UNIFORM CONVERGENCE OF NYSTRÖM DISCRETIZATIONS ON HÖLDER SPACES

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ABSTRACT. We establish that Nyström discretizations of linear Fredholm integral operators on Hölder spaces converge in the operator norm while preserving the consistency order of the quadrature or cubature rule. This allows to employ tools from classical perturbation theory, rather than collective compactness, when studying numerical approximations of integral operators, as well as applications in for instance the field of nonautonomous dynamical systems.

1. Introduction. The Nyström method is a classical and widely applied scheme to discretize integral operators $\mathcal{K}$ defined on the space of continuous functions, where the integral is replaced by a convergent quadrature or cubature rule. Here the approximated operators $\mathcal{K}^n$ converge strongly but not uniformly due to the estimate $\|\mathcal{K}\| \leq \|\mathcal{K} - \mathcal{K}^n\|$ for all $n \in \mathbb{N}$ (e.g. [**5,** pp. 130–131, Lemma 4.7.6]). This led to a versatile convergence theory based on collectively compact operator families [**1**], which is fully satisfactory for various tasks like solving linear and nonlinear integral equations, or eigenvalue problems for integral operators [**1, 2, 5, 8**].

Nyström methods are also convenient and thus frequently used in simulations for integrodifference equations. These discrete-time dynamical systems describe the behavior of iterates of (Hammerstein) integral operators and are popular models in theoretical ecology for the temporal evolution and spatial dispersal of species [**9**]. The perturbation theory needed to show that integrodifference equations and their Nyström discretizations share the same long term behavior often requires uniform convergence (at least up to the author's present abilities). For this reason it is a helpful observation that Nyström approximations do converge in the operator norm, when studying in-

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tegral operators on spaces of Hölder functions, rather than merely on the continuous functions over a compact domain (cf. Thms. 1–3). The suitable Hölder exponent is determined by the consistency order of the employed quadrature rule, but turns out to be of minor importance, since various properties like the spectrum of an integral operator are independent of the particular space (see Cor. 1). Although the prize for this endeavor is essentially to assume smooth (cf. Rem. 1) and not just continuous kernels, such a setting frequently holds in applications [**9,** pp. 17ff].

Our central Thm. 2 is illustrated by means of a convergence result for approximations of the dichotomy spectrum, a crucial tool in the stability theory of nonautonomous dynamical systems [**7,** pp. 82–86].

Notation. Let $\Omega \subset \mathbb{R}^\kappa$ be a nonempty compact set and $\mathbb{K}$ stand for the fields $\mathbb{R}$ or $\mathbb{C}$. We write $\operatorname{diam}\Omega := \sup_{y_1,y_2\in\Omega}|y_1-y_2|$ for the diameter of Ω and $\operatorname{dist}(\Omega,\bar\Omega) := \sup_{y\in\Omega}\inf_{x\in\bar\Omega}|x-y|$ for the Hausdorff semidistance between Ω and a subset $\bar\Omega \subseteq \mathbb{R}^\kappa$.

Given a function $u : \Omega \to \mathbb{K}$, its modulus of continuity is

$$\omega(\delta,u) := \sup_{\substack{x,\bar x\in\Omega,\\ |x-\bar x|\le\delta}} |u(x)-u(\bar x)|\,.$$

One denotes u as α-Hölder with Hölder exponent $\alpha \in (0,1]$, if its Hölder constant

$$[u]_\alpha := \sup_{\substack{x,\bar x\in\Omega\\ x\ne\bar x}} \frac{|u(x)-u(\bar x)|}{|x-\bar x|^\alpha}$$

is finite. We write $C^\alpha(\Omega)$ for the spaces of α-Hölder and $C^0(\Omega)$ for the space of continuous functions equipped with the respective norm

$$\|u\|_\alpha := \begin{cases} \sup_{x\in\Omega}|u(x)|\,, & \alpha = 0,\\ \max\{\sup_{x\in\Omega}|u(x)|\,,[u]_\alpha\}\,, & \alpha\in(0,1]. \end{cases}$$

The embeddings $C^\beta(\Omega) \subseteq C^\alpha(\Omega)$, $0 \le \alpha \le \beta \le 1$, are continuous with

$$[u]_\alpha \le (\operatorname{diam}\Omega)^{\beta-\alpha}[u]_\beta \quad \text{for all } u \in C^\beta(\Omega) \tag{1}$$

and compact in case $\alpha < \beta$. Moreover, $C^\alpha(\Omega) \subseteq C^0(\Omega)$, $\alpha \le 1$, constitutes a dense embedding and such pairs of Banach spaces are compatible in the sense of [**3,** p. 49].

2. Fredholm operators and Nyström discretizations. Let us assume throughout that $\Omega_1 \subset \mathbb{R}^{\kappa_1}$, $\Omega \subset \mathbb{R}^{\kappa}$ are nonempty compact sets. Under well-known assumptions [**5, 8**] on a kernel $k : \Omega_1 \times \Omega \to \mathbb{K}$, the Fredholm integral operator

$$\mathcal{K}u := \int_\Omega k(\cdot, y)u(y)\,\mathrm{d}y \tag{2}$$

defines a compact linear operator $\mathcal{K} : C^0(\Omega) \to C^0(\Omega_1)$.

A quadrature (cubature for $\kappa > 1$) rule is a sequence of linear maps

$$(Q_n) \qquad Q^n : C(\Omega) \to \mathbb{K}, \qquad Q^n u := \sum_{\eta \in \Omega^{(n)}} w_\eta u(\eta) \quad \text{for all } n \in \mathbb{N}$$

determined by finite (nonempty) grids $\Omega^{(n)} \subseteq \Omega$ consisting of nodes $\eta \in \Omega^{(n)}$, as well as the weights $w_\eta \in \mathbb{R}$; note that the dependence of w_η on $n \in \mathbb{N}$ is suppressed for convenience. One denotes (Q_n) as

- convergent, if $\lim_{n\to\infty} Q^n u = \int_\Omega u(y)\,\mathrm{d}y$ holds for $u \in C^0(\Omega)$,
- stable, if

$$W := \sup_{n\in\mathbb{N}} W_n < \infty, \qquad W_n := \sum_{\eta\in\Omega^{(n)}} |w_\eta|\,, \tag{3}$$

- having consistency order $\gamma \in (0, 1]$, if there exists a $c_0 > 0$ with

$$\left| \int_\Omega u(y)\,\mathrm{d}y - Q^n u \right| \le \frac{c_0}{n^\gamma} \|u\|_\gamma \quad \text{for all } u \in C^\gamma(\Omega). \tag{4}$$

In our subsequent analysis the consistency order γ is not required to be the maximal possible one for a given (Q_n). Thanks to [**5,** p. 20, Thm. 1.4.17], convergence implies stability.

When applying quadrature rules (Q_n) to Hölder functions, their consistency order often agrees with the Hölder exponent. For this reason, in actual computations one can restrict to simple rules such as listed in Tab. 1.

Example 1. Let $\Omega = [a, b]$ and $\eta_j := a + j\frac{b-a}{n}$, $j \in \mathbb{Z}$. Both the rectangular rule and the midpoint rule

$$Q^n u := \tfrac{b-a}{n} \sum_{j=1}^n u(\eta_j), \qquad Q^n u := \tfrac{b-a}{n} \sum_{j=1}^n u\big(\tfrac{\eta_j - \eta_{j-1}}{2}\big),$$

rule	$\Omega^{(n)}$	c_0
left/right rectangular	$\{\eta_0,\ldots,\eta_{n-1}\}, \{\eta_1,\ldots,\eta_n\}$	$(b-a)^{1+\gamma}$
midpoint	$\left\{\frac{\eta_1-\eta_0}{2},\ldots,\frac{\eta_n-\eta_{n-1}}{2}\right\}$	$\frac{(b-a)^{1+\gamma}}{2^\gamma}$
trapezoidal	$\{\eta_0,\ldots,\eta_n\}$	$(b-a)^{1+\gamma}$

Table 1. The grids $\Omega^{(n)}$ and the constants c_0 for commonly used quadrature rules: Left/right end rectangular, midpoint and trapezoidal rule

resp., having constant weights $\frac{b-a}{n}$, as well as the trapezoidal rule

$$Q^n u := \tfrac{b-a}{2n}\Big(u(\eta_0) + 2\sum_{j=1}^{n-1} u(\eta_j) + u(\eta_n)\Big),$$

fit in the framework of (Q_n). So do their product versions for numerical cubature. These rules are convergent with respective quadrature errors (cf. [**4,** p. 52, Theorem])

$$\left|\int_a^b u - Q^n u\right| \leq (b-a)\begin{cases}\omega\left(\frac{b-a}{n},u\right), & \text{rectangular,}\\ \omega\left(\frac{b-a}{2n},u\right), & \text{midpoint,}\\ \omega\left(\frac{b-a}{n},u\right), & \text{trapezoidal}\end{cases} \tag{5}$$

for $u \in C^0[a,b]$. Hence, for functions $u \in C^\gamma[a,b]$ we indeed obtain the consistency estimates (4) with rate γ and constants c_0 given in Tab. 1.

A natural way to avoid the integral in (2) is to replace it by a convergent quadrature rule (Q_n) yielding the Nyström methods

$$\mathcal{K}^n u := \sum_{\eta\in\Omega^{(n)}} w_\eta k(\cdot,\eta)u(\eta) \quad \text{for all } n \in \mathbb{N}. \tag{6}$$

Under continuity assumptions on k this yields a sequence of operators $\mathcal{K}^n \in L(C(\Omega), C(\Omega_1))$ being bounded uniformly in n due to (3).

3. Uniform convergence on Hölder spaces. When restricting $\mathcal{K}$ and $\mathcal{K}^n$ to Hölder spaces $C^\alpha(\Omega)$, $\alpha \in (0,1]$ the uniform convergence of $\mathcal{K}^n$ to $\mathcal{K}$ as $n \to \infty$ can be established under justifiable assumptions.

For this purpose, we need a quantitative version of the fact that α-Hölder functions form an algebra.

Lemma 1 (product rule)**.** *Let $\alpha \in (0, 1]$. If $u_1, u_2 : \Omega \to \mathbb{K}$ are α-Hölder, then also their product $u_1 u_2 : \Omega \to \mathbb{K}$ is α-Hölder with*

$$[u_1u_2]_\alpha \le [u_1]_\alpha \|u_2\|_0 + \|u_1\|_0 [u_2]_\alpha, \tag{7}$$

$$\|u_1u_2\|_\alpha \le (\|u_1\|_0 + [u_1]_\alpha) \|u_2\|_\alpha . \tag{8}$$

Proof. First, we obtain from the triangle inequality

$$\begin{aligned} &|(u_1u_2)(x_1) - (u_1u_2)(x_2)| \\ \le &|u_1(x_1) - u_1(x_2)| |u_2(x_1)| + |u_1(x_2)| |u_2(x_1) - u_2(x_2)| \\ \le &(\|u_2\|_0 [u_1]_\alpha + \|u_1\|_0 [u_2]_\alpha) |x_1 - x_2|^\alpha \quad \text{for all } x_1, x_2 \in \Omega, \end{aligned}$$

which implies that u_1u_2 is α-Hölder, as well as the estimate (7). Second, the aforesaid inequality (7) furthermore guarantees

$$\begin{aligned} \|u_1u_2\|_\alpha &= \max \{\|u_1u_2\|_0 , [u_1u_2]_\alpha\} \\ &\le \max \{\|u_1\|_0 \|u_2\|_0 , [u_1]_\alpha \|u_2\|_0 + \|u_1\|_0 [u_2]_\alpha\} \\ &\le \max \{\|u_1\|_0 , [u_1]_\alpha + \|u_1\|_0\} \|u_2\|_\alpha = ([u_1]_\alpha + \|u_1\|_0) \|u_2\|_\alpha \end{aligned}$$

and this establishes our lemma. □

The subsequent result is a specification of [**5,** p. 136, Thm. 4.7.15]:

Theorem 1. *Let $\alpha \in (0, 1]$. If a kernel $k : \Omega_1 \times \Omega \to \mathbb{K}$ satisfies*

(i) *$k(\cdot, y) : \Omega_1 \to \mathbb{K}$ is continuous for all $y \in \Omega$,*
(ii) *$k_2 := \sup_{x\in\Omega_1} [k(x, \cdot)]_\alpha < \infty$,*

then k is uniformly continuous and $\mathcal{K}, \mathcal{K}^n \in L(C^\alpha(\Omega), C^0(\Omega_1))$, $n \in \mathbb{N}$, are compact. If moreover (Q_n) has consistency order $\gamma \in (0, \alpha]$, then for all $n \in \mathbb{N}$ the following norm estimate holds:

$$\|\mathcal{K} - \mathcal{K}^n\|_{L(C^\gamma(\Omega),C^0(\Omega_1))} \le \frac{c_0}{n^\gamma} \left(k_2(\operatorname{diam} \Omega)^{\alpha-\gamma} + \|k\|_0\right) . \tag{9}$$

Proof. As preparation we show that $k : \Omega_1 \times \Omega \to \mathbb{K}$ is continuous. Thereto, choose $(x_0, y_0) \in \Omega_1 \times \Omega$ being the limit of a sequence of points

$(x_l, y_l) \in \Omega_1 \times \Omega$, $l \in \mathbb{N}$. Then continuity of k results from

$$\begin{aligned} 0 &\le |k(x_l, y_l) - k(x_0, y_0)| \\ &\le |k(x_l, y_l) - k(x_l, y_0)| + |k(x_l, y_0) - k(x_0, y_0)| \\ &\overset{(ii)}{\le} k_2 |y_l - y_0|^\alpha + |k(x_l, y_0) - k(x_0, y_0)| \xrightarrow[l\to\infty]{(i)} 0, \end{aligned}$$

and since $\Omega_1 \times \Omega$ is compact, k is uniformly continuous and bounded. The compactness of $\mathcal{K}$ (and $\mathcal{K}^n$) follows as in [**5,** p. 45, Thm. 3.2.6].

Let $u \in C^\alpha(\Omega)$ and $n \in \mathbb{N}$. By (ii) the function $k(x, \cdot)$ is α-Hölder and Lemma 1 implies $k(x, \cdot)u(\cdot) \in C^\alpha(\Omega) \subseteq C^\gamma(\Omega)$ (uniformly in $x \in \Omega_1$). Thanks to the inequality

$$\begin{aligned} |(\mathcal{K} - \mathcal{K}^n)u(x)| &\overset{(2)}{=} \left| \int_\Omega k(x, y)u(y)\,\mathrm{d}y - Q^n k(x, \cdot)u(\cdot) \right| \\ &\overset{(4)}{\le} \frac{c_0}{n^\gamma} \|k(x, \cdot)u(\cdot)\|_\gamma \quad \text{for all } x \in \Omega_1 \end{aligned}$$

it remains to estimate the Hölder norm of the product $k(x, \cdot)u(\cdot)$. Thereto, combining $\|k(x, \cdot)u(\cdot)\|_0 \le \|k\|_0 \|u\|_\gamma$ with

$$\begin{aligned} [k(x, \cdot)u(\cdot)]_\gamma &\overset{(7)}{\le} [k(x, \cdot)]_\gamma \|u\|_0 + \|k\|_0 [u]_\gamma \\ &\overset{(1)}{\le} (\operatorname{diam} \Omega)^{\alpha-\gamma} [k(x, \cdot)]_\alpha \|u\|_0 + \|k\|_0 [u]_\gamma \\ &\overset{(ii)}{\le} (k_2 (\operatorname{diam} \Omega)^{\alpha-\gamma} + \|k\|_0) \|u\|_\gamma \end{aligned}$$

leads to $|(\mathcal{K} - \mathcal{K}^n)u(x)| \le \frac{c_0}{n^\gamma} (k_2(\operatorname{diam} \Omega)^{\alpha-\gamma} + \|k\|_0) \|u\|_\gamma$ for $x \in \Omega_1$. Passing first to the least upper bound over $x \in \Omega_1$, and then over all functions $u \in C^\gamma(\Omega)$ with $\|u\|_\gamma \le 1$, one arrives at (9). □

The next result addresses operators acting between Hölder spaces:

Theorem 2. *Let* $\alpha, \beta \in (0, 1]$. *If a kernel* $k : \Omega_1 \times \Omega \to \mathbb{K}$ *satisfies*

(i) $k_1 := \sup_{y\in\Omega} [k(\cdot, y)]_\beta < \infty$,
(ii) $k_2 := \sup_{x\in\Omega_1} [k(x, \cdot)]_\alpha < \infty$,

then k is $\min\{\alpha,\beta\}$*-Hölder and* $\mathcal{K},\mathcal{K}^n \in L(C^\alpha(\Omega),C^\beta(\Omega_1))$*,* $n\in\mathbb{N}$*, are compact for* $\alpha\le\beta$*. If moreover* (Q_n) *has consistency order* $\gamma\in(0,\alpha]$ *and*

(iii) *there exists a* $k_0\ge 0$ *such that*

$$\begin{aligned}(10)\quad & |k(x_1,y_1)-k(x_2,y_1)-[k(x_1,y_2)-k(x_2,y_2)]| \\ & \le k_0\,|x_1-x_2|^\beta\,|y_1-y_2|^\alpha \quad \text{for all } x_1,x_2\in\Omega_1,\ y_1,y_2\in\Omega,\end{aligned}$$

then for all $n\in\mathbb{N}$ *the following norm estimate holds:*

$$\begin{aligned}(11)\quad \|\mathcal{K}-\mathcal{K}^n\|_{L(C^\gamma(\Omega),C^\beta(\Omega_1))} \le \frac{c_0}{n^\gamma}\max\{&\|k\|_0+k_2(\operatorname{diam}\Omega)^{\alpha-\gamma},\\ &k_0(\operatorname{diam}\Omega)^{\alpha-\gamma}+k_1\}.\end{aligned}$$

Remark 1 (on assumption (iii)). (1) Using the mean value theorem (combined with the fact that absolutely continuous or Lipschitz functions are differentiable almost everywhere) several sufficient conditions can be given such that k satisfies assumption (iii), that is the estimate (10). Here, D_1k or D_2k denote the partial derivative of k w.r.t. the first resp. second variable:

- Ω_1 is convex, $k(\cdot,y)$ is Lipschitz for all $y\in\Omega$ and

  $$k_2' := \operatorname*{ess\,sup}_{x\in\Omega_1}[D_1k(x,\cdot)]_\alpha<\infty, \qquad k_0=(\operatorname{diam}\Omega_1)^{1-\beta}k_2',$$

- Ω is convex, $k(x,\cdot)$ is Lipschitz for all $x\in\Omega_1$ and

  $$k_1' := \operatorname*{ess\,sup}_{y\in\Omega}[D_2k(\cdot,y)]_\beta<\infty, \qquad k_0=(\operatorname{diam}\Omega)^{1-\alpha}k_1',$$

- Ω_1,Ω are convex, $k(\cdot,y)$ is Lipschitz for all $y\in\Omega$, $D_1k(x,\cdot)$ is Lipschitz for almost all $x\in\Omega_1$ and

  $$k_0' := \operatorname*{ess\,sup}_{x\in\Omega_1,y\in\Omega}|D_2D_1k(x,y)|<\infty,$$

  or $k(x,\cdot)$ is Lipschitz for all $x\in\Omega_1$, $D_2k(\cdot,y)$ is Lipschitz for almost all $y\in\Omega$ and

  $$k_0' := \operatorname*{ess\,sup}_{x\in\Omega_1,y\in\Omega}|D_1D_2k(x,y)|<\infty$$

  with $k_0=(\operatorname{diam}\Omega_1)^{1-\beta}(\operatorname{diam}\Omega)^{1-\alpha}k_0'$.

If the respective domains Ω_1, Ω are intervals, then the above Lipschitz assumption can be weakened to absolute continuity.

(2) Degenerate kernels $k(x,y) = \sum_{j=1}^d b_j(x)a_j(y)$ with $a_j \in C^\alpha(\Omega)$, $b_j \in C^\beta(\Omega_1)$ fulfill assumption (iii), because

$$k(x_1,y_1) - k(x_2,y_1) - (k(x_1,y_2) - k(x_2,y_2)) = \sum_{j=1}^{d}(b_j(x_1) - b_j(x_2))(a_j(y_1) - a_j(y_2))$$

implies the estimate (10) with $k_0 = \sum_{j=1}^d [a_j]_\alpha [b_j]_\beta$.

Proof. Since $k : \Omega_1 \times \Omega \to \mathbb{K}$ is uniformly $\min\{\alpha,\beta\}$-Hölder in both arguments it is $\min\{\alpha,\beta\}$-Hölder; hence, $\mathcal{K}, \mathcal{K}^n \in L(C^0(\Omega), C^\beta(\Omega_1))$. For exponents $\alpha \le \beta$, the compactness of $\mathcal{K}, \mathcal{K}^n \in L(C^\alpha(\Omega), C^\beta(\Omega_1))$ follows as in [**5,** p. 57, Rem. 3.4.13 for $\Omega_1 = \Omega$]. Let $x_1, x_2 \in \Omega_1$ be arbitrary and define the kernel function

$$\tilde{k} : \Omega_1^2 \times \Omega \to \mathbb{K}, \qquad \tilde{k}(x_1,x_2,y) := k(x_1,y) - k(x_2,y),$$

which by assumption (ii) is α-Hölder, and therefore also γ-Hölder in the third argument with

$$\begin{aligned}[\tilde{k}(x_1,x_2,\cdot)]_\gamma &\overset{(1)}{\le} (\operatorname{diam}\Omega)^{\alpha-\gamma}[\tilde{k}(x_1,x_2,\cdot)]_\alpha \\ &\overset{(10)}{\le} (\operatorname{diam}\Omega)^{\alpha-\gamma} k_0 \left|x_1 - x_2\right|^\beta . \end{aligned} \tag{12}$$

Moreover, the assumption (i) implies $\left|\tilde{k}(x_1,x_2,y)\right| \le k_1 \left|x_1 - x_2\right|^\beta$ for all $y \in \Omega$ and therefore

$$\|\tilde{k}(x_1,x_2,\cdot)\|_0 \le k_1 \left|x_1 - x_2\right|^\beta . \tag{13}$$

Let $u \in C^\alpha(\Omega) \subseteq C^\gamma(\Omega)$ and $n \in \mathbb{N}$. We know from (i) that $(\mathcal{K} - \mathcal{K}^n)u$ is a β-Hölder function. Furthermore, observe that

$$(\mathcal{K} - \mathcal{K}^n)u(x_1) - (\mathcal{K} - \mathcal{K}^n)u(x_2) = \int_\Omega \tilde{k}(x_1,x_2,y)u(y)\,\mathrm{d}y - Q^n\tilde{k}(x_1,x_2,\cdot)u(\cdot)$$

and Lemma 1 applied to the α-Hölder functions $\tilde{k}(x_1, x_2, \cdot)$ and u yields that $\tilde{k}(x_1, x_2, \cdot)u(\cdot)$ is α-Hölder, and therefore γ-Hölder with

$$|(\mathcal{K} - \mathcal{K}^n)u(x_1) - (\mathcal{K} - \mathcal{K}^n)u(x_2)| \overset{(4)}{\leq} \frac{c_0}{n^\gamma} \left\| \tilde{k}(x_1, x_2, \cdot)u(\cdot) \right\|_\gamma .$$

This, as in the proof of Thm. 1, implies that

$$\begin{aligned} &|(\mathcal{K} - \mathcal{K}^n)u(x_1) - (\mathcal{K} - \mathcal{K}^n)u(x_2)| \\ &\overset{(8)}{\leq} \frac{c_0}{n^\gamma} \left([\tilde{k}(x_1, x_2, \cdot)]_\gamma + \|\tilde{k}(x_1, x_2, \cdot)\|_0 \right) \|u\|_\gamma \\ &\overset{(13)}{\leq} \frac{c_0}{n^\gamma} \left([\tilde{k}(x_1, x_2, \cdot)]_\gamma + k_1 \,|x_1 - x_2|^\beta \right) \|u\|_\gamma \\ &\overset{(12)}{\leq} \frac{c_0}{n^\gamma} \left((\operatorname{diam} \Omega)^{\alpha - \gamma} k_0 + k_1 \right) \|u\|_\gamma \,|x_1 - x_2|^\beta \end{aligned}$$

and since $x_1, x_2 \in \Omega_1$ were arbitrary, the estimate

$$[(\mathcal{K} - \mathcal{K}^n)u]_\beta \leq \frac{c_0}{n^\gamma} \left((\operatorname{diam} \Omega)^{\alpha - \gamma} k_0 + k_1 \right) \|u\|_\gamma$$

results. Combining this with (9) implies that

$$\begin{aligned} &\|(\mathcal{K} - \mathcal{K}^n)u\|_\beta = \max \left\{ \|(\mathcal{K} - \mathcal{K}^n)u\|_0 , [(\mathcal{K} - \mathcal{K}^n)u]_\beta \right\} \\ \leq &\frac{c_0}{n^\gamma} \max \left\{ \|k\|_0 + k_2 (\operatorname{diam} \Omega)^{\alpha - \gamma}, k_0 (\operatorname{diam} \Omega)^{\alpha - \gamma} + k_1 \right\} \|u\|_\gamma \end{aligned}$$

and consequently the claimed norm estimate follows. □

Corollary 1. *If the conditions of Thm. 2 are fulfilled for $\Omega_1 = \Omega$ and all $\alpha = \beta \in [0, 1]$, then the spectra $\sigma(\mathcal{K})$ and $\sigma(\mathcal{K}^n)$ are independent of α.*

Proof. Let $\alpha \in [0, 1]$, $\Omega_1 := \Omega$, set $\mathcal{K}_\alpha := \mathcal{K}|_{C^\alpha(\Omega)}$ and Thm. 2 yields that $\mathcal{K}_\alpha \in L(C^\alpha(\Omega))$ is compact. Since the spaces $C^\alpha(\Omega)$, $C^0(\Omega)$ are compatible and the operators $\mathcal{K}_\alpha$, $\mathcal{K}_0$ are consistent (see [**3,** p. 49]), we conclude from [**3,** pp. 109–110, Thm. 4.2.15] that $\sigma(\mathcal{K}_\alpha) = \sigma(\mathcal{K}_0)$, i.e. all $\mathcal{K}_\alpha$ have identical spectra. The argument for the Nyström discretizations $\mathcal{K}^n \in L(C^\alpha(\Omega))$, $n \in \mathbb{N}$, is verbatim. □

For consistency rates $\gamma < \alpha$ one can dispense assumption (iii):

Theorem 3. *Let $\alpha, \beta \in (0,1]$. If a kernel $k : \Omega_1 \times \Omega \to \mathbb{K}$ satisfies*

(i) $k_1 := \sup_{y \in \Omega} [k(\cdot, y)]_\beta < \infty$,
(ii) $k_2 := \sup_{x \in \Omega_1} [k(x, \cdot)]_\alpha < \infty$

and (Q_n) has consistency order $\gamma \in (0, \alpha)$, then for all $n \in \mathbb{N}$ and $\theta \in \left[\frac{\gamma}{\alpha}, 1\right)$ the following norm estimate holds:

$$\|\mathcal{K} - \mathcal{K}^n\|_{L(C^\gamma(\Omega), C^{(1-\theta)\beta}(\Omega_1))} \le \frac{c_0}{n^\gamma} \max\Big\{\|k\|_0 + k_2 (\operatorname{diam} \Omega)^{\alpha - \gamma},\ k_1\Big(2(\operatorname{diam}\Omega)^{\theta\alpha-\gamma}\tfrac{k_2^\theta}{k_1^\theta} + (\operatorname{diam}\Omega_1)^{\theta\beta}\Big)\Big\}.$$

Proof. We derive a Hölder condition for the kernel $\tilde{k} : \Omega_1^2 \times \Omega \to \mathbb{K}$ from the proof of Thm. 2 and set $\Delta := \left|\tilde{k}(x_1, x_2, y_1) - \tilde{k}(x_1, x_2, y_2)\right|$ for $x_1, x_2 \in \Omega_1$ and $y_1, y_2 \in \Omega$. The two estimates

$$\Delta \le |k(x_1, y_1) - k(x_2, y_1)| + |k(x_1, y_2) - k(x_2, y_2)| \overset{(i)}{\le} 2k_1 |x_1 - x_2|^\beta,$$

$$\Delta \le |k(x_1, y_1) - k(x_1, y_2)| + |k(x_2, y_1) - k(x_2, y_2)| \overset{(ii)}{\le} 2k_2 |y_1 - y_2|^\alpha$$

imply

$$\Delta = \Delta^{1-\theta}\Delta^\theta \le 2k_1^{1-\theta}k_2^\theta |x_1 - x_2|^{(1-\theta)\beta} |y_1 - y_2|^{\theta\alpha} \quad \text{for all } \theta \in (0,1),$$

which allows us to conclude

$$[\tilde{k}(x_1, x_2, \cdot)]_{\theta\alpha} \le 2k_1^{1-\theta}k_2^\theta |x_1 - x_2|^{(1-\theta)\beta} \quad \text{for all } \theta \in (0,1).$$

Referring to the proofs of Thm. 1 and 2, if $\gamma \le \theta\alpha$ holds, then

$$\begin{aligned}
&|(\mathcal{K} - \mathcal{K}^n)u(x_1) - (\mathcal{K} - \mathcal{K}^n)u(x_2)| \\
&\overset{(13)}{\le} \frac{c_0}{n^\gamma}\left([\tilde{k}(x_1, x_2, \cdot)]_\gamma + k_1 |x_1 - x_2|^\beta\right)\|u\|_\gamma \\
&\overset{(1)}{\le} \frac{c_0}{n^\gamma}\left((\operatorname{diam}\Omega)^{\theta\alpha-\gamma}[\tilde{k}(x_1, x_2, \cdot)]_{\theta\alpha} + k_1 |x_1 - x_2|^\beta\right)\|u\|_\gamma \\
&\le \frac{c_0}{n^\gamma}\left(2(\operatorname{diam}\Omega)^{\theta\alpha-\gamma}k_1^{1-\theta}k_2^\theta |x_1 - x_2|^{(1-\theta)\beta} + k_1 |x_1 - x_2|^\beta\right)\|u\|_\gamma \\
&\le \frac{c_0}{n^\gamma}\left(2(\operatorname{diam}\Omega)^{\theta\alpha-\gamma}k_1^{1-\theta}k_2^\theta + k_1(\operatorname{diam}\Omega_1)^{\theta\beta}\right)|x_1 - x_2|^{(1-\theta)\beta}\|u\|_\gamma
\end{aligned}$$

and since $x_1, x_2 \in \Omega_1$ were arbitrary, finally

$$[(\mathcal{K} - \mathcal{K}^n)u]_{(1-\theta)\beta} \le \frac{c_0 k_1}{n^\gamma} \left(2(\operatorname{diam}\Omega)^{\theta\alpha-\gamma}\frac{k_2^\theta}{k_1^\theta} + (\operatorname{diam}\Omega_1)^{\theta\beta}\right) \|u\|_\gamma$$

results. This implies that

$$\begin{aligned} &\|(\mathcal{K} - \mathcal{K}^n)u\|_{(1-\theta)\beta} = \max\left\{\|(\mathcal{K} - \mathcal{K}^n)u\|_0, [(\mathcal{K} - \mathcal{K}^n)u]_{(1-\theta)\beta}\right\} \\ \le &\max\left\{\|(\mathcal{K} - \mathcal{K}^n)u\|_0, c_0 k_1 \tfrac{2(\operatorname{diam}\Omega)^{\theta\alpha-\gamma}k_1^{-\theta}k_2^\theta + (\operatorname{diam}\Omega_1)^{\theta\beta}}{n^\gamma}\right\} \|u\|_\gamma \end{aligned}$$

and combined with (9) follows the claimed norm estimate. □

4. An application to nonautonomous dynamics. Let $\Omega \subset \mathbb{R}^\kappa$ be nonempty, compact, convex and $\tilde{\Omega} := \{x - y \in \mathbb{R}^\kappa : x, y \in \Omega\}$. Suppose that $c_t \in C^2(\tilde{\Omega})$, $t \in \mathbb{Z}$, describes a sequence of convolution kernels. Linearizing for instance a nonautonomous Ricker integrodifference equation (cf. **[9]**)

$$u_{t+1}(x) = \int_\Omega c_t(x-y)u_t(y)e^{-u_t(y)}\,\mathrm{d}y \quad \text{for all } x \in \Omega \tag{14}$$

along the trivial solution yields the linear difference equation

$$u_{t+1} = \mathcal{K}_t u_t, \qquad \mathcal{K}_t u(x) := \int_\Omega c_t(x-y)u(y)\,\mathrm{d}y \tag{15}$$

in $C^0(\Omega)$. Since (15) depends explicitly on time t, stability properties cannot be concluded from the (time-variant) spectrum $\sigma(\mathcal{K}_t) \subset \mathbb{C}$ generically. Nonetheless, an appropriate substitute to infer stability is the dichotomy spectrum $\Sigma(\mathcal{K}) \subseteq (0, \infty)$ (see **[6, 10]**) due to the following features:

- If $\Sigma(\mathcal{K}) \subseteq (0, 1)$, then (15) and hence the trivial solution of (14) are uniformly exponentially stable.
- If a component of $\Sigma(\mathcal{K})$ is contained in $(1, \infty)$, then (15) and the zero solution of (14) are unstable.

In order to determine stability properties of (15), the spectrum $\Sigma(\mathcal{K})$ needs to be approximated numerically. Thereto, it is essential to discretize the Fredholm operator in (15) e.g. using a quadrature formula

(Q_n) yielding the Nyström method

$$\mathcal{K}_t^n u(x) := \sum_{\eta \in \Omega^{(n)}} w_\eta c_t(x-\eta)u(\eta) \quad \text{for all } x \in \Omega$$

(cf. (6)). Because the dichotomy spectrum of the spatially discretized difference equation $u_{t+1} = \mathcal{K}_t^n u_t$ can be computed using methods for finite-dimensional problems **[6]**, it remains to establish a convergence result relating the spectra $\Sigma(\mathcal{K})$ and $\Sigma(\mathcal{K}^n)$.

Theorem 4. *If (Q_n) has consistency order $\alpha \in (0,1]$ and the convolution kernels $c_t \in C^2(\tilde{\Omega})$ satisfy $\sup_{t\in\mathbb{Z}} \{\|c_t\|_\alpha, \|c_t''\|_0\} < \infty$, then*

$$\lim_{n\to\infty} \operatorname{dist}(\Sigma(\mathcal{K}^n), \Sigma(\mathcal{K})) = 0.$$

Proof. We define $k(x,y) := c_t(x-y)$. In the notation of Thm. 2 one has $k_1 = k_2 = [c_t]_\alpha \le \bar{c} := \sup_{t\in\mathbb{Z}} \|c_t\|_\alpha$ and Rem. 1(1) yields the constant $k_0 = (\operatorname{diam}\Omega)^{2-2\alpha} \sup_{t\in\mathbb{Z}} \|c_t''\|_0$ in (iii). Since the dichotomy spectrum is upper-semicontinuous **[10,** Cor. 4], for every given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\sup_{t\in\mathbb{Z}} \|\mathcal{K}_t^n - \mathcal{K}_t\|_{L(C^\alpha(\Omega))} < \delta$ implies the inclusion $\Sigma(\mathcal{K}^n) \subseteq B_\varepsilon(\Sigma(\mathcal{K}))$. Then choosing $N \in \mathbb{N}$ so large that

$$\|\mathcal{K}_t^n - \mathcal{K}_t\|_{L(C^\alpha(\Omega))} \overset{(11)}{\le} \frac{c_0}{n^\alpha}\Big(\bar{c} + \max\Big\{\bar{c}, (\operatorname{diam}\Omega)^{2-2\alpha} \sup_{t\in\mathbb{Z}} \|c_t''\|_0\Big\}\Big) < \delta$$

holds for $n \ge N$ yields $\operatorname{dist}(\Sigma(\mathcal{K}^n), \Sigma(\mathcal{K})) < \varepsilon$ and thus the claim. □

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