Numerical Dynamics of Integrodifference Equations: Basics and Discretization Errors in a C^0-Setting

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Abstract

Besides being interesting infinite-dimensional dynamical systems in discrete time, integrodifference equations successfully model growth and dispersal of populations with nonoverlapping generations, and are often illustrated by simulations. This paper points towards and initiates a mathematical foundation of such simulations using generic methods to numerically discretize (and solve) integral equations. We tackle basic properties of a flexible class of integrodifference equations, as well as of their collocation and degenerate kernel semi-discretizations on the state space of continuous functions over a compact domain. Moreover, various estimates for the global discretization error are provided. Numerical simulations illustrate and confirm our theoretical results.

Keywords: Integrodifference equation, collocation method, degenerate kernel method, piecewise linear splines, global discretization error
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1. Introduction

The numerical analysis of integral equations is a well-studied field (cf. for instance the monographs [1, 2, 5, 11]). It provides efficient algorithms to solve linear and nonlinear equations, as well as eigenvalue problems for various classes of integral operators (Fredholm, Volterra, Hammerstein, Urysohn, etc.). Roughly, these tools allow a classification into Nyström, projection and degenerate kernel methods.

The paper at hand abandons a classical and static perspective of solving, say a fixed integral equation or determining the spectrum of an integral operator numerically. We rather investigate the behavior discrete semi-dynamical systems given by the iterates of nonlinear integral operators — one speaks of *integrodifference equations* – along with their spatial discretization. In particular, the iterates of Hammerstein integral operators serve as successful and increasingly popular models in theoretical ecology (e.g. [10, 14, 9]): If $u : \Omega \to \mathbb{R}$ describes the spatial distribution of a population over a habitat Ω, then the corresponding growth and dispersal between consecutive

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nonoverlapping generations is captured via a Hammerstein operator

$$\mathcal{H}(u)(x) := \int_\Omega k(x,y)g(y,u(y))\,\mathrm{d}y \quad \text{for all } x \in \Omega, \tag{1.1}$$

whereas, when dispersal precedes growth, an adequate model is

$$\mathcal{N}(u)(x) := G\left(x, \int_\Omega k(x,y)u(y)\,\mathrm{d}y\right) \quad \text{for all } x \in \Omega. \tag{1.2}$$

Here, $g, G : \Omega \times \mathbb{R} \to \mathbb{R}$ are growth functions commonly used in theoretical ecology of e.g. Beverton-Holt, Ricker or logistic type (cf. [10, 9]), while $k : \Omega^2 \to \mathbb{R}$ is a probability kernel yielding the dispersal of species. Yet another special case covered are so-called *spatial difference equations* having right-hand sides

$$\mathcal{G}(u)(x) := g(x, u(x)) \quad \text{for all } x \in \Omega \tag{1.3}$$

and which e.g. model populations of sedentary species. Also vector-valued versions of (1.1)–(1.3) are considered to capture multi-species relationships (e.g. [14]).

Such modelling aspects serve as motivation, but are not our main focus. We rather aim to achieve two goals in this paper:

- We abstractly study the iterates of operators like $\mathcal{H}, \mathcal{N}$ or $\mathcal{G}$ on the space of continuous functions over a compact set $\Omega \subset \mathbb{R}^\kappa$ (the *state space*); other function spaces are possible, but require a different analysis. The resulting discrete dynamical systems are called *integrodifference equations* (short IDEs). They are allowed to be nonautonomous, i.e. their right-hand sides (1.1)–(1.3) can depend explicitly on time in an aperiodic way. Such a generalization is well-motivated from applications [9]. Whence, we lay the basics for a qualitative theory of IDEs.

- When simulating the dynamics of IDEs, the problem needs to be discretized, which allows two approaches: For Nyström methods the integral is replaced by a quadrature/cubature rule and one immediately obtains a recursion on a finite-dimensional space. Alternatively, one proceeds in two steps: First, the infinite-dimensional state space is replaced with a finite-dimensional subspace by applying a projection or degenerate kernel method. Since our state space is not a Hilbert space, projection methods will be of collocation type here. This yields a *semi-discretization*, because the resulting operators still contain integrals. Second, one applies a cubature rule in order to approximate these remaining integrals. The resulting full discretizations are obtained by discrete projection (collocation) or discrete degenerate kernel methods (cf. [1, 5, 11]). We therefore prepare preliminary results justifying numerical simulations via semi-discretizations for a general class of IDEs such as (1.1)–(1.3). Nonetheless, an analysis of such full discretizations (i.e. Nyström, discrete collocation and discrete degenerate kernel methods) is reserved for a future paper.

In detail, the structure of this contribution is as follows: Sect. 2 introduces a general class of nonautonomous IDEs containing such with right-hand sides (1.1)–(1.3) as special cases. We provide a flexible framework proving the underlying well-posedness, differentiability, as well as Lipschitz and complete continuity properties of IDEs. This allows to embed such problems into the recent theory of nonautonomous dynamical systems in discrete time (see e.g. [15]). For a subclass, still containing (1.1) and (1.2), a smoothing property is shown, which guarantees that already the first iteration of a continuous initial function inherits the differentiability of the right-hand side. This observation turns out to be helpful in error estimates. Also an interlude on the compactness of linear IDEs and the explicit form of the resulting variational equations is given. Sect. 3 introduces two semi-discretizations for IDEs based on the commonly used collocation and degenerate kernel methods to solve integral equations numerically [5, 1, 11]. They substitute IDEs by difference equations in finite-dimensional subspaces. For these spatially discretized problems, similar questions as in Sect. 2 concerning well-definedness, differentiability, Lipschitz continuity and the associate variational equation are addressed. Furthermore, the local discretization error measuring the discrepancy after one iteration is introduced. Appropriate closeness notions are given by means of (bounded and C^m-) convergence. We provide explicit illustrations for collocation and degenerate kernel approximations based on piecewise linear functions. In general, linear or quadratic convergence orders can be realized only when the state space is restricted to C^1- resp. C^2-functions. Nevertheless, under the smoothing property mentioned above, a relevant special case of IDEs including (1.1), (1.2) yields these convergence orders on the space of continuous functions. In Sect. 4 we provide estimates how the global discretization error between the solutions of the original and of the discretized equation develops over time t. For general IDEs, the global error typically grows exponentially as $t \to \infty$ (cf. Thm. 4.1). Under a contraction condition, however, convergence of the error is established on unbounded intervals. In order to prepare future applications to the numerical dynamics for IDE, we also derive estimates relating the global errors between solution derivatives. An illustration of the obtained results is given in Sect. 5 by means of three examples. They confirm the exponential growth of the global discretization error as $t \to \infty$, the predicted (quadratic) error decay over finite time-intervals, as well as the smoothing property of solutions.

Finally, two appendices conclude the paper. They contain an ambient Grönwall lemma in App. A, which is required to prove error estimates. For the reader's convenience, the concluding App. B presents a rigorous, purposive study of substitution and Urysohn (nonlinear Fredholm) integral operators in a consistent form, rather than referring to diverse sources beyond e.g. [13].

Notation

A *discrete interval* $\mathbb{I}$ is a nonempty intersection of a real interval with the integers $\mathbb{Z}$ and $\mathbb{I}' := \{t \in \mathbb{I} : t+1 \in \mathbb{I}\}$. We write $\mathbb{R}_+ := [0, \infty)$ and $|\cdot|$ for norms on finite-dimensional spaces. In a metric space (M, d), $B_r(x)$ is the open ball with radius $r > 0$ and center x, $\bar{B}_r(x)$ is the closed counterpart and $B_r(A) := \{x \in M : \inf_{a \in A} d(x, a) < r\}$ defines the open r-neighborhood of a nonempty subset $A \subseteq M$.

On the spaces $\mathbb{R}^d$ we throughout use

$$|x| := \max\{|x_1|, \ldots, |x_d|\} \tag{1.4}$$

as norm, which induces the maximum absolute row sum as matrix norm

$$|K| = \max_{i=1}^{p} \sum_{l=1}^{d} |k_{il}| \quad \text{for all } K \in \mathbb{R}^{d\times p}. \tag{1.5}$$

Let X, Y be normed spaces (with norm $\|\cdot\|$) and I_X the identity map on X. The space of bounded j-linear mappings from X^j to Y is denoted by $L_j(X,Y)$, $j \in \mathbb{N}$; $L(X,Y) := L_1(X,Y)$, $L(X) := L(X,X)$, $L_0(X,Y) := Y$ and $GL(X,Y)$ are the invertible elements of $L(X,Y)$. For $T \in L(X,Y)$, $R(T) := TX \subseteq Y$ is the *range*.

If a map $f : X \to Y$ satisfies a Lipschitz condition, then $\operatorname{lip} f$ denotes its smallest Lipschitz constant. When $f : X \times P \to Y$ additionally depends on a parameter from some set P, we write $\operatorname{lip}_1 f := \sup_{p\in P} \operatorname{lip} f(\cdot,p)$ and deploy an analogous notation for Lipschitz constants w.r.t. other variables. The *modulus of continuity* of f is

$$\omega(\delta, f) := \sup_{\|x-\bar{x}\|<\delta} \|f(x) - f(\bar{x})\| \quad \text{for all } \delta > 0$$

and precisely the uniformly continuous functions f satisfy $\lim_{\delta\searrow 0} \omega(\delta, f) = 0$. The class $\mathfrak{N} := \{\Gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \lim_{\rho\searrow 0} \Gamma(\rho) = 0\}$ of limit 0 functions is convenient.

Suppose throughout that $\Omega \subset \mathbb{R}^\kappa$ denotes a nonempty, compact set without isolated points. The space of functions $u : \Omega \to \mathbb{R}$, whose derivatives $D^j u$ up to order $j \le m$ have a continuous extension from the interior $\Omega^\circ \ne \emptyset$ to Ω is denoted by $C^m(\Omega)$, $m \in \mathbb{N}_0$. Since Ω is compact, $C^m(\Omega)$ is a Banach space w.r.t. the norm

$$\|u\|_m := \max_{j=0}^{m} |u|_j, \qquad |u|_j := \max_{x\in\Omega} \left|D^j u(x)\right| \quad \text{for all } 0 \le j \le m;$$

we write $C(\Omega) := C^0(\Omega)$, $\|\cdot\| := \|\cdot\|_0$. The cartesian product $C(\Omega)^d$ is identified with the space of continuous vector-valued functions $u : \Omega \to \mathbb{R}^d$ (similarly for other function spaces). Since Ω is kept fixed throughout, we abbreviate

$$C_d := C(\Omega)^d, \qquad C_d^m := C^m(\Omega)^d.$$

2. Integrodifference equations

Given a discrete interval $\mathbb{I}$, let us first provide an abstract framework in terms of general nonautonomous difference equations

$$\boxed{u_{t+1} = \mathcal{F}_t(u_t)} \tag{I_0}$$

having right-hand sides $\mathcal{F}_t : C_d \to C_d$, $t \in \mathbb{I}'$. For simplicity we assume that (I_0) is globally defined, but our analysis extends to the situation when $\mathcal{F}_t$ is merely given on ambient (e.g. convex) subsets of C_d.

Given an *initial time* $\tau \in \mathbb{I}$, a *forward solution* to (I_0) is a sequence $(\phi_t)_{\tau\le t}$ in C_d satisfying $\phi_{t+1} = \mathcal{F}_t(\phi_t)$ for all $\tau \le t$, $t \in \mathbb{I}'$ and an *entire solution* $(\phi_t)_{t\in\mathbb{I}}$ fulfills this identity on $\mathbb{I}'$. The unique forward solution starting in the *initial state* $u_\tau \in C_d$ is determined by the compositions

$$\varphi(t;\tau,u_\tau) := \begin{cases} \mathcal{F}_{t-1} \circ \ldots \circ \mathcal{F}_\tau(u_\tau), & \tau < t, \\ u_\tau, & t = \tau \end{cases} \tag{2.1}$$

and denoted as *general solution* to (I_0). If the functions $\mathcal{F}_t$, $t \in \mathbb{I}'$, are of class C^1, then

$$D_3\varphi(\tau;\tau,u_\tau) = I_{C_d}, \qquad D_3\varphi(t;\tau,u_\tau) = \prod_{s=\tau}^{t-1} D\mathcal{F}_s(\varphi(s;\tau,u_\tau)) \tag{2.2}$$

hold for all $\tau \leq t$ and $u_\tau \in C_d$ (by mathematical induction). One says (I_0) is *periodic*, if $\mathbb{I} = \mathbb{Z}$ and there exists a *period* $\theta \in \mathbb{N}$ such that $\mathcal{F}_{t+\theta} = \mathcal{F}_t$ for all $t \in \mathbb{Z}$. An *autonomous* equation is 1-periodic, i.e. the right-hand sides of (I_0) do not depend on t.

Being more specific, as standing assumptions, for every $t \in \mathbb{I}'$ the right-hand sides $\mathcal{F}_t$ of (I_0) are composed of two operators:

- If $G_t : \Omega \times \mathbb{R}^d \times \mathbb{R}^p \to \mathbb{R}^d$ are continuous, then the *substitution operators*

  $$\mathcal{G}_t : C_d \times C_p \to C_d, \quad \mathcal{G}_t(u,v)(x) := G_t(x,u(x),v(x)) \quad \text{for all } x \in \Omega$$

  are well-defined and continuous (see App. B.1 for this and more).

- If the *kernel functions* $f_t : \Omega \times \Omega \times \mathbb{R}^d \to \mathbb{R}^p$ are continuous, then the *Urysohn integral operators*

  $$\mathcal{U}_t : C_d \to C_p, \quad \mathcal{U}_t(u)(x) := \int_\Omega f_t(x,y,u(y))\,\mathrm{d}y \quad \text{for all } x \in \Omega \tag{2.3}$$

  are well-defined and completely continuous (see App. B.2).

In the most general form considered here, the difference eqn. (I_0) possesses

$$\mathcal{F}_t(u)(x) := G_t\left(x, u(x), \int_\Omega f_t(x,y,u(y))\,\mathrm{d}y\right) \quad \text{for all } x \in \Omega \tag{2.4}$$

as right-hand sides, or in a brief notation

$$\mathcal{F}_t(u) = \mathcal{G}_t(u, \mathcal{U}_t(u)) \quad \text{for all } u \in C_d. \tag{2.5}$$

We consequently speak of *nonlinear Urysohn eqns*. (I_0) on C_d, which include right-hand sides (1.1)–(1.3). Let us first address well-definedness and smoothness properties:

Proposition 2.1. *The general solution $\varphi(t;\tau,\cdot) : C_d \to C_d$ of (I_0) is well-defined for all $\tau \leq t$, as well as bounded, continuous and uniformly continuous on bounded sets.*

Proof. Let $s \in \mathbb{I}'$. Thanks to Thm. B.1 the substitution operators $\mathcal{G}_s : C_d \times C_p \to C_d$, and due to Thm. B.5 also the Urysohn operators $\mathcal{U}_s : C_d \to C_p$, are well-defined, bounded, continuous and uniformly continuous on any bounded subset $B \subset C_d \times C_p$ resp. $B \subset C_d$. This carries over to the composition $\mathcal{F}_s$ from (2.5) and by (2.1) also to their composition, that is, the general solution. $\square$

More can be said for pure integral operators, where the mapping G_t does not depend on the second variable, i.e. $\mathcal{F}_t$ coincides with a function

$$\bar{\mathcal{F}}_t : C_d \to C_d, \quad \bar{\mathcal{F}}_t(u)(x) := \bar{G}_t\left(x, \int_\Omega f_t(x, y, u(y))\,\mathrm{d}y\right) \quad \text{for all } x \in \Omega, \tag{2.6}$$

where $\bar{G}_t : \Omega \times \mathbb{R}^p \to \mathbb{R}^d$ is continuous.

Corollary 2.2. *Let $\tau < t$. If there exists a $\tau \le t_0 < t$ so that $\mathcal{F}_{t_0} = \bar{\mathcal{F}}_{t_0}$ is of the form* (2.6)*, then $\varphi(t; \tau, \cdot) : C_d \to C_d$ is completely continuous.*

Proof. Due to (2.6) the right-hand side $\bar{\mathcal{F}}_{t_0} = \bar{\mathcal{G}}_{t_0} \circ \mathcal{U}_{t_0} : C_d \to C_d$ is a composition of the completely continuous mapping $\mathcal{U}_{t_0} : C_d \to C_p$ (cf. Thm. B.5) with the bounded, continuous map $\bar{\mathcal{G}}_{t_0} : C_p \to C_d$ (see Thm. B.1), thus completely continuous. The claim follows by the same argument from (2.1), since $\varphi(t; \tau, \cdot)$ is a composition of $\bar{\mathcal{F}}_{t_0}$ with bounded and continuous mappings $\mathcal{F}_s$, $s \ne t_0$. $\square$

Corollary 2.3. *If for every $s \in \mathbb{I}'$, $r > 0$ there exists a function $\lambda_{s,r} : \Omega^2 \to \mathbb{R}_+$ such that $\lambda_{s,r}(x, \cdot)$ is measurable on Ω with $\ell_s(r) := \sup_{x\in\Omega} \int_\Omega \lambda_{s,r}(x, y)\,\mathrm{d}y < \infty$ and*

$$|f_s(x, y, z) - f_s(x, y, \bar{z})| \le \lambda_{s,r}(x, y)\,|z - \bar{z}| \quad \textit{for all } x, y \in \Omega,\ z, \bar{z} \in \bar{B}_r(0),$$

then $B \subseteq \bar{B}_r(0) \subset C_d$ implies $\operatorname{lip} \mathcal{F}_s|_B \le L_s(r)$, where

- $L_s(r) := \operatorname{lip}_{(2,3)} G_s \max\{1, \ell_s(r)\}$*, provided* $\operatorname{lip}_{(2,3)} G_s < \infty$*,*
- $L_s(r) := \operatorname{lip}_2 G_s + \operatorname{lip}_3 G_s \ell_s(r)$*, provided* $\operatorname{lip}_2 G_s, \operatorname{lip}_3 G_s < \infty$*.*

If $\mathcal{F}_s = \bar{\mathcal{F}}_s$ is of the form (2.6), then $L_s(r) = \operatorname{lip}_2 G_s \ell_s(r)$ holds.

Proof. Let $r > 0$ and $s \in \mathbb{I}'$. In Cor. B.6 it is shown that $\operatorname{lip} \mathcal{U}_s|_B \le \ell_s(r)$ holds for subsets $B \subseteq \bar{B}_r(0)$. The assumed global Lipschitz condition on G_s extends to $\mathcal{G}_s$ (cf. Cor. B.2), which yields the assertion by elementary estimates. $\square$

We point out that Cor. 2.3 has immediate implications for the global dynamics of nonautonomous IDEs (I_0).

Remark 2.4 (forward and pullback convergence)**.** Assume that $\mathcal{U}_s$ satisfies a global Lipschitz condition or that $l_s := \sup_{r\ge0} L_s(r) < \infty$, and suppose $u \in C_d$:

- Let $\mathbb{I}$ be unbounded above and $\tau \in \mathbb{I}$. If $(\phi_t)_{\tau\le t}$ is any forward solution to (I_0) and the limit relation $\prod_{s=\tau}^{\infty} l_s = 0$ holds, then mathematical induction yields

  $$\|\varphi(t; \tau, u) - \phi_t\| = \|\varphi(t; \tau, u) - \varphi(t; \tau, \phi_\tau)\| \le \left(\prod_{s=\tau}^{t-1} l_s\right) \|u - \phi_\tau\| \xrightarrow[t\to\infty]{} 0.$$

  Thus, ϕ is globally forward attractive and all solutions to (I_0) are asymptotically forward equivalent; that is, they converge to each other.

- Let $\mathbb{I}$ be unbounded below. If there exists a bounded entire solution $(\phi_t)_{t\in\mathbb{I}}$ of (I_0) and the limit relation $\prod_{s=-\infty}^{t_0} l_s = 0$ for some $t_0 \in \mathbb{I}'$ holds, then (as above)

$$\|\varphi(t;\tau,u) - \phi_t\| \leq \left(\prod_{s=\tau}^{t-1} l_s\right) \|u - \phi_\tau\| \xrightarrow[\tau\to-\infty]{} 0 \quad \text{for all } t \leq t_0.$$

  Hence, ϕ is globally pullback attractive (cf. [15, pp. 62–63, Def. 2.4.4]).

The next results rely on the classes of C_1^m- and C_f^m-functions introduced in App. B:

Proposition 2.5 (differentiability of $\varphi(t;\tau,u)$). *Let $m \in \mathbb{N}_0$. If f_s are of class C_1^m and G_s are C^m-functions for every $s \in \mathbb{I}'$, then $\varphi(t;\tau,u) \in C^m(\Omega)^d$ for all $\tau \leq t$ and $u \in C^m(\Omega)^d$.*

Proof. Let $s \in \mathbb{I}'$, $u \in C^m(\Omega)^d$ and w.l.o.g. $m \in \mathbb{N}$. Combining Thm. B.3 and B.7, it follows from (2.5) and the chain rule [12, p. 337] that $\mathcal{F}_s(u)$ is of class C^m. Again the chain rule and mathematical induction imply the claim with (2.1). □

If the kernel functions f_s are differentiable in the first argument, and G_s are differentiable as well for $s \in \mathbb{I}'$, then IDEs with right-hand sides (2.6) have a smoothing property. This is convenient in error estimates for discretizations (cf. Sect. 3), because continuous initial functions inherit the kernel's smoothness after one iteration.

Corollary 2.6 (smoothing property). *If (I_0) has right-hand sides (2.6) for all $s \in \mathbb{I}'$, then $\varphi(t;\tau,u) \in C^m(\Omega)^d$ for every $\tau < t$ and $u \in C_d$.*

Proof. Given $u \in C_d$, we obtain from Thm. B.7 that $\mathcal{U}_s(u) \in C^m(\Omega)^d$ holds. Then the representation $\mathcal{F}_s(u)(x) \equiv G_s(x, \mathcal{U}_s(u)(x))$ on Ω identifies $\mathcal{F}_s(u)$ as a composition of C^m-functions and the claim follows by induction from (2.1). □

Proposition 2.7 (differentiability of $\varphi(t;\tau,\cdot)$). *Let $m \in \mathbb{N}$. If f_s and G_s are of class C_f^m for every $s \in \mathbb{I}'$, then $\varphi(t;\tau,\cdot) \in C^m(C_d, C_d)$ for all $\tau \leq t$.*

Proof. Let $s \in \mathbb{I}'$. From Thm. B.8 the mapping $\mathcal{U}_s : C_d \to C_p$ is of class C^m, and so is $u \mapsto (u, \mathcal{U}_s(u))$. Then Thm. B.4 and the chain rule [12, p. 337] imply that also the composition $\mathcal{F}_s$ (cf. (2.5)) is m-times continuously differentiable. Using (2.1) this extends to the compositions $\varphi(t;\tau,\cdot)$. □

2.1. Linear integrodifference equations

The linear-homogeneous IDEs being relevant for our purposes read as

$$\boxed{v_{t+1} = \mathcal{L}_t v_t}$$

with coefficients $\mathcal{L}_t := \mathcal{M}_t + \mathcal{K}_t$ being the sum of two operators. For every $t \in \mathbb{I}'$ and

- continuous $M_t : \Omega \to L(\mathbb{R}^d)$, consider the *multiplication operator*

$$(\mathcal{M}_t v)(x) := M_t(x)v(x) \quad \text{for all } x \in \Omega, \tag{2.7}$$

- *kernels* $k_t : \Omega^2 \to L(\mathbb{R}^d)$ such that $k_t(x, \cdot) : \Omega \to L(\mathbb{R}^d)$ is measurable for all $x \in \Omega$, consider the *Fredholm integral operator*

$$(\mathcal{K}_t v)(x) := \int_\Omega k_t(x, y) v(y) \, \mathrm{d}y \quad \text{for all } x \in \Omega. \tag{2.8}$$

Proposition 2.8. *If $k_s : \Omega^2 \to L(\mathbb{R}^d)$ satisfies $\tilde{\ell}_s := \sup_{x\in\Omega} \int_\Omega |k_s(x, y)| \, \mathrm{d}y < \infty$ for all $s \in \mathbb{I}'$, then the* transition operator

$$\Phi(t, \tau) \in L(C_d), \qquad \Phi(t, \tau) := \begin{cases} \mathcal{L}_{t-1} \cdots \mathcal{L}_\tau, & \tau < t, \\ I_{C_d}, & \tau = t \end{cases} \tag{2.9}$$

is well-defined, $\|\Phi(t, \tau)\| \leq \prod_{s=\tau}^{t-1} \ell_s$ for all $\tau \leq t$ and $\ell_s := \tilde{\ell}_s + \max_{x\in\Omega} |M_s(x)|$.

Proof. Let $s \in \mathbb{I}'$. It is not hard to see that every $\mathcal{M}_s \in L(C_d)$ from (2.7) is well-defined and satisfies $\|\mathcal{M}_s\| = \max_{x\in\Omega} |M_s(x)| =: m_s$. In [13, p. 167, Prop. 3.4] it is shown that $\mathcal{K}_s \in L(C_d)$ from (2.8) is continuous and fulfills $\|\mathcal{K}_s\| \leq \tilde{\ell}_s$. Hence, $\mathcal{L}_s \in L(C_d)$ with $\|\mathcal{L}_s\| \leq m_s + \tilde{\ell}_s$. Then induction yields that $\Phi(t, \tau) \in L(C_d)$ is well-defined using (2.9), and moreover one has the norm estimate for $\Phi(t, \tau)$, $\tau \leq t$. □

Corollary 2.9. *Let $\tau < t$. If $k_{t_0} : \Omega^2 \to L(\mathbb{R}^d)$ additionally satisfies*

$$\lim_{x\to\xi} \int_\Omega |k_{t_0}(x, y) - k_{t_0}(\xi, y)| \, \mathrm{d}y \quad \textit{uniformly in } \xi \in \Omega$$

and $\mathcal{M}_{t_0} = 0$ for some $\tau \leq t_0 < t$, then $\Phi(t, \tau) \in L(C_d)$ is compact.

Proof. Due to [13, p. 167, Prop. 3.4] the operator $\mathcal{L}_{t_0} = \mathcal{K}_{t_0}$ is compact. Therefore, [8, p. 215, Prop. 1.2] and (2.9) imply that the composition $\Phi(t, \tau)$ is compact. □

2.2. Variational equations

Suppose $\phi = (\phi_t)_{t\in\mathbb{I}}$ is an entire solution to (I_0). If f_s and G_s are of class C_f^1 for all $s \in \mathbb{I}'$, then $\mathcal{F}_s : C_d \to C_d$ is continuously differentiable due to Prop. 2.7 and

$$\boxed{v_{t+1} = D\mathcal{F}_t(\phi_t) v_t} \tag{V_0}$$

is denoted as *variational equation* of (I_0) along ϕ. It spectral properties determine the stability of ϕ.

Corollary 2.10 (variational equation)**.** *Let $(\phi_t)_{t\in\mathbb{I}}$ be an entire solution to (I_0) and $s \in \mathbb{I}'$. If f_s and G_s are of class C_f^1, then $D\mathcal{F}_s(\phi_s) = \mathcal{M}_s + \mathcal{K}_s$ holds with the summands* (2.7) *and* (2.8) *for all $x \in \Omega$ and $v \in C_d$ explicitly given as*

$$(\mathcal{M}_s v)(x) := D_2 G_s \left(x, \phi_s(x), \int_\Omega f_s(x, y, \phi_s(y)) \, \mathrm{d}y \right) v(x),$$

$$(\mathcal{K}_s v)(x) := D_3 G_s \left(x, \phi_s(x), \int_\Omega f_s(x, y, \phi_s(y)) \, \mathrm{d}y \right) \int_\Omega D_3 f_s(x, y, \phi_s(y)) v(y) \, \mathrm{d}y.$$

If $\mathcal{F}_s = \bar{\mathcal{F}}_s$ is of the form (2.6)*, then $D\mathcal{F}_s(\phi_s) = \mathcal{K}_s \in L(C_d)$ is compact.*

Proof. From Prop. 2.7 we know that $\mathcal{F}_s = \varphi(s+1; s, \cdot) \in C^1(C_d, C_d)$ and the chain rule [12, p. 337] implies $D\mathcal{F}_s(\phi) = D_1\mathcal{G}_s(\phi, \mathcal{U}_s(\phi)) + D_2\mathcal{G}_s(\phi, \mathcal{U}_s(\phi))D\mathcal{U}_s(\phi)$ due to (2.5). Using the explicit derivatives from Thms. B.4 and B.8, we conclude the claim. In case D_2G_s vanishes, then $\mathcal{M}_s = 0$ and $D\mathcal{F}_s(\phi) = D_2\mathcal{G}_s(\phi, \mathcal{U}_s(\phi))D\mathcal{U}_s(\phi)$ is the composition of a bounded (multiplication) operator with the compact derivative $D\mathcal{U}_s(\phi)$ (cf. Thm. B.8) and is therefore compact itself (see [8, p. 215, Prop. 1.2]). □

3. Semi-discretizations of integrodifference equations

Applying a semi-discretization method to an IDE (I_0) yields a family of nonautonomous difference equations

$$\boxed{u_{t+1} = \mathcal{F}_t^n(u_t)} \tag{I_n}$$

with right-hand sides $\mathcal{F}_t^n : C_d \to C_d$, $t \in \mathbb{I}'$, depending on $n \in \mathbb{N}$ and values in a finite-dimensional linear (or affine) subspace of C_d. The same terminology as introduced for (I_0) applies to each (I_n) and particularly its general solution is denoted by φ_n.

The *(local) discretization error* $\varepsilon_t^n : C_d \to C_d$,

$$\varepsilon_t^n(u) := \mathcal{F}_t(u) - \mathcal{F}_t^n(u)$$

captures the distance between solutions to (I_0) and (I_n) starting in the same point after one iteration. Since $n \in \mathbb{N}$ is understood as discretization parameter, $\varepsilon_t^n(u)$ is supposed to become arbitrarily small as $n \to \infty$. To be more precise, we say $\mathcal{F}_t^n$ or $(I_n)_{n\in\mathbb{N}}$ is

- *convergent*, if $\lim_{n\to\infty} \|\varepsilon_t^n(u)\| = 0$ holds for all $t \in \mathbb{I}'$, $u \in C_d$
- *bounded convergent*, if $\lim_{n\to\infty} \sup_{u\in B} \|\varepsilon_t^n(u)\| = 0$ holds for all $t \in \mathbb{I}'$ and bounded $B \subset C_d$
- C^m*-convergent* with $m \in \mathbb{N}_0$, if there is a *convergence function* $\Gamma_0 \in \mathfrak{N}$ and for every bounded $B \subset C^m(\Omega)^d$ (in the C^m-topology) there is a $K(B) \geq 0$ with

$$\|\varepsilon_t^n(u)\| \leq K(B)\Gamma_0(\tfrac{1}{n}) \quad \text{for all } t \in \mathbb{I}',\ n \in \mathbb{N} \text{ and } u \in B. \tag{3.1}$$

  In case there exist $C, \gamma > 0$ with $\Gamma_0(\rho) \leq C\rho^\gamma$ for all $\rho > 0$ one says $(I_n)_{n\in\mathbb{N}}$ has *convergence order* $\gamma > 0$.

Note that C^0-convergence is sufficient for bounded convergence, which in turn implies convergence. Moreover, C^{m-1}-convergence yields C^m-convergence for $m \in \mathbb{N}$.

Under natural assumptions (see below), the right-hand side of (I_n) inherits continuous differentiability from (I_0). This allows to consider the *variational equation*

$$\boxed{v_{t+1} = D\mathcal{F}_t^n(\phi_t)v_t = [\mathcal{M}_t^n + \mathcal{K}_t^n]v_t} \tag{V_n}$$

associated to an entire solution $(\phi_t)_{t\in\mathbb{I}}$ of (I_n). The precise form of (I_n) and (V_n) depends on the particular discretization method studied next:

3.1. Collocation methods

Collocation methods for linear integral equations were thoroughly discussed in [1, pp. 49ff, Chapt. 3], [5, pp. 81ff, Sect. 4.4] and [11, pp. 241, Chapt. 13]. Our straight-forward generalization to nonlinear problems is as follows: Assume that X_n, $n \in \mathbb{N}$, are subspaces of the real-valued continuous functions C_1 with finite dimension $d_n := \dim X_n$ having a basis $\{e_1, \dots, e_{d_n}\}$. We choose distinct *collocation points* $\xi_1, \dots, \xi_{d_n} \in \Omega$ and require the interpolation condition

$$(e_i(\xi_j))_{i,j=1}^{d_n} \in GL(\mathbb{R}^{d_n}). \tag{3.2}$$

Given this, $P_n u = \sum_{j=1}^{d_n} e_j(\cdot)v^j$ define projections $P_n : C_d \to X_n^d$, where the vectors $v^1, \dots, v^{d_n} \in \mathbb{R}^d$ are (by (3.2) uniquely) determined via the interpolation conditions $\sum_{j=1}^{d_n} e_j(\xi_i)v^j = u(\xi_i)$ for all $1 \le i \le d_n$.

A *collocation method* to approximate (I_0) now reads as

$$\mathcal{F}_t^n(u) := P_n\mathcal{F}_t(u) = \sum_{j=1}^{d_n} e_j(\cdot)v_t^j. \tag{3.3}$$

Each evaluation of $\mathcal{F}_t^n$, $t \in \mathbb{I}'$, requires to determine the vectors $v_t^1, \dots, v_t^{d_n} \in \mathbb{R}^d$ from the linear equations

$$\sum_{j=1}^{d_n} e_j(\xi_i)v_t^j = G_t\left(\xi_i, u(\xi_i), \int_\Omega f_t(\xi_i, y, u(y))\,\mathrm{d}y\right) \quad \text{for all } 1 \le i \le d_n.$$

Accordingly, $\mathcal{F}_t^n(u) \in X_n^d$ holds and we obtain merely a semi-discretization (3.3), since d_n $\mathbb{R}^d$-valued integrals remain to be computed.

Theorem 3.1 (collocation methods)**.** *Let $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$. The general solution $\varphi_n(t;\tau,\cdot) : C_d \to X_n^d$ of collocation methods* (3.3) *is well-defined, completely continuous and uniformly continuous on bounded sets for $\tau < t$. In case $\sup_{n\in\mathbb{N}} \|P_n\| < \infty$ these properties hold uniformly in $n \in \mathbb{N}$. Moreover, one has:*

(a) *If $e_1, \dots, e_{d_n} \in C^m(\Omega)$, then $\varphi_n(t;\tau,u) \in C^m(\Omega)^d$ for all $\tau < t$, $u \in C_d$.*
(b) *If f_s, G_s are of class C_f^m for every $s \in \mathbb{I}'$, then $\varphi_n(t;\tau,\cdot) \in C^m(C_d, C_d)$.*

Proof. Let $n \in \mathbb{N}$, $s \in \mathbb{I}'$. By (3.3) the collocation method $\mathcal{F}_s^n$ is a composition of a linear operator $P_n \in L(C_d)$ with $\mathcal{F}_s$. Thus, $\mathcal{F}_s^n$ and $\varphi_n(t;\tau,\cdot)$ inherit the corresponding properties of $\mathcal{F}_s$ resp. $\varphi(t;\tau,\cdot)$ stated in Prop. 2.1. Moreover, since P_n has finite-dimensional range, $\mathcal{F}_s^n$ is completely continuous, and so is $\varphi_n(t;\tau,\cdot)$ for $\tau < t$. The uniformity of these properties in $n \in \mathbb{N}$ for bounded sequences $(P_n)_{n\in\mathbb{N}}$ is equally evident from (3.3).

(a) Given $u \in C_d$, it immediately follows from (3.3) that $\mathcal{F}_t^n(u)$ inherits its smoothness from the basis functions $e_1, \dots, e_n$, and so does $\varphi_n(t;\tau,u)$.

(b) Due to Prop. 2.7 the mapping $\mathcal{F}_s = \varphi(s+1;s,\cdot)$ is of class C^m, and so is $\mathcal{F}_s^n = P_n\mathcal{F}_s$. Then the claim for $\varphi_n(t;\tau,\cdot)$ results from the chain rule [12, p. 337]. □

Corollary 3.2. *In the setting of Cor. 2.3 the following implication holds:*

$$B \subseteq \bar{B}_r(0) \subset C_d \quad \Rightarrow \quad \operatorname{lip} \mathcal{F}_s^n|_B \leq \|P_n\| \, L_s(r) \quad \textit{for all } r > 0,\ s \in \mathbb{I}',\ n \in \mathbb{N}.$$

Proof. Let $r > 0$ and $s \in \mathbb{I}'$. If $u, \bar{u} \in B \subseteq \bar{B}_r(0)$, then

$$\|\mathcal{F}_s^n(u) - \mathcal{F}_s^n(\bar{u})\| \overset{(3.3)}{\leq} \|P_n\| \, \|\mathcal{F}_s(u) - \mathcal{F}_s(\bar{u})\|$$

and the concluding estimate in Cor. 2.3 guarantees the assertion. □

Corollary 3.3 (variational collocation equation)**.** *Let $n \in \mathbb{N}$, $s \in \mathbb{I}'$ and $(\phi_t)_{t \in \mathbb{I}}$ be an entire solution to (I_n) given in* (3.3)*. If f_s and G_s are of class C_f^1, then the derivative $D\mathcal{F}_s^n(\phi_s) \in L(C_d, X_n^d)$ is compact and given by $D\mathcal{F}_s^n(\phi_s) = P_n \mathcal{M}_s + P_n \mathcal{K}_s$.*

Proof. Due to $D\mathcal{F}_s^n(\phi_s) = P_n D\mathcal{F}_s(\phi_s)$ the formula for $D\mathcal{F}_s^n(\phi_s)$ follows by Cor. 2.10 and (3.3). Furthermore, [13, p. 89, Prop. 6.5] shows that $D\mathcal{F}_s^n(\phi_s)$ is compact. □

3.1.1. Lagrangian interpolation

Due to Cor. 3.2 the norms of the projections P_n play a crucial role to preserve uniformity, and in the light of Rem. 2.4 and its implications, for contractiveness under discretization. For *Langrangian interpolation*, that is, when (3.2) is strengthened to

$$e_i(\xi_j) = \delta_{ij} \quad \text{for all } 1 \leq i, j \leq d_n, \tag{3.4}$$

one deduces: According to [1, p. 51, (3.1.8)] the projections $p_n : C_1 \to X_n$ given by $p_n u := \sum_{j=1}^{d_n} e_j(\cdot) u(\xi_j)$ satisfy $\|p_n\| = \max_{x \in \Omega} \sum_{j=1}^{d_n} |e_j(x)|$ and thus (1.4) implies that $\|P_n u\| = \max\{\|p_n u_1\|, \ldots, \|p_n u_d\|\} \leq \max_{x \in \Omega} \sum_{j=1}^{d_n} |e_j(x)| \, \|u\|$, i.e.

$$\|P_n\| \leq \max_{x \in \Omega} \sum_{j=1}^{d_n} |e_j(x)| \quad \text{for all } n \in \mathbb{N}. \tag{3.5}$$

Lagrangian interpolation furthermore allows the explicit representation

$$\mathcal{F}_t^n(u) = \sum_{j=1}^{d_n} G_t\left(\xi_j, u(\xi_j), \int_\Omega f_t(\xi_j, y, u(y)) \, \mathrm{d}y\right) e_j$$

for the right-hand side of (I_n) and

$$\begin{aligned}
\mathcal{M}_t^n v &:= \sum_{j=1}^{d_n} D_2 G_t\left(\xi_j, \phi_t(\xi_j), \int_\Omega f_t(\xi_j, y, \phi_t(y)) \, \mathrm{d}y\right) v(\xi_j) e_j, \\
\mathcal{K}_t^n v &:= \sum_{j=1}^{d_n} D_3 G_t\left(\xi_j, \phi_t(\xi_j), \int_\Omega f_t(\xi_j, y, \phi_t(y)) \, \mathrm{d}y\right) \\
&\qquad \cdot \int_\Omega D_3 f_t(\xi_j, y, \phi_t(y)) v(y) \, \mathrm{d}y e_j \quad \text{for all } t \in \mathbb{I}',\ v \in C_d
\end{aligned}$$

in the variational equation (V_n) (cf. Cor. 3.3). We continue with two special cases:

3.1.2. Piecewise linear interpolation on intervals

Let $\Omega := [a,b]$ and define $\xi_j := a + j\frac{b-a}{n}$ for $0 \le j \le n$. If X_n is the space of continuous, piecewise affine functions over $[a,b]$ with collocation points $\{\xi_0,\dots,\xi_n\}$, then $d_n = n+1$. We choose the basis $\{e_0,\dots,e_n\} \subset C[a,b]$ of *hat functions*

$$e_j : [a,b] \to \mathbb{R}_+, \qquad e_j(x) := \max\left\{1 - n\tfrac{|x-\xi_j|}{b-a}, 0\right\} \quad \text{for all } 0 \le j \le n.$$

They fulfill (3.4) (the interpolation is Lagrangian) and (3.5) implies $\|P_n\| = 1$, $n \in \mathbb{N}$. Hence, for piecewise linear collocation we obtain from Thm. 3.1 and Cor. 3.2:

- $\varphi_n(t;\tau,u) \in C_d$ for all $\tau \le t$, $u \in C_d$.
- $\operatorname{lip} \mathcal{F}_s^n|_B \le L_s(r)$ for $s \in \mathbb{I}'$ and $B \subset \bar{B}_r(0) \subset C_d$, i.e. the Lipschitz conditions are preserved and uniform in $n \in \mathbb{N}$.

The interpolation estimates [6, p. 241, p. 247] yield the local discretization error

$$\|\varepsilon_t^n(u)\| \le \begin{cases} \omega(\frac{b-a}{n}, \mathcal{F}_t(u)), & \mathcal{F}_t(u) \in C[a,b], \\ \frac{b-a}{4n}\omega(\frac{b-a}{n}, \mathcal{F}_t(u)'), & \mathcal{F}_t(u) \in C^1[a,b], \\ \frac{(b-a)^2}{8n^2}\left|\mathcal{F}_t(u)\right|_2, & \mathcal{F}_t(u) \in C^2[a,b]; \end{cases}$$

whence, the convergence order depends on the smoothness of $\mathcal{F}_t(u)$, $t \in \mathbb{I}'$. Let us illuminate this using our above convergence notions: Because $[a,b]$ is compact, the functions $\mathcal{F}_t(u) : [a,b] \to \mathbb{R}^d$ are uniformly continuous. Thus, piecewise linear interpolation yields a convergent method $(I_n)_{n\in\mathbb{N}}$ at least. The question for convergence rates, however, depends on the structure of the right-hand sides for (I_0). For this reason, we suppose that G_t is of class C^m and that f_t is of class C_1^m.

- If $\mathcal{F}_t$ is of the general form (2.4), then Prop. 2.5 implies $\mathcal{F}_t(u) \in C^m[a,b]^d$ for arguments $u \in C^m[a,b]^d$. In case $m = 1$ the coarse estimate

  $$\omega(\tfrac{b-a}{n}, \mathcal{F}_t(u)') \le 2\,\|\mathcal{F}_t(u)'\| \quad \text{for all } n \in \mathbb{N},\ t \in \mathbb{I}'$$

  thus implies (3.1) with convergence function $\Gamma_0(\rho) := \frac{b-a}{2}\rho$ and

  $$K(B) := \sup_{t\in\mathbb{I}'}\sup_{u\in B}\|\mathcal{F}_t(u)'\| \quad \text{for every } C^1\text{-bounded } B \subset C^1[a,b]^d.$$

  Hence, piecewise linear collocation is C^1-convergent of order 1. In case $m = 2$ we obtain (3.1) with the functions $\Gamma_0(\rho) := \frac{(b-a)^2}{8}\rho^2$ and

  $$K(B) := \sup_{t\in\mathbb{I}'}\sup_{u\in B}\|\mathcal{F}_t(u)''\| \quad \text{for every } C^2\text{-bounded } B \subset C^2[a,b]^d$$

  and therefore even C^2-convergence of order 2 holds. The explicit form of the derivatives $\mathcal{F}_t(u)^{(m)}$ shows that $\sup_{u\in B}\left\|\mathcal{F}_t(u)^{(m)}\right\|$ are always finite for every C^m-bounded set $B \subset C^m[a,b]^d$ and $m \in \{1,2\}$. Yet, one has to assume that the further supremum over $t \in \mathbb{I}'$ in the definition of $K(B)$ exists.

- If $G_t = \bar{G}_t$ and consequently $\mathcal{F}_t = \bar{\mathcal{F}}_t$ are of the form (2.6), then the derivatives of $\bar{\mathcal{F}}_t(u)$ do not depend on the derivatives of u. This guarantees that $(I_n)_{n\in\mathbb{N}}$ is C^0-convergent of respective order m with the above $K(B)$ and $\Gamma_0(\rho)$. Here, $\sup_{u\in B} \left\|\mathcal{F}_t(u)^{(m)}\right\|$ exist for every bounded set $B \subset C[a,b]^d$ in the $\|\cdot\|$-norm.

In conclusion, error estimates for general right-hand sides (2.4) require smooth arguments, whereas IDEs given by (2.6) can be treated in a C^0-setting.

3.1.3. Piecewise linear interpolation in $\mathbb{R}^\kappa$

Piecewise linear interpolation extends to higher dimensional domains and we consider a rectangle $\Omega := [a_1, b_1] \times \ldots \times [a_\kappa, b_\kappa]$, where each interval $[a_j, b_j]$, $1 \le j \le \kappa$ may have n subdivisions. If $e_0^j, \ldots, e_n^j : [a_j, b_j] \to \mathbb{R}_+$, $1 \le j \le \kappa$, denote the hat functions as introduced in Sect. 3.1.2, then we define their multivariate version

$$e_\iota(x) := \prod_{j=1}^{\kappa} e_{\iota_j}^j(x_j) \quad \text{for all } x = (x_1, \ldots, x_\kappa) \in \Omega,\ \iota = (\iota_1, \ldots, \iota_\kappa) \in \{0, \ldots, n\}^\kappa$$

and choose $\{e_\iota : \Omega \to \mathbb{R}_+ \mid \iota \in \{0, \ldots, n\}^\kappa\}$ as basis of $X_n \subset C_1$, thus having dimension $d_n = (n+1)^\kappa$. It is not hard to see that $e_\iota(x) \in [0,1]$ for all $x \in \Omega$ and that the interpolation is Lagrangian.

Lemma 3.4. (a) $\sum_{\iota\in\{0,\ldots,n\}^\kappa} e_\iota(x) \equiv 1$ *on* Ω.
(b) *If* $u \in C^2(\Omega)$*, then* $\|u - p_n u\| \le \frac{1}{8n^2} \sum_{j=1}^{\kappa} (b_j - a_j)^2 \left\|D_j^2 u\right\|$.

Proof. (a) We proceed by induction. For $\kappa = 1$ the claim is clear from Sect. 3.1.2. As induction step $\kappa \to \kappa + 1$ we obtain

$$\begin{aligned}
&\sum_{\iota\in\{0,\ldots,n\}^{\kappa+1}} e_\iota(x_1, \ldots, x_{\kappa+1}) \equiv \sum_{(\iota,\iota')\in\{0,\ldots,n\}^{\kappa+1}} e_\iota(x_1, \ldots, x_\kappa) e_{\iota'}^{\kappa+1}(x_{\kappa+1}) \\
&\equiv \sum_{\iota\in\{0,\ldots,n\}^\kappa} e_\iota(x_1, \ldots, x_\kappa) \sum_{\iota'=0}^{n} e_{\iota'}^{\kappa+1}(x_{\kappa+1}) \equiv \sum_{\iota\in\{0,\ldots,n\}^\kappa} e_\iota(x_1, \ldots, x_\kappa) \equiv 1
\end{aligned}$$

on the product $\Omega \times [a_{\kappa+1}, b_{\kappa+1}]$ from the induction hypothesis.
(b) results inductively using [6, p. 267]. □

The partition of unity from Lemma 3.4(a), the fact that the basis functions e_ι are nonnegative and (3.5) imply

$$\|P_n\| = 1 \quad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

Whence, also in the multivariate case the general solution φ_n has the properties stated in Sect. 3.1.2. Furthermore, if $\mathcal{F}_t(u) \in C^2(\Omega)^d$ holds, then Lemma 3.4(b) leads to

$$\|e_t^n(u)\| \overset{(1.4)}{\le} \frac{1}{8n^2} \sum_{j=1}^{\kappa} (b_j - a_j)^2 \left\|D_j^2(\mathcal{F}_t(u))\right\| \quad \text{for all } n \in \mathbb{N},\ t \in \mathbb{I}'.$$

Finally, suppose $\Omega \subset \mathbb{R}^2$ is a compact domain with polygonal boundary. Let T_n be a subdivision of Ω into triangles $K \subset \Omega$, constructed so that no vertex of a triangle lies on the edge of another triangle, and that $\operatorname{diam} K \le \frac{1}{n}$ holds for all $n \in \mathbb{N}$. This means that all triangles in T_n have a diameter $\le \frac{1}{n}$. We equip the space of piecewise linear continuous functions $X_n := \{u \in C_1 : u|_K \text{ is affine linear for all } K \in T_n\}$ with corresponding tent functions as basis. If $\mathcal{F}_t(u) \in C^2(\Omega)^d$ holds, then there exists a $C > 0$ such that (cf. [2, p. 402, Thm. 10.3.8])

$$\|\varepsilon_t^n(u)\| \le \tfrac{C}{n^2} \left|\mathcal{F}_t(u)\right|_2 \quad \text{for all } n \in \mathbb{N},\, t \in \mathbb{I}'.$$

In both cases, $\Omega \subset \mathbb{R}^\kappa$ is rectangular, or $\Omega \subset \mathbb{R}^2$ has polygonal boundary, an analysis as in Sect. 3.1.2 shows that $(I_n)_{n\in\mathbb{N}}$ is C^2-convergent of order 2 for general right-hand sides (2.4). Yet, for IDEs (I_0) with (2.6) one has C^0-convergence of order 2.

3.2. Degenerate kernel methods and Hammerstein integrodifference equations

Due to their prominent role in applications [10, 7, 14, 9], an important special case of general IDEs (I_0) are *Hammerstein integrodifference equations* (HDEs for short)

$$u_{t+1} = \mathcal{F}_t(u_t), \qquad \mathcal{F}_t(u) := \int_\Omega K_t(\cdot, y) g_t(y, u(y))\,\mathrm{d}y + h_t \qquad (H_0)$$

with inhomogeneities $h_t \in C_d$, $t \in \mathbb{I}'$. They fit into our framework of (I_0) with

$$G_t(x, y, z) := z + h_t(x) \quad \text{for all } t \in \mathbb{I}',\, x \in \Omega,\, y, z \in \mathbb{R}^d$$

(independent of the second variable) and kernel functions

$$f_t : \Omega^2 \times \mathbb{R}^d \to \mathbb{R}^d, \qquad f_t(x, y, z) := K_t(x, y) g_t(y, z) \quad \text{for all } t \in \mathbb{I}' \qquad (3.7)$$

having the continuous factors $K_t : \Omega^2 \to \mathbb{R}^{d\times p}$ (the *kernel*), $g_t : \Omega \times \mathbb{R}^d \to \mathbb{R}^p$ (the *growth function*). In particular, HDEs have right-hand sides of the form (2.6) with the corresponding consequences. For all $\tau < t$ and $u \in C_d$ we obtain:

- $\varphi(t; \tau, \cdot) : C_d \to C_d$ is well-defined, continuous and uniformly continuous on bounded sets by Prop. 2.1, while it is completely continuous by Cor. 2.2.
- If $D_1^j K_s : \Omega^2 \to L_j(\mathbb{R}^\kappa, \mathbb{R}^{d\times p})$, $0 \le j \le m$, exist as continuous functions and $h_s \in C^m(\Omega)^d$, $s \in \mathbb{I}'$, then Cor. 2.6 ensures $\varphi(t; \tau, u) \in C^m(\Omega)^d$.

For growth functions g_t of class C_f^1, Thm. B.8 (applied to f_t from (3.7)) yields that the right-hand side $\mathcal{F}_t$, $t \in \mathbb{I}'$, is continuously differentiable and the variational eqn. (V_0) associated to an entire solution $(\phi_t)_{t\in\mathbb{I}}$ has the coefficients

$$D\mathcal{F}_t(\phi_t)v = \int_\Omega K_t(\cdot, y) D_2 g_t(y, \phi_t(y)) v(y)\,\mathrm{d}y \quad \text{for all } s \in \mathbb{I}',\, v \in C_d.$$

Note that the complete continuity of $\mathcal{F}_t$ from Prop. 2.1 implies that $D\mathcal{F}_t(\phi_t)$ is compact due to [13, p. 89, Prop. 6.5].

While collocation methods apply to general right-hand sides (2.3) and in particular to HDEs, we now discuss an approach being tailor-made for (H_0). For linear integral equations, degenerate kernel methods are discussed in [1, pp. 23, Chapt. 2], [5, pp. 65ff, Sect. 4.2] and [11, pp. 195ff, Chapt. 11]. Our nonlinear set-up is a natural extension, where we suppose that $e_1, \dots, e_{d_n} : \Omega \to \mathbb{R}$ are linearly independent, continuous functions and the kernels $K_t : \Omega^2 \to \mathbb{R}^{d \times p}$ are approximated by degenerate kernels

$$K_t^n(x, y) := \sum_{j=1}^{d_n} e_j(x) B_t^j(y) \quad \text{for all } x, y \in \Omega \tag{3.8}$$

with continuous coefficient matrices $B_t^j : \Omega \to \mathbb{R}^{d \times p}$, $1 \leq j \leq d_n$. Given this, a *degenerate kernel method* is a discretization (I_n) of the form

$$\begin{aligned}\mathcal{F}_t^n(u) &:= \int_\Omega K_t^n(\cdot, y) g_t(y, u(y)) \, \mathrm{d}y + h_t \\ &= \sum_{j=1}^{d_n} \int_\Omega B_t^j(y) g_t(y, u(y)) \, \mathrm{d}y e_j + h_t \quad \text{for all } t \in \mathbb{I}'.\end{aligned} \tag{3.9}$$

We point out that also (3.9) is merely a semi-discretization since it remains to evaluate the d_n $\mathbb{R}^d$-valued integrals $\int_\Omega B_t^j(y) g_t(y, u(y)) \, \mathrm{d}y$ in actual simulations.

Theorem 3.5 (degenerate kernel methods)**.** *Let $m \in \mathbb{N}_0$, $n \in \mathbb{N}$. The general solution $\varphi_n(t; \tau, \cdot) : C_d \to C_d$ of degenerate kernel methods* (3.9) *is well-defined, completely continuous and uniformly continuous on bounded sets for $\tau < t$. Moreover, one has:*

- (a) *If $e_1, \dots, e_{d_n} \in C^m(\Omega)$ and $h_s \in C^m(\Omega)^d$ holds for every $s \in \mathbb{I}'$, then $\varphi_n(t; \tau, u) \in C^m(\Omega)^d \cap (h_{t-1} + \operatorname{span}\{e_1, \dots, e_{d_n}\}^d)$ for all $\tau < t$, $u \in C_d$.*
- (b) *If g_s is of class C_f^m, then $\varphi_n(t; \tau, \cdot) \in C^m(C_d, C_d)$.*

Proof. Let $s \in \mathbb{I}'$. For fixed $n \in \mathbb{N}$ and $1 \leq j \leq d_n$ we abbreviate

$$\phi_s^j(x, y, z) := B_s^j(y) g_s(y, z) e_j(x), \quad \tilde{\mathcal{F}}_s^j(u) := \int_\Omega \phi_s^j(\cdot, y, u(y)) \, \mathrm{d}y : \Omega \to \mathbb{R}^d$$

and obtain $\mathcal{F}_s^n(u) = \sum_{j=1}^{d_n} \tilde{\mathcal{F}}_s^j(u) + h_s$ from (3.9). Thus, it suffices to establish the claimed properties for each summand $\tilde{\mathcal{F}}_s^j$. First, all $\phi_s^j : \Omega^2 \times \mathbb{R}^d \to \mathbb{R}^d$ are continuous and therefore Thm. B.5 guarantees that $\tilde{\mathcal{F}}_s^j : C_d \to C_d$ is well-defined, completely continuous and uniformly continuous on every bounded set. These properties transfer to $\mathcal{F}_s^n$, which moreover maps bounded subsets of C_d into bounded sets. Whence, the composition $\varphi_n(t; \tau, \cdot)$, $\tau \leq t$, is well-defined, as well as completely continuous and uniformly continuous on every bounded set.

(a) Given $u \in C_d$ our assumptions imply that every function ϕ_s^j is of class C_1^m and hence Thm. B.7 yields $\tilde{\mathcal{F}}_s^j(u) \in C^m(\Omega)^d$. The claim follows since $\mathcal{F}_s^n(u)$ is the sum of C^m-functions and as above, also $\varphi_n(t; \tau, \cdot)$ is such a function. The remaining inclusion is evident from (3.9).

(b) Our assumptions ensure that ϕ_s^j is of class C_f^m, therefore Thm. B.8 yields that every $\tilde{\mathcal{F}}_s^j$ is m-times continuously differentiable, and the same holds for the sum $\mathcal{F}_s^n$. Thus, the claim results from the chain rule [12, p. 337]. □

Corollary 3.6. *If for every $r > 0$, $s \in \mathbb{I}'$ there is a measurable function $\tilde{\lambda}_{s,r} : \Omega \to \mathbb{R}_+$ with*

$$\ell_s(r) := \sup_{x\in\Omega} \int_\Omega |K_s(x,y)|\,\tilde{\lambda}_{s,r}(y)\,\mathrm{d}y < \infty, \qquad l_s(r) := \int_\Omega \tilde{\lambda}_{s,r}(y)\,\mathrm{d}y < \infty,$$

such that the Lipschitz condition

$$|g_s(y,z) - g_s(y,\bar{z})| \le \tilde{\lambda}_s(y)\,|z-\bar{z}| \quad \textit{for all } y \in \Omega,\ z, \bar{z} \in \bar{B}_r(0)$$

holds, then $B \subseteq \bar{B}_r(0) \subset C_d$ implies $\operatorname{lip} \mathcal{F}_s^n \le \ell_s(r) + l_s(r)\,\|K_s^n - K_s\|$.

Proof. Let $r > 0$ and $s \in \mathbb{I}'$. Given $u, \bar{u} \in B \subseteq \bar{B}_r(0)$ one has the estimate

$$\begin{aligned} |\mathcal{F}_s^n(u)(x) - \mathcal{F}_s^n(\bar{u})(x)| &\overset{(3.9)}{\le} \left| \int_\Omega K_s(x,y)\,[g_s(y,u(y)) - g_s(y,\bar{u}(y))]\,\mathrm{d}y \right| \\ &\quad + \left| \int_\Omega [K_s^n(x,y) - K_s(x,y)]\,[g_s(y,u(y)) - g_s(y,\bar{u}(y))]\,\mathrm{d}y \right| \\ \le \int_\Omega |K_s(x,y)|\,\tilde{\lambda}_{s,r}(y)\,\mathrm{d}y\,\|u-\bar{u}\| &+ \int_\Omega |K_s(x,y) - K_s^n(x,y)|\,\tilde{\lambda}_{s,r}(y)\,\mathrm{d}y\,\|u-\bar{u}\| \\ &\le (\ell_s(r) + l_s(r)\,\|K_s - K_s^n\|)\,\|u-\bar{u}\| \quad \text{for all } x \in \Omega. \end{aligned}$$

The assertion readily results by passing to the supremum over $x \in \Omega$. □

Corollary 3.7 (variational degenerate kernel equation)**.** *Let $n \in \mathbb{N}$, $s \in \mathbb{I}'$ and $(\phi_t)_{t\in\mathbb{I}}$ be an entire solution to (I_n) given in* (3.9)*. If g_s is of class C_f^1, then $D\mathcal{F}_s^n(\phi_s)$ is compact with*

$$D\mathcal{F}_s^n(\phi_s)v = \sum_{j=1}^{d_n} \int_\Omega B_s^j(y) D_2 g_s(y, \phi_s(y)) v(y)\,\mathrm{d}y e_j \quad \textit{for all } v \in C_d.$$

Proof. Our assumptions guarantee that $(x,y,z) \mapsto B_s^j(y) g_s(y,z) e_j(x)$, $1 \le j \le d_n$, are continuous and of class C_f^1. Therefore, it follows from Thm. B.8 that the functions $\tilde{\mathcal{F}}_s^j : C_d \to C_d$ defined in the proof of Thm. 3.5 are of class C^1 with derivative

$$D\tilde{\mathcal{F}}_s^j(u)v := \int_\Omega B_s^j(y) D_2 g_s(y, u(y)) v(y)\,\mathrm{d}y e_j(x) \quad \text{for all } u, v \in C_d.$$

Then the assertion follows due to $D\mathcal{F}_s^n(u) = \sum_{j=1}^{d_n} D\tilde{\mathcal{F}}_s^j(u)$. □

Given a bounded subset $B \subset C_d$, $u \in B$, $n \in \mathbb{N}$ and $t \in \mathbb{I}'$, estimates for the local discretization error in degenerate kernel methods are based on the estimate

$$|\varepsilon_t^n(u)(x)| \le \int_\Omega |[K_t^n(x,y) - K_t(x,y)]\,g_t(y,u(y))|\,\mathrm{d}y \quad \text{for all } x \in \Omega. \tag{3.10}$$

Among various techniques to approximate kernels K_t via K_t^n, we exemplarily consider piecewise linear approximation on $\Omega = [a,b]$, where $e_0, \ldots, e_n : [a,b] \to \mathbb{R}_+$ denote the hat functions introduced in Sect. 3.1.2:

Example 3.8 (piecewise linear degenerate kernels)**.** Suppose $B_t^j : [a,b] \to \mathbb{R}^{d\times p}$ in (3.8) are given by

$$B_t^j(y) := K_t(\xi_j, y) \quad \text{for all } 0 \le j \le n,$$

which corresponds to a piecewise linear interpolation in the first argument of K_t. Thus, using [6, p. 241, p. 247] the interpolation errors

$$|K_t^n(x,y) - K_t(x,y)| \le \begin{cases} \omega(\frac{b-a}{n}, K_t(\cdot,y)), & K_t(\cdot,y) \in C[a,b]^{d\times p}, \\ \frac{b-a}{4n}\omega(\frac{b-a}{n}, D_1K_t(\cdot,y)), & K_t(\cdot,y) \in C^1[a,b]^{d\times p}, \\ \frac{(b-a)^2}{8n^2}\sup_{x\in[a,b]}\left|D_1^2K_t(x,y)\right|, & K_t(\cdot,y) \in C^2[a,b]^{d\times p} \end{cases}$$

follow for all $x, y \in [a,b]$. Combining this with (3.10) implies that the local discretization error fulfills (3.1) with convergence function

$$\Gamma_0(\rho) := \sup_{t\in\mathbb{I}'} \begin{cases} \int_a^b \omega((b-a)\rho, K_t(\cdot,y))\,\mathrm{d}y, & K_t(\cdot,y) \in C[a,b]^{d\times p}, \\ \frac{b-a}{4}\rho\int_a^b \omega((b-a)\rho, D_1K_t(\cdot,y))\,\mathrm{d}y, & K_t(\cdot,y) \in C^1[a,b]^{d\times p}, \\ \frac{(b-a)^2}{8}\rho^2\int_a^b \sup_{x\in[a,b]}\left|D_1^2K_t(x,y)\right|\,\mathrm{d}y, & K_t(\cdot,y) \in C^2[a,b]^{d\times p} \end{cases}$$

and $K(B) := \sup_{t\in\mathbb{I}'}\sup_{u\in B}\sup_{x\in\Omega}|g_t(x,u(x))|$. In particular, provided the above expressions exist and $K_t(\cdot,y)$ is of class C^m, one has C^0-convergence of order m.

Example 3.9 (bilinear degenerate kernels)**.** Let $B_t^j : [a,b] \to \mathbb{R}^{d\times p}$ in (3.8) read as

$$B_t^j(y) := \sum_{j_2=0}^n e_{j_2}(y)K_t(\xi_j, \xi_{j_2}) \quad \text{for all } 0 \le j \le n,$$

that is, K_t is approximated by piecewise linear functions over $[a,b]^2$. For C^2-kernels $K_t : [a,b]^2 \to \mathbb{R}^{d\times p}$ this leads to the error estimate

$$\begin{aligned} |K_t^n(x,y) - K_t(x,y)| &\overset{(1.5)}{=} \max_{j_1=1}^d \sum_{j_2=1}^p |K_t^n(x,y)_{j_1j_2} - K_t(x,y)_{j_1j_2}| \\ &\le \frac{(b-a)^2}{8n^2}\max_{j_1=1}^d \sum_{j_2=1}^p\sum_{l=1}^2 \left\|D_l^2K_t(\cdot)_{j_1j_2}\right\| \text{ for all } x,y\in[a,b], \end{aligned}$$

if we apply [6, p. 267] to each matrix entry. Thanks to (3.10) this guarantees a local discretization error satisfying (3.1), a convergence function

$$\Gamma_0(\rho) := \frac{(b-a)^2}{8}\rho^2\sup_{t\in\mathbb{I}'}\max_{j_1=1}^d\sum_{j_2=1}^p\sum_{l=1}^2\left\|D_l^2K_t(\cdot)_{j_1j_2}\right\|,$$

$K(B) := \sup_{t\in\mathbb{I}'}\sup_{u\in B}\int_a^b |g_t(y,u(y))|\,\mathrm{d}y$ and thus C^0-convergence of order 2.

4. Global discretization error

This section studies how the solutions to general IDEs (I_0) and to their spatial discretizations $(I_n)_{n\in\mathbb{N}}$ are related over time, provided adequate convergence assumptions are satisfied. On finite time intervals, it is shown that convergent methods yield a global discretization error with limit 0 as $n \to \infty$. This situation changes on infinite intervals, where typically exponential error growth occurs. For contractive equations (I_0), nevertheless, convergence does still hold.

Let φ_n denote the general solution of (I_n) and φ stand for the general solution of (I_0). The *global discretization error*

$$\mathcal{E}^n(t;\tau,u) := \varphi(t;\tau,u) - \varphi_n(t;\tau,u) \quad \text{for all } \tau \le t$$

describes the propagation of the local error as time evolves from τ to $t > \tau$; one clearly has $\mathcal{E}^n(t+1;t,u) = \varepsilon_t^n(u)$ for all $t \in \mathbb{I}'$. Our goal is to estimate the global discretization error by the local discretization error ε_t^n. Suitable estimates on its order for concrete methods were derived above in Sect. 3.

For the upcoming results we assume that a discretization family $(I_n)_{n\in\mathbb{N}}$ is *Lipschitz*, i.e. for $s \in \mathbb{I}'$ and bounded sets $B \subset C_d$ there exists a $L_s(B) \ge 0$ such that

$$\|\mathcal{F}_s^n(u) - \mathcal{F}_s^n(\bar{u})\| \le L_s(B)\,\|u - \bar{u}\| \quad \text{for all } n \in \mathbb{N},\ u, \bar{u} \in B.$$

Note that the constants $L_s(B)$ are assumed to be uniform in the parameter n:

- For collocation methods, according to Cor. 3.2 this requires the projections P_n to satisfy $\sup_{n\in\mathbb{N}} \|P_n\| < \infty$. On the one hand, when working with piecewise linear collocation, even $\|P_n\| \equiv 1$ holds, so that (I_0) and all (I_n) have the same Lipschitz constants. On the other hand, polynomial interpolation should be avoided since the optimal growth of $\|P_n\|$ is basically like $\ln n$ (see [1, p. 93]).
- In degenerate kernel methods we learn from Cor. 3.6 that the Lipschitz constants of (H_0) and (3.9) essentially differ by the norm $\|K_t^n - K_t\|$. Boundedness, or even convergence to 0 as $n \to \infty$, in this quantity, yields a Lipschitzian discretization family.

Cors. 3.2 and 3.6 provide conditions under which a Lipschitz condition for $\mathcal{F}_s$ extends to a collocation resp. degenerate kernel discretization $\mathcal{F}_s^n$, $s \in \mathbb{I}'$. Hence, for such methods it suffices to assume a Lipschitz condition for (I_0).

When dealing with nonautonomous difference equations (I_0) it is helpful to introduce the following terminology. A subset $\mathcal{B} \subseteq \mathbb{I} \times C_d$ is called *nonautonomous set* with the *t-fiber* $\mathcal{B}(t) := \{u \in C_d : (t,u) \in \mathcal{B}\}$ for every $t \in \mathbb{I}$. We say $\mathcal{B}$ is bounded, if every fiber $\mathcal{B}(t)$, $t \in \mathbb{I}$, is bounded.

Theorem 4.1. *Let $m \in \mathbb{N}_0$, $\tau, T \in \mathbb{I}$ with $\tau \le T$ fixed, $u \in C_d$ and $\mathcal{B} \subseteq \mathbb{I} \times C_d^m$ denote a nonautonomous set being bounded (in the C^m-topology). If a Lipschitzian discretization familiy $(I_n)_{n\in\mathbb{N}}$ is C^m-convergent and*

$$\varphi(t;\tau,u) \in \mathcal{B}(t) \quad \textit{for all } \tau \le t \le T, \tag{4.1}$$

then for every $\rho > 0$ there exists a $N_0 \in \mathbb{N}$ such that $\varphi_n(t;\tau,u) \in B_\rho(\mathcal{B}(t))$ and

$$\|\mathcal{E}^n(t;\tau,u)\| \leq \Gamma_0(\tfrac{1}{n}) \sum_{s=\tau}^{t-1} K(\mathcal{B}(s)) \prod_{r=s+1}^{t-1} L_r\big(B_\rho(\mathcal{B}(r))\big) \quad \text{for all } n \geq N_0 \tag{4.2}$$

and $\tau \leq t \leq T$.

Remark 4.2. For autonomous IDEs (I_0) and $\mathcal{B} = \mathbb{Z} \times B$ with some C^m-bounded set $B \subset C_d^m$, the constants in (4.2) become uniform in $s \in \mathbb{I}'$. Then the estimate (4.2) for the global discretization error simplifies to

$$\|\mathcal{E}^n(t;\tau,u)\| \leq K(B)\Gamma_0(\tfrac{1}{n}) \frac{L(B_\rho(B))^{t-\tau} - 1}{L(B_\rho(B)) - 1} \quad \text{for all } \tau \leq t \leq T.$$

In the typical case $L(B_\rho(B)) > 1$ this means that for fixed $n \in \mathbb{N}$ the global error $\mathcal{E}^n(t;\tau,u)$ is exponentially growing as $t \to \infty$. For this reason such "classical" error estimates are useless when studying asymptotic properties of (I_0) and relating them to (I_n). Yet, for a finite interval $\tau \leq t \leq T$ one has convergence to 0 as $n \to \infty$.

Proof. Let $\rho > 0$ and $(\tau, u) \in \mathcal{B}$ be given; abbreviate $\varphi(t) := \varphi(t;\tau,u)$. By induction over $t \geq \tau$ we show the existence of a $N_0 = N_0(T) \in \mathbb{N}$ such that (4.2), as well as

$$\varphi_n(t;\tau,u) \in B_\rho(\mathcal{B}(t)) \quad \text{for all } \tau \leq t \leq T,\ n \geq N_0 \tag{4.3}$$

hold. For $t = \tau$ this is trivially true. Suppose that (4.3), (4.2) are satisfied for some fixed $t < T$, set $x_t := \|\mathcal{E}^n(t;\tau,u)\|$, and we obtain

$$\begin{aligned}
x_{t+1} &\stackrel{(4.1)}{\leq} \|\mathcal{F}_t^n(\varphi_n(t)) - \mathcal{F}_t^n(\varphi(t))\| + \|\varepsilon_t^n(\underbrace{\varphi(t)}_{\in \mathcal{B}(t)})\| \\
&\stackrel{(3.1)}{\leq} \|\mathcal{F}_t^n(\underbrace{\varphi_n(t;\tau,u)}_{\stackrel{(4.3)}{\in} B_\rho(\mathcal{B}(t))}) - \mathcal{F}_t^n(\underbrace{\varphi(t)}_{\stackrel{(4.1)}{\in} \mathcal{B}(t)})\| + K(\mathcal{B}(t))\Gamma_0(\tfrac{1}{n}) \\
&\leq L_t\big(B_\rho(\mathcal{B}(t))\big)x_t + K(\mathcal{B}(t))\Gamma_0(\tfrac{1}{n}) \\
&\stackrel{(4.2)}{\leq} L_t\big(B_\rho(\mathcal{B}(t))\big)\Gamma_0(\tfrac{1}{n}) \sum_{s=\tau}^{t-1} K(\mathcal{B}(s)) \prod_{r=s+1}^{t-1} L_r\big(B_\rho(\mathcal{B}(r))\big) + K(\mathcal{B}(t))\Gamma_0(\tfrac{1}{n}) \\
&= \Gamma_0(\tfrac{1}{n}) \sum_{s=\tau}^{t} K(\mathcal{B}(s)) \prod_{r=s+1}^{t} L_r\big(B_\rho(\mathcal{B}(r))\big).
\end{aligned}$$

Due to $\Gamma_0 \in \mathfrak{N}$, for a sufficiently large N_0 it holds $\|\mathcal{E}^n(t+1;\tau,u)\| < \rho$ and hence the desired inclusion $\varphi_n(t+1;\tau,u) \in B_\rho(\mathcal{B}(t+1))$. This concludes the proof. □

Nevertheless, IDEs (I_0) having an ambient contraction property allow uniform estimates on the global discretization error. Here we retreat to C^0-convergence:

Corollary 4.3. *Let $\tau \in \mathbb{I}$, $u \in C_d$. Suppose that there exist reals $K_0 \geq 1$, $\alpha \in (0,1)$ and $\bar{N} \in \mathbb{N}$ with*

$$\prod_{r=s}^{t-1} \operatorname{lip} \mathcal{F}_r \leq K_0 \alpha^{t-s}, \qquad \prod_{r=s}^{t-1} \operatorname{lip} \mathcal{F}_r^n \leq K_0 \alpha^{t-s} \quad \text{for all } s \leq t,\ n \geq \bar{N} \tag{4.4}$$

and that (I_0) has a bounded forward solution. If $(I_n)_{n \in \mathbb{N}}$ is C^0-convergent, then there exist $N_0 \geq \bar{N}$ and $\tilde{K} \geq 0$ with $\|\mathcal{E}^n(t; \tau, u)\| \leq \frac{\tilde{K}}{1-\alpha} \Gamma_0(\frac{1}{n})$ for all $\tau \leq t$, $n \geq N_0$.

It follows from Cor. 3.2 and (3.6) that the second estimate in (4.4) trivially holds for piecewise linear collocation methods. When using degenerate kernels, then Cor. 3.6 allows to conclude the second estimate in (4.4) from the first one, provided $\|K_t^n - K_t\|$ is sufficiently small, which might hold for a sufficiently large $\bar{N}$.

Proof. Given an initial time $\tau \in \mathbb{I}$, let $(\phi_t^*)_{\tau \leq t}$ denote a bounded forward solution to (I_0), i.e. there exists a $R_1 \geq 0$ such that $\|\phi_t^*\| \leq R_1$ holds for all $\tau \leq t$. If $t_1 \in \mathbb{N}_0$ is chosen so large that $K_0 \alpha^t \leq 1$ for all $t_1 \leq t$, then

$$\begin{aligned} \|\varphi(t)\| &\leq \|\varphi(t) - \phi_t^*\| + \|\phi_t^*\| = \|\varphi(t) - \varphi(t; \tau, \phi_\tau^*)\| + \|\phi_t^*\| \\ &\leq \left(\prod_{r=\tau}^{t-1} \operatorname{lip} \mathcal{F}_r \right) \|u - \phi_\tau^*\| + R_1 \stackrel{(4.4)}{\leq} K_0 \alpha^{t-\tau} \|u - \phi_\tau^*\| + R_1 \\ &\leq \|u - \phi_\tau^*\| + R_1 \quad \text{for all } t_1 \leq t - \tau. \end{aligned}$$

This shows $\|\varphi(t)\| \leq \max\left\{R_1 + \|u - \phi_\tau^*\|, \max_{r=\tau}^{\tau+t_1-1} \|\varphi(r)\|\right\} =: R$ for all $\tau \leq t$. Therefore, the assumptions of Thm. 4.1 hold with $\mathcal{B} := \mathbb{I} \times B_R(0)$ and arbitrary times $T \geq \tau$. Whence, setting $\tilde{K} := K_0 K(B_R(0))$ there exists an $N_0 \geq \bar{N}$ such that

$$\begin{aligned} \|\mathcal{E}^n(t; \tau, u)\| &\stackrel{(4.2)}{\leq} \Gamma_0(\tfrac{1}{n}) \sum_{s=\tau}^{t-1} K(B_R(0)) \prod_{r=s+1}^{t-1} \operatorname{lip} \mathcal{F}_r^n \\ &\stackrel{(4.4)}{\leq} \tilde{K} \Gamma_0(\tfrac{1}{n}) \sum_{s=0}^{t-\tau-1} \alpha^s \leq \frac{\tilde{K} \Gamma_0(\frac{1}{n})}{1-\alpha} \end{aligned}$$

holds for all $\tau \leq t$, $n \geq N_0$. □

We next provide an estimate for the global discretization error in the C^m-topology. To keep the technical effort at a reasonable level, let us restrict to Hammerstein integrodifference eqns. (H_0) and their degenerate kernel discretizations:

Corollary 4.4 (Hammerstein equations)**.** *Let $\rho > 0$, $m \in \mathbb{N}_0$ and choose $N_0 \in \mathbb{N}$ from Thm. 4.1. Suppose a Hammerstein eqn. (H_0) fulfills for $\tau \leq s < T$, $0 \leq j \leq m$ that*

(i) $D_1^j K_s : \Omega \times \Omega \to L_j(\mathbb{R}^\kappa, \mathbb{R}^{d \times p})$ exist as continuous functions, $h_s \in C^m(\Omega)^d$,

(ii) for all $r > 0$ there exists a measurable function $\tilde{\lambda}_{s,r} : \Omega \to \mathbb{R}_+$ such that

$$|g_s(y, z) - g_s(y, \bar{z})| \leq \tilde{\lambda}_{s,r}(y) |z - \bar{z}| \quad \text{for all } y \in \Omega,\ z, \bar{z} \in B_r(0)$$

and $\ell_s^j(r) := \sup_{n \in \mathbb{N}} \sup_{y \in \Omega} \int_\Omega \left| D_1^j K_s^n(x, y) \right| \tilde{\lambda}_{s,r}(y) \, \mathrm{d}y < \infty$.

If the functions $e_1, \dots, e_{d_n}$ in a C^0-convergent degenerate kernel discretization (3.9) *are of class C^m and there exists a constant $\bar{K}_1 \geq 0$ such that*

$$\max_{j=0}^{m} \sup_{x\in\Omega} \int_\Omega \left| D_1^j K_t^n(x,y) - D_1^j K_t(x,y) \right| \mathrm{d}y \leq \bar{K}_1 \Gamma_0(\tfrac{1}{n}) \quad \text{for all } n \geq N_0, \tag{4.5}$$

then $\mathcal{E}^n(t;\tau,u) \in C^m(\Omega,\mathbb{R}^d)$ with the error estimate

$$\begin{aligned}\|\mathcal{E}^n(t;\tau,u)\|_m \leq & \Gamma_0(\tfrac{1}{n}) \max_{j=0}^{m} \ell_t^j(R+\rho) \sum_{s=\tau}^{t-1} K(\mathcal{B}(s)) \prod_{r=s+1}^{t-1} L_r\big(B_\rho(\mathcal{B}(r))\big) \\ & + \Gamma_0(\tfrac{1}{n})\bar{K}_1 \sup_{y\in\Omega, z\in\mathcal{B}(t)} |g_t(y,z)| \quad \text{for all } n \geq N_0\end{aligned}$$

and $\tau \leq t \leq T$ hold.

Proof. Let $\tau \in \mathbb{I}$, $\tau \leq t$ and $u, \bar{u} \in C_d$. In order to apply Thm. 4.1 we first show that $(I_n)_{n\in\mathbb{N}}$ is Lipschitz. If we set $r := \max\{\|u\|, \|\bar{u}\|\}$, then

$$\begin{aligned}|\mathcal{F}_s^n(u)(x) - \mathcal{F}_s^n(\bar{u})(x)| &\overset{(3.9)}{=} \left| \int_\Omega K_s^n(x,y)\left[g_s(y,u(y)) - g_s(y,\bar{u}(y))\right] \mathrm{d}y \right| \\ &\leq \int_\Omega |K_s^n(x,y)| \, |g_s(y,u(y)) - g_s(y,\bar{u}(y))| \,\mathrm{d}y \\ &\leq \int_\Omega |K_s^n(x,y)| \, \tilde{\lambda}_{s,r}(y)\,\mathrm{d}y \, \|u-\bar{u}\| \leq \ell_s^0(r)\, \|u-\bar{u}\|\end{aligned}$$

and passing to the supremum over $x \in \Omega$ yields $\|\mathcal{F}_s^n(u) - \mathcal{F}_s^n(\bar{u})\| \leq \ell_s^0(r)\,\|u-\bar{u}\|$ for $s \in \mathbb{I}'$. Hence, the discretization is Lipschitz. From Thm. B.7 and (3.9) we obtain

$$\begin{aligned}\varphi(t+1;\tau,u)^{(j)}(x) &\equiv \int_\Omega D_1^j K_t(x,y) g_t(y,\varphi(t;\tau,u)(y))\,\mathrm{d}y + h_t^{(j)}(x), \\ \varphi_n(t+1;\tau,u)^{(j)}(x) &\equiv \int_\Omega D_1^j K_t^n(x,y) g_t(y,\varphi_n(t;\tau,u)(y))\,\mathrm{d}y + h_t^{(j)}(x)\end{aligned}$$

on Ω for $0 \leq j \leq m$. With the nonautonomous set $\mathcal{B}$ from Thm. 4.1 we choose some $R > 0$ so large that the inclusion $\mathcal{B}(t) \subseteq \bar{B}_R(0)$ holds for $\tau \leq t \leq T$. Hence, $\varphi_n(t;\tau,u) \in \bar{B}_{R+\rho}(0)$ and as above, this yields

$$\begin{aligned}&\left|\varphi_n(t+1;\tau,u)^{(j)}(x) - \varphi(t+1;\tau,u)^{(j)}(x)\right| \\ &\quad\leq \left| \int_\Omega D_1^j K_t^n(x,y)\left[g_t(y,\varphi_n(t;\tau,u)(y)) - g_t(y,\varphi(t;\tau,u)(y))\right] \mathrm{d}y \right| \\ &\qquad + \left| \int_\Omega \left[D_1^j K_t^n(x,y) - D_1^j K_t(x,y)\right] g_t(y,\varphi(t;\tau,u)(y))\,\mathrm{d}y \right| \\ &\quad\leq \ell_t^j(R+\rho)\, \|\varphi_n(t;\tau,u) - \varphi(t;\tau,u)\| \\ &\qquad + \int_\Omega \left| D_1^j K_t^n(x,y) - D_1^j K_t(x,y) \right| \mathrm{d}y \sup_{y\in\Omega, z\in\mathcal{B}(t)} |g_t(y,z)|\end{aligned}$$

$$\overset{(4.5)}{\leq} \ell_t^j(R+\rho)\,\|\mathcal{E}^n(t;\tau,u)\| + \bar{K}_1\Gamma_0(\tfrac{1}{n}) \sup_{y\in\Omega, z\in\mathcal{B}(t)} |g_t(y,z)| \quad \text{for all } n \geq N_0$$

and $0 \leq j \leq m$. Then (4.2) implies the claim. □

We conclude with a local variant, but also a tightening of Thm. 4.1:

Proposition 4.5. *Let $\tau, T \in \mathbb{I}$ with $\tau \leq T$ and $(\phi_t^*)_{\tau \leq t}$ be a solution of (I_0). Suppose there exist a $\rho_0 > 0$ and functions $\Gamma_0^0, \Gamma_0^1, \gamma_0 \in \mathfrak{N}$ such that for all $\tau \leq s \leq T$ there exist $L_s^* \geq 0$ fulfilling*

$$\|\mathcal{F}_s^n(\phi_s^*) - \mathcal{F}_s(\phi_s^*)\| \leq \Gamma_0^0(\tfrac{1}{n}), \qquad \|\mathcal{F}_s^n(u) - \mathcal{F}_s^n(\bar{u})\| \leq L_s^*\,\|u - \bar{u}\| \tag{4.6}$$

for all $u, \bar{u} \in B_{\rho_0}(\phi_s^*)$ and $n \in \mathbb{N}$. Then there is a $N_0 \in \mathbb{N}$ so that for $n \geq N_0$, times $\tau \leq t \leq T$ and initial values $u \in B_{\frac{\rho_0}{2L}}(\phi_\tau^*)$ one has

(a) *$\varphi_n(t;\tau,u) \in B_{\rho_0}(\phi_t^*)$ and*

$$\|\varphi_n(t;\tau,u) - \phi_t^*\| \leq \left(\prod_{r=\tau}^{t-1} L_r^*\right)\|u - \phi_\tau^*\| + \Gamma_0^0(\tfrac{1}{n})\sum_{s=\tau}^{t-1}\prod_{r=s+1}^{t-1} L_r^*,$$

(b) *if additionally for all $\tau \leq s \leq T$, $n \in \mathbb{N}$ the maps $\mathcal{F}_s, \mathcal{F}_s^n$ are of class C^1 with*

$$\|D\mathcal{F}_s^n(u) - D\mathcal{F}_s^n(\phi_s^*)\| \leq \gamma_0(\|u - \phi_s^*\|), \tag{4.7}$$

$$\|D\mathcal{F}_s^n(\phi_s^*) - D\mathcal{F}_s(\phi_s^*)\| \leq \Gamma_0^1(\tfrac{1}{n}) \tag{4.8}$$

for all $u \in B_{\rho_0}(\phi_s^*)$, then

$$\|D_3\varphi_n(t;\tau,u) - D_3\varphi(t;\tau,\phi_\tau^*)\| \leq \sum_{s=\tau}^{t-1}\ell_s\left[\gamma_0(\|\varphi_n(s;\tau,u) - \phi_s^*\|) + \Gamma_0^1(\tfrac{1}{n})\right]\prod_{r=s+1}^{t-1} L_r^*, \tag{4.9}$$

where we abbreviate $L := \max_{t=\tau}^{T}\prod_{r=\tau}^{t-1} L_r^*$ and $\ell_t := \left\|\prod_{s=\tau}^{t-1} D\mathcal{F}_s(\phi_s^*)\right\|$.

Proof. Let $\varphi_n(t) := \varphi_n(t;\tau,u)$, $x_t := \|\varphi_n(t) - \phi_t^*\|$ and we proceed by induction:

(a) For $t = \tau$ the claim obviously holds true. As induction step $t \to t+1$ one has

$$x_{t+1} \leq \|\mathcal{F}_t^n(\varphi_n(t)) - \mathcal{F}_t^n(\phi_t^*)\| + \|\mathcal{F}_t^n(\phi_t^*) - \mathcal{F}_t(\phi_t^*)\| \overset{(4.6)}{\leq} L_t^* x_t + \Gamma_0^0(\tfrac{1}{n}),$$

since $\varphi_n(t) \in B_{\rho_0}(\phi_t^*)$ holds by induction hypothesis, which, in turn, also yields

$$x_{t+1} \leq \left(\prod_{r=\tau}^{t} L_r^*\right)\|u - \phi_\tau^*\| + \Gamma_0^0(\tfrac{1}{n})\sum_{s=\tau}^{t}\prod_{r=s+1}^{t} L_r^* < \tfrac{\rho_0}{2} + \tfrac{\rho_0}{2} = \rho_0$$

for all $\tau \leq t < T$, provided $u \in B_{\rho_0/(2L)}(\phi_\tau^*)$ and one chooses $N_0 \in \mathbb{N}$ so large that the second term in the above sum is bounded above by $\frac{\rho_0}{2}$.

(b) First of all, due to (2.1) the partial derivatives $D_3\varphi, D_3\varphi_n$ exist, because φ, φ_n are compositions of C^1-mappings. It remains to perform the induction step in the proof of the inequality (4.9). Let us write $y_t := \|D_3\varphi_n(t;\tau,u) - D_3\varphi(t;\tau,\phi_\tau^*)\|$. We first observe that the Lipschitz condition (4.6) implies $\|D\mathcal{F}_t^n(u)\| \le L_t^*$ for all $u \in B_{\rho_0}(\phi_t^*)$ (see [15, p. 363, Prop. C.1.1]). Thus,

$$\begin{aligned} y_{t+1} &\overset{(2.2)}{\le} \left\| D\mathcal{F}_t^n(\varphi_n(t)) \prod_{s=\tau}^{t-1} D\mathcal{F}_s^n(\varphi_n(s)) - D\mathcal{F}_t^n(\varphi_n(t)) \prod_{s=\tau}^{t-1} D\mathcal{F}_s(\phi_s^*) \right\| \\ &\qquad + \left\| D\mathcal{F}_t^n(\varphi_n(t)) \prod_{s=\tau}^{t-1} D\mathcal{F}_s(\phi_s^*) - D\mathcal{F}_t(\phi_t^*) \prod_{s=\tau}^{t-1} D\mathcal{F}_s(\phi_s^*) \right\| \\ &\overset{(2.2)}{\le} \|D\mathcal{F}_t^n(\varphi_n(t))\|\, y_t + \|D\mathcal{F}_t^n(\varphi_n(t)) - D\mathcal{F}_t(\phi_t^*)\|\, \ell_t \end{aligned}$$

and the inclusion $\varphi_n(t) \in B_{\rho_0}(\phi_t^*)$ established above yields

$$\begin{aligned} y_{t+1} &\le L_t^* y_t + \|D\mathcal{F}_t^n(\varphi_n(t)) - D\mathcal{F}_t(\phi_t^*)\|\, \ell_t \\ &\le L_t^* y_t + \|D\mathcal{F}_t^n(\varphi_n(t)) - D\mathcal{F}_t^n(\phi_t^*)\|\, \ell_t + \|D\mathcal{F}_t^n(\phi_t^*) - D\mathcal{F}_t(\phi_t^*)\|\, \ell_t \\ &\overset{(4.7)}{\le} L_t^* y_t + \gamma_0(\|\varphi_n(t) - \phi_t^*\|)\ell_t + \|D\mathcal{F}_t^n(\phi_t^*) - D\mathcal{F}_t(\phi_t^*)\|\, \ell_t \\ &\overset{(4.8)}{\le} L_t^* y_t + \left(\gamma_0(\|\varphi_n(t) - \phi_t^*\|) + \Gamma_0^1(\tfrac{1}{n})\right) \ell_t. \end{aligned}$$

Having this at hand, the induction hypothesis guarantees the desired estimate

$$y_{t+1} \le \sum_{s=\tau}^{t} \ell_s \left[\gamma_0(\|\varphi_n(s) - \phi_s^*\|) + \Gamma_0^1(\tfrac{1}{n})\right] \prod_{r=s+1}^{t} L_r^*$$

for all $\tau \le t < T$ and initial values $u \in B_{\rho_0/(2L)}(\phi_\tau^*)$. □

5. Numerical simulations

Let us close by illuminating some results using concrete IDEs and their discretization over one- and two-dimensional domains. Throughout, we work with piecewise linear approximations. In order to obtain full discretizations preserving a quadratic error order, the following simulations replace the remaining integrals by the trapezoidal quadrature rule (see [4, p. 368]). For $\Omega = [a,b]$ this means

$$\int_a^b u(y)\,\mathrm{d}y = \frac{b-a}{2n}\Big(u(\eta_0) + 2\sum_{j=1}^{n-1} u(\eta_j) + u(\eta_n)\Big) - \frac{(b-a)^3}{12n^2} u''(\xi^*)$$

with nodes $\eta_j := a + j\frac{b-a}{n}$, $0 \le j \le n$, $n \in \mathbb{N}$ and some intermediate value $\xi^* \in [a,b]$.

First, we approximate scalar, nonlinear Urysohn IDEs

$$u_{t+1}(x) = G_t\left(x, u_t(x), \int_a^b f_t(x,y,u_t(y))\,\mathrm{d}y\right) \quad \text{for all } x \in [a,b]$$

by piecewise linear collocation. With the hat functions $e_0, \ldots, e_n : [a, b] \to \mathbb{R}$ defined in Sect. 3.1.2, this yields the semi-discretization

$$u_{t+1}(x) = \sum_{i=0}^{n} G_t\left(\eta_i, u_t(\eta_i), \int_a^b f_t(\eta_i, y, u_t(y))\,\mathrm{d}y\right) e_i(x) \quad \text{for all } x \in [a, b]$$

in $C[a, b]$. If we evaluate these functions at $x = \eta_i$, replace the integral by the trapezoidal rule and set $\upsilon_t(i) := u_t(\eta_i)$, one arrives at the explicit recursion[1]

$$\upsilon_{t+1}(i) = G_t\Bigg(\eta_i, \upsilon_t(i), \frac{b-a}{2n}\Bigg(f_t\left(\eta_i, \eta_0, \upsilon_t(0)\right) + 2\sum_{j=1}^{n-1} f_t\left(\eta_i, \eta_j, \upsilon_t(j)\right) + f_t\left(\eta_i, \eta_n, \upsilon_t(n)\right)\Bigg)\Bigg) \quad \text{for all } 0 \le i \le n \tag{5.1}$$

in $\mathbb{R}^{n+1}$, which is our desired full discretization of (I_0).

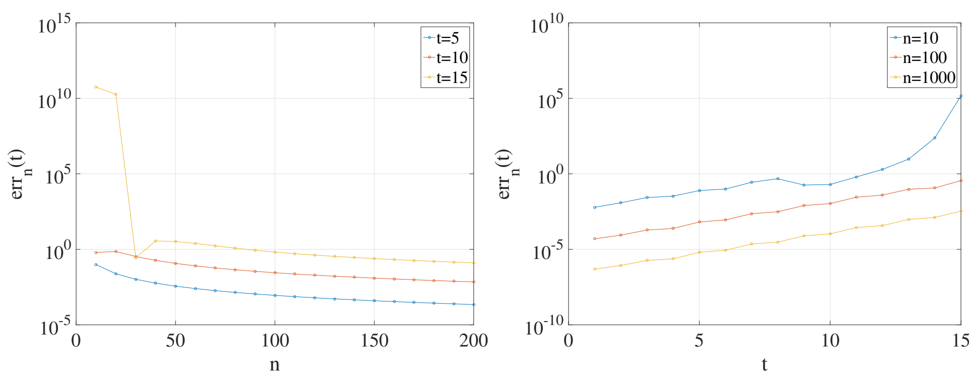

Figure 1: Errors in Exam. 5.1 as a function of the accuracy n (left) and time t (right)

Example 5.1. Let $\Omega = [-\pi, \pi]$ and $\alpha \in \mathbb{R}$. Consider the scalar, autonomous IDE

$$u_{t+1}(x) = \sin x + \alpha \frac{u_t(x)^2 - \pi^2 - x^2}{2} + \alpha \int_{-\pi}^{\pi} |x - y|\, u_t(y)^2\,\mathrm{d}y \tag{5.2}$$

for all $x \in [-\pi, \pi]$, with the time-constant solution $\phi^*(x) = \sin x$ being independent of the parameter α. A full discretization is of the form (5.1) with integration bounds $a = -\pi$, $b = \pi$ and

$$f_t(x, y, z) := |x - y|\, z^2, \qquad G_t(x, u, v) := \sin x + \alpha \tfrac{u^2 - \pi^2 - x^2}{2} + \alpha v.$$

[1] The reader might realize that this is the Nyström discretization [1, 5] of (I_0) based on the trapezoidal rule. We will observe the same phenomenon for the full discretizations in our subsequent examples.

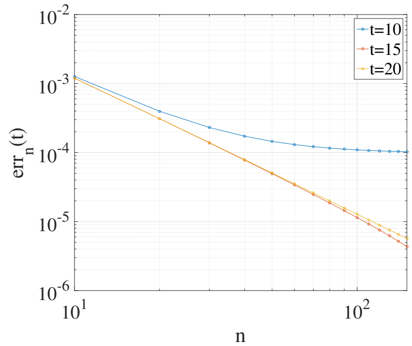

Figure 2: Error in Exam. 5.2 for $t \in \{10, 15, 25\}$ as a function of the accuracy n.

Applying the IDE (5.2) to the initial function $u_0(x) := \phi^*(x)$ yields a fixed point, i.e. $\varphi(t; 0, u_0) = \phi^*$ for all $t \geq 0$. Due to discretization errors, this changes when working with approximations (5.1): Fig. 1 demonstrates for $\alpha = 0.15$ how the error

$$\mathrm{err}_n(t) = \frac{1}{n} \sum_{j=0}^{n} |\varphi_n(t; 0, u_0)(\eta_j) - \phi^*(\eta_j)|$$

develops as function of the accuracy n (left) and in time t (right). In particular, err_{1000} grows exponentially with rate ≈ 1.88; an exponential growth of the global discretization error in t is predicted by Thm. 4.1. Moreover, keeping $t = 15$ fixed, the global error preserves the order of the method. The quadratic decay of $\mathrm{err}_n(15)$ ensured in Sect. 3.1.2 is reflected by the numerical value $\frac{\ln \mathrm{err}_{500}(15) - \ln \mathrm{err}_{250}(15)}{\ln 500 - \ln 250} \approx -2.07$.

Second, the next example is defined on a rectangle $\Omega = [a_1, b_1] \times [a_2, b_2]$ and for $\eta_i^j := a_j + i\frac{b_j - a_j}{n}$, $j = 1, 2$, the trapezoidal cubature rule becomes (see [4, p. 411])

$$\begin{aligned} \int_{a_1}^{b_1} \int_{a_2}^{b_2} u = \frac{(b_2 - a_2)(b_1 - a_1)}{4n^2} \Big(& u(\eta_0^1, \eta_0^2) + u(\eta_0^1, \eta_n^2) + u(\eta_n^1, \eta_0^2) + u(\eta_n^1, \eta_n^2) \\ & + 2 \sum_{j_1=1}^{n-1} \big(u(\eta_{j_1}^1, \eta_0^2) + u(\eta_{j_1}^1, \eta_n^2) \big) + 2 \sum_{j_2=1}^{n-1} \big(u(\eta_0^1, \eta_{j_2}^2) + u(\eta_n^1, \eta_{j_2}^2) \big) \\ & + 4 \sum_{j_1=1}^{n-1} \sum_{j_2=1}^{n-1} u(\eta_{j_1}^1, \eta_{j_2}^2) \Big) + O(\tfrac{1}{n^2}). \end{aligned}$$

Example 5.2. Let $\Omega = [0, 1]^2$, $\alpha \in \mathbb{R}$ and $h(x) := \sqrt{x_1 x_2} - \frac{\pi^2}{12}\alpha(x_1 + x_2)$. We consider the scalar, autonomous and inhomogeneous Urysohn IDE

$$u_{t+1}(x) = \alpha \int_0^1 \int_0^1 \frac{x_1 + x_2}{1 + u_t(y)^2} \, \mathrm{d}y_1 \, \mathrm{d}y_2 + h(x) =: \mathcal{F}(u_t)(x) \tag{5.3}$$

for all $x = (x_1, x_2) \in \Omega$, having the constant solution $\phi^*(x) = \sqrt{x_1 x_2}$. In order to derive a Lipschitz estimate for the right-hand side of (5.3), we obtain

$$\left| \tfrac{1}{1+z^2} - \tfrac{1}{1+\bar{z}^2} \right| \leq \tfrac{3\sqrt{3}}{8} |z - \bar{z}| \quad \text{for all } z, \bar{z} \in \mathbb{R}$$

from the mean value theorem, consequently for every $u, \bar{u} \in C(\Omega)$ it results

$$|\mathcal{F}(u)(x) - \mathcal{F}(\bar{u})(x)| \overset{(5.3)}{\leq} |\alpha| \int_0^1 \int_0^1 \tfrac{3\sqrt{3}(x_1+x_2)}{8} \, \mathrm{d}y_1 \, \mathrm{d}y_2 \, \|u - \bar{u}\|$$
$$\leq |\alpha| \tfrac{3\sqrt{3}}{8} \max_{x_1, x_2 \in [0,1]} (x_1 + x_2) \, \|u - \bar{u}\| = |\alpha| \tfrac{3\sqrt{3}}{4} \, \|u - \bar{u}\|$$

and thus $\operatorname{lip} \mathcal{F} \leq |\alpha| \frac{3\sqrt{3}}{4}$. For $|\alpha| \leq \frac{4\sqrt{3}}{9}$ the IDE (5.3) is contractive. Combining (3.6) with Cor. 3.2 implies that also the semi-discretizations (I_n) via piecewise linear collocation are contractions (with the same constant) and Cor. 4.3 applies. Choosing a fixed parameter $\alpha = 0.75$ the error

$$\operatorname{err}_n(t) = \frac{1}{n^2} \sum_{j_1=0}^{n} \sum_{j_2=0}^{n} \left| \varphi_n(t; 0, u_0)(\eta_{j_1}^1, \eta_{j_2}^2) - \phi^*(\eta_{j_1}^1, \eta_{j_2}^2) \right|$$

for times $t \in \{10, 15, 25\}$ as a function over the accuracy n is illustrated in Fig. 2. It confirms that the error decays quadratically with n.

Third, we finally tackle a scalar, homogeneous Hammerstein IDE

$$u_{t+1} = \int_a^b k(\cdot, y) g_t(y, u_t(y)) \, \mathrm{d}y,$$

whose kernel function $k : [a, b]^2 \to \mathbb{R}$ is approximated by piecewise linear degenerate kernels $k^n(x, y) := \sum_{i_1=0}^{n} \sum_{i_2=0}^{n} k(\eta_{i_1}, \eta_{i_2}) e_{i_1}(x) e_{i_2}(y)$. This yields

$$u_{t+1} = \sum_{i_1=0}^{n} \sum_{i_2=0}^{n} k(\eta_{i_1}, \eta_{i_2}) \int_a^b g_t(y, u_t(y)) e_{i_2}(y) \, \mathrm{d}y e_{i_1}$$

as semi-discretization. Approximating the remaining integrals with the trapezoidal rule leads to a recursion in $\mathbb{R}^{n+1}$ given by

$$\upsilon_{t+1}(i) = \frac{b-a}{n} \Bigg(k(\eta_i, \eta_0) g_t(\eta_0, \upsilon_t(0)) + 2 \sum_{j=1}^{n-1} k(\eta_i, \eta_j) g_t(\eta_j, \upsilon_t(j))$$
$$+ k(\eta_i, \eta_n) g_t(\eta_n, \upsilon_t(n)) \Bigg) \quad \text{for all } 0 \leq i \leq n.$$

Example 5.3 (Beverton-Holt equation)**.** Let $\Omega = [-1, 1]$. We consider the IDE

$$u_{t+1}(x) = \int_{-1}^{1} k(x, y) \frac{a_t(y) \, |u_t(y)|}{1 + |u_t(y)|} \, \mathrm{d}y \tag{5.4}$$

with the Laplace kernel $k(x, y) := \frac{\alpha}{2} e^{-\alpha|x-y|}$, a Beverton-Holt-like growth function $g_t(y, z) := \frac{a_t(y)|z|}{1+|z|}$ having a 2-periodic, space-dependent growth rate

$$a_t(y) := 5 \begin{cases} 1 - \cos y, & t \text{ is odd}, \\ 1 + \sin y, & t \text{ is even}. \end{cases}$$

If the dispersal parameter $\alpha > 0$ is increased, then a 2-periodic solution $(\phi_t^*)_{t\in\mathbb{Z}}$ to (5.4) bifurcates transcritically from the trivial solution, due to [7] it attracts every nonzero initial function (see Fig. 3). For $\alpha = 20$ we illustrate the convergence to this attractive

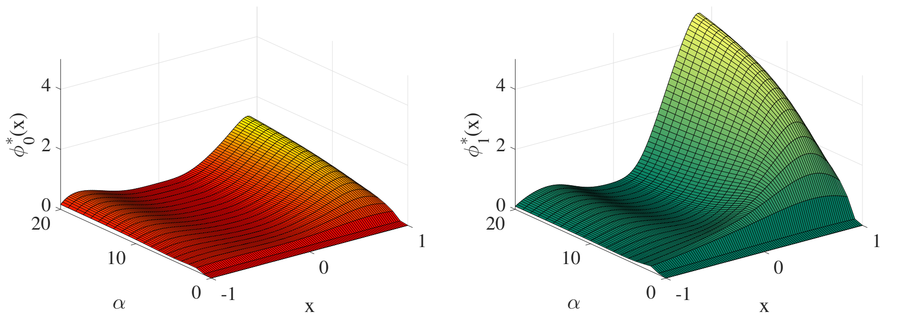

Figure 3: Attractive 2-periodic orbit of (5.4) depending on the dispersal parameter $\alpha \in [0, 20]$ for even instants (red, left) and odd instants (green, right)

2-periodic orbit in Fig. 4, which given a rough initial function u_0, also illustrates the smoothing property of (5.4) guaranteed by Cor. 2.6. We semi-discretize (5.4) using piecewise linear degenerate kernel approximation and the trapezoidal rule to arrive at full discretizations. In the following, the attractive 2-periodic orbit ϕ^* is understood as its approximation for $n = 2000$. The L^1-error

$$\operatorname{err}(n) = \frac{1}{2n}\sum_{j=0}^{n}(|\phi_0^n(\eta_j) - \phi_0^*(\eta_j)| + |\phi_1^n(\eta_j) - \phi_1^*(\eta_j)|)$$

between ϕ^* and its discrete counterpart ϕ^n as function of the discretization parameter n is pictured in Fig. 5 (right). For fixed values $n \in \{250, 500, 1000\}$ the temporal evolution of the global discretization error is depicted in Fig. 5 (left). The errors become stationary for larger values of t, since the solution to (I_n) converge to a fixed point different from the equilibrium of (H_0), that is (5.4).

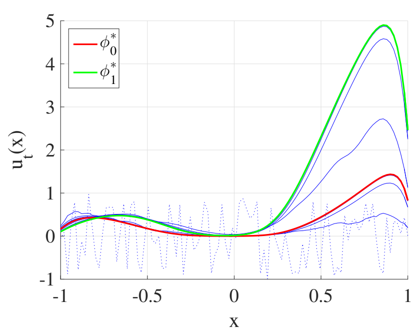

Figure 4: Iterates $u_t = \varphi(t; 0, u_0)$ for $t \geq 0$ resulting from a (rough) initial function u_0 (dotted curve) and graphs of the globally attractive 2-periodic orbit $(\phi_t^*)_{t\in\mathbb{Z}}$ (green and red) for $\alpha = 20$

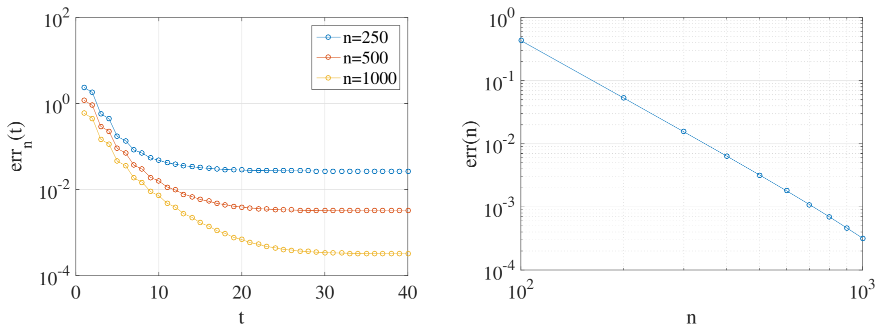

Figure 5: Errors in Exam. 5.3 as functions in time t (left) and accuracy n (right)

A. A Grönwall lemma

The following elementary Grönwall-type lemma is useful in error estimates:

Lemma A.1. *Let $(\lambda_t)_{t\in\mathbb{I}'}$, $(\beta_t)_{t\in\mathbb{I}'}$ be sequences in $\mathbb{R}_+$ resp. $\mathbb{R}$ and $\tau \in \mathbb{I}$. If a real sequence $(x_t)_{t\in\mathbb{I}}$ satisfies $x_{t+1} \le \lambda_t x_t + \beta_t$ for every $t \in \mathbb{I}'$, $\tau \le t$, then*

$$x_t \le x_\tau \prod_{r=\tau}^{t-1} \lambda_r + \sum_{s=\tau}^{t-1} \beta_s \prod_{r=s+1}^{t-1} \lambda_r \quad \textit{for all } \tau \le t,\ \tau, t \in \mathbb{I}.$$

Proof. We proceed by induction. For $t = \tau$ the above inequality becomes $x_\tau \le x_\tau$ due to the usual conventions. In the induction step $t \to t+1$ one has

$$\begin{aligned} x_{t+1} &\le \lambda_t x_t + \beta_t \le \lambda_t x_\tau \prod_{r=\tau}^{t-1} \lambda_r + \lambda_t \sum_{s=\tau}^{t-1} \beta_s \prod_{r=s+1}^{t-1} \lambda_r + \beta_t \\ &= x_\tau \prod_{r=\tau}^{t} \lambda_r + \sum_{s=\tau}^{t} \beta_s \prod_{r=s+1}^{t} \lambda_r \end{aligned}$$

and consequently the claim follows. □

B. Nonlinear operators on $C(\Omega)^d$

Let $\Omega \subset \mathbb{R}^\kappa$ be nonempty, compact without isolated points and suppose $(Y, |\cdot|)$ is a finite-dimensional normed space; we abbreviate $C_d := C(\Omega, \mathbb{R}^d)$.

Given $m \in \mathbb{N}_0$, a function $\phi : \Omega \times \Omega \times \mathbb{R}^d \to Y$ is called of class

- C_1^m, if the partial derivatives $D_1^j \phi : \Omega^2 \times \mathbb{R}^d \to L_j(\mathbb{R}^\kappa, Y)$

exist as continuous functions for all $0 \le j \le m$. Furthermore, ϕ is *of class*

- C_f^m (f for final variable), if the partial derivatives $D_3^j \phi : \Omega^2 \times \mathbb{R}^d \to L_j(\mathbb{R}^d, Y)$ exist as continuous functions and that for all $\varepsilon > 0$, $x, y \in \Omega$ there exists a $\delta > 0$ such that for every $1 \le j \le m$ one has the implication

$$|z_1 - z_2| < \delta \quad \Rightarrow \quad \left| D_3^j \phi(x, y, z_1) - D_3^j \phi(x, y, z_2) \right| < \varepsilon \text{ for all } z_1, z_2 \in \mathbb{R}^d.$$

B.1. Substitution operators

For a continuous function $G : \Omega \times \mathbb{R}^d \to Y$ we define the *substitution operator*

$$\mathcal{G}(u)(x) := G(x, u(x)) \quad \text{for all } x \in \Omega,\ u \in C_d \tag{B.1}$$

and derive the following properties:

Theorem B.1. *The operator $\mathcal{G} : C_d \to C(\Omega, Y)$ is well-defined, bounded, continuous and uniformly continuous on every bounded subset of C_d.*

Throughout, a mapping is called *bounded*, if it maps bounded sets to bounded sets.

Proof. Since continuity is preserved under composition, $\mathcal{G}(u) \in C(\Omega, Y)$ for $u \in C_d$ holds. The continuity and boundedness of $\mathcal{G}$ result from the definition of the sup-norm and the uniform continuity of G on compact subsets of $\Omega \times \mathbb{R}^d$. We vicariously verify the uniform continuity on any bounded $B \subset C_d$. Then there exists a $R > 0$ with $\|u\| \leq R$ for $u \in B$. Since G is uniformly continuous on the compact set $\Omega \times \bar{B}_R(0)$, for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|x_1 - x_2| < \delta$, $|z_1 - z_2| < \delta$ implies the inequality $|G(x_1, z_1) - G(x_2, z_2)| < \frac{\varepsilon}{2}$. For all $u_1, u_2 \in B$ with $\|u_1 - u_2\| < \delta$ this yields $|G(x, u_1(x)) - G(x, u_2(x))| < \frac{\varepsilon}{2}$ for $x \in \Omega$. Finally $\|\mathcal{G}(u_1) - \mathcal{G}(u_2)\| < \varepsilon$ holds due to (B.1), by passing to the maximum over $x \in \Omega$. That is, the substitution operator $\mathcal{G}$ is uniformly continuous on B. □

We say that a mapping $G : \Omega \times \mathbb{R}^d \to Y$ is *uniformly Lipschitz*, if for every $r > 0$ there exists a real $\tilde{\lambda}(r) \geq 0$ such that

$$|G(x, z) - G(x, \bar{z})| \leq \tilde{\lambda}(r)\, |z - \bar{z}| \quad \text{for all } x \in \Omega,\ z, \bar{z} \in \bar{B}_r(0).$$

Corollary B.2. *If $G : \Omega \times \mathbb{R}^d \to Y$ is uniformly Lipschitz, then $\mathcal{G}$ is Lipschitz; more precisely, $B \subseteq \bar{B}_r(0) \subseteq C_d$ implies* $\operatorname{lip} \mathcal{G}|_B \leq \tilde{\lambda}(r)$ *for all $r > 0$.*

Proof. Given $r > 0$, choose continuous functions $u, \bar{u} \in \bar{B}_r(0)$. This leads to

$$|\mathcal{G}(u)(x) - \mathcal{G}(\bar{u})(x)| \overset{\text{(B.1)}}{=} |G(x, u(x)) - G(x, \bar{u}(x))| \leq \tilde{\lambda}(r)\, \|u - \bar{u}\| \quad \text{for all } x \in \Omega$$

and passing to the maximum over Ω guarantees $\|\mathcal{G}(u) - \mathcal{G}(\bar{u})\| \leq \tilde{\lambda}(r)\, \|u - \bar{u}\|$. □

Theorem B.3 (differentiability of $\mathcal{G}(u)$)**.** *Let $m \in \mathbb{N}$. If u and G are of class C^m, then $\mathcal{G}(u) \in C^m(\Omega, Y)$ holds with the first order derivative*

$$\mathcal{G}(u)'(x) = D_1 G(x, u(x)) + D_2 G(x, u(x)) u'(x) \quad \textit{for all } x \in \Omega.$$

Proof. This immediately follows from the basic chain rule, e.g. [12, p. 337]. □

Theorem B.4 (differentiability of $\mathcal{G}$)**.** *Let $m \in \mathbb{N}_0$. If G is of class C_f^m, then $\mathcal{G}$ is m-times continuously differentiable with derivatives*

$$(D^j \mathcal{G}(u) h_1 \cdots h_j)(x) = D_2^j G(x, u(x)) h_1(x) \cdots h_j(x) \quad \textit{for all } 0 \leq j \leq m,$$

$x \in \Omega$ and functions $u, h_1, \ldots, h_j \in C_d$.

Proof. Let $u, h \in C_d$ be fixed. In case $m = 0$ the claim results from Thm. B.1.

(I) For every $1 \le j \le m$ we consider the mappings $r_j : C_d \to \mathbb{R}_+$,

$$r_j(h) := \sup_{x \in \Omega} \sup_{\vartheta \in [0,1]} \left| D_2^j G(x, u(x) + \vartheta h(x)) - D_2^j G(x, u(x)) \right|$$

and show $\lim_{h \to 0} r_j(h) = 0$. Given $\varepsilon > 0$, $x_0 \in \Omega$, $1 \le j \le m$, our assumption implies that there exists a $\delta(x_0) > 0$ so that $\left| D_2^j G(x_0, z_1) - D_2^j G(x_0, z_2) \right| < \frac{\varepsilon}{2}$ holds for $z_1, z_2 \in \mathbb{R}^d$ with $|z_1 - z_2| < \delta(x_0)$. Since $\left\{ B_{\delta(x)}(x) : x \in \Omega \right\}$ is an open cover of the compact set Ω, there exist finitely many $x_1, \ldots, x_n \in \Omega$ with $\Omega \subseteq \bigcup_{i=1}^n B_{\delta(x_i)}(x_i)$. We define $\delta_0 := \min \{\delta(x_i) : 1 \le i \le n\} > 0$ and for every $x \in \Omega$ one finds an index $\iota \in \{1, \ldots, n\}$ such that $x \in B_{\delta(x_\iota)}(x_\iota)$ holds. Hence, $|z_1 - z_2| < \delta_0 \le \delta(x_\iota)$ leads to $\left| D_2^j G(x, z_1) - D_2^j G(x, z_2) \right| < \frac{\varepsilon}{2}$ for all $x \in \Omega$. Thus, in case $\|h\| < \delta_0$ it is

$$\left| D_2^j G(x, u(x) + \vartheta h(x)) - D_2^j G(x, u(x)) \right| < \frac{\varepsilon}{2} \quad \text{for all } \vartheta \in [0,1],\ x \in \Omega.$$

By passing over to the least upper bound for $\vartheta \in [0,1]$ and then over $x \in \Omega$, this implies the desired inequality $|r_j(h)| < \varepsilon$, whenever $h \in B_{\delta_0}(0)$ holds.

(II) For $0 \le j < m$ we define the mappings

$$\begin{aligned} F_j &: \Omega \times C_d \to L_j(\mathbb{R}^d, Y), & F_j(x, u) &:= D_2^j G(x, u(x)), \\ F_j' &: \Omega \times C^2(\Omega)^d \to L_j(\mathbb{R}^d, Y), & F_j'(x, u, h) &:= D_2^{j+1} G(x, u(x)) h(x). \end{aligned}$$

It is easy to see that $h \mapsto F_j'(\cdot, u, h)$ are linear and bounded. Given this, for all $x \in \Omega$ we obtain from the mean value theorem (cf. [12, p. 341, Thm. 4.2]) that

$$\begin{aligned} &\left| F_j(x, u + h) - F_j(x, u) - F_j'(x, u, h) \right| \\ &= \left| D_2^j G(x, u(x) + h(x)) - D_2^j G(x, u(x)) - D_2^{j+1} G(x, u(x)) h(x) \right| \\ &= \left| \int_0^1 \left[D_2^{j+1} G(x, u(x) + \vartheta h(x)) - D_2^{j+1} G(x, u(x)) \right] \mathrm{d}\vartheta \, h(x) \right| \\ &\le \sup_{\vartheta \in [0,1]} \left| D_2^{j+1} G(x, u(x) + \vartheta h(x)) - D_2^{j+1} G(x, u(x)) \right| \|h\| \le r_{j+1}(h) \|h\| . \end{aligned} \tag{B.2}$$

(III) Passing over to the supremum over all $x \in \Omega$ in the inequality

$$\begin{aligned} &|\mathcal{G}(u + h)(x) - \mathcal{G}(u)(x) - F_0'(x, u, h)| \\ &\qquad \overset{(B.1)}{=} |F_0(x, u + h) - F_0(x, u) - F_0'(x, u, h)| \overset{(B.2)}{\le} r_1(h) \|h\| \end{aligned}$$

implies $\|\mathcal{G}(u + h) - \mathcal{G}(u) - F_0'(\cdot, u, h)\| \le r_1(h) \|h\|$. Hence, due to the limit relation $\lim_{h \to 0} r_1(h) = 0$ and its uniqueness, one obtains the explicit Fréchet derivative $D\mathcal{G}(u)h = F_0'(\cdot, u, h)$. It is standard to show that $u \mapsto D\mathcal{G}(u)$ is continuous (see the proof of Thm. B.1), i.e. $\mathcal{G} : C_d \to C(\Omega, Y)$ is of class C^1.

(IV) We proceed by mathematical induction and assume $\mathcal{G}$ is of class C^j, $j < m$. It follows for all $x \in \Omega$ that

$$\left|D^j \mathcal{G}(u+h)(x) - D^j\mathcal{G}(u)(x) - F_j'(x,u,h)\right|$$
$$= \left|F_j(x,u+h) - F_j(x,u) - F_j'(x,u,h)\right| \stackrel{\text{(B.2)}}{\leq} r_{j+1}(h)\,\|h\|\,.$$

Hence, $D^j\mathcal{G}$ is differentiable in u with the derivative $h \mapsto F_j'(\cdot,u,h)$. As in the proof of Thm. B.1 one establishes continuity of $D^{j+1}\mathcal{G}$. □

B.2. Urysohn integral operators

Let us study *Urysohn integral operators*

$$\mathcal{U}(u) := \int_\Omega \phi(\cdot,y,u(y))\,\mathrm{d}y \quad \text{for all } u \in C_d \tag{B.3}$$

associated to a continuous *kernel function* $\phi : \Omega^2 \times \mathbb{R}^d \to Y$. Our subsequent measure-theoretical terminology always refers to the Lebesgue measure λ_κ on $\mathbb{R}^\kappa$.

Theorem B.5. *The operator $\mathcal{U} : C_d \to C(\Omega,Y)$ is well-defined, completely continuous and uniformly continuous on each bounded subset of C_d.*

Proof. The special case $Y = \mathbb{R}^d$ is shown in [13, p. 166, Prop. 3.2]. The reader might verify that our situation of a general finite-dimensional normed space Y requires no additional arguments yet. □

Corollary B.6. *If for every $r > 0$ there exists a function $\lambda_r : \Omega^2 \to \mathbb{R}_+$ so that $\lambda_r(x,\cdot)$ is measurable on Ω with $\ell(r) := \sup_{x\in\Omega}\int_\Omega \lambda_r(x,y)\,\mathrm{d}y < \infty$ and*

$$|\phi(x,y,z) - \phi(x,y,\bar z)| \leq \lambda_r(x,y)\,|z-\bar z| \quad \text{for all } x,y\in\Omega,\ z,\bar z \in \bar B_r(0),$$

then $\mathcal{U}$ is Lipschitz, i.e. $B \subseteq \bar B_r(0) \subset C_d$ implies $\operatorname{lip}\mathcal{U}|_B \leq \ell(r)$.

Proof. Given $r > 0$, choose $u, \bar u \in \bar B_r(0)$, which implies

$$|\mathcal{U}(u)(x) - \mathcal{U}(\bar u)(x)| \stackrel{\text{(B.3)}}{\leq} \int_\Omega |\phi(x,y,u(y)) - \phi(x,y,\bar u(y))|\,\mathrm{d}y$$
$$\leq \int_\Omega \lambda_r(x,y)\,|u(y)-\bar u(y)|\,\mathrm{d}y \leq \sup_{x\in\Omega}\int_\Omega \lambda_r(x,y)\,\mathrm{d}y\,\|u-\bar u\| \leq \ell(r)\,\|u-\bar u\|\,.$$

Passing over to the supremum over $x \in \Omega$ shows $\|\mathcal{U}(u) - \mathcal{U}(\bar u)\| \leq \ell(r)\,\|u-\bar u\|$. □

Theorem B.7 (differentiability of $\mathcal{U}(u)$)**.** *Let $m \in \mathbb{N}_0$. If $\phi : \Omega^2 \times \mathbb{R}^d \to Y$ is of class C_1^m, then $\mathcal{U}(u) \in C^m(\Omega,Y)$ with the derivatives*

$$\mathcal{U}(u)^{(j)} = \int_\Omega D_1^j\phi(\cdot,y,u(y))\,\mathrm{d}y \quad \text{for all } 0 \leq j \leq m,\ u \in C_d.$$

Proof. Let $u \in C_d$ be fixed and define $r := \|u\|$ for $u \neq 0$ and $r := 1$ else. For $m = 0$ the claim results from Thm. B.5.

(I) For $0 \le j \le m$ we consider the Urysohn integral operators

$$\mathcal{U}_j(u) := \int_\Omega D_1^j \phi(\cdot, y, u(y))\,\mathrm{d}y$$

with continuous kernel functions. As quoted in the proof of Thm. B.5 one obtains the inclusion $\mathcal{U}_j(u) \in C(\Omega, L_j(\mathbb{R}^\kappa, Y))$. We establish that $\mathcal{U}(u) : \Omega \to Y$ is of class C^m.

(II) Let us abbreviate $F_j(x, y) := D_1^j \phi(x, y, u(y))$ and due to our assumption, the functions $F_j : \Omega^2 \to L_j(\mathbb{R}^\kappa, Y)$ are continuous. First, $y \mapsto F_j(x, y)$ is integrable for all $x \in \Omega$, since Ω is compact. Second, the partial derivative $D_1 F_j = F_{j+1}$ exists by assumption for $0 \le j < m$. Third, because $D_1^{j+1}\phi$ is bounded on the compact product $\Omega^2 \times \bar{B}_r(0)$, also F_{j+1} is bounded on Ω^2 and thus integrable. Then differentiability of parameter integrals from e.g. [3, p. 90, 16.3 Cor.] implies the derivatives

$$\mathcal{U}_j(u)' = \int_\Omega D_1 F_j(\cdot, y)\,\mathrm{d}y = \int_\Omega D_1^{j+1}\phi(\cdot, y, u(y))\,\mathrm{d}y = \mathcal{U}_{j+1}(u)$$

for all $0 \le j < m$. Now induction yields $\mathcal{U}(u)^{(m)} = \mathcal{U}_0(u)^{(m)} = \mathcal{U}_m(u)$, where the mapping $\mathcal{U}_m(u) : \Omega \to L_m(\mathbb{R}^\kappa, Y)$ is continuous due to (I). Hence, also the mth order derivative $\mathcal{U}(u)^{(m)}$ is continuous and the claim follows. □

Theorem B.8 (differentiability of $\mathcal{U}$)**.** *Let $m \in \mathbb{N}_0$. If $\phi : \Omega^2 \times \mathbb{R}^d \to Y$ is of class C_f^m, then $\mathcal{U} : C_d \to C(\Omega, Y)$ is of class C^m with the derivatives*

$$D^j\mathcal{U}(u)h_1 \dots h_j = \int_\Omega D_3^j \phi(\cdot, y, u(y))h_1(y)\dots h_j(y)\,\mathrm{d}y \quad \textit{for all } 0 \le j \le m$$

and $D\mathcal{U}(u) \in L(C_d, C(\Omega, Y))$ is compact for all functions $u, h_1, \dots, h_j \in C_d$.

Proof. Let $u, h \in C_d$ be fixed throughout.

(I) For each $1 \le j \le m$ let us define the mappings $r_j : C_d \to \mathbb{R}_+$,

$$r_j(h) := \lambda_\kappa(\Omega) \sup_{x,y\in\Omega} \sup_{\vartheta\in[0,1]} \left| D_3^j\phi(x, y, u(y) + \vartheta h(y)) - D_3^j\phi(x, y, u(y)) \right|$$

and establish $\lim_{h\to 0} r_j(h) = 0$. We abbreviate $\xi := (x, y)$ and given $\varepsilon > 0$, $\xi_0 \in \Omega^2$ our assumption yields a $\delta(\xi_0) > 0$ so that $\left| D_3^{j+1}\phi(\xi_0, z_1) - D_3^{j+1}\phi(\xi_0, z_2) \right| < \frac{\varepsilon}{2\lambda_\kappa(\Omega)}$ holds for $z_1, z_2 \in \mathbb{R}^d$, $|z_1 - z_2| < \delta(\xi_0)$. Because $\left\{B_{\delta(\xi)}(\xi) : \xi \in \Omega^2\right\}$ is an open cover of the compact product Ω^2, there exist finitely many $\xi_1, \dots, \xi_n \in \Omega^2$ with

$$\Omega^2 \subseteq \bigcup_{i=1}^n B_{\delta(\xi_i)}(\xi_i).$$

We define $\delta_0 := \min\{\delta(\xi_i) : 1 \le i \le n\} > 0$ and for every $\xi \in \Omega^2$ one finds an index $\iota \in \{1, \dots, n\}$ such that $\xi \in B_{\delta(\xi_\iota)}(\xi_\iota)$ holds. Hence, $|z_1 - z_2| < \delta_0 \le \delta(\xi_\iota)$ yields $\left| D_3^{j+1}\phi(\xi, z_1) - D_3^{j+1}\phi(\xi, z_2) \right| < \frac{\varepsilon}{2\lambda_\kappa(\Omega)}$ for $\xi \in \Omega^2$. If $\|h\| < \delta_0$ and $x, y \in \Omega$, then

$$\left| D_3^{j+1}\phi(x, y, u(y) + \vartheta h(y)) - D_3^{j+1}\phi(x, y, u(y)) \right| < \frac{\varepsilon}{2\lambda_\kappa(\Omega)} \quad \text{for all } \vartheta \in [0, 1]$$

holds. Passing over to the supremum for $\vartheta \in [0,1]$, $x, y \in \Omega$ leads to $|r_{j+1}(h)| < \varepsilon$.

(II) For $0 \le j < m$ we introduce

$$F_j : \Omega \times C_d \to L_j(\mathbb{R}^d, Y), \qquad F_j(x,u) := \int_\Omega D_3^j \phi(x,y,u(y))\,\mathrm{d}y,$$
$$F_j' : \Omega \times C^2(\Omega)^d \to L_j(\mathbb{R}^d, Y), \quad F_j'(x,u,h) := \int_\Omega D_3^{j+1} \phi(x,y,u(y))h(y)\,\mathrm{d}y.$$

As readily seen, the mappings $h \mapsto F_j'(\cdot,u,h)$ are linear and bounded. From the mean value theorem [12, p. 341, Thm. 4.2] we arrive at

$$\begin{aligned}
&\left|F_j(x,u+h) - F_j(x,u) - F_j'(x,u,h)\right| \\
&= \left|\int_\Omega D_3^j\phi(x,y,u(y)+h(y)) - D_3^j\phi(x,y,u(y)) - D_3^{j+1}\phi(x,y,u(y))h(y)\,\mathrm{d}y\right| \\
&= \left|\int_\Omega \int_0^1 \left[D_3^{j+1}\phi(x,y,u(y)+\vartheta h(y)) - D_3^{j+1}\phi(x,y,u(y))\right]\,\mathrm{d}\vartheta\, h(y)\,\mathrm{d}y\right| \\
&\le \int_\Omega \sup_{\vartheta\in[0,1]} \left|D_3^{j+1}\phi(x,y,u(y)+\vartheta h(y)) - D_3^{j+1}\phi(x,y,u(y))\right|\,\mathrm{d}y\,\|h\| \\
&\le r_{j+1}(h)\,\|h\| \quad \text{for all } x \in \Omega.
\end{aligned} \tag{B.4}$$

(III) Passing to the supremum over $x \in \Omega$ in the inequality

$$|\mathcal{U}(u+h)(x) - \mathcal{U}(u)(x) - F_0'(x,u,h)| \overset{\text{(B.3)}}{=} |F_0(x,u+h) - F_0(x,u) - F_0'(x,u,h)| \overset{\text{(B.4)}}{\le} r_1(h)\,\|h\|$$

yields $\|\mathcal{U}(u+h) - \mathcal{U}(u) - F_0'(\cdot,u,h)\| \le r_1(h)\,\|h\|$. Hence, due to its uniqueness and $\lim_{h\to 0} r_1(h) = 0$, one has the Fréchet derivative $D\mathcal{U}(u)h = F_0'(\cdot,u,h)$ and as in Thm. B.5, $u \mapsto D\mathcal{U}(u)$ is continuous, i.e. $\mathcal{U} : C_d \to C(\Omega,Y)$ is of class C^1.

(IV) We now proceed by mathematical induction and assume that the Urysohn integral operator $\mathcal{U}$ is of class C^j, $j < m$. It results for every $x \in \Omega$ that

$$\left|D^j\mathcal{U}(u+h)(x) - D^j\mathcal{U}(u)(x) - F_j'(x,u,h)\right| = \left|F_j(x,u+h) - F_j(x,u) - F_j'(x,u,h)\right| \overset{\text{(B.4)}}{\le} r_{j+1}(h)\,\|h\|.$$

Hence, $D^j\mathcal{U}$ is differentiable in u with the derivative $h \mapsto F_j'(\cdot,u,h)$. As in the proof of Thm. B.5 one shows that $D^{j+1}\mathcal{U}$ is also continuous.

(V) Combined with Thm. B.5 for $m \in \mathbb{N}$ we established that $\mathcal{U} : C_d \to C(\Omega,Y)$ is a completely continuous C^1-mapping. Thus, [13, p. 89, Prop. 6.5] finally implies that the derivative $D\mathcal{U}(u)$ is compact. □

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