$ (cf., e.g., [Ama90]; note that one has $\operatorname{Lip} r_k \leq 1$ in a Hilbert space setting) one obtains from (4.1), (4.2) the estimates

$$\|K'(k)\left[F_\rho(k,x,y) - F_\rho(k,x,\bar{y})\right]\|_{X_{k+1}} \leq 2l_0(\rho)\|y-\bar{y}\|_{X_{k+1}} \quad \text{for all } x \in X_k, y, \bar{y} \in X_{k+1},$$
$$\left\|P'_\pm(k)\left[F_\rho(k,x,y) - F_\rho(k,\bar{x},y)\right]\right\|_{Y_{k+1}} \leq 2l_2^\pm(\rho)\|x-\bar{x}\|_{X_k} \quad \text{for all } x, \bar{x} \in X_k, y \in X_{k+1},$$
$$\left\|P'_\pm(k)\left[F_\rho(k,x,y) - F_\rho(k,x,\bar{y})\right]\right\|_{Y_{k+1}} \leq 2l_3^\pm(\rho)\|y-\bar{y}\|_{X_{k+1}} \quad \text{for all } x \in X_k, y, \bar{y} \in X_{k+1},$$

respectively. Having this at hand, we can focus on the modified difference equation

$$y' = A(k)y + \theta K'(k)F_\rho(k,y,y'); \tag{4.8}$$

it satisfies $(H)_1$–$(H)_2$ and the parametrized contraction mapping principle (see, e.g., [Aul98, Theorem 6.1]) yields that also $(H)_0$ holds for (4.8). By the spectral gap condition (4.5), Proposition 2.1 and Theorem 2.5 apply and there exists an invariant fiber bundle $\tilde{\mathcal{W}}_\theta$ of the modified equation (4.8), which is graph of a function $\tilde{w}_\theta$ over $\mathcal{P}_-$ possessing the asymptotic phase $\pi(\kappa,\cdot)$. Furthermore, let $\tilde{\varphi}$ denote the general solution of (4.8). We now show how to derive from $\tilde{\mathcal{W}}_\theta$ a positively invariant nonautonomous set $\mathcal{W}_\theta$ for the initial equation (2.1).

Since $\mathcal{A}$ is uniformly pullback absorbing, there exists a $N = N(\mathcal{U}_\rho) \in \mathbb{N}$ such that

$$\varphi(k; k-n, \mathcal{U}_\rho(k-n)) \subseteq \mathcal{A}(k) \quad \text{for all } k \in \mathbb{Z},\, n \geq N \tag{4.9}$$

and we define the nonautonomous set $\mathcal{B}_1 \subseteq \mathcal{X}$ by its fibers

$$\mathcal{B}_1(k) := \bigcup_{n \geq N} \varphi(k; k-n, \mathcal{U}_\rho(k-n)) \quad \text{for all } k \in \mathbb{Z}.$$

Then (4.9) implies $\mathcal{B}_1 \subseteq \mathcal{A}$, $\bar{\mathcal{B}}_1 \subseteq \mathcal{U}_\rho$ and

$$\begin{aligned} \varphi(k; l, \mathcal{B}_1(l)) &= \varphi\Big(k; l, \bigcup_{n \geq N} \varphi(l; l-n, \mathcal{U}_\rho(l-n))\Big) \overset{(1.2)}{\subseteq} \bigcup_{n \geq N} \varphi(k; l-n, \mathcal{U}_\rho(l-n)) \\ &= \bigcup_{n \geq N+k-l} \varphi(k; k-n, \mathcal{U}_\rho(k-n)) \subseteq \mathcal{B}_1(k) \quad \text{for all } l \leq k, \end{aligned} \tag{4.10}$$

which yields $\varphi(k;l,\cdot)|_{\mathcal{B}_1(l)} = \tilde{\varphi}(k;l,\cdot)|_{\mathcal{B}_1(l)}$ for all $l \leq k$, and $\mathcal{B}_1$ is also uniformly pullback attracting for the initial equation (2.1). Now define $\mathcal{W}^*_\theta := \tilde{\mathcal{W}}_\theta \cap \mathcal{B}_1$ and we obtain

$$\begin{aligned} \varphi(k; \kappa, \mathcal{W}^*_\theta(\kappa)) &= \tilde{\varphi}(k; \kappa, \mathcal{W}^*_\theta(\kappa)) \subseteq \tilde{\varphi}(k; \kappa, \tilde{\mathcal{W}}_\theta(\kappa)) \cap \tilde{\varphi}(k; \kappa, \mathcal{B}_1(\kappa)) \\ &\subseteq \tilde{\mathcal{W}}_\theta(k) \cap \mathcal{B}_1(k) = \mathcal{W}^*_\theta(k) \quad \text{for all } k \in \mathbb{Z}^+_\kappa, \end{aligned}$$

so that $\mathcal{W}^*_\theta$ is positively invariant w.r.t. the initial equation (2.1) and the modified equation (4.8).

Choose $\varepsilon > 0$ so small that the open ε-neighborhood $\mathcal{N}_\varepsilon(\mathcal{B}_1) := \{(k,x) : k \in \mathbb{Z}, \operatorname{dist}(x, \mathcal{B}_1(k)) < \varepsilon\}$ of $\mathcal{B}_1$ is contained in $\mathcal{U}_\rho$ and set $\mathcal{W}^\varepsilon_\theta := \tilde{\mathcal{W}}_\theta \cap \mathcal{N}_\varepsilon(\mathcal{B}_1)$. Then $\mathcal{W}^\varepsilon_\theta$ is an open neighborhood of $\mathcal{W}^*_\theta$ in $\tilde{\mathcal{W}}_\theta$ and due to the uniform continuity of $\tilde{\varphi}(k;\kappa,\cdot)$ in $k - \kappa \leq N$ (see the Lipschitz estimate (2.20) in Corollary 2.2), we obtain the existence of a $\delta > 0$ such that the open δ-neighborhood $\mathcal{W}^\delta_\theta$ of $\mathcal{W}^*_\theta$ in $\tilde{\mathcal{W}}_\theta$ satisfies

$$\tilde{\varphi}(k; \kappa, \mathcal{W}^\delta_\theta(\kappa)) \subseteq \mathcal{W}^\varepsilon_\theta(k) \quad \text{for all } k - \kappa \leq N. \tag{4.11}$$

Consequently, by (4.9) and (4.11) we obtain $\varphi(k;\kappa,\mathcal{W}_\theta^\delta(\kappa))\subseteq\mathcal{W}_\theta^\varepsilon(k)$ and $\varphi(k;\kappa,\mathcal{W}_\theta^\delta(\kappa))=\tilde\varphi(k;\kappa,\mathcal{W}_\theta^\delta(\kappa))$ for all $k\in\mathbb{Z}_\kappa^+$. Let us show that $\mathcal{W}_\theta$, fiber-wise defined by

$$\mathcal{W}_\theta(k):=\bigcup_{n\geq 0}\varphi(k;k-n,\mathcal{W}_\theta^\delta(k-n))\quad\text{for all } k\in\mathbb{Z}$$

is the desired positively invariant nonautonomous set for (2.1). By definition, we readily see the inclusion $\varphi(k;\kappa,\mathcal{W}_\theta(\kappa))\subseteq\mathcal{W}_\theta(k)$ for all $k\in\mathbb{Z}_\kappa^+$, i.e., $\mathcal{W}_\theta$ is positively invariant w.r.t. (2.1).

(a) Thanks to Corollary 2.2(b) and Corollary 2.3 we get that $\tilde\varphi(k;\kappa,\cdot)|_{\tilde{\mathcal{W}}_\theta(\kappa)}:\tilde{\mathcal{W}}_\theta(\kappa)\to\tilde{\mathcal{W}}_\theta(k)$ is a homeomorphism (indeed a Lipeomorphism), so that it sends open subsets of $\tilde{\mathcal{W}}_\theta(\kappa)$ into open sets of $\tilde{\mathcal{W}}_\theta(k)$. Thus, $\varphi(k;\kappa,\mathcal{W}_\theta^\delta(\kappa))=\tilde\varphi(k;\kappa,\mathcal{W}_\theta^\delta(\kappa))$ is open in $\mathcal{W}_\theta^\delta(k)$ for $k\in\mathbb{Z}_\kappa^+$, and therefore $\mathcal{W}_\theta(k)$ and $\mathcal{W}_\theta$ are open in $\tilde{\mathcal{W}}_\theta(k)$ and $\tilde{\mathcal{W}}_\theta$, respectively. Due to the fact that $I_{X_k}+\tilde w_\theta(k,\cdot):\mathcal{P}_-(k)\to\tilde{\mathcal{W}}_\theta(k)$ is a homeomorphism (note $\operatorname{Lip}_2\tilde w_\theta<1$), also the set $\mathcal{O}_\theta\subseteq\mathcal{X}$, fiber-wise given by

$$\mathcal{O}_\theta(k):=\left[I_{X_k}+\tilde w_\theta(k,\cdot)\right]^{-1}(\mathcal{W}_\theta(k))\quad\text{for all } k\in\mathbb{Z}$$

is open in $\mathcal{P}_-$. If we define $w_\theta:=\tilde w_\theta|_{\mathcal{O}_\theta}$, then w_θ is graph of a function w_θ with $w_\theta(\kappa,\cdot):\mathcal{O}_\theta(\kappa)\to\mathcal{P}_+(\kappa)$ satisfying $\operatorname{Lip}_2 w_\theta<1$, i.e., the assertion (a_1) holds. In addition, the statement (a_2) instantly follows from the corresponding properties for $\tilde w_\theta$ guaranteed by Proposition 2.1. We, nevertheless, postpone the verification of (a_3) to the end of the present proof.

(b) Let $(\kappa,\xi)\in\mathcal{X}$ and $\gamma\in\bar\Gamma$. Choose a bounded set $\mathcal{B}\subseteq\mathcal{X}$ such that $(\kappa,\xi)\in\mathcal{B}$ and from the above we know that there exists $N_1=N_1(\mathcal{B})\in\mathbb{Z}_0^+$ such that $\varphi(k;k-n,\mathcal{B}(k-n))\subseteq\mathcal{B}_1(k)$ for all $k\in\mathbb{Z}$, $n\geq N_1$. In particular, this yields $\xi_0:=\varphi(\kappa+N_1;\kappa,\xi)\in\mathcal{B}_1(\kappa+N_1)$, thanks to (4.10) one has $\varphi(k;\kappa+N_1,\xi_0)=\tilde\varphi(k;\kappa+N_1,\xi_0)$,

$$\varphi(k;\kappa,\xi)\overset{(1.2)}{=}\varphi(k;\kappa+N_1,\xi_0)=\tilde\varphi(k;\kappa+N_1,\xi_0)\quad\text{for all } k\geq\kappa+N_1. \tag{4.12}$$

Due to the asymptotic phase of $\tilde{\mathcal{W}}_\theta$ (cf. Theorem 2.5) there exists a point $\eta_0\in\tilde{\mathcal{W}}_\theta(\kappa+N_1)$ such that

$$\left\|\tilde\varphi(k;\kappa+N_1,\xi_0)-\tilde\varphi(k;\kappa+N_1,\eta_0)\right\|_{X_k}\leq C\gamma^{k-\kappa}\quad\text{for all } k\geq\kappa+N_1, \tag{4.13}$$

where the constant $C\geq 0$ depends boundedly on κ, ξ, γ. Now we choose another bounded set $\hat{\mathcal{B}}\subseteq\mathcal{X}$ such that $(\kappa+N_1,\eta_0)\in\hat{\mathcal{B}}$. Again, there exists $N_2=N_2(\mathcal{B})\in\mathbb{Z}_0^+$ with $\varphi(k;k-n,\hat{\mathcal{B}}(k-n))\subseteq\mathcal{B}_1(k)$ for all $k\in\mathbb{Z}$, $n\geq N_2$, and in particular $\eta:=\varphi(\kappa+N_1+N_2,\eta_0)\in\mathcal{B}_1(\kappa+N_1+N_2)$. Then, the positive invariance of $\mathcal{B}_1$ from (4.10) implies $\varphi(k;\kappa+N_1,\eta_0)=\tilde\varphi(k;\kappa+N_1,\eta_0)$ and therefore

$$\varphi(k;\kappa+N_1+N_2,\eta_0)\overset{(1.2)}{=}\varphi(k;\kappa+N_1,\eta_0)=\tilde\varphi(k;\kappa+N_1,\eta_0)\quad\text{for all } k\geq\kappa+N_1+N_2. \tag{4.14}$$

Setting $\kappa_0:=\kappa+N_1+N_2$, inserting (4.12) and (4.14) into the estimate (4.13) gives us the claim (b).

It remains to establish assertion (a_3). Thereto, one has to find a C^m-modification F_ρ of F such that (4.8) meets the assertions of Proposition 2.1(c). Denoting the right-hand side of inequality (4.7) by L, we choose a real $s>1$ sufficiently close to 1 such that $\frac{1}{1+2s}\in\left(L,\frac{1}{3}\right)$ holds. Then the cut-off function ϑ from Lemma A.1 satisfies $|D\vartheta(t)|\leq s$ for all $t\in\mathbb{R}$ and we define $r_k:X_k\to X_k$ by $r_k(x):=\vartheta(\|x\|_{X_k}/\rho)$. Since X_k are C^m-Banach spaces, also the functions r_k are of class C^m and we have

$$\|Dr_k(x)\|_{L(X_k)}\leq\frac{1}{\rho}\left|D\vartheta\left(\frac{\|x\|_{X_k}}{\rho}\right)\right|\|x\|_{X_k}+\left|\vartheta\left(\frac{\|x\|_{X_k}}{\rho}\right)\right|\quad\text{for all } x\in B_{2\rho}(X_k),$$

yielding $\operatorname{Lip} r_k\leq 1+2s$ for all $k\in\mathbb{Z}$. Then the modified nonlinearities $F_\rho(k,x,y):=F(k,r_k(x),r_{k+1}(y))$ are of class C^m, one has $F_\rho(k,x,y)=F(k,x,y)$ for $(k,x)\in\mathcal{U}_\rho\times\mathcal{U}_\rho'$, they satisfy

$$\left\|P_\pm'(k)\left[F_\rho(k,x,y)-F_\rho(k,\bar x,y)\right]\right\|_{Y_{k+1}}\leq(1+2s)l^\pm(2\rho)\left\|x-\bar x\right\|_{X_k}\quad\text{for all } x,\bar x\in X_k, y\in X_{k+1},$$
$$\left\|P_\pm'(k)\left[F_\rho(k,x,y)-F_\rho(k,x,\bar y)\right]\right\|_{Y_{k+1}}\leq(1+2s)l^\pm(2\rho)\left\|y-\bar y\right\|_{X_{k+1}}\quad\text{for all } x\in X_k, y,\bar y\in X_{k+1},$$

as well as the further assumptions of Proposition 2.1(c). Hence, due to the choice of s, the gap condition is satisfied for (4.8). Thus, the function w_θ, as defined above, is of class C^m. This finishes the proof. □

Another important feature of inertial manifolds is that they contain the universal attractor of a dissipative equation. The existence of an attractor is implied by a more easily determinable absorbing set. As we will see next, this feature fits well into our theory. A nonautonomous set $\mathcal{A}^* \subseteq \mathcal{X}$ is called *pullback attractor* of (2.1), if it is bounded with compact fibers $\mathcal{A}^*(k)$, $k \in \mathbb{Z}$, invariant w.r.t. (2.1) and pullback attracting (cf. Corollary 2.6). The existence of a pullback absorbing set from Hypothesis $(H)_5$ has a striking consequence on the long-term behavior of the difference equation (2.1). To be more precise, under Hypothesis $(H)_5$ there exists a uniquely determined pullback attractor $\mathcal{A}^*$ of (2.1), whose fibers are given by (see [Kl00])

$$\mathcal{A}^*(k) = \bigcap_{m\geq 0} \operatorname{cl}_{X_k} \bigcup_{n\geq m} \varphi(k; k-n, \mathcal{B}(k-n)) \quad \text{for all } k \in \mathbb{Z}.$$

Corollary 4.2 (pullback attractors)**.** *Assume $X_k = X$ for all $k \in \mathbb{Z}$, $\lambda - \sigma < 1$ and that the sequences $(C_\kappa^\pm)_{\kappa\in\mathbb{Z}}$ from* (2.10) *are backward tempered. Then every bounded and w.r.t. equation* (2.1) *invariant set $\mathcal{B} \subseteq \mathcal{X}$ with $\mathcal{B}(k) \subseteq \bigcup_{n\geq N} \varphi(k; k-n, \mathcal{U}_\rho(k-n))$ for all $k \in \mathbb{Z}$ satisfies $\mathcal{B} \subseteq \mathcal{W}_\theta$; and in particular the inertial fiber bundle $\mathcal{W}_\theta$ contains the unique pullback attractor $\mathcal{A}^*$ of* (2.1)*, i.e., $\mathcal{A}^* \subseteq \mathcal{W}_\theta$.*

Proof. Let $\theta \in \Theta$ and $k \in \mathbb{Z}$ be arbitrary. Then $\mathcal{B} \subseteq \mathcal{B}_1$ and the invariance of $\mathcal{B}$ leads to

$$\begin{aligned} h(\mathcal{B}(k), \tilde{\mathcal{W}}_\theta(k)) &= h(\varphi(k; k-n, \mathcal{B}(k-n)), \tilde{\mathcal{W}}_\theta(k)) \\ &= h(\tilde{\varphi}(k; k-n, \mathcal{B}(k-n)), \tilde{\mathcal{W}}_\theta(k)) \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

due to Corollary 2.6. Hence $\mathcal{B}(k) \subseteq \operatorname{cl}_X \tilde{\mathcal{W}}_\theta(k)$, but $\mathcal{B} \subseteq \mathcal{B}_1$ and $\mathcal{W}_\theta \supseteq \tilde{\mathcal{W}}_\theta \cap \mathcal{N}_\delta(\mathcal{B}_1)$ implies the desired inclusion $\mathcal{B}(k) \subseteq \mathcal{W}_\theta(k)$. Obviously, this holds for the special case $\mathcal{B} = \mathcal{A}^*$. □

5. Discretization of Evolutionary PDEs

In this final section we provide criteria that various discretizations of evolutionary PDEs possess inertial fiber bundles. Basically, these criteria reduce to an application of Theorem 4.1 and our primary purpose is to establish "persistence" of attractive invariant manifolds; their convergence for better refinements of the discretization will be postponed to upcoming papers. A nonautonomous counterpart of the time-h-map had been discussed in [Pöt07, Subsections 5.1 and 5.3]. Here we consider temporal-, as well as full discretizations and strongly benefit from preparations obtained in [EMR90, FJ$^+$91, Lor97]. However, due to space limitations we reduce the lengthy presentation and sketch some arguments only.

Basic for our discretization schemes is an appropriate discrete set of time steps. Given two bounds $0 < h \leq H$, this will be a real sequence $(t_k)_{k\in\mathbb{Z}}$ satisfying $t_{k+1} - t_k \in [h, H]$ for all $k \in \mathbb{Z}$.

5.1. **Temporally discretized Allen-Cahn equation.** In the beginning of this subsection we are interested in the question how pseudo-hyperbolicity of analytic semigroups is preserved under implicit Euler discretization. For the autonomous case this follows essentially from perturbation theory for linear operators. The present variable step-size setting, however, requires a roughness argument for dichotomies.

Before tackling a nonautonomous Allen-Cahn equation, we start by discussing a general situation of analytical semigroups. Consider a linear autonomous evolutionary equation

$$u_t + Bu = 0 \tag{5.1}$$

on some ambient Banach space Y, where B is a positive sectorial operator on Y generating an analytic semigroup $(e^{-Bt})_{t\geq 0}$ (cf. [SY02, p. 79, Lemma 36.1]). We suppose the spectrum $\sigma(B) \subseteq \mathbb{C}$ allows a decomposition $\sigma(B) = \sigma_- \cup \sigma_+$ into closed disjoint spectral sets σ_-, σ_+ such that σ_- is bounded and that we can choose reals $0 < \alpha < \beta$,

$$\sup_{\lambda\in\sigma_-} \Re\lambda < \alpha < \beta < \inf_{\lambda\in\sigma_+} \Re\lambda. \tag{5.2}$$

Let $Q_\pm$ denote the complementary spectral projections corresponding to the spectral sets $\sigma_\pm$, $Y_\pm := Q_\pm Y$ and $B_\pm := B|_{Y_\pm}$. Then the subspaces $Y_\pm$ are invariant under B and e^{-Bt}; moreover, $B_- \in L(Y_-)$, $\sigma(B_-) = \sigma_-$ and B_+ is sectorial with $\sigma(B_+) = \sigma_+$, $D(B_+) = D(B) \cap Y_+$ (cf. [Hen81, p. 30, Theorem 1.5.2]).

We define the fractional power spaces $X^\nu := D(B^\nu) \subseteq Y$ for $\nu \in [0,1]$. Then a variable step-size implicit Euler discretization of (5.1) is a linear nonautonomous difference equation of the form (2.2) with

$$A(k) := [I_Y + (t_{k+1} - t_k)B]^{-1} \quad \text{for all } k \in \mathbb{Z};$$

in addition, we define $K(k) := [I_Y + (t_{k+1} - t_k)B]^{-1}$.

Lemma 5.1. *If the above assumptions on* (5.1) *are satisfied, then there exist* $q \in \left[\frac{\alpha}{\beta}, 1\right)$ *and* $H_0 > 0$ *such that for all* $H \in (0, H_0]$ *and* $h \in (qH, H]$ *the following holds: There are complementary projections* $P_-(k), P_+(k)$ *on* Y *as in Hypothesis* $(H)_1$ *and constants* $K_1^+, K_3^+, K_1^-, K_2^- > 0$,

$$(1 + h\beta)^{-1} < \Lambda < \lambda < (1 + H\alpha)^{-1}$$

with $\|P_+(k) - Q_+\| = O(H)$ *as* $H \to 0$ *and*

$$\|\Phi(k,l)P_+(l)\|_{L(X^\nu)} \leq K_1^+ \Lambda^{k-l} \quad \text{for all } l \leq k,$$
$$\|\Phi(k,l)P_+(l)K(l)\|_{L(Y,X^\nu)} \leq K_3^+ h^{-\nu}(k-l+1)^{-\nu}\Lambda^{k-l} \quad \text{for all } l \leq k,$$
$$\left\|\bar{\Phi}(k,l)P_-(l)\right\|_{L(X^\nu)} \leq K_1^- \lambda^{k-l} \quad \text{for all } k \leq l,$$
$$\left\|\bar{\Phi}(k,l)P_-(l)\right\|_{L(Y,X^\nu)} \leq K_2^- \lambda^{k-l} \quad \text{for all } k < l.$$

Proof. Since B is a positive operator, there exists an $a > 0$ such that $\Re\lambda \geq a$ for all $\lambda \in \sigma(B)$. Hence, we have $[0, H] \subseteq \rho(B)$, provided that $H > 0$ is chosen according to $aH < 1$. This implies the well-definedness of $A(k), K(k) \in L(Y)$ for all $k \in \mathbb{Z}$. Our further proof is subdivided into two parts. Thereto, we choose $\delta \in [h, H]$, define $A_\delta := K_\delta := [I_Y + \delta B]^{-1}$ and consider the autonomous equation

$$y' = A_\delta y. \tag{5.3}$$

(I) In this step we progressively verify that (5.3) satisfies the dichotomy estimates (2.6)–(2.9). As a first observation, the spectral mapping theorem (cf., e.g., [Con90, p. 204]) implies

$$\sigma(A_\delta) \setminus \{0\} = \left\{(1+\delta\lambda)^{-1} \in \mathbb{C} : \lambda \in \sigma_-\right\} \dot\cup \left\{(1+\delta\lambda)^{-1} \in \mathbb{C} : \lambda \in \sigma_+\right\}.$$

Now we investigate the consequences of (5.2) to this spectral decomposition for $A_\delta \in L(Y)$. We observe that the mapping $\Re_\delta : \mathbb{C} \to \mathbb{R}$, $\Re_\delta(z) := \frac{|1+\delta z|-1}{\delta}$ satisfies $\lim_{\delta \searrow 0} \Re_\delta(z) = \Re z$ and $\Re z \leq \Re_\delta z$ for all $z \in \mathbb{C}$. Therefore, with $\alpha' \in \left(\sup_{\lambda\in\sigma_-} \Re\lambda, \alpha\right)$ we obtain from the compactness of σ_- that there exists a small $H > 0$ (and in turn a small $0 < \delta \leq H$) such that $\sup_{\lambda\in\sigma_-} \Re_\delta(z) < \alpha'$ and consequently

$$(1 + \delta\alpha)^{-1} < (1 + \delta\alpha')^{-1} \leq \inf_{\lambda\in\sigma_-} |1 + \delta\lambda|^{-1}.$$

On the other hand, with $\beta' \in \left(\beta, \inf_{\lambda\in\sigma_+} \Re\lambda\right)$ one has $\beta' < \Re\lambda \leq \Re_\delta(\lambda)$ for all $\lambda \in \sigma_+$, which implies

$$\sup_{\lambda\in\sigma_+} |1 + \delta\lambda|^{-1} \leq (1 + \delta\beta')^{-1} < (1 + \delta\beta)^{-1}.$$

Having these two relations on $\sigma(A_\delta)$ at hand, an equivalent re-norming of the Banach space X^ν (cf., for instance, [Ioo79, p. 6, Technical lemma 1]) implies two dichotomy estimates

$$\left\|A_\delta^k Q_+\right\|_{L(X^\nu)} \leq (1 + \delta\beta)^{-k} \quad \text{for all } k \geq 0, \qquad \left\|A_\delta^k Q_-\right\|_{L(X^\nu)} \leq (1 + \delta\alpha)^{-k} \quad \text{for all } k \leq 0,$$

representing autonomous formulations of (2.6), (2.8), respectively. A further consequence of the spectral decomposition (5.2) are the estimates (cf. [SY02, p. 97, Theorem 37.5])

$$\left\|e^{-Bt}Q_+\right\|_{L(Y,X^\nu)} \leq Ct^{-\nu}e^{-\beta t} \quad \text{for all } t > 0, \qquad \left\|e^{-Bt}Q_-\right\|_{L(Y,X^\nu)} \leq Ce^{-\alpha t} \quad \text{for all } t < 0 \tag{5.4}$$

with some $C > 0$, and referring to the definition of powers of operators (cf. [SY02, p. 95, (37.8)]) one has

$$\left\|[I_Y - \delta(-B)]^{-(k+1)} Q_+ y\right\|_{X^\nu} = \frac{1}{k!\delta^{k+1}} \left\|\int_0^\infty s^k e^{-s/\delta} e^{-Bs} Q_+ y \, ds\right\|_{X^\nu}$$

$$\stackrel{(5.4)}{\leq} \frac{C}{k!\delta^{k+1}}\int_0^\infty s^{k-\nu}e^{-s/\delta-\beta s}\,ds\,\|y\|_Y \leq 2C\delta^{-\nu}\frac{\Gamma(k+1-\nu)}{k!}\,(1+\delta\beta)^{-k}\,\|y\|_Y \quad \text{for all } k\geq 0,\ y\in Y.$$

To simplify this expression we observe that well-known properties of the Gamma function imply

$$\Gamma(x) \leq \frac{(k+1)^x(k+1)!}{x(x+1)\cdots(x+k+1)}\frac{x+k+1}{k+1} \quad \text{for all } x>0,\ k\in\mathbb{Z}_0^+$$

and after an easy computation we arrive at the desired relation

$$\left\|A_\delta^k K_\delta Q_+\right\|_{L(Y,X^\nu)} \leq \frac{2C}{1-\nu}\delta^{-\nu}(k+1)^{-\nu}\,(1+\delta\beta)^{-k} \quad \text{for all } k\in\mathbb{Z}_0^+,$$

which represents (2.7). Similarly one deduces an autonomous version of the estimate (2.9). Therefore, we have shown that (5.3) satisfies Hypothesis $(H)_1$, provided $0<\delta\leq H$ are sufficiently small.

(II) In the above first step we derived the assertion for constant step sizes $\delta>0$. Now we demonstrate how to get rid of this limitation. The resolvent map $R(\cdot,-B)$ of $-B$ is related to A_δ via

$$A_\delta = \delta^{-1}R(\delta^{-1},-B) = I_Y - BR(\delta^{-1},-B) \tag{5.5}$$

and since B is a positive sectorial operator there exists a constant $M>0$ (cf. [SY02, p. 78]) such that

$$\left\|R(\delta^{-1},-B)\right\|_{L(X^\nu)} \leq \frac{M}{|\delta^{-1}+a|} \leq HM, \qquad \|A_\delta\|_{L(X^\nu)} \leq M \quad \text{for all } \delta\in[0,H]\,.$$

Using (5.5) and the resolvent equation $R(\lambda,-B)-R(\mu,-B) = (\mu-\lambda)R(\lambda,-B)R(\mu,-B)$ this implies the estimate $\|A_\delta - A_{\bar\delta}\|_{L(X^\nu)} \leq M(1+M)\big(\frac{H}{h}-1\big)$ for all $\delta,\bar\delta\in[h,H]$ and, note $A(k)=A_{t_{k+1}-t_k}$, also

$$\|A(k)-A_\delta\|_{L(X^\nu)} \leq M(1+M)\big(\tfrac{H}{h}-1\big) \quad \text{for all } k\in\mathbb{Z},\ \delta\in[h,H]\,.$$

Thus, choosing the parameter $q<1$ (and in turn the quotient $\frac{H}{h}>1$) close to 1, we can make the difference $\|A(k)-A_\delta\|_{L(X^\nu)}$ arbitrarily small. Then a roughness theorem for exponential dichotomies as in [Hen81, p. 232, Theorem 7.6.7] implies that also (2.2) admits an exponential dichotomy as in $(H)_1$. Finally, the relation $\|P_+(k)-Q_+\| = O(H)$ as $H\to 0$ is a byproduct of the above roughness theorem for discrete dichotomies. □

Let $\tau\in\mathbb{R}$ and $\Omega\subseteq\mathbb{R}$ be a bounded interval of length $|\Omega|$, on which an Allen-Cahn equation

$$\begin{aligned} u_t - \Delta u + g(t,x,u) = 0 &\quad \text{in } (\tau,\infty)\times\Omega,\\ u|_{t=\tau} = u_0, \quad u = 0 &\quad \text{on } (\tau,\infty)\times\partial\Omega \end{aligned} \tag{5.6}$$

is considered in [EMR90]. The above remarks on (5.1) apply when B is the negative Laplacian $B=-\Delta$ on $Y=L^2(\Omega)$. In this case, the spectrum of B consists of real simple eigenvalues $\nu_n = \frac{\pi^2}{|\Omega|^2}n^2$ satisfying

$$0<\nu_1\leq\nu_2\leq\ldots, \qquad \lim_{n\to\infty}\nu_n=\infty;$$

thus, we can choose spectral sets $\sigma_- = \{\nu_1,\ldots,\nu_n\}$ and $\sigma_+ = \{\nu_{n+1},\nu_{n+2},\ldots\}$ for some $n\in\mathbb{N}$. The appropriate interpolation space is $X^{1/2}=H_0^1(\Omega)$, and concerning the nonlinearity we impose $g:\mathbb{R}\times\Omega\times\mathbb{R}\to\mathbb{R}$ is continuous, D_3g exists, is continuous and for every bounded $I\subset\mathbb{R}$ the set $D_3g(\mathbb{R}\times\Omega\times I)$ is bounded, and there exist constants $c_1,\ldots,c_5>0$, $p>2$ and an increasing function $a_0:[0,\infty)\to[0,\infty)$ such that

$$\limsup_{|u|\to\infty}\frac{g(t,x,u)}{u}\geq 0, \qquad \operatorname{sgn} u\; g(t,x,u)\geq c_1|u|^{p-1}-c_2,$$

$$|g(0,x,u)|\leq c_3|u|^{p-1}+c_4, \qquad |g(t,x,u)|\leq a_0(|u|)$$

and $u\mapsto g(t,x,u)+c_5u$ is increasing.

Given $\kappa\in\mathbb{Z}$ and an initial value u_0, as temporal discretization of (5.6) we consider the following recursion

$$\frac{u^{k+1}-u^k}{t_{k+1}-t_k} - \Delta u^{k+1} + g(t,x,u^{k+1}) = 0 \quad \text{in } \Omega, \tag{5.7}$$

$$u^\kappa = u_0, \quad u^k = 0 \quad \text{on } \partial\Omega,$$

which can be brought into the form (2.1) with

$$A(k) = K(k) := \left[I - (t_{k+1} - t_k)\Delta\right]^{-1}, \qquad \big(F(k, y')\big)(x) := \tfrac{t_{k+1}-t_k}{\theta} g(t_{k+1}, x, y')$$

and constant state spaces $X_k \equiv H_0^1(\Omega)$, $Y_k \equiv L^2(\Omega)$ for $k \in \mathbb{Z}$. We gradually verify that the assumptions of Theorem 4.1 hold and begin with $(H)_0$.

Lemma 5.2. *If the estimate $H < \min\{1, c_5\}$ holds, then* (2.1) *is well-defined on $\mathbb{Z} \times H_0^1(\Omega)$ and the general solution $\varphi(k; \kappa, \cdot) : H_0^1(\Omega) \to H_0^1(\Omega)$, $\kappa \le k$, is continuous.*

Proof. The constant step-size proof in [EMR90, Theorem 4.3] lifts to our time-dependent setting. □

Lemma 5.3. *The nonlinearity $F : \mathbb{Z} \times H_0^1(\Omega) \to L^2(\Omega)$ is well-defined and there exists a $L : [0, \infty) \to [0, \infty)$ such that for all $r > 0$ one has*

$$\|F(k,u) - F(k,v)\|_{L^2(\Omega)} \le L(r) \left|\tfrac{H}{\theta}\right| \|u - v\|_{H_0^1(\Omega)} \quad \textit{for all } k \in \mathbb{Z},\ u, v \in B_r(H_0^1(\Omega)).$$

Proof. Thanks to the continuous embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$, we can proceed as in [SY02, p. 270ff]. □

Lemma 5.4. *There exists a real number $\rho > 0$ such that for all bounded nonautonomous sets $\mathcal{B} \subseteq \mathbb{Z} \times H_0^1(\Omega)$ and $H < \min\{1, c_5\}$ there is an $M(H) \in \mathbb{Z}_0^+$ such that*

$$\varphi(k; k - m, \mathcal{B}(k - m)) \subseteq B_\rho(H_0^1(\Omega)) \quad \textit{for all } k \in \mathbb{Z},\ m \ge M(H).$$

Proof. The proof for constant step-sizes $t_{k+1} - t_k$ in [EMR90, Lemma 7.6] is based on techniques like the discrete uniform Gronwall inequality (cf. [EMR90, Lemma 8.2]), which carry over to our setting. □

Having all preparations collected we arrive at the following discrete inertial manifold theorem:

Theorem 5.5 (discrete Allen-Cahn equation)**.** *Choose $\omega \in (0,1)$ and $N \in \mathbb{N}$. There exist $H_0 < \min\{1, c_5\}$ and $q \in (\omega, 1)$ such that for $H \in (0, H_0]$ and $h \in (qH, H]$ the following holds: If we choose Λ_n, λ_n according to*

$$(1 + h\nu_{n+1})^{-1} < \Lambda_n < \lambda_n < (1 + H\nu_n)^{-1} \tag{5.8}$$

and if there exists a positive integer $n \le N$ satisfying the following spectral gap condition

$$HL(\rho)\left(\frac{2MK_2^-}{\lambda_n - \Lambda_n} + \frac{K_3^+}{\sqrt{\omega H}} \operatorname{Li}_{1/2}\left(\tfrac{2\Lambda_n}{\Lambda_n + \lambda_n}\right) + \max\left\{\frac{2MK_2^-}{\lambda_n - \Lambda_n}, \frac{K_3^+}{\sqrt{\omega H}} \operatorname{Li}_{1/2}\left(\tfrac{2\Lambda_n}{\Lambda_n + \lambda_n}\right)\right\}\right) < \frac{1}{\Lambda_n + \lambda_n}, \tag{5.9}$$

then the temporal discretization (5.7) *of the Allen-Cahn equation* (5.6) *possesses a n-dimensional inertial fiber bundle $\mathcal{W} \subseteq \mathbb{Z} \times H_0^1(\Omega)$ as in Theorem 4.1.*

Proof. The proof verifies the assumptions of Theorem 4.1, where we set $\theta = H$. Above all, the inequalities (4.1) and (4.3) are only imposed to guarantee that the modified equation (4.8) is well-defined (and continuous) on $\mathbb{Z} \times H_0^1(\Omega)$. Since (5.7) is an implicit Euler discretization of (5.6), this fact can be seen similarly to [BG99, Lemma 3.6] – provided H_0 is small. The remaining hypothesis of Theorem 4.1 hold as follows:

Lemma 5.2 guarantees that $(H)_0$ is true. It is evident that for $q < 1$ sufficiently close to 1 one can choose real numbers λ_n, Λ_n as in (5.8). Thus, referring to Lemma 5.1, we know that also $(H)_1$ is satisfied with growth rates $\Lambda_n < \lambda_n$ for $n \le N$; in particular, due to $\omega H < h$ one has

$$\|\Phi(k,l)P_+(l)K(l)\|_{L(L^2(\Omega), H_0^1(\Omega))} \le K_3^+(\omega H)^{-1/2}(k - l + 1)^{-1/2}\Lambda_n^{k-l} \quad \text{for all } l \le k.$$

Our Lemma 5.3 implies the Lipschitz estimates needed in $(H)_4$, where (2.10) follows from $|g(t,x,0)| \le a_0(0)$ for all $t \in \mathbb{R}$, $x \in \Omega$. Finally, Lemma 5.4 yields the pullback dissipativity $(H)_5$. In addition, for sufficiently small $H_0 > 0$ the estimates (4.4) and (4.6) hold true. Then it is easy to see that (5.9) implies (2.12). □

5.2. **Fully discretized complex Ginzburg-Landau equation.** Consider the nonautonomous complex Ginzburg-Landau equation with cubic nonlinearity satisfying 1-periodic boundary conditions

$$\begin{aligned} u_t - \mu_1(t)u + (1+i\nu)u_{xx} - (1+i\mu_2(t))\,|u|^2\,u = 0 \quad &\text{in } (\tau,\infty)\times\mathbb{R}, \\ u|_{t=\tau} = u_0, \quad u(t,x) = u(t,x+1) \quad &\text{on } (\tau,\infty)\times\mathbb{R}, \end{aligned} \tag{5.10}$$

and a given initial time $\tau \in \mathbb{R}$. The instability parameter $\mu_1 : \mathbb{R} \to \mathbb{R}$, as well as the dispersion parameter $\mu_2 : \mathbb{R} \to \mathbb{R}$ are assumed to be continuous functions satisfying

$$\mu_1(t) \in (R_0, R_1]\,, \qquad \mu_2(t) \in [-R_2, R_2] \quad \text{for all } t \in \mathbb{R}$$

with bounds $R_0, R_1, R_2 > 0$, and $\nu \in \mathbb{R}$. It is shown in [CV01, p. 118] that the problem (5.10) is well-posed in the space L^2, has regular global solutions and a uniform attractor (under the assumption $R_2 \le \sqrt{3}$).

Let us turn to an appropriate full discretization of the initial-boundary value problem (5.10). Here we largely benefit from — and closely follow — previous work in [Lor97], who considered the autonomous case of (5.10) and constant step-size schemes. Concerning the spatial discretization we subdivide the periodicity interval $[0,1]$ into $N \ge 3$ uniform subintervals of length $1/N$. Thus, the state space for a finite-difference approximation of (5.10) respecting periodic boundary conditions is the set of N-periodic sequences in $\mathbb{C}$, which will be canonically identified with $\mathbb{C}^N$. On this set we introduce the difference operators δ_-, δ_+ with components

$$\delta_+ x_j := \begin{cases} x_{j+1} - x_j & \text{for } j = 1,\dots,N-1 \\ x_1 - x_N & \text{for } j = N \end{cases}, \qquad \delta_- x_j := \begin{cases} x_1 - x_N & \text{for } j = 1 \\ x_{j-1} - x_j & \text{for } j = 2,\dots,N \end{cases},$$

and a finite-difference version of the second order spatial derivative is given by the product $\delta_+\delta_- : \mathbb{C}^N \to \mathbb{C}^N$ with corresponding matrix representation

$$A_N := N^2 \begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{N\times N},$$

where entries not explicitly stated are assumed to be zero. This reduces the initial-boundary value problem (5.10) to a finite-dimensional problem in the space $\mathbb{C}^N$, endowed with the weighted inner product

$$\langle x, y\rangle := \frac{1}{N}\sum_{j=1}^{N} x_i \bar{y}_i \quad \text{for all } x, y \in \mathbb{C}^N.$$

Lemma 5.6 (properties of A_N)**.** *The matrix $A_N \in \mathbb{R}^{N\times N}$ has eigenvalues $\nu_n \in \mathbb{R}$ with corresponding eigenvectors $e_n \in \mathbb{C}^N$ given by*

$$\nu_n = 4N^2 \sin^2\left(\tfrac{n\pi}{N}\right), \qquad e_n = \left(\exp\left(2\pi \tfrac{ijn}{N}\right)\right)_{j=0}^{N-1} \quad \textit{for all } n = 1,\dots,N,$$

respectively, where $e_1,\dots,e_N$ are orthonormal w.r.t. $\langle\cdot,\cdot\rangle$. Moreover, one has:

(a) $0 < \nu_n < \nu_{n+1}$ *for all* $n = 1,\dots,\left\lfloor \frac{N}{2}\right\rfloor$,

(b) $\nu_{n+1} - \nu_n \ge 2\sqrt{3}N^2 \sin\left(\frac{\pi}{N}\right)$, *if* $N \ge 5$ *and* $n \ge 1$ *with* $\frac{N-3}{6} \le n \le \frac{2N-3}{6}$.

Proof. Concerning the eigenvalues ν_n and pairwise orthogonal eigenvectors e_n, we refer to the reference given in [Lor97, Lemma 1.1]. Assertion (a) is a direct consequence of monotonicity properties for the sine function. In order to prove (b), we remark that elementary trigonometric identities yield

$$\nu_{n+1} - \nu_n = 4N^2 \sin\left(\frac{(2n+1)\pi}{N}\right) \sin\left(\frac{\pi}{N}\right) \quad \text{for all } n = 1,\dots,N-1$$

and that we have the estimate $\sin x \geq \frac{\sqrt{3}}{2}$ for all $x \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$. Provided $N \geq 5$ there exist positive integers n such that $\frac{\pi}{3} \leq \frac{(2n+1)\pi}{N} \leq \frac{2\pi}{3}$ and the claim follows. □

Having such quantitative information on the matrix A_N at hand, we can proceed by establishing an ambient space setting for our discretization scheme. Since $\{e_1, \dots, e_N\}$ is an orthonormal basis of $\mathbb{C}^N$ one can define discrete Sobolev spaces H_N^{2s} as follows: We equip $\mathbb{C}^N$ with inner products

$$\langle x, y\rangle_{H_N^{2s}} := \sum_{j=1}^N (1+\nu_j)^{2s}\langle x, e_j\rangle\overline{\langle y, e_j\rangle} \quad \text{for all } s > 0,\ x \in \mathbb{C}^N,$$

set $L_N^2 := H_N^0$ and remark that $\|\cdot\|_{L_N^2}$ is related to the above inner product on $\mathbb{C}^N$ and $\|\cdot\|_{H_N^1}$ via

$$\|x\|_{L_N^2} = \sqrt{\langle x, x\rangle}, \qquad \|x\|_{L_N^2} \leq \|x\|_{H_N^1} \quad \text{for all } x \in \mathbb{C}^N, \tag{5.11}$$

respectively. If we introduce a semi-norm $|x|_{A_N} := \sqrt{\langle x, A_N x\rangle}$ on $\mathbb{C}^N$, then the H_N^1-norm satisfies

$$\|x\|_{H_N^1} = \sqrt{\|x\|_{L_N^2}^2 + |x|_{A_N}^2} \quad \text{for all } x \in \mathbb{C}^N \tag{5.12}$$

and using the above difference operators δ_-, δ_+ one easily deduces the relation

$$\langle x, A_N x\rangle = -\frac{1}{N}\sum_{j=1}^N x_j\overline{\delta_+\delta_- x_j} = \frac{1}{N}\sum_{j=1}^N \delta_+ x_j\overline{\delta_+ x_j} \quad \text{for all } x \in \mathbb{C}^N. \tag{5.13}$$

Since the nonlinear term in (5.10) does not contain spatial derivatives, define $G_N : \mathbb{C}^N \to \mathbb{C}^N$ by

$$G_N(x) := \left(|x_1|^2 x_1, \dots, |x_N|^2 x_N\right)^T.$$

Our full discretization of (5.10) will consist of such a finite-difference approximation in space (represented by the matrix A_N), which leads to the nonautonomous ODE in $\mathbb{C}^N$,

$$\dot{x} + \tilde{A}_N x = G(t, x) \tag{5.14}$$

and a fully implicit variable step-size Euler method for (5.14), i.e., we arrive at the implicit recursion

$$\frac{x^{k+1} - x^k}{t_{k+1} - t_k} + \tilde{A}_N x^{k+1} = G(t_{k+1}, x^{k+1}) \tag{5.15}$$

with $\tilde{A}_N := (1 + i\nu)\left[I_{\mathbb{C}^N} + A_N\right]$ and $G(t, x) := (\mu_1(t) + 1 + i\nu)x - (1 + i\mu_2(t))G_N(x)$. In order to embed this into our notational framework given by (2.1) we write (5.15) in the familiar form

$$x' = A(k)x + \theta K'(k)F(k, x') \tag{5.16}$$

with abbreviations $K(k) :\equiv I_{\mathbb{C}^N}$,

$$A(k) := \left[I_{\mathbb{C}^N} + (t_{k+1} - t_k)\tilde{A}_N\right]^{-1}, \qquad F(k, x') := \frac{t_{k+1} - t_k}{\theta}\left[I_{\mathbb{C}^N} + (t_{k+1} - t_k)\tilde{A}_N\right]^{-1} G(t_{k+1}, x').$$

Lemma 5.7. *The mapping $A : \mathbb{Z} \to \mathbb{C}^{N\times N}$ is well-defined, has invertible values and with complementary orthogonal projections $P_n^\pm \in \mathbb{C}^{N\times N}$ given by*

$$P_n^+ x := \sum_{j=n+1}^{N-(n+1)} \langle x, e_j\rangle e_j, \qquad P_n^- x := x - P_n x$$

one has the following properties for all integers $k \in \mathbb{Z}$, $n = 1, \dots, \left\lfloor \frac{N-2}{2} \right\rfloor$:

(a) $A(k)P_n^\pm = P_n^\pm A(k)$,

(b) $\|A(k)P_n^+\|_{L(H_N^1)} \leq |1 + (1 + i\nu)(t_{k+1} - t_k)(1 + \nu_{n+1})|^{-1}$,

(c) $\|A(k)P_n^-\|_{L(H_N^1)} \geq |1 + (1 + i\nu)(t_{k+1} - t_k)(1 + \nu_n)|^{-1}$.

Proof. Let $k \in \mathbb{Z}$ and n be an integer with $n = 1, \dots, \lfloor \frac{N-2}{2} \rfloor$. By the spectral mapping theorem (cf., e.g., [Con90, p. 204]) one derives the explicit relation $\sigma(A(k)) = \{\upsilon_j(k) \in \mathbb{C} : j = 1, \dots, N\}$ with

$$\upsilon_j(k) := [1 + (t_{k+1} - t_k)(1 + i\nu)(1 + \nu_j)]^{-1}$$

and consequently $0 \notin \sigma(A(k))$. Thus, $A(k) \in \mathbb{C}^{N \times N}$ is an invertible matrix. For later use we introduce the discrete sets $\mathbb{I}_n^+ := \{n+1, \dots, N-(n+1)\}$, $\mathbb{I}_n^- := \{1, \dots, N\} \setminus \mathbb{I}_n^+$ and choose $x \in \mathbb{C}^N$ with $x = \sum_{j=1}^N x_j e_j$ and $x_j = \langle x, e_j \rangle$. We instantly obtain assertion (a) from

$$A(k)P_n^{\pm} x = \sum_{j \in \mathbb{I}_n^{\pm}} x_j A(k) e_j = \sum_{j \in \mathbb{I}_n^{\pm}} \upsilon_j(k) x_j e_j = P_n^{\pm} \sum_{j=1}^N x_j \upsilon_j(k) e_j = P_n^{\pm} A(k) \sum_{j=1}^N x_j e_j = P_n^{\pm} A(k) x.$$

In addition, due to $|\upsilon_j(k)| \leq |\upsilon_{n+1}(k)|$ for all $j \in \mathbb{I}_n^+$ one has the forward estimate

$$\left\| A(k) P_n^+ x \right\|_{H_N^1}^2 = \sum_{j \in \mathbb{I}_n^+} (1 + \nu_j) |\upsilon_j(k)|^2 |x_j|^2 \leq |\upsilon_{n+1}(k)|^2 \|x\|_{H_N^1}^2 ,$$

which implies (b), and our claim (c) follows from the corresponding backward estimate

$$\left\| A(k) P_n^- x \right\|_{H_N^1}^2 = \sum_{j \in \mathbb{I}_n^-} (1 + \nu_j) |\upsilon_j(k)|^2 |x_j|^2 \geq |\upsilon_n(k)|^2 \|x\|_{H_N^1}^2 ,$$

since we have $|\upsilon_j(k)| \geq |\upsilon_n(k)|$ for all $j \in \mathbb{I}_n^-$. □

After these preparations we verify the assumptions of Theorem 4.1 step by step. First of all, we need to investigate wether (5.16) is well-defined.

Lemma 5.8. *Choose $\rho \in \mathbb{R}$ so large that*

$$\rho > 2\sqrt{R_1} \exp\left[2R_1 + 216(1+\nu^2)(1 + 216(1+\nu^2)R_1^2)R_1^2\right]. \tag{5.17}$$

Then for all $(\kappa, \xi) \in \mathbb{Z} \times \mathbb{C}^N$ the following holds:

(a) *The difference equation* (5.16) *possesses a forward solution $\phi : \mathbb{Z}_\kappa^+ \to \mathbb{C}^N$ satisfying $\phi(\kappa) = \xi$.*

(b) *There exists an $N_0(\xi) \in \mathbb{Z}_0^+$ such that the solution values $\phi(k)$ are uniquely determined for $k \geq \kappa + N_0$, provided the temporal step-sizes satisfy*

$$2\left(R_1 + 6\sqrt{1+\nu^2}\right)\rho^2 H < 1. \tag{5.18}$$

(c) *For each uniformly bounded set $\mathcal{B} \subseteq \mathbb{Z} \times H_N^1$ there exists an $N_1(\mathcal{B}) \in \mathbb{Z}_0^+$ such that the general solution $\varphi(k; \kappa, \cdot) : \mathcal{B}(\kappa) \to \mathbb{C}^N$ is well-defined and continuous for $k - \kappa \geq N_1(\mathcal{B})$, provided* (5.18) *holds.*

Proof. (a) The existence of forward solutions can be shown using a criterion from [FJ+91, Proposition 4.8]. For details see [Lor97, Lemma 3.1], whose techniques carry over to our time-depending setting.

(b) This can be derived along the lines of [Lor97, Lemma 3.2].

(c) Since H_N^1 is finite-dimensional, this follows from (b). □

Lemma 5.9. *For every $r > 0$ and $u, v \in B_r(H_N^1)$ the nonlinearity $F : \mathbb{Z} \times \mathbb{C}^N \to \mathbb{C}^N$ satisfies*

$$\left\| P_n^{\pm} [F(k,u) - F(k,v)] \right\|_{H_N^1} \leq L(r) \left| \tfrac{H}{\theta} \right| \|u - v\|_{H_N^1} \quad \text{for all } n = 1, \dots, \lfloor \tfrac{N-2}{2} \rfloor,\ k \in \mathbb{Z}$$

with $L(r) := \sqrt{2(1 + R_1^2 + \nu^2) + 360(1 + R_2^2)r^4}$.

Proof. Let $r > 0$, $u, v \in B_r(H_N^1)$ and $n \in \left\{1, \dots, \lfloor \frac{N-2}{2} \rfloor\right\}$. We proceed in two steps:

(I) We begin to derive a Lipschitz condition for the function $G_N : \mathbb{C}^N \to \mathbb{C}^N$. Here, our approach is based on relation (5.12). The mean value inequality leads to

$$\left| |u_j|^2 u_j - |v_j|^2 v_j \right| \leq 2 \sup_{t \in [0,1]} |u_j + t(v_j - u_j)|^2 |u_j - v_j| \quad \text{for all } j = 1, \dots, N.$$

From [Lor97, Lemma 1.2] we borrow the embedding relation $|u_j| \leq \sqrt{3}\,\|u\|_{H_N^1}$ and arrive at

$$\left||u_j|^2 u_j - |v_j|^2 v_j\right| \leq 6r^2 |u_j - v_j| \quad \text{for all } j = 1, \dots, N,\ u, v \in B_r(H_N^1), \tag{5.19}$$

which, in turn, equips us with the first L_N^2-estimate

$$\|G_N(u) - G_N(v)\|_{L_N^2}^2 = \frac{1}{N}\sum_{j=1}^N \left||u_j|^2 u_j - |v_j|^2 v_j\right|^2 \leq 36r^4 \|u - v\|_{L_N^2}^2 \quad \text{for all } u, v \in B_r(H_N^1). \tag{5.20}$$

Moreover, for notational convenience we identify u_{N+1} with u_1 (and v_{N+1} with v_1) to obtain

$$\begin{aligned}
\left|\delta_+(|u_j|^2 u_j) - \delta_+(|v_j|^2 v_j)\right|^2 &\leq \left(\left||u_{j+1}|^2 u_{j+1} - |v_{j+1}|^2 v_{j+1}\right| + \left||u_j|^2 u_j - |v_j|^2 v_j\right|\right)^2 \\
&\overset{(5.19)}{\leq} \left(6r^2 |u_{j+1} - v_{j+1}| + 6r^2 |u_j - v_j|\right)^2 \\
&\leq 72r^4 \left(|u_{j+1} - v_{j+1}|^2 + |u_j - v_j|^2\right) \quad \text{for all } j = 1, \dots, N
\end{aligned}$$

from the elementary inequality

$$(x + y)^2 \leq 2x^2 + 2y^2 \quad \text{for all } x, y \in \mathbb{R}. \tag{5.21}$$

Therefore, with relation (5.13) for the seminorm $|\cdot|_{A_N}$ we get

$$\begin{aligned}
|G_N(u) - G_N(v)|_{A_N}^2 &= \frac{1}{N}\sum_{j=1}^N \delta_+(G_N(u) - G_N(v))_j \overline{\delta_+(G_N(u) - G_N(v))_j}^2 \\
&= \frac{1}{N}\sum_{j=1}^N |\delta_+(G_N(u) - G_N(v))_j|^2 \leq 144r^4 \|u - v\|_{L_N^2}^2
\end{aligned}$$

and combining this with (5.20) we obtain from (5.12) that

$$\|G_n(u) - G_N(v)\|_{H_N^1}^2 \leq 180r^4 \|u - v\|_{L_N^2}^2 \overset{(5.11)}{\leq} 180r^4 \|u - v\|_{H_N^1}^2 \quad \text{for all } u, v \in B_r(H_N^1). \tag{5.22}$$

(II) Now we aim at a Lipschitz estimate for the full nonlinearity F. By definition, adopting the notation from Lemma 5.7 and its proof one has

$$\left\|P_n^{\pm}\left[F(k, u) - F(k, v)\right]\right\|_{H_N^1}^2 \leq \left|\tfrac{H}{\theta}\right|^2 \sum_{j \in \mathbb{I}_n^{\pm}} (1 + \nu_j)\, |v_j(k)|^2\, |\langle G(t_{k+1}, u) - G(t_{k+1}, v), e_j\rangle|^2$$

and referring again to the basic inequality (5.21) we proceed to

$$\begin{aligned}
\left\|P_n^{\pm}\left[F(k, u) - F(k, v)\right]\right\|_{H_N^1}^2 &\leq 2\left|\tfrac{H}{\theta}\right|^2 \left(1 + R_1^2 + \nu^2\right) \sum_{j \in \mathbb{I}_n^{\pm}} (1 + \nu_j)\, |\langle u - v, e_j\rangle|^2 \\
&\quad + 2\left|\tfrac{H}{\theta}\right|^2 \left(1 + R_2^2\right) \sum_{j \in \mathbb{I}_n^{\pm}} (1 + \nu_j)\, |\langle G_N(u) - G_N(v), e_j\rangle|^2 \\
&\overset{(5.22)}{\leq} 2\left|\tfrac{H}{\theta}\right|^2 \left[(1 + R_1^2 + \nu^2) + 180(1 + R_2^2)r^4\right] \|u - v\|_{H_N^1}^2 .
\end{aligned}$$

Taking the square root of this estimate yields the assertion. □

Lemma 5.10. *Suppose $\rho \in \mathbb{R}$ satisfies* (5.17). *If the temporal step-sizes satisfy*

$$4R_1\left[1 + 108(1 + \nu^2)(1 + 216(1 + \nu^2)R_1^2)R_1\right] H < 1, \tag{5.23}$$

then the nonautonomous set $\mathbb{Z} \times \bar{B}_\rho(H_N^1)$ is uniformly pullback absorbing in H_N^1 and positively invariant.

Proof. Let $(\kappa,\xi)\in\mathbb{Z}\times\mathbb{C}^N$. It can be shown as in [Lor97, Theorem 3.2] that there exists an $N_1(\xi)\in\mathbb{Z}_0^+$ such that a solution ϕ of (5.16) exists with $\phi(\kappa)=\xi$ and $\phi(k)\in B_\rho(H_N^1)$ for $k\geq\kappa+N_1$. Since the integer N_1 depends only on the H_N^1-norm of ξ, this implies our claim. □

After all these preparations we eventually arrive at

Theorem 5.11 (discrete Ginzburg-Landau equation)**.** *Choose $\omega\in(0,1)$ fixed, the radius of the absorbing ball $\rho>0$ so large that* (5.17) *holds, $N\geq 5$ so large that*

$$N^2\sin\left(\tfrac{\pi}{N}\right)\geq\tfrac{2\sqrt{3}}{\omega}L(\rho) \tag{5.24}$$

and define $n:=\left\lceil\frac{N-3}{6}\right\rceil$. In addition, choose

$$q\in\left(\max\left\{\omega,\tfrac{1+\nu_n}{1+\nu_{n+1}},\tfrac{2\nu_n}{\nu_n+\nu_{n+1}},\sqrt{\tfrac{\nu+\nu_n^2}{\nu+\nu_{n+1}^2}}\right\},1\right] \tag{5.25}$$

and $H>0$ so small that beyond (5.18)*,* (5.23) *also the estimates*

$$L(\rho)H<1,\qquad\qquad qH\leq\tfrac{\sqrt{\omega^{-1}(1+\nu_{n+1})^2+\nu^2}-1-\nu_{n+1}}{(1+\nu_{n+1})^2+\nu^2} \tag{5.26}$$

are satisfied. Then for temporal step-sizes with $h\in[qH,H]$ the following holds:

(a) *The full finite difference discretization* (5.16) *of the complex Ginzburg-Landau equation* (5.10) *possesses a $(2n+1)$-dimensional inertial fiber bundle $\mathcal{W}\subseteq\mathbb{Z}\times\bar{B}_\rho(H_N^1)$ as in Theorem 4.1,*
(b) *there exists a unique pullback attractor $\mathcal{A}^*$ for* (5.16)*, which satisfies $\mathcal{A}^*\subseteq\mathcal{W}$,*

where the constant $L(\rho)>0$ is defined in Lemma 5.9.

Remark 5.1. The formulation of Theorem 5.11 is quantitative in the sense that the fiber-dimension of the inertial fiber bundle $\mathcal{W}$ can actually be computed for given values of ν and bounds $R_1,R_2>0$. Figure 1 illustrates the dependence of $\dim\mathcal{W}(k)$ on these parameters. Related estimates for the continuous problem (5.10) (and constant functions μ_1,μ_2) are given in [DLT96].

Proof. First of all, Lemma 5.6(a) ensures that the interval for the balancing parameter q in (5.25) in nonempty. Unfortunately, we cannot apply Theorem 4.1 directly, since the forward solutions of (5.16) need not to be unique in forward time (cf. Lemma 5.8). However, this problem can be circumvented as follows:

Choose $\rho>0$ so large that (5.17) holds. We modify the nonlinearity of (5.16) as in the proof of Theorem 4.1 and directly employ Proposition 2.1 and Theorem 2.5 to the modified equation

$$x'=A(k)x+\theta K'(k)F_\rho(k,x'). \tag{5.27}$$

Thereto, we verify the corresponding assumptions with constant state spaces $X_k=Y_k=H_N^1$ for an appropriate spatial discretization with $N\geq 5$. Since H_N^1 is a Hilbert space, we are in the scope of Remark 4.1(1).

$(H)_0$ For $H>0$ so small that the left estimate in (5.26) holds, we prove just as in Theorem 4.1 that the general solution of (5.27) is well-defined (and continuous) on $\mathbb{Z}\times H_N^1$. Note that $\operatorname{Lip}_2 F_\rho\leq L(\rho)$ due to Lemma 5.9.

$(H)_1$ We define constant projectors $P_\pm(k):\equiv P_n^\pm$ for some positive integer $n\leq\frac{N-2}{2}$ and observe using Lemma 5.7(a) that (2.3)–(2.5) hold with $\bar{C}=1$. Introducing the growth rates

$$\Lambda_n:=\left|1+(1+i\nu)h(1+\nu_{n+1})\right|^{-1},\qquad\qquad\lambda_n:=\left|1+(1+i\nu)H(1+\nu_n)\right|^{-1},$$

an elementary computation shows $\Lambda_n<\lambda_n$, provided the step-sizes are balanced according to (5.25). Then Lemma 5.7 implies the desired dichotomy estimates

$$\|\Phi(k,l)P_+(l)\|_{L(H_N^1)}\leq\prod_{j=l}^{k-1}\left\|A(j)P_n^+\right\|_{L(H_N^1)}\leq\Lambda_n^{k-l}\quad\text{for all } l\leq k$$

and also $\|\Phi(k,l)P_-(l)\|_{L(H_N^1)}\leq\prod_{j=k}^{l-1}\|A(j)P_n^+\|_{L(H_N^1)}\leq\lambda_n^{k-l}$ for all $k\leq l$, verifying $(H)_1$.

$(H)_2$ Clearly, $F(k,0)\equiv 0$ on $\mathbb{Z}$ implies (2.10) and Lemma 5.9 yields (2.11) with $L_3^\pm=L(\rho)\left|\frac{H}{\theta}\right|$.

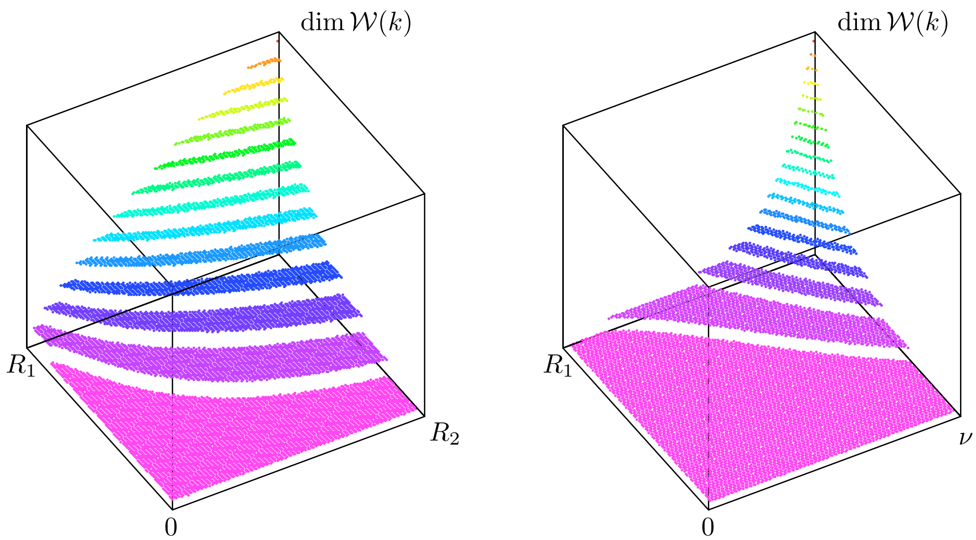

FIGURE 1. Dimension of the inertial fiber bundle $\mathcal{W}$ from Theorem 5.11: Left: $\dim\mathcal{W}(k)$ over (R_1,R_2)-plane for $\nu=1$, $R_1\in[0.001,0.05]$, $R_2\in[0.25,15]$ yielding $3\leq\dim\mathcal{W}(k)\leq 29$. Right: $\dim\mathcal{W}(k)$ over (R_1,ν)-plane for $\nu\in[0.0125,1.5]$, $R_1\in[0.02,0.04]$ and $R_2=1$ yielding $3\leq\dim\mathcal{W}(k)\leq 29$.

The remaining goal for an application of Theorem 2.5 is to verify the estimate (2.12), which reduces to

$$3(\lambda_n-\bar\sigma)L(\rho)H<\bar\sigma \quad\text{for all } \bar\sigma\in\left(\sigma,\tfrac{\lambda_n-\Lambda_n}{2}\right). \tag{5.28}$$

Our approach is to show that there exist large N,n such that the spectral gap condition (5.28) holds. We begin with a preparatory elementary estimate. For real numbers $0\leq\chi\leq\psi\leq\omega^{-1}-1$ one has $\sqrt{1+\chi}\leq\sqrt{1+\psi}\leq\omega^{-1/2}$, thus $\sqrt{1+\psi}(\sqrt{1+\psi}+\sqrt{1+\chi})\leq\frac{2}{\omega}$ and this implies

$$\frac{\omega}{2}(\psi-\chi)\leq\frac{\psi-\chi}{\sqrt{1+\psi}(\sqrt{1+\psi}+\sqrt{1+\chi})}=1-\frac{\sqrt{1+\chi}}{\sqrt{1+\psi}}\quad\text{for all } 0\leq\chi\leq\psi\leq\omega^{-1}-1. \tag{5.29}$$

Abbreviating $\chi:=2\tau_n+\tau_n^2+\nu^2H^2$ and $\psi:=2q\tau_{n+1}+q^2\tau_{n+1}^2+q^2\nu^2H^2$ we obtain $0<\chi<\psi$ from $q>\frac{1+\nu_n}{1+\nu_{n+1}}$ (cf. (5.25)). Moreover, we get from the right relation in (5.26) that $\psi\leq\omega^{-1}-1$, thus

$$\frac{\lambda_n-\Lambda_n}{\lambda_n}=1-\frac{\sqrt{1+\chi}}{\sqrt{1+\psi}}\overset{(5.29)}{\geq}\frac{\omega}{2}(\psi-\chi).$$

For the difference $\psi-\chi$ one can establish the relation $\psi-\chi\geq 2H(q\nu_{n+1}-\nu_n)$, provided $q^2\nu_{n+1}^2-\nu_n^2\geq\nu(1-q^2)$ holds, but this is given by (5.25). The assumption $q>\frac{2\nu_n}{\nu_n+\nu_{n+1}}$ (cf. (5.25)) implies $q\nu_{n+1}-\nu_n\geq\frac{q}{2}(\nu_{n+1}-\nu_n)$ and combining the above three estimates finally yields

$$\frac{\lambda_n-\Lambda_n}{H\lambda_n}>\frac{q\omega}{2}(\nu_{n+1}-\nu_n)\geq\frac{\omega^2}{2}(\nu_{n+1}-\nu_n)$$

due to $q\geq\omega$ (cf. (5.25)). This brings us into the position to apply Lemma 5.6(c) guaranteeing

$$\frac{\lambda_n-\Lambda_n}{H\lambda_n}>\sqrt{3}\omega^2N^2\sin\left(\tfrac{\pi}{N}\right)\quad\text{for } n=\left\lceil\tfrac{N-3}{6}\right\rceil.$$

Keeping in mind that $\varphi : [2,\infty) \to [4,\infty)$, $\varphi(t) := t^2 \sin\left(\frac{\pi}{t}\right)$ is strictly increasing with $\lim_{t\to\infty} \varphi(t) = \infty$, we are able to fulfill (5.24), i.e., there exists a minimal integer $N \geq 5$ such that

$$\frac{\lambda_n - \Lambda_n}{H\lambda_n} > \frac{\sqrt{3}\omega^2}{2} N^2 \sin\left(\tfrac{\pi}{N}\right) \geq 3L(\rho) \quad \text{for } n = \left\lceil \tfrac{N-3}{6} \right\rceil,$$

this implies $3(\lambda_n - \bar{\sigma})L(\rho)H < 3\lambda_n L(\rho)H \leq \frac{\lambda_n - \Lambda_n}{2}$ for $n = \left\lceil \frac{N-3}{6} \right\rceil$, $\bar{\sigma} \in (0, \lambda_n)$ and consequently there exists a real $\sigma \in \left(\bar{\sigma}, \frac{\lambda_n - \Lambda_n}{2}\right)$ such that (5.28) is satisfied.

(a) Having verified all assumptions of Theorem 2.5 we know that the implicit difference equation (5.27) in $\mathbb{C}^N$ has a global inertial fiber bundle $\tilde{\mathcal{W}}$, if $N \geq 5$ satisfies (5.24). Each fiber $\tilde{\mathcal{W}}(k)$, $k \in \mathbb{Z}$, is a graph over $P_n^- \mathbb{C}^N$ and by definition of the projector P_n^- in Lemma 5.7 we have $\dim \tilde{\mathcal{W}}(k) = \dim P_n^- \mathbb{C}^N = 2n + 1$ for all $k \in \mathbb{Z}$. Since the nonautonomous set $\mathbb{Z} \times \bar{B}_\rho(H_N^1)$ is uniformly pullback absorbing for the original equation (5.16) (see Lemma 5.10), all solutions of (5.16) eventually enter the ball $\bar{B}_\rho(H_N^1)$. We define $\mathcal{W} := \tilde{\mathcal{W}} \cap \mathcal{U}_\rho \subseteq \mathbb{Z} \times H_N^1$ and the positive invariance of $\mathbb{Z} \times \bar{B}_\rho(H_N^1)$ implies that $\mathcal{W}$ is an inertial fiber bundle for (5.16).

(b) Because of Lemma 5.10 there exists a compact pullback absorbing set $\mathbb{Z} \times \bar{B}_\rho(H_N^1)$ for (5.16) and [Kl00, Theorem 3.6] guarantees the existence of unique pullback attractor $\mathcal{A}^* \subseteq \mathbb{Z} \times B_\rho(H_N^1)$. Clearly, $\lambda_n < 1$ and $F(k, 0) \equiv 0$ on $\mathbb{Z}$ implies that Corollary 4.2 can be applied, which yields $\mathcal{A}^* \subseteq \mathcal{W}$.

Therewith, Theorem 5.11 is established. □

Appendix A. An Optimal Cut-Off Function

The existence of smooth cut-off functions is of importance to deduce local results from global ones. To keep the paper self-contained we present a tool aiming for a kind of optimality in their Lipschitz constant.

Lemma A.1. *For every real $s > 1$ there exists a function $\vartheta \in C^\infty(\mathbb{R})$ such that $\vartheta(t) \equiv 1$ on $(-\infty, 1]$, $\vartheta(t) \in [0,1]$ for $t \in [1,2]$, $\vartheta(t) \equiv 0$ on $[2,\infty)$ and $D\vartheta(t) \in [-s, 0]$ for all $t \in \mathbb{R}$.*

Proof. For reals $r > 0$ consider the C^∞-bump function $\omega_r : \mathbb{R} \to \mathbb{R}$,

$$\omega_r(x) := \begin{cases} \exp\left(-\frac{r}{1-4x^2}\right) & \text{for } |x| < \frac{1}{2} \\ 0 & \text{for } |x| \geq \frac{1}{2} \end{cases}$$

(cf. [AMR88, p. 94]). Then $\vartheta_r : \mathbb{R} \to \mathbb{R}$ given by $\vartheta_r(x) := \int_{-\infty}^x \omega_r / \int_{-\infty}^\infty \omega_r$ is an increasing C^∞-function with $\vartheta_r(x) = 0$ for $x \leq -\frac{1}{2}$, $\vartheta_r(x) = 1$ for $x \geq \frac{1}{2}$ and the derivative $D\vartheta_r(x) = \omega_r(x) / \int_{-\infty}^\infty \omega_r$. From the properties of ω_r we see that $\min_{x\in\mathbb{R}} D\vartheta_r(x) = 0$ and $m(r) := \max_{x\in\mathbb{R}} D\vartheta_r(x) = \exp(-r) / \int_{-\infty}^\infty \omega_r$. It is not difficult to prove that $m : (0,\infty) \to \mathbb{R}$ is a strictly increasing continuous function with $\lim_{r\searrow 0} m(r) = 1$. Thus, for every $s > 1$ there exists a $r^* > 0$ such that $m(r^*) \leq s$, therefore $D\vartheta_{r^*}(x) \in [0, s]$ for all $x \in \mathbb{R}$. Then the function ϑ given by $\vartheta(x) := \vartheta_{r^*}(\frac{3}{2} - x)$ satisfies the assertions of our lemma. □

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