ATTRACTIVE INVARIANT FIBER BUNDLES

CHRISTIAN PÖTZSCHE

Abstract. This work on implicit nonautonomous difference equations (iterations) is devoted to an abstract technical result on the existence of attractive invariant manifolds. We investigate their existence, smoothness and their asymptotic phase property using invariant foliations. Such results lay the basic foundation of further investigations on discretization methods of evolutionary differential equations.

1. Introduction

1.1. **Introduction and Motivation:** The ultimate goal of the qualitative theory of dynamical systems is to describe the asymptotic behavior of a given evolutionary equation, i.e. to determine its global attractor. However, since it is usually difficult to obtain the detailed structure of an attractor — not only for infinite-dimensional systems —, a mathematically justified simplification of the given problem is desirable. One of the key techniques to simplify a dynamical system is to embed the attractor into a lower dimensional surface and thus to reduce the dimension of its state space to arrive at an equation sharing the same essential dynamical features as the original one. A rigorous mathematical formulation of this idea is based on the concept of attractive invariant manifolds. Here, *invariance* means that solutions of the system starting on the manifold cannot leave it, and *attractive* means that the dynamics on such an invariant set affects the behavior of the system in a whole neighborhood, so that an overview over the whole system can be given. The flow on the manifold forms a (reduced) system on its own, which in many cases is finite-dimensional and can (hopefully) be investigated more easily. Therefore, in a way, attractive invariant manifolds capture the essential behavior of a dynamical system.

The ubiquitous invariant manifold theory started over a century ago with the classical works of Poincaré [Poi86], Hadamard [Had01] and Perron [Per30], and its development has continued to this day. The use of attractive invariant manifolds is a fairly traditional matter as well, beginning with Pliss's study on local center manifolds leading to his celebrated reduction principle (cf. [Pli64]), which is a main tool to investigate the stability of equilibria in a nonhyperbolic situation. While Pliss's original work is helpful in a local analysis near invariant sets, other approaches in the same vain led to reduction principles of a global nature involving, e.g. slow manifolds of singularly perturbed problems (e.g. [Bar69]) or inertial manifolds of evolutionary equations (e.g. [FST88]).

While the above references are concerned with dynamical systems generated by differential equations, the same questions also occur in the field of discrete dynamical systems generated by maps or difference equations. Theorems on invariant manifolds for maps have been proved in a variety of settings, ranging from stable/unstable manifolds of fixed points, local center manifolds (cf. [Ioo79]) to general results of Hirsch, Pugh & Shub [HPS76] or results on nonautonomous equations (see, e.g. [Aul98]). Attractive invariant manifolds of maps were considered by Kirchgraber [KS78], and his results have been generalized by Nipp & Stoffer [NS92], including the proof of differentiability properties. Beyond that, global smooth center-unstable manifolds related to fixed points were treated in Chow & Lu [CL88]. Further applications of perturbation results for attractive invariant manifolds for maps naturally come from discretization theory of inertial manifolds (see [DG91, JS95, JST98, vDL99, Kob94, Kob95, Kob99]).

Date: November 19, 2008.

2000 *Mathematics Subject Classification.* 37D10, 39A11, 37B55, 37C60, 65M06, 65Q05.

Key words and phrases. Invariant fiber bundles, inertial manifold, attractive invariant manifold, asymptotic phase, discretization, nonautonomous.

Research supported by the Deutsche Forschungsgemeinschaft.

Apart from [Aul98], all the above references work with maps, i.e. autonomous difference equations, where the right-hand side does not depend explicitly on time. Such an approach has certain limitations, since the obtained results can be applied only to constant step-size discretizations of autonomous evolutionary equations. This is rather restrictive, since step-size control is widely used in order to counteract a possible "stiff" behavior, and from a modelling perspective, nonautonomous equations frequently yield more realistic descriptions of certain phenomena.

Motivated by this deficit, the present paper was initiated by some work on the discretization theory of nonautonomous evolutionary equations. In such a set-up, it frequently occurs that systems under consideration possess global attractors of finite dimension or attractive invariant manifolds. To prove that the latter property is preserved under discretization with numerical methods and to address convergence issues later on, one first of all needs corresponding existence results for attractive invariant manifolds of nonautonomous difference equations.

The article at hand is the first one in a series on that topic. Our general philosophy is to provide flexible quantitative results applicable to a large class of discretizations, and to gather technical background for later reference. More precisely, we lay the basics on persistence of attractive invariant manifolds under discretizations. Our nonautonomous difference equations set-up is sufficiently flexible to obtain invariant manifolds including discrete versions of inertial manifolds, as well as of pseudo-unstable manifolds related to equilibria. To capture schemes being adaptive in spatial variables (cf., e.g. [EJ95]), our state spaces can vary in time. The assumptions on the linear part admit non-smooth initial data (and corresponding blow-ups). All this works for implicit equations, since stability issues in the numerical solution of PDEs demand the use of such methods. In addition, the nonlinearities of our systems are allowed to be unbounded in time, as long as the growth rate is dominated by their linear part. Beyond existence matters, we include assertions concerning smoothness, invariant foliations and a strong asymptotic attraction property, namely an asymptotic phase. The latter property guarantees that each solution is attracted exponentially by a unique solution on the invariant manifold starting at the same time.

Classically, there are basically two approaches to construct such invariant manifolds. The geometrically intuitive graph transform [KS78, NS92, JS95, JST98, vDL99] and the Lyapunov-Perron method [DG91, Kob94, Kob95, Kob99] with a more functional analytical background. In an autonomous framework, both approaches feature certain advantages. In our nonautonomous situation, however, it seems more canonical to use a Lyapunov-Perron technique and additionally this approach has the benefit that we can refer to earlier results in a similar setting. Actually, we state that a C^m-smoothness of the right-hand side is shared by the invariant manifold, provided a certain gap condition is satisfied.

Relating this work to earlier results, [KS78, JS95, vDL99] work in a Lipschitzian setting, the C^1-case is considered in [JST98] and higher-order smoothness assertions can be found in [NS92]. The existence of an asymptotic phase is shown in [KS78, NS92] and without proof in [Kob94, Kob95, Theorem 2.1] or [Kob99], while [DG91, JS95, JST98, vDL99] derive a weaker exponential attraction property. To derive such an asymptotic phase for a given attractive set one typically uses invariant foliations over the set. In the framework of not necessarily invertible mappings, existence and C^1-smoothness results for invariant manifolds can be found in [CHT97] using the Lyapunov-Perron technique. Related references concerning invariant manifolds of nonautonomous difference equations (so-called invariant fiber bundles) include [Aul98, APS02, PS04], where pseudo-stable and -unstable fiber bundles and their smoothness is addressed. A construction of invariant foliations, however different from ours, can be found in [AW03].

To conclude this introduction we give an outline of this paper: We state our basic setting and assumptions in Section 2. Having this at hand, the existence and smoothness result on attractive invariant manifolds (invariant fiber bundles) for nonautonomous difference equations is derived in Section 3. We continue our investigations in Section 4 deriving an invariant foliation over the invariant fiber bundle, and its asymptotic phase. So far, our results are global in nature and supposed to hold under Lipschitz conditions on the whole state space. This strong assumption is weakened to construct a discrete counterpart of an inertial manifold in the follow-up [Pöt06a]. A straightforward application of such results to simplify nonlinear higher-order difference equations has been given in [Pöt06b]. Finally, Section 5 is intended to illuminate our results and hypotheses by various discretization approaches to evolutionary equations including temporal and spatial schemes.

1.2. **Basic Notation and Nonautonomous Sets.** We write $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$ for the set of integers, a *discrete interval* is the intersection of a real interval with $\mathbb{Z}$, in particular, $\mathbb{Z}_\kappa^+ := \{k \in \mathbb{Z} : \kappa \leq k\}$, $\mathbb{Z}_\kappa^- := \{k \in \mathbb{Z} : k \leq \kappa\}$ for some $\kappa \in \mathbb{Z}$, and $\mathbb{N} := \mathbb{Z}_1^+$. The Banach spaces X, Y of this paper are real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$), and their norm is denoted by $\|\cdot\|_X, \|\cdot\|_Y$, respectively. We write $L(X, Y)$ for the space of bounded linear operators between X and Y, $L(X) := L(X, X)$, and I_X for the identity on X. The space of bounded n-linear operators from X to Y is denoted by $L_n(X, Y)$, $n \in \mathbb{N}$.

For a map $F : X \times Z \to Y$, where Z denotes a nonempty set, we define the Lipschitz constants

$$\begin{aligned} \operatorname{Lip} F(\cdot, z) &:= \inf\left\{L \geq 0 : \|F(x, z) - F(\bar{x}, z)\|_Y \leq L \|x - \bar{x}\|_X \text{ for all } x, \bar{x} \in X\right\}, \\ \operatorname{Lip}_1 F &:= \sup_{z \in Z} \operatorname{Lip} F(\cdot, z), \end{aligned}$$

provided they exist. If the set Z has a metric structure, one defines the Lipschitz constant w.r.t. the second variable $\operatorname{Lip}_2 F$ analogously, and proceeds correspondingly, if F depends on more than two variables. Moreover, $f : X \to Y$ is said to be of *class* C^m, if it is m-times continuously Fréchet differentiable.

In order to provide a flexible set-up covering many discretization schemes, we are dealing with time-dependent state spaces, and thereto let X_k, $k \in \mathbb{Z}$, be a sequence of Banach spaces and $\mathcal{X} := \bigcup_{k \in \mathbb{Z}} X_k$. For $\kappa \in \mathbb{Z}$ and reals $\gamma > 0$ we introduce the spaces of exponentially bounded sequences

$$\mathcal{X}_{\kappa,\gamma}^+ := \left\{ \phi : \mathbb{Z}_\kappa^+ \to \mathcal{X} \;\middle|\; \begin{array}{l} \phi(k) \in X_k \text{ for all } k \in \mathbb{Z}_\kappa^+ \text{ and} \\ \sup_{k \in \mathbb{Z}_\kappa^+} \|\phi(k)\|_{X_k} \gamma^{\kappa - k} < \infty \end{array} \right\},$$

$$\mathcal{X}_{\kappa,\gamma}^- := \left\{ \phi : \mathbb{Z}_\kappa^- \to \mathcal{X} \;\middle|\; \begin{array}{l} \phi(k) \in X_k \text{ for all } k \in \mathbb{Z}_\kappa^- \text{ and} \\ \sup_{k \in \mathbb{Z}_\kappa^-} \|\phi(k)\|_{X_k} \gamma^{\kappa - k} < \infty \end{array} \right\}.$$

It is handy to use the abbreviation $\mathcal{X}_{\kappa,\gamma}^\pm$ for either $\mathcal{X}_{\kappa,\gamma}^+$ or $\mathcal{X}_{\kappa,\gamma}^-$; accordingly we proceed with $\mathbb{Z}_\kappa^\pm$ and our further notation. As an elementary, yet important observation we state without proof

Lemma 1.1 (quasibounded functions). *Let $\kappa \in \mathbb{Z}$ and $\gamma > 0$. Then $\mathcal{X}_{\kappa,\gamma}^\pm$ is a Banach space w.r.t. the norm $\|\phi\|_{\kappa,\gamma}^\pm := \sup_{k \in \mathbb{Z}_\kappa^\pm} \|\phi(k)\|_{X_k} \gamma^{\kappa - k}$. A sequence $\phi : \mathbb{I} \to \bigcup_{k \in \mathbb{I}} X_k$ defined on a discrete interval $\mathbb{I}$, which is unbounded according to $\mathbb{Z}_\kappa^\pm \subseteq \mathbb{I}$, is called* $\gamma^\pm$-quasibounded, *if $\phi|_{\mathbb{Z}_\kappa^\pm} \in \mathcal{X}_{\kappa,\gamma}^\pm$ holds.*

For (not necessarily invertible) linear operators $A(k) : X_k \to X_{k+1}$, $k \in \mathbb{Z}$, we define the associate *evolution operator* $\Phi(k, \kappa) : X_\kappa \to X_k$, $\kappa, k \in \mathbb{Z}$, $\kappa \leq k$, as the linear mapping given by

$$\Phi(k, \kappa) := \begin{cases} I_{X_\kappa} & \text{for } k = \kappa \\ A(k-1) \cdots A(\kappa) & \text{for } k > \kappa \end{cases},$$

and if $A(k)$ is invertible for $k < \kappa$, then $\Phi(k, \kappa) := A(k)^{-1} \cdots A(\kappa - 1)^{-1}$ for $k < \kappa$.

Let $\mathbb{I}$ stand for a discrete interval. Given a sequence $\phi : \mathbb{I} \to \mathcal{X}$, we define $\phi'(k) := \phi(k+1)$ for all $k \in \mathbb{I}$ such that $k + 1 \in \mathbb{I}$, and use a similar notation for operator- or set-valued sequences.

Numerical schemes to approximate evolutionary equations are typically recursive formulas (iterations) and for (numerical) stability reasons they are frequently assumed to be implicit. To denote such (ordinary) difference equations, we prefer the notation

$$x' = f(k, x, x') \tag{1.1}$$

rather than the more established ones $x(k+1) = f(k, x(k), x(k+1))$ or $x_{k+1} = f(k, x_k, x_{k+1})$, with *right-hand side* $f(k, \cdot) : X_k \times X_{k+1} \to X_{k+1}$, $k \in \mathbb{Z}$; advantages of this notation are explained in [Aul98]. A sequence $\phi : \mathbb{I} \to \mathcal{X}$ satisfying $\phi(k) \in X_k$ for $k \in \mathbb{I}$ and the *solution identity* $\phi'(k) = f(k, \phi(k), \phi'(k))$ for all $k \in \mathbb{I}$ with $k + 1 \in \mathbb{I}$ is called *solution* of (1.1).

We often write $\mathcal{X} := \{(k, x) : k \in \mathbb{Z}, x \in X_k\}$ for the *extended state space* of (1.1). If the difference equation (1.1) is *well-defined on $\mathcal{X}$ in forward time*, i.e. if for all $\kappa \in \mathbb{Z}$, $\xi \in X_\kappa$ there exists a unique forward solution $\phi : \mathbb{Z}_\kappa^+ \to \mathcal{X}$ of (1.1) satisfying $\phi(\kappa) = \xi$, we introduce the *general solution* $\varphi(\cdot; \kappa, \xi) := \phi$ of (1.1). It is easy to see that the so-called *cocycle property*

$$\varphi(k; \kappa, \xi) = \varphi(k; l, \varphi(l; \kappa, \xi)) \quad \text{for all } k \geq l \geq \kappa \tag{1.2}$$

holds for φ. Note that (1.2) is also known as 2-*parameter semigroup* or *process* property.

Due to the implicit nature of (1.1), solutions need not to exist or to be unique. We do not dwell on this fact here and only remark that even in the explicit case (f does not depend on the third argument), $\varphi(k;\kappa,\cdot)$ in general does not exist for $k<\kappa$.

Finally, a subset $\mathcal{A}\subseteq\mathcal{X}$ is called a *nonautonomous set* with *k-fiber* $\mathcal{A}(k):=\{x\in X_k:(k,x)\in\mathcal{A}\}$ for $k\in\mathbb{Z}$. Such a set $\mathcal{A}$ is called *positively invariant* w.r.t. (1.1), if the inclusion $\varphi(k;\kappa,\mathcal{A}(\kappa))\subseteq\mathcal{A}(k)$ holds for $\kappa\le k$, and it is called *invariant*, if one has equality $\varphi(k;\kappa,\mathcal{A}(\kappa))=\mathcal{A}(k)$ for $\kappa\le k$. Moreover, we denote (1.1) as difference equation *in* $\mathcal{A}$, if $\mathcal{A}$ is positively invariant. A nonautonomous set $\mathcal{A}$ is called a *vector bundle*, if each fiber $\mathcal{A}(k)$ is a linear subspace of X_k. The *cartesian product* of two nonautonomous sets $\mathcal{A},\mathcal{B}\subseteq\mathcal{X}$ is defined as $\mathcal{A}\times\mathcal{B}:=\{(k,a,b)\in\mathbb{Z}\times\mathfrak{X}\times\mathfrak{X}:\ a\in\mathcal{A}(k),b\in\mathcal{B}(k)\}$.

2. Preliminaries and Assumptions

In this section we motivate the kind of difference equations under consideration and give the precise hypotheses on the abstract framework we shall use throughout this paper. The generality and scope of these results will be illustrated by examples in Section 5.

Let Y_k, $k\in\mathbb{Z}$, be a further sequence of Banach spaces. The paper is concerned with θ-dependent nonautonomous difference equations of the form

$$y'=A(k)y+\theta K'(k)F(k,y,y'), \tag{2.1}$$

where $A(k):Y_k\to Y_{k+1}$, $K(k)\in L(Y_k)$ are linear operators and $F(k,\cdot):Y_k\times Y_{k+1}\to Y_{k+1}$ denotes the nonlinearity for $k\in\mathbb{Z}$. As indicated in our introductory remarks on (1.1), we have used a compact and convenient notation in (2.1), which classically reads as

$$y_{k+1}=A(k)y_k+\theta K(k+1)F(k,y_k,y_{k+1}).$$

A priori, (2.1) is a difference equation in $\mathcal{Y}:=\{(k,y):\ k\in\mathbb{Z},y\in Y_k\}$. Here, $\theta\in\Theta$ denotes a parameter from a nonempty set $\Theta\subseteq\mathbb{F}$. Typically, it will be related to the maximal step-size in time discretizations, or to the ratio of temporal and spatial step-size in full discretizations. Nevertheless, in this paper we do not focus on the dependence of (2.1) on the parameter θ and include it only for future applications (see also Remark 2.1(1)).

Now we formulate our main assumptions in a quite quantitative fashion. At first glance they seem very technical, but basically mean that the linear part of (2.1), given by

$$y'=A(k)y, \tag{2.2}$$

possesses an exponential dichotomy and the nonlinearity F is Lipschitzian.

Hypothesis. *Let X_k,Y_k, $k\in\mathbb{Z}$, be Banach spaces with the embedding*

$$X_k\subseteq Y_k\quad\textit{for all } k\in\mathbb{Z}, \tag{2.3}$$

let $\nu\ge0$, $0<\Lambda<\lambda$, K_1^-,K_2^-, $K_1^+,K_2^+,K_3^+>0$, $L_2^-,L_2^+,L_3^-,L_3^+>0$ be reals and assume:

- (H_0) *The difference equation (2.1) is well-defined on $\mathcal{X}$ in forward time with general solution φ.*
- (H_1) *There exist complementary projections $P_-(k),P_+(k)$ on Y_k with $P_-(k)\in L(Y_k)$, $P_-(k)Y_k\subseteq X_k$, $P_+(k)X_k\subseteq X_k$,*

$$P_-'(k)A(k)=A(k)P_-(k),\qquad K(k)P_-(k)=P_-(k)K(k)\quad\textit{for all } k\in\mathbb{Z}, \tag{2.4}$$

one has the inclusions

$$A(k)P_+(k)X_k\subseteq X_{k+1},\qquad P_+(k)K(k)Y_k\subseteq X_k\quad\textit{for all } k\in\mathbb{Z}, \tag{2.5}$$

the mappings

$$A(k)|_{P_-(k)X_k}:P_-(k)X_k\to P_-'(k)X_{k+1} \tag{2.6}$$

are invertible with associate evolution operator $\bar\Phi(k,\kappa)$, we have

$$\bar C:=\sup_{k\in\mathbb{Z}}\|K(k)\|_{L(Y_k)}<\infty \tag{2.7}$$

and for all $k,l\in\mathbb{Z}$ one finally has the exponential dichotomy estimates

$$\|\Phi(k,l)P_+(l)\|_{L(X_l,X_k)}\le K_1^+\Lambda^{k-l}\quad\textit{for all } l\le k, \tag{2.8}$$

$$\|\Phi(k,l)P_+(l)K(l)\|_{L(Y_l,X_k)} \le (K_2^+ + K_3^+ |\theta|^{-\nu} (k-l+1)^{-\nu})\Lambda^{k-l} \quad \textit{for all } l \le k, \tag{2.9}$$

$$\left\|\bar\Phi(k,l)P_-(l)\right\|_{L(X_l,X_k)} \le K_1^- \lambda^{k-l} \quad \textit{for all } k \le l, \tag{2.10}$$

$$\left\|\bar\Phi(k,l)P_-(l)\right\|_{L(Y_l,X_k)} \le K_2^- \lambda^{k-l} \quad \textit{for all } k < l, \tag{2.11}$$

where $\Phi(k,l)$ is the evolution operator for A.

(H_2) *For one (and hence every) $\kappa \in \mathbb{Z}$ the constants*

$$C_\kappa^\pm := \sup_{k<\kappa} \left\|P'_\pm(k)F(k,0,0)\right\|_{Y_{k+1}} \lambda^{\kappa-k} \tag{2.12}$$

are finite, we have the global Lipschitz estimates

$$\begin{aligned} \left\|P'_\pm(k)\left[F(k,x,y)-F(k,\bar x,y)\right]\right\|_{Y_{k+1}} &\le L_2^\pm \|x-\bar x\|_{X_k} \quad \textit{for all } k\in\mathbb{Z},\ x,\bar x\in X_k,\ y\in X_{k+1},\\ \left\|P'_\pm(k)\left[F(k,x,y)-F(k,x,\bar y)\right]\right\|_{Y_{k+1}} &\le L_3^\pm \|y-\bar y\|_{X_{k+1}} \quad \textit{for all } k\in\mathbb{Z},\ x\in X_k,\ y,\bar y\in X_{k+1} \end{aligned} \tag{2.13}$$

and we require the spectral gap condition*: There exist reals $0<\sigma<\sigma_{\max}\le\frac{\lambda-\Lambda}{2}$ such that*

$$|\theta|\,\Sigma(\bar\sigma) < 1 \quad \textit{for all } \theta\in\Theta,\ \bar\sigma\in(\sigma,\sigma_{\max}) \tag{2.14}$$

holds with a function $\Sigma : (\sigma,\sigma_{\max}) \to \mathbb{R}^+$ to be specified later, but depending on the dichotomy data and $L_2^\pm, L_3^\pm$; we then define the nonempty interval $\bar\Gamma := [\Lambda+\sigma, \lambda-\sigma]$.

(H_3) *The Fréchet derivatives $D_2^n F(k,\cdot) : X_k \times X_{k+1} \to L_n(X_k\times X_{k+1}, Y_{k+1})$, $k\in\mathbb{Z}$, exist, are continuous and one has the global boundedness*

$$\sup_{k\in\mathbb{Z}} \sup_{(x,y)\in X_k\times X_{k+1}} \left\|D^n_{(2,3)}F(k,x,y)\right\|_{L_n(X_k\times X_{k+1},Y_{k+1})} < \infty \quad \textit{for all } n\in\{1,\dots,m\}.$$

Remark 2.1. (1) As a matter of notational convenience we suppress a possible dependence of the functions A, K and F on the parameter θ. Nevertheless, all our assertions persist, as long as the estimates (2.7), (2.8)–(2.11), (2.12), (2.13) are uniform in $\theta\in\Theta$. Moreover, also the spaces X_k, Y_k can depend on θ.

(2) The left relation in (2.4) implies that the sets $\mathcal{P}_\pm := \{(k,x)\in\mathcal{X} : x\in P_\pm(k)X_k\}$ are positively invariant w.r.t. (2.2), and the *regularity condition* (2.6) guarantees invariance of the vector bundle $\mathcal{P}_-$. We denote $\mathcal{P}_+$ as *pseudo-stable* and $\mathcal{P}_-$ as *pseudo-unstable vector bundle* of (2.2); the motivation for this terminology lies in a possible dynamical characterization of these sets following from the dichotomy estimates (2.8)–(2.11). One should note that the operator $A(k) : Y_k \to Y_{k+1}$ defining the linear part of (2.1) is not assumed to be continuous so far. However, (2.8)–(2.11) imply the boundedness of the mappings $P_\pm(k) : X_k \to X_k$, $P_+(k)K(k) : Y_k \to X_k$ and

$$A(k)|_{\mathcal{P}_-(k)}^{-1} : \mathcal{P}'_-(k) \to \mathcal{P}_-(k), \qquad A(k)|_{\mathcal{P}_+(k)} : \mathcal{P}_+(k) \to X_{k+1},$$

$$A(k)|_{P_+(k)K(k)Y_k} : P_+(k)K(k)Y_k \to X_{k+1}.$$

(3) The estimates (2.9), (2.11) are not present in the established notions of an exponential dichotomy for linear difference equations (cf., for instance [Hen81, p. 229, Definition 7.6.4]), in particular due to the singularity for $\theta\to 0$ in (2.9). Nevertheless, our approach is well-motivated by diverse applications in discretization theory sketched in Subsections 5.2 and 5.4.

(4) Hypothesis (H_0) holds, if $K'(k)F(k,x,\cdot) : X_{k+1} \to X_{k+1}$ satisfies a global Lipschitz condition

$$|\theta|\,\mathrm{Lip}\,K'(k)F(k,x,\cdot) < 1 \quad \text{for all } k\in\mathbb{Z},\ \theta\in\Theta,\ x\in X_k; \tag{2.15}$$

this is an easy consequence of the contraction mapping principle.

The following corollary justifies the operator norms used in the dichotomy estimates (2.8)–(2.11).

Corollary 2.1. *Under Hypothesis (H_1) the following is true:*

(a) *$A(k)\mathcal{P}_-(k) \subseteq \mathcal{P}'_-(k)$ for $k\in\mathbb{Z}$; hence the linear map in* (2.6) *is well-defined and satisfies*

$$\bar\Phi(k,l)P_-(l) = P_-(k)\bar\Phi(k,l) \quad \textit{for all } k,l\in\mathbb{Z}, \tag{2.16}$$

(b) *$\bar\Phi(k,l)P_-(l)Y_l \subseteq X_k$ for $k\le l$,*

(c) *$\Phi(k,l)P_+(l)K(l)Y_l \subseteq X_k$ for $l\le k$,*

(d) *if* (2.3) *is a continuous embedding and* (2.11) *holds also for* $k = l$, *then* (2.10), (2.11) *are satisfied, if and only if there exists a* $K_0^- > 0$ *with* $\left\|\bar{\Phi}(k,l)P_-(l)\right\|_{L(Y_l,Y_k)} \leq K_0^- \lambda^{k-l}$ *for all* $k \leq l$.

Proof. The proof of (a)–(c) is simple; assertion (d) can be shown with Banach's isomorphism theorem. □

3. Invariant Fiber Bundles

In this section we derive the existence of an invariant nonautonomous set $\mathcal{W}_\theta \subseteq \mathcal{X}$ for (2.1), which generalizes the pseudo-unstable vector bundle $\mathcal{P}_-$ to a nonlinear setting. Due to our global assumption (H_2), each of the fibers $\mathcal{W}_\theta(\kappa)$, $\kappa \in \mathbb{Z}$, will be a submanifold of X_κ given by the graph of a globally Lipschitzian mapping over $\mathcal{P}_-(\kappa)$. A proof of this fact requires technical preparations.

Above all, we introduce the *polylogarithm* (cf. [Lew82, pp. 236–238]), which is the strictly increasing unbounded continuous function $\mathrm{Li}_\nu : [0,1) \to \mathbb{R}$, $\nu \in [0,\infty)$ with $\mathrm{Li}_\nu(0) = 0$, given by

$$\mathrm{Li}_\nu(x) := \sum_{n=1}^{\infty} n^{-\nu} x^n; \tag{3.1}$$

it can be interpreted as discrete version of the Gamma function frequently occurring in Lyapunov-Perron constructions of inertial manifolds for evolutionary differential equations (cf., e.g., [Kob94, Kob95]).

For additional notational convenience, consider the *Green's function* for A, given by

$$G(k,\kappa) := \begin{cases} -\bar{\Phi}(k,\kappa)P_-(\kappa) & \text{for } k < \kappa \\ \Phi(k,\kappa)P_+(\kappa) & \text{for } k \geq \kappa \end{cases}. \tag{3.2}$$

We begin our analysis with a perturbation result for linear equations, which essentially goes back to [Hen81, p. 230, Theorem 7.6.5]. It is concerned with the admissibility of spaces of quasibounded sequences.

Lemma 3.1 (admissibility)**.** *Let* $\theta \in \Theta$, $\kappa \in \mathbb{Z}$, $C \geq 0$, *assume* (H_1) *holds, let* $\gamma \in (\Lambda, \lambda)$ *and consider the nonautonomous linear difference equation*

$$y' = A(k)y + \theta K'(k)r(k) \tag{3.3}$$

with a sequence $r(k) \in Y_{k+1}$ *for all* $k \in \mathbb{Z}$. *Then the following holds:*

(a) *If one chooses* $\delta \in [\gamma, \infty)$ *and the inequality*

$$\|r(k)\|_{Y_{k+1}} \leq C\gamma^{k-\kappa} \quad \text{for all } k \in \mathbb{Z}_\kappa^+ \tag{3.4}$$

holds, then for any $\xi \in X_\kappa$ *there exists a unique solution* $\phi : \mathbb{Z}_\kappa^+ \to \mathfrak{X}$ *of* (3.3), *satisfying* $\phi \in \mathfrak{X}_{\kappa,\delta}^+$ *and* $P_+(\kappa)\phi(\kappa) = P_+(\kappa)\xi$. *It is given by*

$$\phi(k) := \Phi(k,\kappa)P_+(\kappa)\xi + \theta \sum_{n=\kappa}^{\infty} G(k,n+1)K'(n)r(n) \quad \text{for all } k \in \mathbb{Z}_\kappa^+$$

and satisfies the estimate $\|\phi\|_{\kappa,\delta}^+ \leq K_1^+ \|\xi\|_{X_\kappa} + |\theta| C \left(\frac{K_2^+}{\gamma-\Lambda} + \frac{|\theta|^{-\nu} K_3^+}{\Lambda} \mathrm{Li}_\nu\left(\frac{\Lambda}{\gamma}\right) + \frac{\bar{C}K_2^-}{\lambda-\gamma}\right)$.

(b) *If one chooses* $\delta \in (0, \gamma]$ *and the inequality*

$$\|r(k)\|_{Y_{k+1}} \leq C\gamma^{k-\kappa} \quad \text{for all } k < \kappa \tag{3.5}$$

holds, then for any $\xi \in Y_\kappa$ *there exists a unique solution* $\phi : \mathbb{Z}_\kappa^- \to \mathfrak{X}$ *of* (3.3), *satisfying* $\phi \in \mathfrak{X}_{\kappa,\delta}^-$ *and* $P_-(\kappa)\phi(\kappa) = P_-(\kappa)\xi$. *It is given by*

$$\phi(k) := \bar{\Phi}(k,\kappa)P_-(\kappa)\xi + \theta \sum_{n=-\infty}^{\kappa-1} G(k,n+1)K'(n)r(n) \quad \text{for all } k \in \mathbb{Z}_\kappa^-$$

and satisfies the backward estimate

$$\|\phi\|_{\kappa,\delta}^- \leq K_1^- \|P_-(\kappa)\xi\|_{X_\kappa} + |\theta| C \left(\frac{K_2^+}{\gamma - \Lambda} + \frac{|\theta|^{-\nu} K_3^+}{\Lambda} \mathrm{Li}_\nu\left(\tfrac{\Lambda}{\gamma}\right) + \frac{\bar{C}K_2^-}{\lambda - \gamma}\right). \tag{3.6}$$

Proof. Let $\theta \in \Theta$ and $\kappa \in \mathbb{Z}$ be given. Since both assertions of Lemma 3.1 can be shown in a very similar fashion, we present only the proof of (b).

For points $\xi \in Y_\kappa$ it is easy to verify that $\phi : \mathbb{Z}_\kappa^- \to \mathcal{X}$ is a solution of (3.3); here the inclusions (2.5) imply $\phi(k) \in X_k$ for all $k \in \mathbb{Z}_\kappa^-$. Moreover, due to (2.16) one gets $P_-(\kappa)\phi(\kappa) = P_-(\kappa)\xi$. Now we establish that ϕ is δ^--quasibounded for $\delta \in (0, \gamma]$. Therefore, (2.7), (2.10), (2.11), (3.5) implies

$$\begin{aligned} \|P_-(k)\phi(k)\|_{X_k} &\stackrel{(2.16)}{\leq} \left\|\bar\Phi(k,\kappa)P_-(\kappa)\right\|_{L(X_\kappa,X_k)} \|P_-(\kappa)\xi\|_{X_\kappa} \\ &\qquad + |\theta| \sum_{n=k}^{\kappa-1} \left\|\bar\Phi(k,n+1)P_-'(n)\right\|_{L(Y_{n+1},X_k)} \|K'(n)\|_{L(Y_{n+1})} \|r(n)\|_{Y_{n+1}} \\ &\leq K_1^- \lambda^{k-\kappa} \|P_-(\kappa)\xi\|_{X_\kappa} + |\theta| \frac{C\bar C K_2^-}{\lambda} \frac{\lambda^k}{\gamma^\kappa} \sum_{n=k}^{\kappa-1} \left(\frac{\gamma}{\lambda}\right)^n \quad \text{for all } k \in \mathbb{Z}_\kappa^-, \end{aligned}$$

and accordingly one gets from (2.16), (2.9), (3.5) that

$$\|P_+(k)\phi(k)\|_{X_k} \leq |\theta| \frac{C}{\Lambda} \left(K_2^+ \frac{\Lambda^k}{\gamma^\kappa} \sum_{n=-\infty}^{k-1} \left(\frac{\gamma}{\Lambda}\right)^n + |\theta|^{-\nu} K_3^+ \gamma^{k-\kappa} \operatorname{Li}_\nu\left(\tfrac{\Lambda}{\gamma}\right)\right) \quad \text{for all } k \in \mathbb{Z}_\kappa^-,$$

which, using $\|\phi(k)\|_{X_k} \leq \|P_-(k)\phi(k)\|_{X_k} + \|P_+(k)\phi(k)\|_{X_k}$, leads to

$$\|\phi(k)\|_{X_k} \delta^{\kappa-k} \leq K_1^- \|P_-(\kappa)\xi\|_{X_\kappa} + |\theta| C \left(\frac{K_2^+}{\gamma - \Lambda} + \frac{|\theta|^{-\nu} K_3^+}{\Lambda} \operatorname{Li}_\nu\left(\tfrac{\Lambda}{\gamma}\right) + \frac{\bar C K_2^-}{\lambda - \gamma}\right) \quad \text{for all } k \in \mathbb{Z}_\kappa^-.$$

This implies $\phi \in \mathcal{X}_{\kappa,\delta}^-$, as well as the estimate (3.6). To deduce uniqueness of ϕ, let $\bar\phi \in \mathcal{X}_{\kappa,\delta}^-$ be another solution of (3.3) satisfying $P_-(\kappa)\bar\phi(\kappa) = P_-(\kappa)\xi$. Then the difference $\phi - \bar\phi$ is a δ^--quasibounded solution of (2.2) with $P_-(\kappa)\left[\phi(\kappa) - \bar\phi(\kappa)\right] = 0$. Using the dichotomy estimates (2.10), (2.8) one shows that the trivial solution is the only δ^--quasibounded solution of (2.2) in $\mathcal{P}_+$, i.e. $\bar\phi = \phi$. $\square$

Let $(\kappa, \xi) \in \mathcal{Y}$ and $\gamma > 0$. For given $\phi \in \mathcal{X}_{\kappa,\gamma}^-$, we formally define the sequence-valued operator

$$\mathcal{T}_\kappa(\phi; \xi) := \bar\Phi(\cdot,\kappa)P_-(\kappa)\xi + \theta \sum_{n=-\infty}^{\kappa-1} G(\cdot, n+1)K'(n)F(n, \phi(n), \phi'(n)). \tag{3.7}$$

Similarly to this operator, we frequently encounter sequence-valued maps $\phi : Z \to \mathcal{X}_{\kappa,\gamma}^\pm$, defined on some set Z. For an efficient notation we use the sometimes imprecise abbreviation $\phi(k,z) := \big(\phi(z)\big)(k) \in X_k$.

Lemma 3.2. *Let $\theta \in \Theta$, $\kappa \in \mathbb{Z}$, assume (H_1)–(H_2) and let $\gamma \in (\Lambda, \lambda)$, $\delta \in (0, \gamma]$ be reals. Then the mapping $\mathcal{T}_\kappa : \mathcal{X}_{\kappa,\gamma}^- \times Y_\kappa \to \mathcal{X}_{\kappa,\delta}^-$ is well-defined with*

$$\|\mathcal{T}_\kappa(\phi;\xi)\|_{\kappa,\delta}^- \leq K_1^- \|P_-(\kappa)\xi\|_{X_\kappa} + |\theta|\,\Gamma_\kappa(\gamma) + |\theta|\,\ell(\gamma)\,\|\phi\|_{\kappa,\gamma}^-, \tag{3.8}$$

$$\|P_+(\kappa)\mathcal{T}_\kappa(\kappa,\phi;\xi)\|_{X_\kappa} \leq |\theta|\,\ell^+(\gamma)\left(C_\kappa^+ + L^+(\gamma)\,\|\phi\|_{\kappa,\gamma}^-\right) \tag{3.9}$$

for all $\phi \in \mathcal{X}_{\kappa,\gamma}^-$, $\xi \in Y_\kappa$, and we have Lipschitz estimates

$$\operatorname{Lip}_1 P_+(\kappa)\mathcal{T}_\kappa(\kappa,\cdot) \leq |\theta|\, L^+(\gamma)\ell^+(\gamma), \qquad \operatorname{Lip}_1 \mathcal{T}_\kappa \leq |\theta|\,\ell(\gamma), \qquad \operatorname{Lip}_2 \mathcal{T}_\kappa \leq K_2^- \tag{3.10}$$

with the constants $L^\pm(\gamma) := L_2^\pm + \gamma L_3^\pm$,

$$\Gamma_\kappa(\gamma) := C_\kappa^+\ell^+(\gamma) + C_\kappa^-\ell^-(\gamma), \qquad \ell(\gamma) := L^+(\gamma)\ell^+(\gamma) + L^-\ell^-(\gamma),$$

$$\ell^+(\gamma) := \frac{K_2^+}{\gamma - \Lambda} + \frac{|\theta|^{-\nu} K_3^+}{\Lambda} \operatorname{Li}_\nu\left(\tfrac{\Lambda}{\gamma}\right), \qquad \ell^-(\gamma) := \frac{\bar C K_2^-}{\lambda - \gamma}.$$

Proof. Let $\theta \in \Theta$ and $\kappa \in \mathbb{Z}$. We begin with preparatory estimates. For $\phi \in \mathcal{X}_{\kappa,\gamma}^-$, using the triangle inequality, from (2.13), (2.12) one has $\left\|P_\pm'(n)F(n,\phi(n),\phi'(n))\right\|_{Y_{n+1}} \leq \left(C_\kappa^\pm + L^\pm(\gamma)\,\|\phi\|_{\kappa,\gamma}^-\right)\gamma^{n-\kappa}$ for

$n < \kappa$. From this, using the relations (2.4), we obtain almost identically to the proof of Lemma 3.1(b) that for each pair $(\kappa, \xi) \in \mathcal{Y}$ one gets

$$\|P_-(k)\mathcal{T}_\kappa(k,\phi;\xi)\|_{X_k}\delta^{\kappa-k} \le K_1^- \|P_-(\kappa)\xi\|_{X_\kappa} + |\theta|\left(C_\kappa^- + L^-(\gamma)\|\phi\|_{\kappa,\gamma}^-\right)\frac{\bar{C}K_2^-}{\lambda-\gamma},$$

$$\|P_+(k)\mathcal{T}_\kappa(k,\phi;\xi)\|_{X_k}\delta^{\kappa-k} \le |\theta|\left(C_\kappa^+ + L^+(\gamma)\|\phi\|_{\kappa,\gamma}^-\right)\left(\frac{K_2^+}{\gamma-\Lambda} + \frac{|\theta|^{-\nu}K_3^+}{\Lambda}\operatorname{Li}_\nu\left(\tfrac{\Lambda}{\gamma}\right)\right) \quad \text{for all } k \in \mathbb{Z}_\kappa^-.$$

Combining these two estimates, one has $\mathcal{T}_\kappa(\phi;\xi) \in \mathcal{X}_{\kappa,\delta}^-$ and (3.8). The relation (3.9) follows from the latter estimate above by setting $k = \kappa$. To prove the Lipschitz estimates in (3.10), let $\phi, \bar{\phi} \in \mathcal{X}_{\kappa,\delta}^-$ and $\xi, \bar{\xi} \in Y_\kappa$. We obtain from (2.5), (2.16), (2.7), (2.11), (2.13) that

$$\left\|P_-(k)\left[\mathcal{T}_\kappa(k,\phi;\xi) - \mathcal{T}_\kappa(k,\bar{\phi};\xi)\right]\right\|_{X_k} \le |\theta|\frac{\bar{C}K_2^- L^-(\gamma)}{\lambda}\frac{\lambda^k}{\gamma^\kappa}\sum_{n=k}^{\kappa-1}\left(\frac{\gamma}{\lambda}\right)^n \left\|\phi-\bar{\phi}\right\|_{\kappa,\gamma}^- \quad \text{for all } k \in \mathbb{Z}_\kappa^-,$$

and (2.16), (2.9), (2.13) implies

$$\begin{aligned} &\left\|P_+(k)\left[\mathcal{T}_\kappa(k,\phi;\xi) - \mathcal{T}_\kappa(k,\bar{\phi};\xi)\right]\right\|_{X_k} \\ \le\ & |\theta|\frac{L^+(\gamma)}{\Lambda}\left(K_2^+\frac{\Lambda^k}{\gamma^\kappa}\sum_{n=-\infty}^{k-1}\left(\frac{\gamma}{\Lambda}\right)^n + |\theta|^{-\nu}K_3^+\gamma^{k-\kappa}\operatorname{Li}_\nu\left(\tfrac{\Lambda}{\gamma}\right)\right)\left\|\phi-\bar{\phi}\right\|_{\kappa,\gamma}^- \quad \text{for all } k \in \mathbb{Z}_\kappa^-. \end{aligned}$$

Setting $k = \kappa$ immediately yields the first relation in (3.10). Multiplying both above estimates with $\delta^{\kappa-k}$ and applying the triangle inequality to estimate $\mathcal{T}_\kappa(\phi;\xi) - \mathcal{T}_\kappa(\bar{\phi};\xi)$, gives us the middle relation of (3.10). Finally, using (3.7), (2.11), the remaining Lipschitz estimate in (3.10) follows from

$$\left\|\mathcal{T}_\kappa(k,\phi;\xi) - \mathcal{T}_\kappa(k,\phi;\bar{\xi})\right\|_{X_k}\delta^{\kappa-k} \le K_2^-\left\|\xi-\bar{\xi}\right\|_{Y_\kappa} \quad \text{for all } k \in \mathbb{Z}_\kappa^-.$$

□

By virtue of the so-called *Lyapunov-Perron operator* $\mathcal{T}_\kappa$, we can now characterize the exponentially bounded solutions of difference equation (2.1) quite easily as its fixed points, and solve the corresponding problem using the contraction mapping principle. Nonetheless, our approach differs from [DG91] or the continuous counterpart in [SY02, pp. 569ff, Chapter 8], and imposes somewhat weaker assumptions on the size of $|\theta| L^\pm(\gamma)$.

Lemma 3.3. *Let $\theta \in \Theta$, $(\kappa, \xi) \in \mathcal{Y}$, $\gamma \in (\Lambda, \lambda)$, $\phi \in \mathcal{X}_{\kappa,\gamma}^-$ and assume (H_1)–(H_2). Then for the mapping $\mathcal{T}_\kappa(\cdot;\xi) : \mathcal{X}_{\kappa,\gamma}^- \to \mathcal{X}_{\kappa,\gamma}^-$ the following two statements are equivalent:*

(a) *ϕ solves the difference equation* (2.1) *with $P_-(\kappa)\phi(\kappa) = P_-(\kappa)\xi$,*
(b) *ϕ is a solution of the fixed point equation*

$$\phi = \mathcal{T}_\kappa(\phi;\xi). \tag{3.11}$$

Proof. Let $\theta \in \Theta$, $(\kappa, \xi) \in \mathcal{Y}$, $\gamma \in (\Lambda, \lambda)$. We define a sequence $r(k) := F(k, \phi(k), \phi'(k))$ and using (2.7), (2.13) one has $\|r(k)\|_{Y_{k+1}} \le \left(\sup_{k<\kappa}\|F(k,0,0)\|_{Y_{k+1}}\lambda^{\kappa-k} + (\operatorname{Lip}_2 F + \gamma \operatorname{Lip}_3 F)\|\phi\|_{\kappa,\gamma}^-\right)\gamma^{k-\kappa}$ for $k < \kappa$. Together with (2.12) this yields that r satisfies an estimate of the form (3.5).

(a) ⇒ *(b)* Let $\phi : \mathbb{Z}_\kappa^- \to \mathcal{X}$ be a γ^--quasibounded solution of (2.1) with $P_-(\kappa)\phi(\kappa) = P_-(\kappa)\xi$. Then ϕ also solves the linear inhomogeneous equation

$$y' = A(k)y + \theta K'(k)F(k,\phi(k),\phi'(k)) \tag{3.12}$$

and Lemma 3.1(b) implies assertion (b).

(b) ⇒ *(a)* A fixed point of $\mathcal{T}_\kappa(\cdot,\xi)$ is a solution of (3.12), and therefore of the nonlinear difference equation (2.1) satisfying $P_-(\kappa)\phi(\kappa) = P_-(\kappa)\xi$. □

Lemma 3.4. *Let $\theta \in \Theta$, $\kappa \in \mathbb{Z}$, assume Hypotheses (H_1)–(H_2) with $\sigma_{\max} = \frac{\lambda-\Lambda}{2}$, Σ given by*

$$\Sigma(\sigma) := L^-(\lambda-\sigma)\tfrac{\bar{C}K_2^-}{\sigma} + L^+(\lambda-\sigma)\left(\tfrac{K_2^+}{\sigma} + |\theta|^{-\nu}K_3^+\operatorname{Li}_\nu\left(\tfrac{\Lambda}{\Lambda+\sigma}\right)\right) \tag{3.13}$$

and choose $\gamma \in \bar{\Gamma}$*. Then the mapping* $\mathcal{T}_\kappa(\cdot;\xi) : \mathfrak{X}^-_{\kappa,\gamma} \to \mathfrak{X}^-_{\kappa,\gamma}$ *possesses a unique fixed point* $\phi_\kappa(\xi) \in \mathfrak{X}^-_{\kappa,\gamma}$ *for* $\xi \in Y_\kappa$*. Moreover, the fixed point mapping* $\phi_\kappa : Y_\kappa \to \mathfrak{X}^-_{\kappa,\gamma}$ *satisfies* $\phi_\kappa(\xi) = \phi_\kappa(P_-(\kappa)\xi)$ *and one has:*

(a) $\phi_\kappa : Y_\kappa \to \mathfrak{X}^-_{\kappa,\gamma}$ *is linearly bounded, i.e. for all* $(\kappa,\xi) \in \mathcal{Y}$ *it is*

$$\|\phi_\kappa(\xi)\|^-_{\kappa,\gamma} \le \frac{K_1^-}{1-|\theta|\,\ell(\gamma)} \|P_-(\kappa)\xi\|_{X_\kappa} + \frac{|\theta|\,\Gamma_\kappa(\gamma)}{1-|\theta|\,\ell(\gamma)}, \tag{3.14}$$

$$\|P_+(\kappa)\phi_\kappa(\kappa,\xi)\|_{X_\kappa} \le |\theta|\,\ell^+(\gamma)\left[C_\kappa^+ + \frac{L^+(\gamma)}{1-|\theta|\,\ell(\gamma)}\left(K_1^-\,\|P_-(\kappa)\xi\|_{X_\kappa} + |\theta|\,\Gamma_\kappa(\gamma)\right)\right], \tag{3.15}$$

(b) ϕ_κ *is globally Lipschitzian with*

$$\operatorname{Lip}\phi_\kappa \le \frac{K_2^-}{1-|\theta|\,\ell(\gamma)}, \qquad \operatorname{Lip} P_+(\kappa)\phi_\kappa \le |\theta|\,K_1^-\frac{L^+(\gamma)\ell^+(\gamma)}{1-|\theta|\,\ell(\gamma)}, \tag{3.16}$$

(c) *if additionally* (H_3), $\Lambda < \lambda^m$ *with* $m \in \mathbb{N}$, *and* $\sigma_{\max} = \min\left\{\frac{\lambda-\Lambda}{2}, \lambda\left(1-\sqrt[m]{\frac{\lambda+\Lambda}{\lambda+\lambda^m}}\right)\right\}$ *hold, then for* $\gamma \in [\Lambda+\sigma, \lambda-\sigma)$ *the mapping* $\phi_\kappa : Y_\kappa \to \mathfrak{X}^-_{\kappa,\gamma}$ *is of class* C^m *with globally bounded derivatives,*

where the constants $L^\pm(\gamma)$, $\Gamma_\kappa(\gamma)$, $\ell(\gamma), \ell^+(\gamma)$ *are defined in Lemma 3.2.*

Proof. Let $\theta \in \Theta$ and $(\kappa,\xi) \in \mathcal{Y}$ be given. The spectral gap condition (2.14) implies $|\theta|\,\ell(\gamma) < 1$ for all $\gamma \in \bar{\Gamma}$. Therefore, from the middle estimate (3.10) in Lemma 3.2 we know that $\mathcal{T}_\kappa(\cdot;\xi)$ is a contraction on the Banach space $\mathfrak{X}^-_{\kappa,\gamma}$ and the contraction mapping theorem implies the existence of a unique fixed point $\phi_\kappa(\xi) \in \mathfrak{X}^-_{\kappa,\gamma}$. The relation $\phi_\kappa(\xi) = \phi_\kappa(P_-(\kappa)\xi)$ follows from the fact $\mathcal{T}_\kappa(\cdot;\xi) = \mathcal{T}_\kappa(\cdot;P_-(\kappa)\xi)$, and consequently the fixed points of the two contractions coincide.

(a) Thanks to $|\theta|\,\ell(\gamma) < 1$, the estimate (3.14) follows from

$$\|\phi_\kappa(\xi)\|^-_{\kappa,\gamma} \stackrel{(3.11)}{=} \|\mathcal{T}_\kappa(\phi_\kappa(\xi);\xi)\|^-_{\kappa,\gamma} \stackrel{(3.8)}{\le} K_1^-\,\|P_-(\kappa)\xi\|_{X_\kappa} + |\theta|\,\Gamma_\kappa(\gamma) + |\theta|\,\ell(\gamma)\,\|\phi_\kappa(\xi)\|^-_{\kappa,\gamma} \quad \text{for all } \xi \in Y_\kappa$$

and the estimate (3.15) is a consequence of (3.14) and

$$\|P_+(\kappa)\phi_\kappa(\kappa,\xi)\|_{X_\kappa} \stackrel{(3.11)}{=} \|P_+(\kappa)\mathcal{T}_\kappa(\kappa,\phi_\kappa(\xi);\xi)\|_{X_\kappa} \stackrel{(3.9)}{\le} |\theta|\,\ell^+(\gamma)\left(C_\kappa^+ + L^+(\gamma)\,\|\phi_\kappa(\xi)\|^-_{\kappa,\gamma}\right).$$

(b) Next we derive the Lipschitz estimates in (3.16). Let $\xi, \bar{\xi} \in Y_\kappa$ and from (3.11), (3.10) we obtain

$$\left\|\phi_\kappa(\xi) - \phi_\kappa(\bar{\xi})\right\|^-_{\kappa,\gamma} \le K_2^-\,\left\|\xi-\bar{\xi}\right\|_{Y_\kappa} + |\theta|\,\ell(\gamma)\,\left\|\phi_\kappa(\xi)-\phi_\kappa(\bar{\xi})\right\|^-_{\kappa,\gamma},$$

which yields the left relation in (3.16). Similarly, from (3.11), (3.7), (3.10) one has

$$\left\|P_+(\kappa)\left[\phi_\kappa(\kappa,\xi) - \phi_\kappa(\kappa,\bar{\xi})\right]\right\|_{X_\kappa} \le |\theta|\,L^+(\gamma)\ell^+(\gamma)\,\left\|\phi_\kappa(\xi)-\phi_\kappa(\bar{\xi})\right\|^-_{\kappa,\gamma},$$

and this in connection with the left estimate for ϕ_κ in (3.16), leads to the right assertion in (3.16).

(c) The rigorous proof of the fact that $\phi_\kappa : Y_\kappa \to \mathfrak{X}^+_{\kappa,\gamma}$ is of class C^m with globally bounded derivatives, is based on an involved induction argument. It essentially follows the ideas of [PS04] and we refer to this reference for further details. One formally differentiates (3.11) w.r.t. $\xi \in X_\kappa$ and shows that the formal derivatives are the actual derivatives using an argument similar to the proof of Lemma 4.3(b). □

Invariant fiber bundles generalize invariant manifolds to nonautonomous difference equations. In order to be more precise, we call a nonautonomous set $\mathcal{W}_\theta$ an *invariant fiber bundle* (IFB for short) of (2.1), if it is *invariant* and each fiber $\mathcal{W}_\theta(k)$ is a submanifold of X_k for $k \in \mathbb{Z}$.

Theorem 3.5 (existence of IFBs)**.** *Let* $\theta \in \Theta$ *and assume Hypotheses* (H_0)*–*(H_2) *with* $\sigma_{\max} = \frac{\lambda-\Lambda}{2}$ *and* Σ *given by* (3.13)*. Then the set*

$$\mathcal{W}_\theta := \left\{ (\kappa,\xi) \in \mathcal{X} \;\middle|\; \begin{array}{l} \text{there exists a solution } \phi : \mathbb{Z} \to \mathfrak{X} \text{ of } (2.1) \\ \text{with } \phi(\kappa) = \xi \in X_\kappa \text{ and } \phi|_{\mathbb{Z}_\kappa^-} \in \mathfrak{X}^-_{\kappa,\gamma} \end{array} \right\}$$

is an IFB of (2.1)*, which is independent of* $\gamma \in \bar{\Gamma}$ *and possesses the representation*

$$\mathcal{W}_\theta = \{(\kappa, \eta + w_\theta(\kappa,\eta)) \in \mathcal{X} : (\kappa,\eta) \in \mathcal{P}_-\} \tag{3.17}$$

as graph of a uniquely determined mapping $w_\theta(\kappa,\cdot):Y_\kappa\to X_\kappa$ *with* $w_\theta(\kappa,\xi)=w_\theta(\kappa,P_-(\kappa)\xi)\in\mathcal{P}_+(\kappa)$ *for all* $(\kappa,\xi)\in\mathcal{Y}$ *and satisfying the* invariance equation

$$\begin{aligned} w_\theta(\kappa+1,\eta_1)&=A(\kappa)w_\theta(\kappa,\eta)+\theta P'_+(\kappa)K'(\kappa)F(\kappa,\eta+w_\theta(\kappa,\eta),\eta_1+w_\theta(\kappa+1,\eta_1)),\\ \eta_1&=A(\kappa)\eta+\theta K'(\kappa)F(\kappa,\eta+w_\theta(\kappa,\eta),\eta_1+w_\theta(\kappa+1,\eta_1))\end{aligned}\tag{3.18}$$

for all $(\kappa,\eta)\in\mathcal{P}_-$, $\eta_1\in\mathcal{P}'_-(\kappa)$*. Furthermore, for all* $\gamma\in\bar\Gamma$ *and* $\theta\in\Theta$ *it holds:*

(a) $w_\theta(\kappa,\cdot):Y_\kappa\to X_\kappa$ *is linearly bounded*

$$\|w_\theta(\kappa,\xi)\|_{X_\kappa}\le|\theta|\,\ell^+(\gamma)\left[C_\kappa^+ +\frac{L^+(\gamma)}{1-|\theta|\,\ell(\gamma)}\left(|\theta|\,\Gamma_\kappa(\gamma)+K_1^-\,\|P_-(\kappa)\xi\|_{X_\kappa}\right)\right]\quad\textit{for all } (\kappa,\xi)\in\mathcal{Y},\tag{3.19}$$

(b) $w_\theta(\kappa,\cdot)$ *is globally Lipschitzian with*

$$\operatorname{Lip}_2 w_\theta\le|\theta|\,K_1^-\frac{L^+(\gamma)\ell^+(\gamma)}{1-|\theta|\,\ell(\gamma)},\tag{3.20}$$

(c) *assume Hypothesis* (H_3) *is satisfied with* $\Lambda<\lambda^m$, $m\in\mathbb{N}$, *and if the spectral gap condition* (2.14) *holds with* $\sigma_{\max}=\min\left\{\frac{\lambda-\Lambda}{2},\lambda\left(1-\sqrt[m]{\frac{\lambda+\Lambda}{\lambda+\lambda^m}}\right)\right\}$, *then the partial derivatives* $D_2^n w_\theta(\kappa,\cdot):Y_\kappa\to L_n(Y_\kappa,X_\kappa)$ *exist, are continuous, and globally bounded for* $n\in\{1,\dots,m\}$,

Remark 3.1. (1) It is instructive to relate Theorem 3.5 to a more classical situation — in particular to explicit equations. Under the assumption $F(k,0,0)\equiv0$ on $\mathbb{Z}$, the relations (2.12) are trivially fulfilled and $\mathcal{W}_\theta$ is the pseudo-unstable fiber bundle of (2.1) (cf. [Aul98, APS02, PS04] for the situation of a decoupled linear part and constant state spaces). To be more precise, in the hyperbolic case $\Lambda<1<\lambda$, Theorem 3.5 gives us the unstable fiber bundle of 0, and in the nonhyperbolic cases $\lambda<1$ or $1<\Lambda$ we get the center-unstable or the strongly unstable fiber bundle of 0, respectively. Likewise, these nonautonomous sets reduce to the classical invariant manifolds in an autonomous situation.

(2) A result dual to Theorem 3.5 can be shown for the nonautonomous set

$$\mathcal{W}_\theta^+:=\left\{(\kappa,\xi)\in\mathcal{X}:\,\varphi(\cdot;\kappa,\xi)\in\mathcal{X}_{\kappa,\gamma}^+\right\},$$

if one assumes $A(k)\in L(X_k,X_{k+1})$ for all $k\in\mathbb{Z}$ and that

$$\sup_{k\in\mathbb{Z}_\kappa^+}\max\left\{\left\|P'_-(k)F(k,0,0)\right\|_{Y_{k+1}},\left\|P'_+(k)F(k,0,0)\right\|_{Y_{k+1}}\right\}\Lambda^{\kappa-k}<\infty\quad\text{for one }\kappa\in\mathbb{Z}$$

holds instead of (2.12). Then $\mathcal{W}_\theta^+$ generalizes the pseudo-stable fiber bundle (cf. [Aul98, APS02, PS04]).

(3) To provide an intuition for the crucial spectral gap condition (2.14) observe the following: Assume a more classical situation in which the linear part (2.2) is autonomous and generates a continuous discrete semigroup $(A^k)_{k\in\mathbb{Z}_0^+}$ on a common state space $X=X_k$. Then the exponential dichotomy assumptions (2.8)–(2.11) are satisfied, if the spectrum $\sigma(A)$ allows a decomposition $\sigma(A)=\sigma_+\cup\sigma_-$ into disjoint spectral sets $\sigma_+,\sigma_-\subseteq\mathbb{C}$ such that $\max_{z\in\sigma_-}|z|<\Lambda<\lambda<\inf_{z\in\sigma_+}|z|$ (cf., for example [Ioo79]). Hence, in order to satisfy the spectral gap condition (2.14) in this setting, the limit relations

$$\lim_{\theta\to0}|\theta|\,\Sigma(\bar\sigma)=0\quad\text{for all }\bar\sigma\in(\sigma,\sigma_{\max}),\qquad\qquad\lim_{\bar\sigma\to\infty}|\theta|\,\Sigma(\bar\sigma)=0\quad\text{for all }\theta\in\Theta,$$

offer two possible points of view:

- For a given spectral gap $\lambda-\Lambda$ and arbitrary $0<\sigma<\sigma_{\max}<\frac{\lambda-\Lambda}{2}$ one can choose $\theta\in\Theta$ so small that (2.14) is fulfilled.
- On the other side, for fixed $\theta\in\Theta$, the spectral gap $\lambda-\Lambda>0$ has to be sufficiently large such that there exist $0<\sigma<\sigma_{\max}<\frac{\lambda-\Lambda}{2}$ satisfying (2.14).

Which of these perspectives is favorable, depends on the application one has in mind.

Proof. Let $\theta\in\Theta$, $(\kappa,\xi)\in\mathcal{Y}$ and $\gamma\in\bar\Gamma$.

We want to show that $\mathcal{W}_\theta$ is an IFB of (2.1). By definition, for each pair of initial values $(\kappa,\xi_0)\in\mathcal{W}_\theta$ there exists a solution $\phi\in\mathcal{X}_{\kappa,\gamma}^-$ of (2.1) with $\phi(\kappa)=\xi_0$. Due to the uniqueness of forward solutions guaranteed by (H_0), we have $\phi=\varphi(\cdot;l,\phi(l))$; accordingly $\varphi(\cdot;l,\phi(l))$ is a γ^-quasibounded solution and this yields the inclusion $\varphi(l;\kappa,\xi)\in\mathcal{W}_\theta(l)$ for all $l\in\mathbb{Z}_\kappa^+$. Conversely, let $\xi_1\in\mathcal{W}'_\theta(\kappa)$. Then there exists a

γ^--quasibounded solution $\phi : \mathbb{Z} \to \mathcal{X}$ of (2.1) with $\phi'(\kappa) = \xi_1$. Obviously, $\xi_0 := \phi(\kappa) \in \mathcal{W}_\theta(\kappa)$ and (H_0) yields $\xi_1 = \varphi(\kappa+1; \kappa, \xi_0)$, i.e., we have the inclusion $\xi_1 \in \varphi(\kappa+1; \kappa, \mathcal{W}_\theta(\kappa))$.

The spectral gap condition (2.14) and the middle estimate (3.10) from Lemma 3.10 give us

$$|\theta|\, \ell(\gamma) \le |\theta| \left(L^+(\gamma)\ell^+(\Lambda+\sigma) + L^-(\gamma)\ell^-(\lambda-\sigma)\right) < 1 \quad \text{for all } \gamma \in \bar{\Gamma}; \tag{3.21}$$

therefore, Lemma 3.4 implies that the mapping $\mathcal{T}_\kappa(\cdot;\xi) : \mathfrak{X}^-_{\kappa,\gamma} \to \mathfrak{X}^-_{\kappa,\gamma}$ possesses a unique fixed point $\phi_\kappa(\xi) \in \mathfrak{X}^-_{\kappa,\gamma}$. This fixed point is independent of the growth constant $\gamma \in \bar{\Gamma}$ because one has the inclusion $\mathfrak{X}^-_{\kappa,\lambda-\sigma} \subseteq \mathfrak{X}^-_{\kappa,\gamma}$ and every $\mathcal{T}_\kappa(\cdot;\xi) : \mathfrak{X}^-_{\kappa,\gamma} \to \mathfrak{X}^-_{\kappa,\gamma}$ possesses the same fixed point as the restriction $\mathcal{T}_\kappa(\cdot;\xi)\big|_{\mathfrak{X}^-_{\kappa,\lambda-\sigma}}$. Furthermore, the fixed point is a solution of the nonautonomous difference equation (2.1) satisfying $P_-(\kappa)\phi_\kappa(\xi)(\kappa) = P_-(\kappa)\xi$ (cf. Lemma 3.3). Now we define

$$w_\theta(\kappa,\xi) := P_+(\kappa)\phi_\kappa(\kappa,\xi) \tag{3.22}$$

and have $w_\theta(\kappa, x_0) \in \mathcal{P}_+(\kappa)$. In addition, (3.7) and the relation $\phi_\kappa(\xi) = \phi_\kappa(P_-(\kappa)\xi)$ in Lemma 3.4 imply

$$w_\theta(\kappa,\xi) \overset{(3.22)}{=} P_+(\kappa)\phi_\kappa(\kappa,\xi) = P_+(\kappa)\phi_\kappa(\kappa, P_-(\kappa)\xi) = w_\theta(\kappa, P_-(\kappa)\xi).$$

We now verify the representation (3.17) and the invariance equation (3.18).

($\subseteq$) Let $(\kappa, x_0) \in \mathcal{W}_\theta$, i.e. there exists a γ^--quasibounded solution $\phi : \mathbb{Z} \to \mathcal{X}$ of (2.1) with $\phi(\kappa) = x_0$. Then ϕ satisfies $P_-(\kappa)\phi(\kappa) = P_-(\kappa)x_0$ and is consequently the unique fixed point of $\mathcal{T}_\kappa(\cdot; x_0)$, i.e., $\phi = \phi_\kappa(x_0)$ (see Lemma 3.3). This, and $\phi_\kappa(\xi) = \phi_\kappa(P_-(\kappa)\xi)$ (cf. Lemma 3.4), implies

$$x_0 = \phi_\kappa(\kappa, x_0) = P_-(\kappa)\phi_\kappa(\kappa, x_0) + P_+(\kappa)\phi_\kappa(\kappa, x_0) = P_-(\kappa)x_0 + P_+(\kappa)\phi_\kappa(\kappa, P_-(\kappa)x_0).$$

So, setting $\xi := P_-(\kappa)x_0$, we have $x_0 = \xi + P_+(\kappa)\phi_\kappa(\kappa,\xi) = \xi + w_\theta(\kappa,\xi)$ by (3.22) and the first inclusion of (3.17) is verified.

($\supseteq$) On the other hand, let $x_0 \in X_\kappa$ be of the form $x_0 = \xi + w_\theta(\kappa,\xi)$ for some $\xi \in \mathcal{P}_-(\kappa)$. Then

$$x_0 \overset{(3.22)}{=} \xi + P_+(\kappa)\phi_\kappa(\kappa,\xi) = P_-(\kappa)\phi_\kappa(\kappa,\xi) + P_+(\kappa)\phi_\kappa(\kappa,\xi) = \phi_\kappa(\kappa,\xi)$$

and therefore, $\phi = \phi_\kappa(\xi)$ is a γ^--quasibounded solution of (2.1) with $\phi(\kappa) = x_0$.

With points $(\kappa, \xi_0) \in \mathcal{W}_\theta$ the invariance of $\mathcal{W}_\theta$ implies the relation $\varphi(k;\kappa,\xi_0) = P_-(k)\varphi(k;\kappa,\xi_0) + w_\theta(k, P_-(k)\varphi(k;\kappa,\xi_0))$, multiplication with $P_+(k)$ yields $P_+(k)\varphi(k;\kappa,\xi_0) = w_\theta(k, P_-(k)\varphi(k;\kappa,\xi_0))$ for $k \in \mathbb{Z}^+_\kappa$, and setting $k = \kappa+1$ finally yields the invariance equation (3.18).

(*a*) We obtain (3.19) from Lemma 3.4 using

$$\|w_\theta(\kappa,\xi)\|_{X_\kappa} \overset{(3.22)}{=} \|P_+(\kappa)\phi_\kappa(\kappa,\xi)\|_{X_\kappa} \overset{(3.15)}{\le} |\theta|\,\ell^+(\gamma) \left[C^+_\kappa + \frac{L^+(\gamma)}{1-|\theta|\,\ell(\gamma)}\left(|\theta|\,\Gamma_\kappa(\gamma) + K^-_1 \|P_-(\kappa)\xi\|_{Y_\kappa}\right)\right].$$

(*b*) To prove the claimed Lipschitz estimate (3.20), consider $\xi, \bar{\xi} \in Y_\kappa$ and corresponding fixed points $\phi_\kappa(\xi), \phi_\kappa(\bar{\xi}) \in \mathfrak{X}^-_{\kappa,\gamma}$ of $\mathcal{T}_\kappa(\cdot;\xi)$ and $\mathcal{T}_\kappa(\cdot;\bar{\xi})$, respectively. One gets from Lemma 3.4(b) that

$$\left\|w_\theta(\kappa,\xi) - w_\theta(\kappa,\bar{\xi})\right\|_{X_\kappa} \overset{(3.22)}{=} \left\|P_+(\kappa)\left[\phi_\kappa(\kappa,\xi) - \phi_\kappa(\kappa,\bar{\xi})\right]\right\|_{X_\kappa} \overset{(3.16)}{\le} |\theta|\, K^-_1 \frac{L^+(\gamma)\ell^+(\gamma)}{1-|\theta|\,\ell(\gamma)} \left\|\xi - \bar{\xi}\right\|_{X_\kappa}.$$

(*c*) We have the identity $w_\theta(\kappa,\xi) = \phi_\kappa(\kappa,\xi)$ (see (3.22)) for $(\kappa,\xi) \in \mathcal{Y}$ and by well-known properties of the evaluation map (cf. [APS02, Lemma 3.4]) it follows from Lemma 3.4(c) that $w_\theta(\kappa,\cdot) : Y_\kappa \to X_\kappa$ is m-times continuously differentiable and possesses globally bounded derivatives. □

4. Invariant Foliation and Asymptotic Phase

In this section we investigate the attraction properties of the IFB $\mathcal{W}_\theta$ from Theorem 3.5 using certain invariant fibers. These fibers serve as leaves for an invariant foliation of the extended state space $\mathcal{X}$ and enable us to construct an asymptotic phase for $\mathcal{W}_\theta$. This means that $\mathcal{W}_\theta$ is not only exponentially attracting, but solutions are also synchronized with corresponding solutions on the IFB $\mathcal{W}_\theta$.

Our strategy in the first part of this section is largely parallel to the previous. Nonetheless, the present assumptions are stronger than in Section 3, and actually the continuity of the general solution $\varphi(k;\kappa,\cdot) : X_\kappa \to X_k$ will play a crucial role. We remark that the construction of IFBs, as well as of invariant foliations, can be put in a common framework by studying general "Lyapunov-Perron equations" (cf. [CHT97] for the autonomous case).

For this, some additional assumptions beyond (H_1)–(H_3) are required:

Hypothesis. *Let $m \in \mathbb{N}$, assume for all $\kappa \in \mathbb{Z}$ that:*

- (H_0') *In addition to (H_0) we suppose $\varphi(k;\kappa,\cdot) : X_\kappa \to X_k$ is continuous for $k \in \mathbb{Z}_\kappa^+$.*
- (H_1') *In addition to (H_1) we suppose $A(\kappa) \in L(X_\kappa, X_{\kappa+1})$.*
- (H_2') *In addition to (H_2) we suppose $K'(\kappa)F(\kappa,\cdot) : X_\kappa \times X_{\kappa+1} \to X_{\kappa+1}$ is continuous.*
- (H_3') *In addition to (H_3) the Fréchet derivatives $D_2^n F(\kappa,\cdot) : X_\kappa \times X_{\kappa+1} \to L_n(X_\kappa \times X_{\kappa+1}, X_{\kappa+1})$ exist and are continuous for $n \in \{1,\dots,m\}$.*

Remark 4.1. Hypothesis (H_0') can be replaced by (H_1')–(H_2') and inequality (2.15), which can easily be deduced using the parametrized contraction mapping principle (see, for instance, [Aul98, Theorem 6.1]).

We introduce an appropriate Lyapunov-Perron operator to construct invariant fibers. Thereto, let $(\kappa,\eta,\xi) \in \mathcal{P}_+ \times \mathcal{X}$ and $\gamma > 0$. For $\psi \in \mathcal{X}_{\kappa,\gamma}^+$, we formally define the mapping

$$\begin{aligned}\mathcal{S}_\kappa(\psi;\eta,\xi) :=& \Phi(\cdot,\kappa)\left[\eta - P_+(\kappa)\xi\right] + \theta \sum_{n=\kappa}^{\infty} G(\cdot,n+1)K'(n)\cdot \\ &\cdot \left[F(n,\psi(n)+\varphi(n;\kappa,\xi),\psi'(n)+\varphi(n+1;\kappa,\xi)) - F(n,\varphi(n;\kappa,\xi),\varphi(n+1;\kappa,\xi))\right]. \end{aligned} \tag{4.1}$$

Lemma 4.1. *Let $\theta \in \Theta$, $\kappa \in \mathbb{Z}$, assume $(H_0),(H_1')$–(H_2') and let $0 < \gamma \le \delta$, $\gamma \in (\Lambda,\lambda)$ be reals. Then the mapping $\mathcal{S}_\kappa : \mathcal{X}_{\kappa,\gamma}^+ \times \mathcal{P}_+(\kappa) \times X_\kappa \to \mathcal{X}_{\kappa,\delta}^+$ is well-defined with*

$$\|\mathcal{S}_\kappa(\psi;\eta,\xi)\|_{\kappa,\delta}^+ \le K_1^+ \|\eta - P_+(\kappa)\xi\|_{X_\kappa} + |\theta|\,\ell(\gamma)\,\|\psi\|_{\kappa,\gamma}^+, \tag{4.2}$$

$$\|P_-(\kappa)\mathcal{S}_\kappa(\kappa,\psi;\eta,\xi)\|_{X_\kappa} \le |\theta|\,L^-(\gamma)\ell^-(\gamma)\,\|\psi\|_{\kappa,\gamma}^+ \tag{4.3}$$

for all $\psi \in \mathcal{X}_{\kappa,\delta}^+$, $\eta \in \mathcal{P}_+(\kappa)$, $\xi \in X_\kappa$ and we have the Lipschitz estimates

$$\operatorname{Lip}_1 P_-(\kappa)\mathcal{S}_\kappa(\kappa,\cdot) \le |\theta|\,L^-(\gamma)\ell^-(\gamma), \qquad \operatorname{Lip}_1 \mathcal{S}_\kappa \le |\theta|\,\ell(\gamma), \qquad \operatorname{Lip}_2 \mathcal{S}_\kappa \le \frac{K_1^+}{1-|\theta|\,\ell(\gamma)}, \tag{4.4}$$

where the constants $L^-(\gamma)$, $\ell(\gamma),\ell^-(\gamma)$ are defined in Lemma 3.2.

Proof. Let $\theta \in \Theta$, $\gamma \in (\Lambda,\lambda)$, $\psi \in \mathcal{X}_{\kappa,\gamma}^+$ and $(\kappa,\eta,\xi) \in \mathcal{P}_+ \times \mathcal{X}$ be given. Then (H_2') implies $\varphi(k;\kappa,\xi) \in X_k$ for all $k \in \mathbb{Z}_\kappa^+$. First, we show that the sequence $\mathcal{S}_\kappa(\psi;\eta,\xi)$ is δ^+-quasibounded for $\delta \in [\gamma,\infty)$. Thereto, from (2.16), (2.7), (2.11), (2.13) one has

$$\|P_-(k)\mathcal{S}_\kappa(k,\psi;\xi,\eta)\|_{X_k} \le |\theta|\,\frac{\bar{C}K_2^- L^-(\gamma)}{\lambda}\frac{\lambda^k}{\gamma^\kappa}\sum_{n=k}^{\infty}\left(\frac{\gamma}{\lambda}\right)^n \|\psi\|_{\kappa,\gamma}^+ \quad \text{for all } k \in \mathbb{Z}_\kappa^+, \tag{4.5}$$

and accordingly (2.4), (2.8), (2.9), (2.13) imply

$$\begin{aligned}&\|P_+(k)\mathcal{S}_\kappa(k,\psi;\eta,\xi)\|_{X_k}\\ \le\;& K_1^+\Lambda^{k-\kappa}\|\eta - P_+(\kappa)\xi\|_{X_\kappa} + |\theta|\,\frac{L^+(\gamma)}{\Lambda}\left(K_2^+\frac{\Lambda^k}{\gamma^\kappa}\sum_{n=\kappa}^{k-1}\left(\frac{\gamma}{\Lambda}\right)^n + |\theta|^{-\nu}K_3^+\gamma^{k-\kappa}\operatorname{Li}_\nu\left(\tfrac{\Lambda}{\gamma}\right)\right)\|\psi\|_{\kappa,\gamma}^+\end{aligned}$$

for all $k \in \mathbb{Z}_\kappa^+$, which, using the triangle inequality leads to

$$\|\mathcal{S}_\kappa(k,\psi;\eta,\xi)\|\,\delta^{\kappa-k} \le K_1^+\|\eta - P_+(\kappa)\xi\|_{X_\kappa} + |\theta|\,\ell(\gamma)\,\|\psi\|_{\kappa,\gamma}^+ \quad \text{for all } k \in \mathbb{Z}_\kappa^+.$$

This implies $\mathcal{S}_\kappa(\psi;\eta,\xi) \in \mathcal{X}_{\kappa,\delta}^+$, as well as the estimate (4.2). Moreover, if we set $k=\kappa$, then (4.3) is a consequence of (4.5). Next we derive the Lipschitz estimates (4.4). Let $\psi,\bar{\psi} \in \mathcal{X}_{\kappa,\gamma}^+$, $\eta,\bar{\eta} \in \mathcal{P}_+(\kappa)$ and fix $\xi \in X_\kappa$. We obtain from (2.4), (2.16), (2.7), (2.11), (2.13) that

$$\left\|P_-(k)\left[\mathcal{S}_\kappa(k,\psi;\eta,\xi) - \mathcal{S}_\kappa(k,\bar{\psi};\eta,\xi)\right]\right\|_{X_k} \le |\theta|\,\frac{\bar{C}K_2^- L^-(\gamma)}{\lambda}\frac{\lambda^k}{\gamma^\kappa}\sum_{n=k}^{\infty}\left(\frac{\gamma}{\lambda}\right)^n\left\|\psi - \bar{\psi}\right\|_{\kappa,\gamma}^+ \tag{4.6}$$

for all $k \in \mathbb{Z}_\kappa^+$, and (2.16), (2.9), (2.13) have the consequence

$$\begin{aligned}
&\left\| P_+(k) \left[\mathcal{S}_\kappa(k, \psi; \eta, \xi) - \mathcal{S}_\kappa(k, \bar{\psi}; \eta, \xi) \right] \right\|_{X_k} \\
\leq\ & |\theta| \frac{L^+(\gamma)}{\Lambda} \left(K_2^+ \frac{\Lambda^k}{\gamma^\kappa} \sum_{n=\kappa}^{k-1} \left(\frac{\gamma}{\Lambda} \right)^n + |\theta|^{-\nu} K_3^+ \gamma^{k-\kappa} \operatorname{Li}_\nu \left(\tfrac{\Lambda}{\gamma} \right) \right) \|\psi - \bar{\psi}\|_{\kappa,\gamma}^+ \quad \text{for all } k \in \mathbb{Z}_\kappa^+.
\end{aligned} \tag{4.7}$$

Multiplying both above estimates with $\delta^{\kappa-k}$ and applying the triangle inequality to estimate the difference $\mathcal{S}_\kappa(\psi; \eta, \xi) - \mathcal{S}_\kappa(\bar{\psi}; \eta, \xi)$, gives us the middle relation in (4.4); moreover, setting $k = \kappa$ in (4.6) leads to the first estimate in (4.4). Finally, the remaining right Lipschitz estimate in (4.4) follows from

$$\|\mathcal{S}_\kappa(k, \psi; \eta, \xi) - \mathcal{S}_\kappa(k, \psi; \bar{\eta}, \xi)\|_{X_k} \delta^{\kappa-k} \leq K_1^+ \|\eta - \bar{\eta}\|_{X_\kappa} \quad \text{for all } k \in \mathbb{Z}_\kappa^+,$$

which we get from (3.7), (2.8), and we are done. □

The following lemma provides a dynamical interpretation of the operator $\mathcal{S}_\kappa$.

Lemma 4.2. *Let $\theta \in \Theta$, $(\kappa, \eta, \xi) \in \mathcal{P}_+ \times \mathcal{X}$, $\gamma \in (\Lambda, \lambda)$, $\psi \in \mathfrak{X}_{\kappa,\gamma}^+$ and assume $(H_0), (H_1')$–(H_2'). Then for the mapping $\mathcal{S}_\kappa(\cdot; \eta, \xi) : \mathfrak{X}_{\kappa,\gamma}^+ \to \mathfrak{X}_{\kappa,\gamma}^+$ the following two statements are equivalent:*

(a) *There exists a $\zeta \in X_\kappa$ such that $\psi = \varphi(\cdot; \kappa, \zeta) - \varphi(\cdot; \kappa, \xi)$ and*

$$P_+(\kappa)\psi(\kappa) = \eta - P_+(\kappa)\xi, \tag{4.8}$$

(b) *ψ is a solution of the fixed point equation*

$$\psi = \mathcal{S}_\kappa(\psi; \eta, \xi). \tag{4.9}$$

Proof. Let $\theta \in \Theta$, $(\kappa, \xi) \in \mathcal{X}$ and assume $\psi \in \mathfrak{X}_{\kappa,\gamma}^+$. For all $k \in \mathbb{Z}_\kappa^+$ we define the inhomogeneity

$$r(k) := F(k, \psi(k) + \varphi(k; \kappa, \xi), \psi'(k) + \varphi(k+1; \kappa, \xi)) - F(k, \varphi(k; \kappa, \xi), \varphi(k+1; \kappa, \xi)),$$

and due to $\|r(k)\|_{Y_{k+1}} \leq (\operatorname{Lip}_2 F + |\gamma| \operatorname{Lip}_3 F) \|\psi\|_{\kappa,\gamma}^+ \gamma^{k-\kappa}$ for all $k \in \mathbb{Z}_\kappa^+$ (cf. (2.13)), an estimate of the form (3.4) holds for the sequence r.

(a) ⇒ *(b)* Let $\eta \in \mathcal{P}_+(\kappa)$ and assume there exists a $\zeta \in X_\kappa$ such that $\psi = \varphi(\cdot; \kappa, \zeta) - \varphi(\cdot; \kappa, \xi)$ is γ^+-quasibounded and $P_+(\kappa)\psi(\kappa) = \eta - P_+(\kappa)\xi$. Then ψ is a γ^+-quasibounded solution of the inhomogeneous equation $y' = A(k)y + \theta K'(k) r(k)$ and Lemma 3.1(a) implies that ψ is a fixed point of $\mathcal{S}_\kappa(\cdot; \xi, \eta)$.

(b) ⇒ *(a)* Conversely, assume $\psi \in \mathfrak{X}_{\kappa,\gamma}^+$ satisfies (4.9) for some $\eta \in \mathcal{P}_+(\kappa)$, $\xi \in X_\kappa$. Then define $\zeta := P_-(\kappa)[\xi + \psi(\kappa)] + \eta$ and set $\phi := \psi + \varphi(\cdot; \kappa, \xi)$. Hence,

$$\begin{aligned}
\phi(\kappa) &= \psi(\kappa) + \xi \overset{(4.9)}{=} P_-(\kappa)\psi(\kappa) + P_+(\kappa)\mathcal{S}_\kappa(\psi; \eta, \xi)(\kappa) + \xi \\
&\overset{(4.1)}{=} P_-(\kappa)\psi(\kappa) + \eta - P_+(\kappa)\xi + \xi = P_-(\kappa)[\psi(\kappa) + \xi] + \eta = \zeta
\end{aligned} \tag{4.10}$$

and ϕ also solves (2.1). Due to the uniqueness of forward solutions guaranteed by (H_0), this gives us $\phi = \varphi(\cdot; \kappa, \zeta)$, i.e. $\psi = \varphi(\cdot; \kappa, \zeta) - \varphi(\cdot; \kappa, \xi)$. Finally, one has

$$P_+(\kappa)\psi(\kappa) \overset{(4.10)}{=} P_+(\kappa)[\zeta - \xi] = P_+(\kappa)[\eta - \xi] = \eta - P_+(\kappa)\xi.$$

□

Lemma 4.3. *Let $\theta \in \Theta$, $\kappa \in \mathbb{Z}$, assume Hypotheses (H_0')–(H_2') with $\sigma_{\max} = \frac{\lambda - \Lambda}{2}$, Σ given by (3.13) and choose $\gamma \in \bar{\Gamma}$. Then the mapping $\mathcal{S}_\kappa(\cdot; \eta, \xi) : \mathfrak{X}_{\kappa,\gamma}^+ \to \mathfrak{X}_{\kappa,\gamma}^+$ possesses a unique fixed point $\psi_\kappa(\eta, \xi) \in \mathfrak{X}_{\kappa,\gamma}^+$ for each $(\kappa, \eta, \xi) \in \mathcal{P}_+ \times \mathcal{X}$. Moreover, for the fixed point mapping $\psi_\kappa : \mathcal{P}_+(\kappa) \times X_\kappa \to \mathfrak{X}_{\kappa,\gamma}^+$ one has:*

(a) *$\psi_\kappa : \mathcal{P}_+(\kappa) \times X_\kappa \to \mathfrak{X}_{\kappa,\gamma}^+$ is linearly bounded*

$$\|\psi_\kappa(\eta, \xi)\|_{\kappa,\gamma}^+ \leq \frac{K_1^+}{1 - |\theta| \, \ell(\gamma)} \|\eta - P_+(\kappa)\xi\|_{X_\kappa}, \tag{4.11}$$

$$\|P_-(\kappa)\psi_\kappa(\kappa, \eta, \xi)\|_{X_\kappa} \leq |\theta| \, K_1^+ \frac{L^-(\gamma)\ell^-(\gamma)}{1 - |\theta| \, \ell(\gamma)} \|\eta - P_+(\kappa)\xi\|_{X_\kappa} \quad \textit{for all } (\kappa, \eta, \xi) \in \mathcal{P}_+ \times \mathcal{X}, \tag{4.12}$$

(b) *one has the Lipschitz estimates*

$$\operatorname{Lip}_1 \psi_\kappa \le \frac{K_1^+}{1-|\theta|\,\ell(\gamma)}, \qquad \operatorname{Lip}_1 P_-(\kappa)\psi_\kappa(\kappa,\cdot) \le |\theta|\,K_1^+ \frac{L^-(\gamma)\ell^-(\gamma)}{1-|\theta|\,\ell(\gamma)} \quad \text{for all } \kappa\in\mathbb{Z}, \tag{4.13}$$

and $\psi_\kappa : \mathcal{P}_+(\kappa)\times X_\kappa \to \mathfrak{X}^+_{\kappa,\gamma}$ *is continuous for* $\gamma\in(\Lambda+\sigma,\lambda-\sigma]$,

(c) *if additionally* (H_3'), $\Lambda^m<\lambda$ *with* $m\in\mathbb{N}$ *and* $\sigma_{\max} = \min\left\{\frac{\lambda-\Lambda}{2}, \Lambda\left(\sqrt[m]{\frac{\Lambda+\lambda}{\Lambda+\Lambda^m}}-1\right)\right\}$ *hold, then for* $\gamma\in(\Lambda+\sigma,\lambda-\sigma]$ *the mapping* $\psi_\kappa : \mathcal{P}_+(\kappa)\times X_\kappa \to \mathfrak{X}^+_{\kappa,\gamma}$ *is of class* C^1, *m-times continuously partially differentiable w.r.t. the first variable and possesses globally bounded partial derivatives,*

where the constants $L^-(\gamma)$, $\ell(\gamma)$, $\ell^-(\gamma)$ *are defined in Lemma 3.2.*

Proof. Let $\theta\in\Theta$ and $(\kappa,\eta,\xi)\in\mathcal{P}_+\times\mathcal{X}$. From (4.4) in Lemma 4.1 we know that $\mathcal{S}_\kappa(\cdot;\eta,\xi)$ is a contraction on $\mathfrak{X}^+_{\kappa,\gamma}$ and Banach's theorem implies the existence of a unique fixed point $\psi_\kappa(\eta,\xi)\in\mathfrak{X}^+_{\kappa,\gamma}$.

(a) Thanks to $|\theta|\,\ell(\gamma)<1$, the estimate (4.11) follows from

$$\|\psi_\kappa(\eta,\xi)\|^+_{\kappa,\gamma} \overset{(4.9)}{=} \|\mathcal{S}_\kappa(\psi_\kappa(\eta,\xi);\eta,\xi)\|^+_{\kappa,\gamma} \overset{(4.2)}{\le} K_1^+\,\|\eta-P_+(\kappa)\xi\|_{X_\kappa} + |\theta|\,\ell(\gamma)\,\|\psi_\kappa(\eta,\xi)\|^+_{\kappa,\gamma},$$

and similarly we get

$$\|P_-(\kappa)\psi_\kappa(\kappa,\eta,\xi)\|_{X_\kappa} \overset{(4.9)}{=} \|P_-(\kappa)\mathcal{S}_\kappa(\kappa,\psi_\kappa(\eta,\xi);\eta,\xi)\|_{X_\kappa} \overset{(4.3)}{\le} |\theta|\,L^-(\gamma)\ell^-(\gamma)\,\|\psi_\kappa(\eta,\xi)\|^+_{\kappa,\gamma};$$

thanks to (4.11) this implies (4.12).

(b) Next we derive the Lipschitz estimates in (4.13). Thereto, let $\eta,\bar\eta\in\mathcal{P}_+(\kappa)$, fix $\xi\in X_\kappa$, and from the estimates (4.9), (4.4) we obtain

$$\|\psi_\kappa(\eta,\xi)-\psi_\kappa(\bar\eta,\xi)\|^+_{\kappa,\gamma} \overset{(4.4)}{\le} K_1^+\,\|\eta-\bar\eta\|_{X_\kappa} + |\theta|\,\ell(\gamma)\,\|\psi_\kappa(\eta,\xi)-\psi_\kappa(\bar\eta,\xi)\|^+_{\kappa,\gamma},$$

yielding the left relation in (4.13). Similarly, using the triangle inequality and (4.9), (4.1), (4.4) one has

$$\|P_-(\kappa)\left[\psi_\kappa(\kappa,\eta,\xi)-\psi_\kappa(\kappa,\bar\eta,\xi)\right]\|_{X_\kappa} \le |\theta|\,L^-(\gamma)\ell^-(\gamma)\,\|\psi_\kappa(\eta,\xi)-\psi_\kappa(\bar\eta,\xi)\|^+_{\kappa,\gamma}$$

leading to the remaining right assertion in (4.13). To close the proof of part (b), one has to show the continuity of $\psi_\kappa : \mathcal{P}_+(\kappa)\times X_\kappa\to\mathfrak{X}^+_{\kappa,\gamma}$ for arbitrary $\gamma\in(\Lambda+\sigma,\lambda-\sigma]$. In order to prove the continuity of $\psi_\kappa(\eta_0,\cdot) : X_\kappa\to\mathfrak{X}^+_{\kappa,\gamma}$, it suffices to show for arbitrary but fixed $(\kappa,\eta_0)\in\mathcal{P}_+$ the following limit relation:

$$\lim_{\xi\to\xi_0} \|\psi_\kappa(\eta_0,\xi)-\psi_\kappa(\eta_0,\xi_0)\|^+_{\kappa,\gamma} = 0 \tag{4.14}$$

(cf. Lemma A.1). To obtain a short notation, we suppress the dependence on the fixed $\eta_0\in\mathcal{P}_+(\kappa)$ from now on and define mappings $H_k : X_k\times X_{k+1}\times X_\kappa\to Y_{k+1}$ by

$$H_k(x,y,\xi) := F(k,x+\varphi(k;\kappa,\xi),y+\varphi(k+1;\kappa,\xi)) - F(k,\varphi(k;\kappa,\xi),\varphi(k+1;\kappa,\xi))$$

and $\bar H_k(\zeta,\xi) := H_k(\psi_\kappa(k,\zeta),\psi_\kappa(k+1,\zeta),\xi)$ for $k\in\mathbb{Z}^+_\kappa$. Note that H_k and $\bar H_k(\zeta,\cdot)$ are continuous due to (H_2'). For any parameter $\xi_0\in X_\kappa$ we obtain from (4.9), similarly to (4.6) and (4.7), the estimate

$$\begin{aligned}
&\|\psi_\kappa(k;\xi)-\psi_\kappa(k;\xi_0)\|_{X_k}\\
\overset{(4.1)}{\le}\; &\|\Phi(k,\kappa)P_+(\kappa)\|_{L(X_\kappa,X_k)}\,\|\xi-\xi_0\|_{X_\kappa}\\
&+|\theta|\,\bar C K_2^- \sum_{n=k}^{\infty}\lambda^{k-n-1}\left\|P_-'(n)\left[\bar H_n(\xi,\xi)-\bar H_n(\xi_0,\xi_0)\right]\right\|_{Y_{n+1}}\\
&+|\theta|\,K_2^+ \sum_{n=\kappa}^{k-1}\Lambda^{k-n-1}\left\|P_+'(n)\left[\bar H_n(\xi,\xi)-\bar H_n(\xi_0,\xi_0)\right]\right\|_{Y_{n+1}}\\
&+|\theta|^{1-\nu}\,K_3^+ \sum_{n=\kappa}^{k-1}(k-n)^{-\nu}\Lambda^{k-n-1}\left\|P_+'(n)\left[\bar H_n(\xi,\xi)-\bar H_n(\xi_0,\xi_0)\right]\right\|_{Y_{n+1}} \quad \text{for all } k\in\mathbb{Z}^+_\kappa.
\end{aligned}$$

Subtraction and addition of $\bar{H}_n(\xi_0, \xi)$ in the corresponding norms leads to

$$\|\psi_\kappa(k;\xi) - \psi_\kappa(k;\xi_0)\|_{X_k} \le \sum_{i=0}^{6} S_i \quad \text{for all } k \in \mathbb{Z}_\kappa^+,$$

where (cf. (2.8) and (2.13)) $S_0 := K_1^+ \Lambda^{k-\kappa} \|\xi - \xi_0\|_{X_\kappa}$,

$$\begin{aligned}
S_1 &:= |\theta|\, \bar{C} K_2^- \sum_{n=k}^{\infty} \lambda^{k-n-1} L_2^- \|\psi_\kappa(n,\xi) - \psi_\kappa(n,\xi_0)\|_{X_n} \\
&\quad + |\theta|\, \bar{C} K_2^- \sum_{n=k}^{\infty} \lambda^{k-n-1} L_3^- \|\psi_\kappa(n+1,\xi) - \psi_\kappa(n+1,\xi_0)\|_{X_n}, \\
S_2 &:= |\theta|\, K_2^+ \sum_{n=\kappa}^{k-1} \Lambda^{k-n-1} L_2^+ \|\psi_\kappa(n,\xi) - \psi_\kappa(n,\xi_0)\|_{X_n} \\
&\quad + |\theta|\, K_2^+ \sum_{n=\kappa}^{k-1} \Lambda^{k-n-1} L_3^+ \|\psi_\kappa(n+1,\xi) - \psi_\kappa(n+1,\xi_0)\|_{X_n}, \\
S_3 &:= |\theta|^{1-\nu} K_3^+ \sum_{n=\kappa}^{k-1} (k-n)^{-\nu} \Lambda^{k-n-1} L_2^+ \|\psi_\kappa(n,\xi) - \psi_\kappa(n,\xi_0)\|_{X_n} \\
&\quad + |\theta|^{1-\nu} K_3^+ \sum_{n=\kappa}^{k-1} (k-n)^{-\nu} \Lambda^{k-n-1} L_3^+ \|\psi_\kappa(n+1,\xi) - \psi_\kappa(n+1,\xi_0)\|_{X_n}, \\
S_4 &:= |\theta|\, \bar{C} K_2^- \sum_{n=k}^{\infty} \lambda^{k-n-1} \left\| P_-'(n) \left[\bar{H}_n(\xi_0,\xi) - \bar{H}_n(\xi_0,\xi_0)\right] \right\|_{Y_{n+1}}, \\
S_5 &:= |\theta|\, K_2^+ \sum_{n=\kappa}^{k-1} \Lambda^{k-n-1} \left\| P_+'(n) \left[\bar{H}_n(\xi_0,\xi) - \bar{H}_n(\xi_0,\xi_0)\right] \right\|_{Y_{n+1}}, \\
S_6 &:= |\theta|^{1-\nu} K_3^+ \sum_{n=\kappa}^{k-1} (k-n)^{-\nu} \Lambda^{k-n-1} \left\| P_+'(n) \left[\bar{H}_n(\xi_0,\xi) - \bar{H}_n(\xi_0,\xi_0)\right] \right\|_{Y_{n+1}}.
\end{aligned}$$

Now we obtain the estimate

$$\|\psi_\kappa(k;\xi) - \psi_\kappa(k;\xi_0)\|_{X_k} \gamma^{\kappa-k} \le K_1^+ \|\xi - \xi_0\|_{X_\kappa} + \sum_{i=1}^{3} S_i \gamma^{\kappa-k} + |\theta|\, \ell(\gamma) \|\psi_\kappa(\xi) - \psi_\kappa(\xi_0)\|_{\kappa,\gamma}^+$$

for $k \in \mathbb{Z}_\kappa^+$. Hence, by passing over to the least upper bound for $k \in \mathbb{Z}_\kappa^+$, we get

$$\|\psi_\kappa(\xi) - \psi_\kappa(\xi_0)\|_{\kappa,\gamma}^+ \le K_1^+ \|\xi - \xi_0\|_{X_\kappa} + \frac{\max\left\{|\theta|\, \bar{C} K_2^-, |\theta|\, K_2^+, |\theta|^{1-\nu} K_3^+\right\} \gamma^\kappa}{1 - |\theta|\, \ell(\gamma)} \sup_{k \in \mathbb{Z}_\kappa^+} U(k,\xi)$$

with the mapping

$$\begin{aligned}
U(k,\xi) :=& \frac{\lambda^{k-1}}{\gamma^k} \sum_{n=k}^{\infty} \lambda^{-n} \left\| P_-'(n) \left[\bar{H}_n(\xi_0,\xi) - \bar{H}_n(\xi_0,\xi_0)\right] \right\|_{Y_{n+1}} \\
&+ \frac{\Lambda^{k-1}}{\gamma^k} \sum_{n=\kappa}^{k-1} \Lambda^{-n} \left\| P_+'(n) \left[\bar{H}_n(\xi_0,\xi) - \bar{H}_n(\xi_0,\xi_0)\right] \right\|_{Y_{n+1}} \\
&+ \frac{\Lambda^{k-1}}{\gamma^k} \sum_{n=\kappa}^{k-1} (k-n)^{-\nu} \Lambda^{-n} \left\| P_+'(n) \left[\bar{H}_n(\xi_0,\xi) - \bar{H}_n(\xi_0,\xi_0)\right] \right\|_{Y_{n+1}}.
\end{aligned} \tag{4.15}$$

Therefore, it suffices to prove the limit relation

$$\lim_{\xi\to\xi_0}\sup_{k\in\mathbb{Z}_\kappa^+} U(k,\xi)=0 \tag{4.16}$$

in order to show the limit relation (4.14). We proceed indirectly. Assume (4.16) does not hold. Then there exists an $\varepsilon>0$ and a sequence $(\xi_i)_{i\in\mathbb{N}}$ in X_κ with $\lim_{i\to\infty}\xi_i=\xi_0$ and $\sup_{k\in\mathbb{Z}_\kappa^+}U(k,\xi_i)>\varepsilon$ for $i\in\mathbb{N}$. This implies the existence of a sequence $(k_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}_\kappa^+$ such that

$$U(k_i,\xi_i)>\varepsilon \quad \text{for all } i\in\mathbb{N}. \tag{4.17}$$

From now on we assume $\gamma>\Lambda+\sigma$, choose a fixed growth rate $\delta\in(\Lambda+\sigma,\gamma)$ and remark that the inequality $\frac{\delta}{\gamma}<1$ will play an important role below. Because of Hypothesis (H_2') and the inclusion $\psi_\kappa(\xi)\in\mathcal{X}_{\kappa,\delta}^+$ we get $\left\|P_\pm'(n)\bar{H}_n(\xi_0,\xi)\right\|_{Y_{n+1}}\le L^\pm(\gamma)\left\|\psi_\kappa(\xi_0)\right\|_{\kappa,\delta}^+\delta^{n-\kappa}$ for all $n\in\mathbb{Z}_\kappa^+$ (cf. (2.13)) and the triangle inequality leads to

$$U(k,\xi)\le\frac{2\left\|\psi_\kappa(\xi_0)\right\|_{\kappa,\delta}^+}{\delta^\kappa}\left(\frac{L^-(\gamma)}{\lambda-\delta}+\frac{L^+(\gamma)}{\delta-\Lambda}+L^+(\gamma)\operatorname{Li}_\nu\left(\tfrac{\Lambda}{\delta}\right)\right)\left(\frac{\delta}{\gamma}\right)^k \quad \text{for all } k\in\mathbb{Z}_\kappa^+.$$

Because of $\frac{\delta}{\gamma}<1$, passing over to the limit $k\to\infty$ yields $\lim_{k\to\infty}U(k,\xi)=0$ uniformly in $\xi\in X_\kappa$, and taking into account (4.17) the sequence $(k_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}_\kappa^+$ has to be bounded above, i.e. there exists an integer $K>\kappa$ with $k_i\le K$ for all $i\in\mathbb{N}$. Hence, we can deduce

$$\begin{aligned}U(k,\xi_i)\overset{(4.15)}{\le}{}&\frac{\lambda^{K-1}}{\gamma^K}\sum_{n=\kappa}^{\infty}\lambda^{-n}\left\|P_-'(n)\left[\bar{H}_n(\xi_0,\xi_i)-\bar{H}_n(\xi_0,\xi_0)\right]\right\|_{Y_{n+1}}\\&+\frac{\Lambda^{\kappa-1}}{\gamma^\kappa}\sum_{n=\kappa}^{K}\Lambda^{-n}\left\|P_+'(n)\left[\bar{H}_n(\xi_0,\xi_i)-\bar{H}_n(\xi_0,\xi_0)\right]\right\|_{Y_{n+1}}\\&+\frac{\Lambda^{\kappa-1}}{\gamma^\kappa}\sum_{n=\kappa}^{K}(k-n)^{-\nu}\Lambda^{-n}\left\|P_+'(n)\left[\bar{H}_n(\xi_0,\xi_i)-\bar{H}_n(\xi_0,\xi_0)\right]\right\|_{Y_{n+1}}\end{aligned}$$

for all $i\in\mathbb{N}$, where the last two finite sums tends to zero for $i\to\infty$ by the continuity of H_n. Continuity properties of H_n also imply $\lim_{i\to\infty}\bar{H}_n(\xi_0,\xi_i)=\bar{H}_n(\xi_0,\xi_0)$ and with the dominated convergence theorem of Lebesgue[1] we get the convergence of the infinite sum to zero for $i\to\infty$. Thus we derived the relation $\lim_{i\to\infty}U(k_i,\xi_i)=0$, which obviously contradicts (4.17). Consequently, we have shown the continuity of $\psi_\kappa(\eta_0,\cdot):X_\kappa\to\mathcal{X}_{\kappa,\gamma}^+$ and the proof of (b) is finished.

(c) A technically quite involved argument yields the rigorous proof of the differentiability assertion on $\psi_\kappa:\mathcal{P}_+(\kappa)\times X_\kappa\to\mathcal{X}_{\kappa,\gamma}^+$. The precise verification essentially follows ideas of [PS04] and we refer the interested reader to this reference for details. □

Proposition 4.4 (invariant fibers). *Let $\theta\in\Theta$ and assume Hypotheses (H_0')–(H_2') with $\sigma_{\max}=\frac{\lambda-\Lambda}{2}$ and Σ given by (3.13). Then for all $(\kappa,\xi)\in\mathcal{X}$ the so-called* fiber *through (κ,ξ), given by*

$$\mathcal{V}_{\xi,\theta}(\kappa):=\left\{\zeta\in X_\kappa:\varphi(\cdot;\kappa,\zeta)-\varphi(\cdot;\kappa,\xi)\in\mathcal{X}_{\kappa,\gamma}^+\right\}$$

is independent of $\gamma\in\bar{\Gamma}$, positively invariant w.r.t. (2.1), i.e.,

$$\varphi(k;\kappa,\mathcal{V}_{\xi,\theta}(\kappa))\subseteq\mathcal{V}_{\varphi(k;\kappa,\xi),\theta}(k)\quad \textit{for all } k\in\mathbb{Z}_\kappa^+ \tag{4.18}$$

and possesses the representation

$$\mathcal{V}_{\xi,\theta}=\{(\kappa,\eta+v_\theta(\kappa,\eta,\xi)):(\kappa,\eta)\in\mathcal{P}_+\} \tag{4.19}$$

as graph of a uniquely determined mapping $v_\theta(\kappa,\cdot):X_\kappa\times X_\kappa\to X_\kappa$ satisfying

$$v_\theta(\kappa,\eta,\xi)\in\mathcal{P}_-(\kappa)\quad \textit{for all } (\kappa,\eta,\xi)\in\mathcal{P}_+\times\mathcal{X} \tag{4.20}$$

and the invariance equation

$$v_\theta(\kappa+1,\eta_1,\xi_1)=A(\kappa)v_\theta(\kappa,\eta,\xi)+\theta P_-'(\kappa)K'(\kappa)F(\kappa,\eta+v_\theta(\kappa,\eta,\xi),\eta_1+v_\theta(\kappa+1,\eta_1,\xi_1)),$$

[1] To apply this result from integration theory, one has to write the infinite sum as an integral over piecewise-constant functions and use the Lipschitz estimate on H_n, which is implied by (2.13), to get an integrable majorant.

$$\begin{aligned} \eta_1 &= A(\kappa)\eta + \theta P_+'(\kappa)K'(\kappa)F(\kappa,\eta+v_\theta(\kappa,\eta,\xi),\eta_1+v_\theta(\kappa+1,\eta_1,\xi)), \\ \xi_1 &= A(\kappa)\xi+\theta K'(\kappa)F(\kappa,\xi,\xi_1) \end{aligned} \tag{4.21}$$

for all $(\kappa,\eta,\xi)\in\mathcal{P}_+\times\mathcal{X}$. *Furthermore, for all* $\gamma\in\bar\Gamma$ *and* $\theta\in\Theta$ *it holds:*

(a) $v_\theta(\kappa,\cdot):\mathcal{P}_+(\kappa)\times X_\kappa\to X_\kappa$ *is continuous and linearly bounded*

$$\|v_\theta(\kappa,\eta,\xi)\|_{X_\kappa}\le\|P_-(\kappa)\xi\|_{X_\kappa}+|\theta|\,K_1^+\frac{L^-(\gamma)\ell^-(\gamma)}{1-|\theta|\,\ell(\gamma)}\|\eta-P_+(\kappa)\xi\|_{X_\kappa}\quad \textit{for all } (\kappa,\eta,\xi)\in\mathcal{P}_+\times\mathcal{X}, \tag{4.22}$$

(b) $v_\theta(\kappa,\cdot,\xi)$ *is globally Lipschitzian with* $\operatorname{Lip}_2 v_\theta\le|\theta|\,K_1^+\frac{L^-(\gamma)\ell^-(\gamma)}{1-|\theta|\ell(\gamma)}$,

(c) *assume Hypothesis* (H_3') *is satisfied with* $\Lambda^m<\lambda$, $m\in\mathbb{N}$, *and if the spectral gap condition* (2.14) *holds with* $\sigma_{\max}:=\min\left\{\frac{\lambda-\Lambda}{2},\lambda\left(\sqrt[m]{\frac{\Lambda+\lambda}{\Lambda+\Lambda^m}}\right)-1\right\}$, *then* $v_\theta(\kappa,\cdot):\mathcal{P}_+(\kappa)\times X_\kappa\to X_\kappa$ *is of class* C^1, *the partial derivatives* $D_2^n v_\theta(\kappa,\cdot):\mathcal{P}_+(\kappa)\times X_\kappa\to L_n(\mathcal{P}_+(\kappa),X_\kappa)$ *exist, are continuous, and globally bounded for* $n\in\{1,\dots,m\}$.

Remark 4.2. (1) In case the difference equation (2.1) is explicit with $F(k,0,0)\equiv 0$ on $\mathbb{Z}$, then $\mathcal{V}_{\xi,\theta}(\kappa)$, $\xi\in\mathcal{W}_\theta(\kappa)$, is the pseudo-stable foliation over the pseudo-unstable fiber bundle $\mathcal{W}_\theta$ of (2.1), with $\mathcal{V}_{0,\theta}$ being the pseudo-stable fiber bundle of 0.

(2) If the general solution φ of (2.1) exists on $\mathbb{Z}\times\mathbb{Z}\times\mathcal{X}$ one can obtain a dual result to Proposition 4.4 for the fibers $\mathcal{V}^-_{\xi,\theta}(\kappa):=\left\{\zeta\in X_\kappa:\ \varphi(\cdot;\kappa,\zeta)-\varphi(\cdot;\kappa,\xi)\in\mathfrak{X}^-_{\kappa,\gamma}\right\}$.

Proof. Let $\theta\in\Theta$, $(\kappa,\eta,\xi)\in\mathcal{P}_+\times\mathcal{X}$ and $\gamma\in\bar\Gamma$.

We show the invariance assertion (4.18) for $\mathcal{V}_{\xi,\theta}(\kappa)$. Let $x_0\in\varphi(k;\kappa,\mathcal{V}_{\xi,\theta}(\kappa))$ for some $k\in\mathbb{Z}^+_\kappa$, and by definition this is equivalent to the existence of a $\zeta\in X_\kappa$ with $x_0=\varphi(k;\kappa,\zeta)$ and a difference $\varphi(\cdot;\kappa,\zeta)-\varphi(\cdot;\kappa,\xi)\in\mathfrak{X}^+_{\kappa,\gamma}$. Therefore,

$$\varphi(\cdot;k,x_0)-\varphi(\cdot;k,\varphi(k;\kappa,\xi))=\varphi(\cdot;k,\varphi(k;\kappa,\zeta))-\varphi(\cdot;k,\varphi(k;\kappa,\xi))\overset{(1.2)}{=}\varphi(\cdot;\kappa,\zeta)-\varphi(\cdot;\kappa,\xi),$$

i.e. $x_0\in\mathcal{V}_{\varphi(k;\kappa,\xi),\theta}(k)$ for all $k\in\mathbb{Z}^+_\kappa$.

Due to the spectral gap condition (2.14) and the middle estimate (4.4) in Lemma 4.1 we have (3.21). Hence, Lemma 4.3 implies that $\mathcal{S}_\kappa(\cdot;\eta,\xi):\mathfrak{X}^+_{\kappa,\gamma}\to\mathfrak{X}^+_{\kappa,\gamma}$ has a unique fixed point $\psi_\kappa(\eta,\xi)\in\mathfrak{X}^+_{\kappa,\gamma}$, which is independent of $\gamma\in\bar\Gamma$, because one has $\mathfrak{X}^+_{\kappa,\Lambda+\sigma}\subseteq\mathfrak{X}^+_{\kappa,\gamma}$ and every mapping $\mathcal{S}_\kappa(\cdot;\eta,\xi):\mathfrak{X}^+_{\kappa,\gamma}\to\mathfrak{X}^+_{\kappa,\gamma}$ possesses the same fixed point as the restriction $\mathcal{S}_\kappa(\cdot;\eta,\xi)\big|_{\mathfrak{X}^+_{\kappa,\Lambda+\sigma}}$. Furthermore, the fixed point is of the form $\psi_\kappa(\eta,\xi)=\varphi(\cdot;\kappa,\zeta)-\varphi(\cdot;\kappa,\xi)$ with $\zeta\in X_\kappa$ (cf. Lemma 4.2). We define

$$v_\theta(\kappa,\eta,\xi):=P_-(\kappa)\left[\xi+\psi_\kappa(\kappa,\eta,\xi)\right] \tag{4.23}$$

and evidently have $v_\theta(\kappa,x_0)\in\mathcal{P}_-(\kappa)$. Let us verify the representation (4.19).

($\subseteq$) Let $\zeta\in\mathcal{V}_{\xi,\theta}(\kappa)$, i.e., $\psi=\varphi(\cdot;\kappa,\zeta)-\varphi(\cdot;\kappa,\xi)\in\mathfrak{X}^+_{\kappa,\gamma}$. Then Lemma 4.2 implies

$$\begin{aligned}\zeta &= \psi(\kappa)+\xi=P_-(\kappa)\psi(\kappa)+P_+(\kappa)\psi(\kappa)+\xi\\ &\overset{(4.8)}{=} P_-(\kappa)\psi(\kappa)+\eta-P_+(\kappa)\xi+\xi=P_-(\kappa)\psi(\kappa)+\eta+P_-(\kappa)\xi,\end{aligned}$$

hence $P_+(\kappa)\zeta=\eta$, and $\zeta=P_+(\kappa)\zeta+P_-(\kappa)\left[\xi+\psi_\kappa(\kappa,\eta,\xi)\right]$. Thus, ζ is contained in the graph of $v_\theta(\kappa,\cdot,\xi)$ over $\mathcal{P}_+(\kappa)$.

($\supseteq$) On the other hand, let $\zeta\in X_\kappa$ be of the form $\zeta=\eta+v_\theta(\kappa,\eta,\xi)$ with some $\eta\in\mathcal{P}_+(\kappa)$. Then (4.1) and (4.9) imply $P_+(\kappa)\psi_\kappa(\eta,\xi)=\eta-P_+(\kappa)\xi$, which yields $\zeta=\eta+P_-(\kappa)\left[\xi+\psi_\kappa(\kappa,\eta,\xi)\right]=\xi+\psi_\kappa(\kappa,\eta,\xi)$, and consequently $\varphi(\cdot;\kappa,\zeta)-\varphi(\cdot;\kappa,\xi)\in\mathfrak{X}^+_{\kappa,\gamma}$, i.e., $\zeta\in\mathcal{V}_{\xi,\theta}(\kappa)$.

To establish invariance equation (4.21) we observe that (4.19) and the positive invariance (4.18) imply

$$\varphi(k;\kappa,\eta+v_\theta(\kappa,\eta,\xi))=P_+(k)\varphi(k;\kappa,\eta+v_\theta(\kappa,\eta,\xi))+v_\theta(k,P_+(k)\varphi(k;\kappa,\eta+v_\theta(\kappa,\eta,\xi)),\varphi(k;\kappa,\xi))$$

for all $k\in\mathbb{Z}^+_\kappa$. Multiplying this relation with $P_-(k)$, setting $k=\kappa+1$, and keeping the inclusion (4.20) in mind, this yields (4.21).

(a) We obtain (4.22) from Lemma 4.3, since (4.23), (4.12) imply

$$\|v_\theta(\kappa,\eta,\xi)\|_{X_\kappa}\le\|P_-(\kappa)\xi\|_{X_\kappa}+|\theta|\,K_1^+\frac{L^-(\gamma)\ell^-(\gamma)}{1-|\theta|\,\ell(\gamma)}\|\eta-P_+(\kappa)\xi\|_{X_\kappa}.$$

(*b*) To prove the claimed Lipschitz estimate, consider $\eta, \bar\eta \in \mathcal{P}_+(\kappa)$, a fixed $\xi \in X_\kappa$ and the corresponding fixed points $\psi_\kappa(\eta,\xi), \psi_\kappa(\bar\eta,\xi) \in \mathfrak{X}^+_{\kappa,\gamma}$ of $\mathcal{S}_\kappa(\cdot;\eta,\xi)$ and $\mathcal{S}_\kappa(\cdot;\bar\eta,\xi)$, respectively. One gets from Lemma 4.3(b) that

$$\|v_\theta(\kappa,\eta,\xi) - v_\theta(\kappa,\bar\eta,\xi)\| \overset{(4.23)}{=} \|P_-(\kappa)\left[\psi_\kappa(\kappa,\eta,\xi) - \psi_\kappa(\kappa,\bar\eta,\xi)\right]\| \overset{(4.13)}{\le} |\theta|\, K_1^- \frac{L^-(\gamma)\ell^-(\gamma)}{1-|\theta|\,\ell(\gamma)} \|\eta - \bar\eta\|_{X_\kappa}.$$

Under Hypothesis (H_0') we know from Lemma 4.3(b) that $\psi_\kappa : \mathcal{P}_+(\kappa) \times X_\kappa \to \mathfrak{X}^+_{\kappa,\gamma}$ is continuous, and by definition in (4.23) we get the continuity of $v_\theta(\kappa,\cdot)$.

(*c*) We have the identity $v_\theta(\kappa,\xi) = P_-(\kappa)\left[\xi + \psi_\kappa(\kappa,\eta;\xi)\right]$ (see (4.23)) for $(\kappa,\eta,\xi) \in \mathcal{P}_+ \times \mathcal{X}$, and using well-known properties of the evaluation map (cf. [APS02, Lemma 3.4]) it follows from Lemma 4.3(c) that the mapping $v_\theta(\kappa,\cdot) : \mathcal{P}_+(\kappa) \times X_\kappa \to X_\kappa$ admits the claimed differentiability properties. □

In a more descriptive way, the subsequent asymptotic phase property is sometimes referred as "exponential tracking" of the IFB $\mathcal{W}_\theta$. It states that convergence to $\mathcal{W}_\theta$ is actually "in phase" with solutions on the IFB $\mathcal{W}_\theta$, and for that reason we speak of an *asymptotic phase*. The proof relies on a geometric argument, which demands a stronger spectral gap condition.

Theorem 4.5 (asymptotic phase)**.** *Let $\theta \in \Theta$, $\kappa \in \mathbb{Z}$ and assume Hypotheses (H_0')–(H_2') with $\sigma_{\max} = \frac{\lambda-\Lambda}{2}$ and Σ given by*

$$\begin{aligned}\Sigma(\sigma) := L^-(\lambda-\sigma)\tfrac{\bar C K_2^-}{\sigma} + L^+(\lambda-\sigma)\left(\tfrac{K_2^+}{\sigma} + |\theta|^{-\nu} K_3^+ \operatorname{Li}_\nu\left(\tfrac{\Lambda}{\Lambda+\sigma}\right)\right) \\ + \max\left\{L^-(\lambda-\sigma)\tfrac{\bar C K_2^-}{\sigma}, L^+(\lambda-\sigma)\left(\tfrac{K_2^+}{\sigma} + |\theta|^{-\nu} K_3^+ \operatorname{Li}_\nu\left(\tfrac{\Lambda}{\Lambda+\sigma}\right)\right)\right\}.\end{aligned} \tag{4.24}$$

Then the IFB $\mathcal{W}_\theta$ from Theorem 3.5 possesses an asymptotic phase, *i.e. for every $\kappa \in \mathbb{Z}$ there exists a retraction $\pi(\kappa,\cdot) : X_\kappa \to \mathcal{W}_\theta(\kappa)$ onto $\mathcal{W}_\theta(\kappa) \subseteq X_\kappa$ with the property:*

$$\|\varphi(k;\kappa,\xi) - \varphi(k;\kappa,\pi(\kappa,\xi))\|_{X_\kappa} \le \frac{K_1^+}{1-|\theta|\,\ell(\gamma)}\left(\|P_+(\kappa)\xi\|_{X_\kappa} + \tilde C_\kappa^+(\xi,\gamma)\right)\gamma^{k-\kappa} \quad \text{for all } k \in \mathbb{Z}_\kappa^+ \tag{4.25}$$

and all $\xi \in X_\kappa$ with $\gamma \in \bar\Gamma$. Geometrically, $\pi(\kappa,\xi)$ is given as the unique intersection

$$\mathcal{W}_\theta(\kappa) \cap \mathcal{V}_{\xi,\theta}(\kappa) = \{\pi(\kappa,\xi)\} \quad \text{for all } \xi \in X_\kappa \tag{4.26}$$

and one has:

(a) *$\pi(\kappa,\cdot) : X_\kappa \to \mathcal{W}_\theta(\kappa)$ is continuous and linearly bounded*

$$\|\pi(\kappa,\xi)\|_{X_\kappa} \le \tilde C_\kappa^+(\xi,\gamma) + \tilde C_\kappa^-(\xi,\gamma) \quad \text{for all } \xi \in X_\kappa \tag{4.27}$$

and, therefore, it maps bounded subsets of X_κ on bounded subsets of $\mathcal{W}_\theta(\kappa)$,

(b) *$\varphi(k;\kappa,\cdot) \circ \pi(\kappa,\cdot) = \pi(k,\cdot) \circ \varphi(k;\kappa,\cdot)$ for $k \in \mathbb{Z}_\kappa^+$,*

(c) *if Hypothesis (H_3') is satisfied for $m = 1$, then $\pi(\kappa,\cdot) : X_\kappa \to X_\kappa$ is of class C^1,*

where the constants $L^\pm(\gamma)$, $\ell(\gamma)$, $\ell^\pm(\gamma)$ are defined in Lemma 3.2 and $\tilde\ell(\gamma) := \frac{L^+(\gamma)\ell^+(\gamma)}{1-|\theta|\ell(\gamma)}\frac{L^-(\gamma)\ell^-(\gamma)}{1-|\theta|\ell(\gamma)}$,

$$\tilde C_\kappa^+(\xi,\gamma) := |\theta|\,\frac{\ell^+(\gamma)C_\kappa^+ + \frac{L^+(\gamma)\ell^+(\gamma)}{1-|\theta|\ell(\gamma)}\left(|\theta|\,\Gamma_\kappa(\gamma) + K_1^-\|P_-(\kappa)\xi\|_{X_\kappa}\right) + |\theta|\,K_1^+K_1^-\tilde\ell(\gamma)\,\|P_+(\kappa)\xi\|_{X_\kappa}}{1-|\theta|^2 K_1^+K_1^-\tilde\ell(\gamma)},$$

$$\tilde C_\kappa^-(\xi,\gamma) := \frac{\|P_-(\kappa)\xi\|_{X_\kappa} + |\theta|\,K_1^+\frac{L^-(\gamma)\ell^-(\gamma)}{1-|\theta|\ell(\gamma)}\left(\|P_+(\kappa)\xi\|_{X_\kappa} + h\ell^+(\gamma)C_\kappa^+\right) + |\theta|^3 K_1^+\tilde\ell(\gamma)\Gamma_\kappa(\gamma)}{1-|\theta|^2 K_1^+K_1^-\tilde\ell(\gamma)}.$$

Remark 4.3. The fact that the gap condition (2.14) holds with a function Σ given in (4.24), implies that the mappings w_θ, v_θ are globally Lipschitzian in their second argument, i.e. we have

$$\operatorname{Lip}_2 w_\theta < 1, \qquad \operatorname{Lip}_2 v_\theta < 1. \tag{4.28}$$

Proof. Let $\theta \in \Theta$, $\gamma \in \bar\Gamma$ and fix $(\kappa,\xi) \in \mathcal{X}$. As first observation we point out that (4.28) implies

$$|\theta|^2 K_1^+K_1^-\tilde\ell(\gamma) < 1 \quad \text{for all } \gamma \in \bar\Gamma. \tag{4.29}$$

We show that there exists one and only one $\zeta \in \mathcal{W}_\theta(\kappa) \cap \mathcal{V}_{\xi,\theta}(\kappa)$. Thereto, note that $\zeta \in \mathcal{W}_\theta(\kappa) \cap \mathcal{V}_{\xi,\theta}(\kappa)$ if and only if $\zeta = P_-(\kappa)\zeta + w_\theta(\kappa, P_-(\kappa)\zeta)$ and $\zeta = P_+(\kappa)\zeta + v_\theta(\kappa, P_+(\kappa)\zeta, \xi)$, which is equivalent to

$$P_+(\kappa)\zeta = w_\theta(\kappa, P_-(\kappa)\zeta) \quad \text{and} \quad P_-(\kappa)\zeta = v_\theta(\kappa, P_+(\kappa)\zeta, \xi). \tag{4.30}$$

Due to Theorem 3.5(b) and Proposition 4.4(b) we know from (4.28) that $\operatorname{Lip}_2 w_\theta \cdot \operatorname{Lip}_2 v_\theta < 1$ and Lemma A.2(a) applies to the equations (4.30). Consequently, there exist two uniquely determined functions $q_\kappa : X_\kappa \to \mathcal{P}_+(\kappa)$, $p_\kappa : X_\kappa \to \mathcal{P}_-(\kappa)$ satisfying the equations (4.30), i.e.,

$$q_\kappa(\xi) = w_\theta(\kappa, p_\kappa(\xi)) \quad \text{and} \quad p_\kappa(\xi) = v_\theta(\kappa, q_\kappa(\xi), \xi) \quad \text{on } X_\kappa. \tag{4.31}$$

Therefore, $\pi(\kappa, \xi) := p_\kappa(\xi) + q_\kappa(\xi)$ is the unique element in the intersection $\mathcal{W}_\theta(\kappa) \cap \mathcal{V}_{\xi,\theta}(\kappa)$. We now derive the estimate (4.25). From (4.31), (3.19), (4.22) we get

$$\begin{aligned} \|q_\kappa(\xi)\|_{X_\kappa} \leq \; & |\theta|\, \ell^+(\gamma) C_\kappa^+ + |\theta| \frac{L^+(\gamma)\ell^+(\gamma)}{1 - |\theta|\, \ell(\gamma)} \left(|\theta|\, \Gamma_\kappa(\gamma) + K_1^- \|P_-(\kappa)\xi\|_{X_\kappa}\right) \\ & + |\theta|^2 K_1^+ K_1^- \tilde{\ell}(\gamma) \|P_+(\kappa)\xi\|_{X_\kappa} + |\theta|^2 K_1^+ K_1^- \tilde{\ell}(\gamma) \|q_\kappa(\xi)\|_{X_\kappa}. \end{aligned} \tag{4.32}$$

Since by definition, $\pi(\kappa, \xi) \in \mathcal{V}_{\xi,\theta}(\kappa)$ for $\xi \in X_\kappa$, it follows from Lemma 4.2 that one obtains $\varphi(\cdot; \kappa, \xi) - \varphi(\cdot; \kappa, \pi(\kappa, \xi)) = \psi_\kappa(P_+(\kappa)\pi(\kappa, \xi), \xi)$ and Lemma 4.3 together with (4.11) implies

$$\|\varphi(k; \kappa, \xi) - \varphi(k; \kappa, \pi(\kappa, \xi))\|_{\kappa,\gamma}^+ \leq \frac{K_1^+}{1 - |\theta|\, \ell(\gamma)} \left(\|q_\kappa(\xi)\|_{X_\kappa} + \|P_+(\kappa)\xi\|_{X_\kappa}\right).$$

This gives us (4.25), if we resolve (4.32) w.r.t. the value of the norm $\|q_\kappa(\xi)\|_{X_\kappa}$. Then Proposition 4.4(c) yields the continuity of the mapping $v_\theta(\kappa, \cdot) : \mathcal{P}_+(\kappa) \times X_\kappa \to \mathcal{P}_-(\kappa)$ and Lemma A.2(b) implies that also $\pi(\kappa, \cdot) : X_\kappa \to X_\kappa$ is continuous.

(a) It remains to derive the estimate (4.27). From (4.31), (4.22), (3.19) we get

$$\begin{aligned} \|p_\kappa(\xi)\|_{X_\kappa} \leq \|P_-(\kappa)\xi\|_{X_\kappa} \; & + \; |\theta|\, K_1^+ \frac{L^-(\gamma)\ell^-(\gamma)}{1 - |\theta|\, \ell(\gamma)} \|P_+(\kappa)\xi\|_{X_\kappa} + |\theta|\, K_1^+ \frac{L^-(\gamma)\ell^-(\gamma)}{1 - |\theta|\, \ell(\gamma)} h\ell^+(\gamma) C_\kappa^+ \\ & + \; |\theta|^3 K_1^+ \tilde{\ell}(\gamma) \Gamma_\kappa(\gamma) + |\theta|^2 K_1^+ K_1^- \tilde{\ell}(\gamma) \|p_\kappa(\xi)\|_{X_\kappa} \end{aligned}$$

and, thanks to (4.29), one can resolve this inequality, as well as inequality (4.32) w.r.t. the values of the norms $\|p_\kappa(\xi)\|_{X_\kappa}$ and $\|q_\kappa(\xi)\|_{X_\kappa}$, respectively; this yields (4.27).

(b) The (positive) invariance of $\mathcal{W}_\theta$ and $\mathcal{V}_{\xi,\theta}(\kappa)$ implies

$$\begin{aligned} \varphi(k; \kappa, \pi(\kappa, \xi)) \; & \overset{(4.26)}{\in} \; \varphi(k; \kappa, \mathcal{W}_\theta(\kappa) \cap \mathcal{V}_{\xi,\theta}(\kappa)) \subseteq \varphi(k; \kappa, \mathcal{W}_\theta(\kappa)) \cap \varphi(k; \kappa, \mathcal{V}_{\xi,\theta}(\kappa)) \\ & \overset{(4.18)}{\subseteq} \; \mathcal{W}_\theta(k) \cap \mathcal{V}_{\varphi(k;\kappa,\xi),\theta}(k) \overset{(4.26)}{=} \{\pi(k, \varphi(k; \kappa, \xi))\} \quad \text{for all } k \in \mathbb{Z}_\kappa^+. \end{aligned}$$

(c) The continuous differentiability of $\pi(\kappa, \cdot) : X_\kappa \to X_\kappa$ under Hypothesis (H_3') immediately follows from a C^1-version of Lemma A.2, which can be derived using [Hen81, p. 13]. □

As an immediate consequence of Proposition 4.5 we obtain that for each $(\kappa, \xi) \in \mathcal{W}_\theta$ the fibers $\mathcal{V}_{\xi,\theta}(\kappa)$ are mutually disjoint and form a foliation of X_κ.

Corollary 4.6 (invariant foliation over $\mathcal{W}_\theta$)**.** *The invariant fibers $\mathcal{V}_{\xi,\theta}(\kappa)$ from Proposition 4.4 are leaves of a* positively invariant foliation *over each fiber of the IFB $\mathcal{W}_\theta$ from Theorem 3.5, i.e. for $\kappa \in \mathbb{Z}$ we have*

$$X_\kappa = \bigcup_{\xi \in \mathcal{W}_\theta(\kappa)} \mathcal{V}_{\xi,\theta}(\kappa), \qquad \mathcal{V}_{\xi_1,\theta}(\kappa) \cap \mathcal{V}_{\xi_2,\theta}(\kappa) = \emptyset \quad \textit{for all } \xi_1, \xi_2 \in \mathcal{W}_\theta(\kappa),\ \xi_1 \neq \xi_2. \tag{4.33}$$

Proof. Let $\theta \in \Theta$, $(\kappa, \xi) \in \mathcal{X}$. The positive invariance of the fibers $\mathcal{V}_{\xi,\theta}(\kappa)$ is stated in (4.18). Thanks to relation (4.25) we know $\varphi(\cdot; \kappa, \xi) - \varphi(\cdot; \kappa, \pi(\kappa, \xi)) \in \mathfrak{X}_{\kappa,\gamma}^+$ and thus Proposition 4.4 implies $\xi \in \mathcal{V}_{\pi(\kappa,\xi),\theta}(\kappa)$. Since $\xi \in X_\kappa$ was arbitrary, we established the left relation in (4.33). The remaining pair-wise disjointness in (4.33) follows from $\emptyset = \{\xi_1\} \cap \{\xi_2\} = \mathcal{V}_{\xi_1,\theta}(\kappa) \cap \mathcal{V}_{\xi_2,\theta}(\kappa)$ for all $\xi_1, \xi_2 \in \mathcal{W}_\theta(\kappa)$ with $\xi_1 \neq \xi_2$. □

5. Examples: Discretization of Evolutionary Equations

In applied sciences, investigations on the behavior of solutions to nonlinear evolutionary equations are mainly computational. For that purpose such problems are discretized. In a first step, one possibly obtains a time discretization, yielding an iteration in a suitable function space over the spacial domain. Of high practical relevance, of course, are also full discretizations leading to recursions in finite-dimensional spaces. In this section we discuss a straightforward time discretization and briefly mention two further discretization approaches. Due to the technical effort, a deeper analysis is postponed to future papers.

Basic for our discretization schemes is an appropriate discrete set of time steps. Given two bounds $0 < h \leq H$, this will be a real sequence $(t_k)_{k\in\mathbb{Z}}$ satisfying $t_{k+1} - t_k \in [h, H]$ for all $k \in \mathbb{Z}$.

5.1. **Discretized Evolutionary Processes.** Our results are intended to hold for discrete dynamical systems given by time-h-maps of autonomous evolutionary equations, like reaction diffusion equations, FDEs or ODEs. Information on the time-h-map is of crucial importance in discretization theory, since it enables us to relate approximate solutions (obtained, e.g. by numerical schemes) to the original solution of the differential equation. In fact, we can deal with nonautonomous problems as follows:

Let V, W be Banach spaces with $V \subseteq W$. Working in a general framework intended to include abstract formulations of PDEs and FDEs (see also [CHT97, Section 5.2] for the autonomous case), we consider an abstract nonautonomous evolutionary equation

$$u_t + B(t)u = f(t, u) \tag{5.1}$$

on the space W, so that the following holds:

(G_1) For each Banach space $X \in \{V, W\}$ the linear equation $u_t + B(t)u = 0$ generates an evolution operator $U(t, s)$ on X, i.e., $(U(t, s))_{s\leq t}$ is a family of bounded linear operators on X such that
   (i) $U(s, s) = I_X$, $U(t, s)U(s, \tau) = U(t, \tau)$ for all $\tau \leq s \leq t$,
   (ii) the mapping $U : \{(t, s) \in \mathbb{R}^2 : s < t\} \to L(X)$ is strongly continuous and there exist reals $a > 0$, $\nu \geq 0$, $M_1, M_2 > 0$ with

$$\|U(t, s)\|_{L(X)} \leq M_1 e^{a(t-s)} \quad \text{for all } s \leq t, \quad \|U(t, s)\|_{L(W,V)} \leq M_2(t - s)^{-\nu} e^{a(t-s)} \quad \text{for all } s < t, \tag{5.2}$$

   (iii) there exist reals $K_1^\pm, K_2^\pm, K_3^+ > 0$, $\beta_1 < \beta_2$, a projection-valued mapping $Q_- : \mathbb{R} \to L(X)$ satisfying $Q_-(t)U(t, s) = U(t, s)Q_-(s)$ for all $s \leq t$, $U(t, s)Q_-(s)W \subseteq V$ for all $s < t$ such that $\bar{U}(t, s) := U(t, s)|_{Q_-(s)X} : Q_-(s)X \to Q_-(t)X$ is invertible with inverse $\bar{U}(s, t)$, and one has the dichotomy estimates with $Q_+(t) = I_X - Q_-(t)$:

$$\begin{aligned}
\|U(t, s)Q_+(s)\|_{L(V)} &\leq K_1^+ e^{-\beta_2(t-s)} \quad \text{for all } s \leq t,\\
\|U(t, s)Q_+(s)\|_{L(W,V)} &\leq (K_2^+ + K_3^+(t - s)^{-\nu})e^{-\beta_2(t-s)} \quad \text{for all } s < t,\\
\left\|\bar{U}(t, s)Q_-(s)\right\|_{L(V)} &\leq K_1^- e^{-\beta_1(t-s)} \quad \text{for all } t \leq s,\\
\left\|\bar{U}(t, s)Q_-(s)\right\|_{L(W,V)} &\leq K_2^- e^{-\beta_1(t-s)} \quad \text{for all } t \leq s.
\end{aligned} \tag{5.3}$$

There exist several sufficient conditions for (G_1)(i)–(ii) to hold, which are well-documented in the literature and we refer to [EN00, p. 478] for an overview. In the autonomous situation of a constant operator B, the usual theory of C_0-semigroups applies (cf., e.g. [EN00]). The dichotomy assumption $(G_1)(iii)$ is more subtle even in the case of ODEs, but related to spectral properties, if $B(t)$ is constant or periodic in time. On the nonlinearity we suppose

(G_2) $f : \mathbb{R} \times V \to W$ satisfies $L := \operatorname{Lip}_2 f < \infty$ and for each $(\tau, u_0) \in \mathbb{R} \times V$ there exists a unique continuous $u : \{(t, \tau, u_0) \in \mathbb{R}^2 \times V : \tau < t\} \to V$ such that $u(\cdot; \tau, u_0)$ solves the integral equation

$$u(t) = U(t, \tau)u_0 + \int_\tau^t U(t, s)f(s, u(s; \tau, u_0))\, ds \quad \text{for all } \tau \leq t. \tag{5.4}$$

Criteria for the existence of such mild solutions can be found, for instance, in [SY02, pp. 224ff].

Now we arrived at a position to introduce our nonautonomous counterpart of a time-h-map. Thereto, define functions $A : \mathbb{Z} \to L(W)$, $K : \mathbb{Z} \to L(W)$, $F : \mathbb{Z} \times V \to W$ by

$$A(k) := U(t_{k+1}, t_k), \qquad K(k) :\equiv I_W, \qquad F(k, x) := \frac{1}{\theta} \int_{t_k}^{t_{k+1}} U(t_{k+1}, s) f(s, u(s; t_k, x))\, ds.$$

Then, noticing that $u(\cdot; t_k, x)$ is a mild solution of (5.1) satisfying the variation of constants formula (5.4), the general solution φ of (2.1) is given by $\varphi(k; \kappa, u_0) = u(t_k; t_\kappa, u_0)$ for $\kappa \le k$. Moreover, (2.1) is well-defined in forward time on $\mathbb{Z} \times V$ and we assume constant state spaces $X_k = V$, $Y_k = W$.

Lemma 5.1. *If Hypotheses (G_1)–(G_2) hold with $\nu \in [0, 1)$ and*

$$\sup_{k<\kappa} \left\| \int_{t_k}^{t_{k+1}} U(t_{k+1}, s) f(s, u(s; t_k, 0))\, ds \right\|_W e^{-\beta_1 hk} < \infty \quad \text{for one } \kappa \in \mathbb{Z}, \tag{5.5}$$

then the nonlinearity $F : \mathbb{Z} \times V \to W$ is well-defined and satisfies

$$|\theta| \operatorname{Lip} F(k, \cdot) \le M_1^2 E_{1-\nu} \left(\sqrt[1-\nu]{L M_2 \Gamma(1-\nu)} H \right) L \int_{t_k}^{t_{k+1}} e^{a(t_{k+1}-s)\, ds} \quad \text{for all } k \in \mathbb{Z},$$

where Γ denotes the Gamma function and E_r is the Mittag-Leffler function (cf. [SY02, p. 624ff]*) given by $E_r(x) = \sum_{n=0}^{\infty} \frac{x^{rn}}{\Gamma(1+rn)}$.*

Proof. Using the Gronwall-Henry inequality from [SY02, p. 625, Lemma D.4], one deduces a Lipschitz condition for the solution $u(t; s, \cdot) : V \to V$, $s \le t$. With this estimate available, the assertion follows easily from (5.4) and (G_2). □

Proposition 5.2. *Suppose Hypotheses (G_1)–(G_2) hold with* (5.5)*, $\nu \in [0, 1)$ and that V is continuously embedded into W, where $N \ge 0$ is chosen such that $\|v\|_W \le N \|v\|$ for all $v \in V$. Then there exists an $H_0 > 0$ such that the nonautonomous difference equation* (2.1) *obtained as discretization of* (5.1) *possesses an attractive invariant fiber bundle (as in Theorem 3.5) with asymptotic phase (as in Theorem 4.5), provided the step-sizes satisfy $H \le H_0$, are balanced according to*

$$\beta_1 h < \beta_2 H, \tag{5.6}$$

and the following spectral gap condition is satisfied:

$$8M_1^2 \left[(K_2^-)^2 N + \left(1 + K_2^- N\right) K_2^+ \right] L < \beta_2 - \tfrac{h}{H} \beta_1. \tag{5.7}$$

Proof. From assumption (G_2) we know that (H_0') is valid. Under inequality (5.6) it is easy to see that Hypothesis (H_1') holds with projector $P_-(k) = Q_-(t_k)$, real constants $\bar{C} = 1$ and growth rates $\Lambda := e^{-\beta_2 H}$, $\lambda := e^{-\beta_1 h}$. Moreover, from (5.3) one obtains the estimate $\|P_-(k)\|_{L(W)} \le N K_2^-$ and then Lemma 5.1 guarantees that (H_2') is satisfied. Within this set-up, the spectral gap condition (2.14) reduces to

$$\begin{aligned} 2 \left[(K_2^-)^2 N + \left(1 + K_2^- N\right) \left(K_2^+ + K_3^+ \operatorname{Li}_\nu \left(\frac{e^{-\beta_2 H}}{e^{-\beta_2 H} + \sigma} \right) \frac{e^{-\beta_1 h} - e^{-\beta_2 H}}{2(qH)^\nu} \right) \right] \cdot \\ \cdot M_1^2 E_{1-\nu} \left(\sqrt[1-\nu]{L M_2 \Gamma(1-\nu)} H \right) L < \frac{e^{-\beta_1 q H} - e^{-\beta_2 H}}{4H}. \end{aligned}$$

and by continuity it is easy to see from (5.7) this this inequality holds true for $h, H > 0$ close to 0 and σ close to $\frac{\lambda - \Lambda}{2}$. The assertion follows from Theorem 4.5. □

5.2. **Crank-Nicholson Time Discretizations.** The discretization approach from the previous Subsection 5.1 is motivated from a theoretical perspective, because it yields values of true solutions u for (5.1) evaluated at discrete points t_k. From an applied point of view, however, this is not helpful since the nonlinearity F depends on the unknown solution u. To circumvent this deficit we briefly discuss another method of more practical importance.

We retreat to a special case of the general framework from Subsection 5.1, where V is continuously embedded into W and the evolutionary family $(U(t, s))_{s \le t}$ is given by a strongly continuous semigroup with generator $-B$ via $U(t,s) = e^{-B(t-s)}$. Then a generalized *Crank-Nicholson* discretization of (5.1) is a linearly implicit recursion of the form

$$\frac{u_{k+1} - u_k}{t_{k+1} - t_k} = -\vartheta B u_k - (1-\vartheta) B u_{k+1} + f(t_k, u_k)$$

with parameter $\vartheta \in \mathbb{R}$. Choosing the maximal step-size $H > 0$ so small that $I_W + (t_{k+1} - t_k)\vartheta B$ is invertible for all $k \in \mathbb{Z}$, one can transform this implicit recursion into an explicit difference equation (2.1) with functions $A : \mathbb{Z} \to L(W)$, $K : \mathbb{Z} \to L(W)$, $F : \mathbb{Z} \times V \to W$,

$$A(k) := [I_W + (t_{k+1} - t_k)\vartheta B]^{-1} [I_W - (t_{k+1} - t_k)(1-\vartheta) B], \qquad K(k) := [I_W + (t_{k+1} - t_k)\vartheta B]^{-1}$$

and $F(k,y) := f(t_k, y)$. We can proceed as in [Kob95, Theorems 2.3 and 2.4] to show that this Crank-Nicholson discretization (2.1) satisfies Hypothesis (H_0)–(H_2) for appropriate (small) $0 < h \le H$, provided $\vartheta = 1$ (i.e. one works with the explicit Euler method), or $\vartheta \in \left(\frac{1}{2}, 1\right)$ and V, W are Hilbert spaces. It is worth to point out that (2.9) holds with $K_3^+ \neq 0$ in this setting.

5.3. **Discretized Parabolic Equations.** In particular reaction diffusion equations allow an abstract formulation as nonautonomous evolutionary equations with time-invariant linear part of the form

$$u_t + Bu = f(t,u) \tag{5.8}$$

on a Banach space $(Z, |\cdot|)$, so that the following holds:

(G_1^*) B is a sectorial operator on Z and there exist two reals $\beta_1 < \beta_2$ such that the spectrum $\sigma(B)$ can be separated into spectral sets (closed, nonempty)

$$\sigma_- := \{\lambda \in \sigma(B) : \Re\lambda < \beta_1\}, \qquad \sigma_+ := \{\lambda \in \sigma(B) : \Re\lambda > \beta_2\}.$$

The spectral projection associated with σ_- is denoted by Q_-.

This assumption guarantees that $-B$ generates an analytic semigroup $(e^{-Bt})_{t\ge0}$ on Z. Thus, for $\alpha \in \mathbb{R}$ one can define fractional powers B^α and corresponding interpolation spaces $Z^\alpha := D(B^\alpha)$ equipped with norms $|x|_\alpha := |B^\alpha x|$ (cf., e.g. [SY02, p. 92ff, Section 3.7]). Then $(e^{-Bt})_{t\ge0}$ defines also an analytic semigroup on Z^α. The spectral splitting of B and an equivalent re-norming of Z^α implies the estimates

$$\left\|e^{-Bt} Q_-\right\|_{L(Z^\alpha)} \le e^{-\beta_1 t} \quad \text{for all } t \le 0, \qquad \left\|e^{-Bt} Q_+\right\|_{L(Z^\alpha)} \le e^{-\beta_2 t} \quad \text{for all } t \ge 0 \tag{5.9}$$

with the complementary projection $Q_+ := I_Z - Q_-$. Beyond that, there exist reals $M \ge 0$ so that

$$\left\|e^{-Bt}\right\|_{L(Z)} \le M, \qquad \left\|e^{-Bt}\right\|_{L(Z, Z^{\alpha-\beta})} \le M t^{\beta-\alpha} \quad \text{for all } t \in (0,1] \tag{5.10}$$

(cf. [SY02, p. 92ff, Section 3.7]). Concerning the nonlinearity f suppose

(G_2^*) There exist $\alpha, \beta \in \mathbb{R}$, $\alpha - \beta \in [0,1)$ such that $f : \mathbb{R} \times Z^\alpha \to Z^\beta$ is locally Lipschitz and $\operatorname{Lip}_2 f < \infty$.

Then the abstract parabolic equation (5.8) admits unique mild solutions $u(\cdot; \tau, u_0) : [\tau, \infty) \to Z^\alpha$ satisfying $u(\tau; \tau, u_0) = u_0$ for all $\tau \in \mathbb{R}$, $u_0 \in Z^\alpha$ (cf. [SY02, p. 239, Theorem 47.7]) and $u(t; \tau, \cdot) : Z^\alpha \to Z^\alpha$ is continuous. Thus, if we define functions $A : \mathbb{Z} \to L(Z^\alpha)$, $K : \mathbb{Z} \to L(Z^\alpha)$, $F : \mathbb{Z} \times Z^\alpha \to Z^\alpha$ by

$$A(k) := e^{-B(t_{k+1} - t_k)}, \qquad K(k) :\equiv I_{Z^\alpha}, \qquad F(k,x,y) := \frac{1}{\theta} \int_{t_k}^{t_{k+1}} e^{-B(t_{k+1} - s)} f(s, u(s; t_k, x))\, ds,$$

then the general solution φ of (2.1) satisfies $\varphi(k; \kappa, u_0) = u(t_k; t_\kappa, u_0)$ for integers $\kappa \le k$ and thus (2.1) is well-defined in forward time on $\mathbb{R} \times Z^\alpha$, where we assume constant state spaces $X_k = Y_k = Z^\alpha$.

Lemma 5.3. *If Hypotheses (G_1^*)–(G_2^*) hold with $H \le \min\left\{1, \sqrt[1-\alpha+\beta]{\frac{1-\alpha+\beta}{2M \operatorname{Lip}_2 f}}\right\}$ and*

$$\sup_{k<\kappa} \left| \int_{t_k}^{t_{k+1}} e^{-B(t_{k+1}-s)} f(s, u(s; t_k, 0))\, ds \right|_\alpha e^{-\beta_1 hk} < \infty \quad \text{for one } \kappa \in \mathbb{Z}, \tag{5.11}$$

then the nonlinearity $F : \mathbb{Z} \times Z^\alpha \to Z^\alpha$ is well-defined and satisfies

$$|\theta| \operatorname{Lip} F(k, \cdot) \le \frac{2M^2}{1-\alpha+\beta} \operatorname{Lip}_2 f (t_{k+1} - t_k)^{1-\alpha+\beta} \quad \text{for all } k \in \mathbb{Z}.$$

Proof. Using (5.10) this can be shown along the lines of [CHT97, Proposition 6.1]. $\square$

Proposition 5.4. *If Hypothesis* (G_1^*)–(G_2^*) *hold with* (5.6), (5.11) *and* $H \le \min\left\{1, \sqrt[1-\alpha+\beta]{\frac{1-\alpha+\beta}{2M\operatorname{Lip}_2 f}}\right\}$, *then the nonautonomous difference equation* (2.1) *obtained as discretization of* (5.8) *possesses an attractive invariant fiber bundle (as in Theorem 3.5) with asymptotic phase (as in Theorem 4.5), provided the following spectral gap condition is satisfied:*

$$\frac{8M^2}{1-\alpha+\beta}\operatorname{Lip}_2 f\, H^{1-\alpha+\beta} < \frac{e^{-\beta_1 h} - e^{-\beta_2 H}}{2}. \tag{5.12}$$

Proof. An easy calculation shows that Hypothesis (H_1) holds with projector $P_-(k) = Q_-(t_k)$, real constants $\bar{C} = 1$, $K_1^\pm = K_2^\pm = 1$, $K_3^+ = 0$ and growth rates $\Lambda := e^{-\beta_2 H}$, $\lambda := e^{-\beta_1 h}$. Then Lemma 5.3 guarantees that Theorem 4.5 can be applied, since (5.12) implies the gap condition (2.14). ☐

5.4. **Finite Difference Full Discretizations.** A temporal and finite difference spatial discretization of the Kuramoto-Sivashinsky equation

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0$$

with spatially 1-periodic boundary conditions $u(x,t) = u(x+1,t)$ is considered in [Kob94, Section 5]. Referring to [SY02, p. 321ff], this equation fits in the abstract set-up of (5.8) with $Bu := -u_{xxxx}$ and $f(t,u) := -u_{xx} - uu_x$, where we choose Z to be the space of 1-periodic odd L^2-functions.

Having this at hand, for some (possibly large) positive integer N the scalar Laplacian u_{xx} is discretized by $\Delta_\delta := \delta^{-2}(\operatorname{diag}(1,-2,1) + E)$, where E is a matrix having the entry 1 in the lower left and the upper right corner, and 0 elsewhere, for $\delta = 1/N$. An implicit Euler method for time discretization of the resulting stiff ODE $u_t = -\Delta_\delta^2 u + f_\delta(u)$ (see [Kob94] for details) leads to a recursion of the form (2.1) with

$$A(k) = K(k) := \left[I - (t_{k+1} - t_k)\Delta_\delta^2\right]^{-1}, \qquad F(k,x,y) := \tfrac{t_{k+1}-t_k}{\theta} f_\delta(y).$$

Preserving periodicity conditions, the appropriate space setting is $X_k = Y_k := \left\{x \in \mathbb{R}^{N+1} : x_1 = x_{N+1}\right\}$ equipped with the norms $\|x\|_{Y_k} := \frac{1}{\delta^2}\sqrt{\sum_{n=1}^N x_k^2}$, $\|x\|_{X_k} := \|\Delta_\delta x\|_{Y_k}$; note that both spaces depend on the spatial discretization parameter δ. Then Hypotheses (H_0)–(H_2) can be verified following the approach given in [Kob94, Section 5].

Acknowledgement

This paper was written while the author was a postdoctoral associate at the School of Mathematics, University of Minnesota in Minneapolis MN. Its hospitality, the financial support of the Deutsche Forschungsgemeinschaft, as well as the mathematical expertise of Prof. George R. Sell are strongly appreciated.

Appendix A

In order to keep this paper largely self-contained we present basic results needed. Let Q, P be metric spaces and let M be a topological space fulfilling the first axiom of countability.

Lemma A.1. *If the mapping* $f : Q \times M \to P$ *satisfies* $\operatorname{Lip}_1 f < \infty$, *and if* $f(q,\cdot) : M \to P$ *is continuous for all* $q \in Q$, *then* f *is continuous itself.*

Proof. We leave the easy proof to the interested reader. ☐

Lemma A.2. *If* Q *is a complete metric space, and if the mappings* $f : Q \times M \to P$, $g : P \times M \to Q$ *satisfy* $\operatorname{Lip}_1 f \operatorname{Lip}_1 g < 1$, *then the following holds:*

(a) *For each* $x \in M$ *there exist unique points* $q^*(x) \in Q$, $p^*(x) \in P$ *such that* $q^*(x) \equiv f(p^*(x), x)$, $p^*(x) \equiv g(q^*(x), x)$ *on* M,

(b) *if* $f(q,\cdot) : Q \to P$, $g(p,\cdot) : P \to Q$ *are continuous for each* $p \in P$, $q \in Q$, *then the mappings* $q^* : M \to Q$, $p^* : M \to P$ *are also continuous.*

Proof. We define the mapping $h : Q \times M \to Q$ by $h(q,x) := g(f(q,x),x)$.

(a) Due to $\operatorname{Lip}_1 f \operatorname{Lip}_1 g < 1$ we have $\operatorname{Lip}_1 h < 1$ and the contraction principle implies the existence of a unique fixed point $q^*(x) \in Q$ of $h(\cdot,x)$ for all $x \in M$. The claim follows, if we set $p^*(x) := f(q^*(x),x)$.

(b) By Lemma A.1 the mappings f, g and h are continuous. Then the continuity of $q^* : M \to Q$, $p^* : M \to P$ is a consequence of the parametrized contraction principle (cf., e.g., [Aul98, Theorem 6.1] for a Banach spaces version, which instantly adapts to our situation of metric spaces). $\square$

References

[Aul98] B. Aulbach, *The fundamental existence theorem on invariant fiber bundles*, J. Difference Equ. Appl. **3** (1998), 501–537.

[APS02] B. Aulbach, C. Pötzsche & S. Siegmund, *A smoothness theorem for invariant fiber bundles*, J. Dyn. Differ. Equations **14(3)** (2002), 519–547.

[AW03] B. Aulbach & T. Wanner, *Invariant foliations and decoupling of non-autonomous difference equations*, J. Difference Equ. Appl. **9(5)** (2003), 459–472.

[Bar69] J.S. Baris, *Integral manifolds of irregularly perturbed differential systems*, Ukr. Math. J. **20** (1969), 379–387.

[CHT97] X.-Y. Chen, J.K. Hale & B. Tan, *Invariant foliations for C^1 semigroups in Banach spaces*, J. Differ. Equations **139** (1997), 283–318.

[CL88] S.-N. Chow & K. Lu, *C^k centre unstable manifolds*, Proc. R. Soc. Edinb., Sect. A **108(3/4)** (1988), 303–320.

[DG91] F. Demengel & J.-M. Ghidaglia, *Intertial manifolds for partial differential evolution equations under time-discretization: Existence, convergence, and applications*, J. Math. Anal. Appl. **155** (1991), 177–225.

[vDL99] J.L.M. van Dorsselaer & C. Lubich, *Inertial manifolds of parabolic differential equations under higher-order discretizations*, IMA J. Numer. Anal. **19** (1999), 455–471.

[EN00] K.J. Engel & R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Math. 194, Springer, Berlin, 2000.

[EJ95] K. Eriksson & C. Johnson, *Adaptive finite element methods for parabolic problems IV: Nonlinear problems*, SIAM J. Numer. Anal. **32(6)** (1995), 1729–1749.

[FST88] C. Foias, G.R. Sell & R. Témam, *Inertial manifolds for nonlinear evolutionary equations*, J. Differ. Equations **73(2)** (1988), 309–353.

[Had01] J. Hadamard, *Sur l'iteration et les solutions asympotiques des equations differentielles*, Bull. Soc. Math. France **39** (1901), 224–228.

[Hen81] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Math. 840, Springer, Berlin, 1981.

[HPS76] M.W. Hirsch, C.C. Pugh & M. Shub, *Invariant Manifolds*, Lecture Notes in Math. 583, Springer, Berlin, 1981.

[Ioo79] G. Iooss, *Bifurcation of maps and applications*, North-Holland, Amsterdam, 1979.

[JST98] D.A. Jones, A.W. Stuart & E.S. Titi, *Persistence of invariant sets for dissipative evolution equations*, J. Math. Anal. Appl. **219** (1998), 479–502.

[JS95] D.A. Jones & A.M. Stuart, *Attractive invariant manifolds under approximation. Inertial manifolds*, J. Differ. Equations **123** (1995), 588–637.

[Kob94] K. Kobayasi, *Inertial manifolds for discrete approximations of evolution equations: Convergence and approximations*, Adv. Math. Sci. Appl. **3** (1993/94), 161–189.

[Kob95] ———, *Convergence and approximation of inertial manifolds for evolution equations*, Differ. Integral Equ. **8** (1995), 1117–1134.

[Kob99] ———, *C^1-approximations of inertial manifolds via finite differences*, Proc. Am. Math. Soc. **127(4)** (1999), 1143–1150.

[KS78] U. Kirchgraber und E. Stiefel, *Methoden der analytischen Störungsrechnung und ihre Anwendungen* (in german), B.G. Teubner, Stuttgart, 1978.

[Lew82] L. Lewin, *Polylogarithms and associated functions*, North Holland, New York, 1982.

[NS92] K. Nipp & D. Stoffer, *Attractive invariant manifolds for maps: Existence, smoothness and continuous dependence on the map*, Research Report No. 92-11, Seminar für Angewandte Mathematik, ETH Zürich, 1992.

[Per30] O. Perron, *Die Stabilitätsfrage bei Differentialgleichungen*, Math. Z. **32** (1930), 703–728.

[Poi86] H. Poincaré, *Mémoire sur les courbes définie par une équation différentielle, IV*, J. Math. Pures Appl. **2** (1886), 151–217.

[Pli64] V.A. Pliss, *A reduction principle in the theory of stability of motion* (in russian), Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 1297–1324.

[Pöt06a] C. Pötzsche, *Discrete inertial manifolds*, submitted, 2006.

[Pöt06b] ———, *Dissipative delay endomorphisms and asymptotic equivalence*, submitted, 2006.

[PS04] C. Pötzsche & S. Siegmund, *C^m-smoothness of invariant fiber bundles*, Topol. Methods Nonlinear Anal. **24(1)** (2004), 107–146.

[SY02] G.R. Sell & Y. You, *Dynamics of evolutionary equations*, Applied Math. Sciences 143, Springer, Berlin, 2002.

Christian Pötzsche, Technische Universität München, Zentrum Mathematik, Boltzmannstrasse 3, D-85748 Garching, Germany

E-mail address: `christian.poetzsche@ma.tum.de`

URL: `www.math.umn.edu/~poetzsch`