A sharp oscillation criterion for a linear delay differential equation

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Abstract

It is well-known that for the oscillation of all solutions of the linear delay differential equation

$$x'(t) + p(t)x(t-\tau) = 0, \qquad t \geq t_0,$$

with $p \in C([t_0, \infty), \mathbb{R}^+)$ and $\tau > 0$ it is necessary that

$$B := \limsup_{t\to\infty} A(t) \geq \frac{1}{e}, \quad \text{where} \quad A(t) := \int_{t-\tau}^{t} p(s)\, ds.$$

Our main result shows that if the function A is slowly varying at infinity (in additive form), then under mild additional assumptions $B > \frac{1}{e}$ implies the oscillation of all solutions of the above linear delay differential equation. The applicability of the obtained results and the importance of the slowly varying assumption on A are illustrated by examples.

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1. Introduction

Consider the linear delay differential equation

$$x'(t) + p(t)x(t-\tau) = 0, \qquad t \geq t_0, \tag{1.1}$$

where $t_0 \in \mathbb{R}$, $\tau > 0$ and $p\colon [t_0, \infty) \to \mathbb{R}^+$ is a nonnegative continuous function. By a solution of (1.1), we mean a function x which is continuous on $[t_1 - \tau, \infty)$ for some $t_1 \geq t_0$, differentiable on (t_1, ∞) and satisfies (1.1) for $t > t_1$. The solution x of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros. Otherwise, it is called *nonoscillatory*.

The study of the oscillatory properties of the solutions of delay differential equations has received much attention, see the monographs [2] and [3] and the references therein. In [7], Koplatadze and Chanturija established oscillation criteria for Eq. (1.1) in terms of the lower and upper limit of the function

$$A(t) := \int_{t-\tau}^{t} p(s)\, ds, \qquad t \geq t_0 + \tau, \tag{1.2}$$

as $t \to \infty$. More precisely, they proved the following result.

Theorem 1 ([7]).

(i) If

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)\, ds > \frac{1}{e}, \tag{1.3}$$

then all solutions of (1.1) are oscillatory.

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(ii) If

$$B := \limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds < \frac{1}{e}, \tag{1.4}$$

or, more generally,

$$\int_{t-\tau}^{t} p(s)\,ds \le \frac{1}{e} \quad \textit{for all large } t, \tag{1.5}$$

then Eq. (1.1) *has a nonoscillatory solution.*

For further related results and a survey of oscillation criteria available for Eq. (1.1), see [5, 8, 11].

Note that if p is τ-periodic, then the function A in (1.2) is a constant. As an immediate consequence of Theorem 1, we obtain the following result.

Corollary 2. *If $p\colon [t_0,\infty) \to \mathbb{R}^+$ is a nonnegative, continuous and τ-periodic function, then for the oscillation of all solutions of* (1.1) *it is necessary and sufficient that*

$$\int_{t_0}^{t_0+\tau} p(s)\,ds > \frac{1}{e}. \tag{1.6}$$

According to Theorem 1, for the oscillation of all solutions of (1.1) it is necessary that $B \ge \frac{1}{e}$, where B is defined in (1.4). Thus, if all solutions of (1.1) are oscillatory, then B cannot be smaller than $\frac{1}{e} \approx 0.367879441$. Our aim is to establish a sufficient condition for the oscillation of all solutions of (1.1) such that the value of B is as close to $\frac{1}{e}$ as possible [5–11]. We will show that if the function A in (1.2) is slowly varying at infinity, then under mild additional assumptions the 'almost necessary' condition $B > \frac{1}{e}$ is sufficient for the oscillation of all solutions of (1.1). Recall (see, e.g., [1]) that a function $f\colon [t_0,\infty) \to \mathbb{R}$ is *slowly varying at infinity* if for every $s \in \mathbb{R}$,

$$f(t+s) - f(t) \longrightarrow 0 \quad \text{as } t \to \infty. \tag{1.7}$$

As noted in [10], a continuous function $f\colon [t_0,\infty) \to \mathbb{R}$ is slowly varying at infinity if and only if there exists $t_1 \ge t_0$ such that on the interval $[t_1,\infty)$ the function f can be decomposed into the sum

$$f(t) = g(t) + h(t), \qquad t \ge t_1, \tag{1.8}$$

where $g\colon [t_1,\infty) \to \mathbb{R}$ is a continuous function which tends to a finite limit as $t \to \infty$ and $h\colon [t_1,\infty) \to \mathbb{R}$ is a continuously differentiable function such that $h'(t) \to 0$ as $t \to \infty$. Since $A'(t) = p(t) - p(t-\tau)$ for $t \ge t_0 + \tau$, this implies that the condition

$$p(t+\tau) - p(t) \longrightarrow 0 \quad \text{as } t \to \infty \tag{1.9}$$

is sufficient for A to be slowly varying at infinity. Functions p satisfying (1.9) are sometimes called *S-asymptotically τ-periodic* [4].

In a recent paper [10], one of the authors established the following oscillation criterion (see Myshkis [9]) for Eq. (1.1).

Theorem 3. *Suppose that the coefficient $p\colon [t_0,\infty) \to \mathbb{R}^+$ is nonnegative, continuous, and slowly varying at infinity with $\liminf_{t\to\infty} p(t) > 0$. Then*

$$\tau \limsup_{t\to\infty} p(t) > \frac{1}{e}$$

implies that all solutions of (1.1) *are oscillatory.*

Clearly, condition (1.9) is less restrictive than the slowly varying condition on p. In this sense, our main result (see Theorem 4 below) may be viewed as a generalization of Theorem 3 to a more general class of linear delay differential equations.

The paper is organized as follows. In Section 2, we formulate and prove our main oscillation criterion. In Section 3, we give examples which illustrate the applicability of the obtained results and the importance of the slowly varying assumption on A.

2. Main result

Our main result is the following theorem.

Theorem 4. *Let $p\colon [t_0,\infty) \to \mathbb{R}^+$ be a nonnegative, bounded and uniformly continuous function such that*

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds > 0. \tag{2.1}$$

Assume also that the function A in (1.2) is slowly varying at infinity. Then

$$B = \limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds > \frac{1}{e} \tag{2.2}$$

implies that all solutions of (1.1) are oscillatory.

The uniform continuity of $p\colon [t_0,\infty) \to \mathbb{R}^+$ in Theorem 4 is satisfied if, e.g., p is differentiable and its derivative is bounded on (t_0,∞).

It can be shown using the decomposition (1.8) that if $f\colon [t_0,\infty) \to \mathbb{R}$ is continuous and slowly varying at infinity, then f is uniformly continuous on $[t_0,\infty)$. This follows from the fact that both g and h in (1.8) are uniformly continuous. Indeed, the uniform continuity of g follows from the Bolzano–Cauchy criterion for the existence of a finite limit at infinity and from the uniform continuity of g on compact intervals, while the uniform continuity of h is a consequence of the boundedness of its derivative. This, together with [10, Lemma 2], implies that if the coefficient p in (1.1) is bounded, then under the assumptions of Theorem 3 all assumptions of Theorem 4 are satisfied. Thus, in case of bounded coefficients Theorem 3 is a special case of Theorem 4.

In the proof of Theorem 4, we need the following auxiliary result from [6].

Lemma 5 ([6, Lemma 2])**.** *Suppose that $p\colon [t_0,\infty) \to \mathbb{R}^+$ is a nonnegative continuous function satisfying (2.1). If x is an eventually positive solution of (1.1), then for all sufficiently large T,*

$$\sup_{t\ge T} \frac{x(t-\tau)}{x(t)} < \infty. \tag{2.3}$$

Now we give a proof of Theorem 4.

Proof of Theorem 4. Suppose, for the sake of contradiction, that the assumptions of Theorem 4 hold and Eq. (1.1) has a nonoscillatory solution x. Without loss of generality, we may assume that x is eventually positive. Otherwise, we pass to the solution $-x$ of (1.1). By Lemma 5, if T is sufficiently large, then

$$x(t) > 0, \qquad t \ge T-\tau, \tag{2.4}$$

and

$$K = \sup_{t\ge T} \frac{x(t-\tau)}{x(t)} < \infty. \tag{2.5}$$

Recall the definition of B in (2.2). In view of (1.2), there exists a sequence $t_n \to \infty$, $t_n \ge T$ for $n \in \mathbb{N}$, such that

$$A(t_n) \longrightarrow B, \quad \text{as } n \to \infty. \tag{2.6}$$

Define

$$y_n(t) = \frac{x(t_n+t)}{x(t_n)} \quad \text{for } t \ge -\tau \text{ and } n \in \mathbb{N}. \tag{2.7}$$

From (1.1), we find that

$$x'(t_n+t) + p(t_n+t)x(t_n+t-\tau) = 0 \tag{2.8}$$

and hence

$$y_n'(t) + p(t_n+t)y_n(t-\tau) = 0 \tag{2.9}$$

for all $t \ge 0$ and $n \in \mathbb{N}$. From (2.5) and (2.8), we obtain for $t \ge 0$ and $n \in \mathbb{N}$,

$$x'(t_n+t) = -p(t_n+t)\frac{x(t_n+t-\tau)}{x(t_n+t)}x(t_n+t) \ge -Kp(t_n+t)x(t_n+t) \ge -KMx(t_n+t),$$

where $M = \sup_{t\ge T} p(t)$. Hence

$$-Ly_n(t) \le y_n'(t) \le 0 \quad \text{for all } t \ge 0 \text{ and } n \in \mathbb{N}, \tag{2.10}$$

where $L = KM$. Taking into account that y is positive on $[-\tau, \infty)$, we have

$$-L \le \frac{y_n'(t)}{y_n(t)} \le 0 \quad \text{for all } t \ge 0 \text{ and } n \in \mathbb{N}. \tag{2.11}$$

Integration from 0 to t yields

$$-Lt \le \ln \frac{y_n(t)}{y_n(0)} \le 0 \quad \text{for all } t \ge 0 \text{ and } n \in \mathbb{N}. \tag{2.12}$$

Since $y_n(0) = 1$ for all $n \in \mathbb{N}$, we obtain

$$e^{-Lt} \le y_n(t) \le 1 \quad \text{for all } t \ge 0 \text{ and } n \in \mathbb{N}. \tag{2.13}$$

Integrating (2.9) from τ to t, we get

$$y_n(t) = y_n(\tau) - \int_\tau^t p(t_n + s) y_n(s - \tau)\, ds \quad \text{for all } t \ge 0 \text{ and } n \in \mathbb{N}. \tag{2.14}$$

Define

$$q_n(t) = p(t_n + t) \quad \text{for } t \ge 0 \text{ and } n \in \mathbb{N}. \tag{2.15}$$

By virtue of (2.10) and (2.13), the functions y_n and their derivatives are uniformly bounded on $[0, \infty)$. This, together with the boundedness and uniform continuity of p, implies that the functions y_n and q_n are uniformly bounded and equicontinuous on $[0, \infty)$. By the application of the Arzelà–Ascoli theorem, we conclude that there exist subsequences $\{y_{n_k}\}_{k\in\mathbb{N}}$ and $\{q_{n_k}\}_{k\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$, respectively, such that the limits

$$y(t) = \lim_{k\to\infty} y_{n_k}(t) \quad \text{and} \quad q(t) = \lim_{k\to\infty} q_{n_k}(t) \tag{2.16}$$

exist for all $t \ge 0$ and the convergence is uniform on every finite subinterval of $[0, \infty)$. Writing $n = n_k$ in (2.13) and (2.14) and letting $k \to \infty$, we obtain

$$e^{-Lt} \le y(t) \le 1 \quad \text{for all } t \ge 0 \text{ and } n \in \mathbb{N}, \tag{2.17}$$

and

$$y(t) = y(\tau) - \int_\tau^t q(s) y(s - \tau)\, ds \quad \text{for all } t \ge \tau \text{ and } n \in \mathbb{N}. \tag{2.18}$$

This implies that y is a positive solution of the delay differential equation

$$y'(t) + q(t) y(t - \tau) = 0. \tag{2.19}$$

Clearly, q is nonnegative and continuous on $[0, \infty)$. By virtue of (1.2) and (2.16), we have for all $t \ge \tau$,

$$\begin{aligned}
\int_{t-\tau}^t q(s)\, ds &= \lim_{k\to\infty} \int_{t-\tau}^t q_{n_k}(s)\, ds = \lim_{k\to\infty} \int_{t-\tau}^t p(t_{n_k} + s)\, ds \\
&= \lim_{k\to\infty} \int_{t_{n_k}+t-\tau}^{t_{n_k}+t} p(u)\, du = \lim_{k\to\infty} A(t_{n_k} + t) = \lim_{k\to\infty} A(t_{n_k}) = B,
\end{aligned}$$

where the last and the last but one equality follows from (2.6) and the slowly varying property of A, respectively. From this we obtain that

$$\liminf_{t\to\infty} \int_{t-\tau}^t q(s)\, ds = B > \frac{1}{e}.$$

By Theorem 1 (i), being a solution of (2.19), y is oscillatory, which contradicts (2.17). □

As noted in Section 1, if p is an S-asymptotically τ-periodic continuous function, then A is slowly varying at infinity. Thus, Theorem 4 yields the following corollaries.

Corollary 6. *Let $p\colon [t_0, \infty) \to \mathbb{R}^+$ be a nonnegative, bounded and uniformly continuous function such that (2.1) holds. If p is S-asymptotically τ-periodic, then (2.2) implies that all solutions of (1.1) are oscillatory.*

Corollary 7. *Suppose that* $p\colon [t_0,\infty)\to\mathbb{R}^+$ *is a nonnegative, bounded function, which can be decomposed into a sum*

$$p(t) = q(t) + r(t), \qquad t \ge t_0, \tag{2.20}$$

where $q\colon [t_0,\infty)\to\mathbb{R}$ *is a continuous* τ*-periodic function and* $r\colon [t_0,\infty)\to\mathbb{R}$ *is continuous and slowly varying at infinity. If*

$$\int_{t_0}^{t_0+\tau} q(s)\,ds + \tau \liminf_{t\to\infty} r(t) > 0 \quad \text{and} \quad \int_{t_0}^{t_0+\tau} q(s)\,ds + \tau \limsup_{t\to\infty} r(t) > \frac{1}{e}, \tag{2.21}$$

then all solutions of (1.1) *are oscillatory.*

Proof. Since q is continuous and periodic, it is uniformly continuous on $[t_0,\infty)$. Further, r is continuous and slowly varying at infinity and therefore, according to the remark below Theorem 4, r is also uniformly continuous on $[t_0,\infty)$. Hence $p = q + r$ is uniformly continuous on $[t_0,\infty)$. It is easily seen that p is S-asymptotically τ-periodic. Combining the periodicity of q with [10, Lemma 2] we obtain that

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds = \int_{t_0}^{t_0+\tau} q(s)\,ds + \tau \liminf_{t\to\infty} r(t). \tag{2.22}$$

By a similar argument we have

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds = \int_{t_0}^{t_0+\tau} q(s)\,ds + \tau \limsup_{t\to\infty} r(t). \tag{2.23}$$

The result follows from Corollary 6. □

3. Examples

In this section we illustrate the applicability of the above oscillation criteria (Example 8), and show the importance of the slowly varying assumption on A (Example 9).

Example 8. Consider the equation

$$x'(t) + \left(\frac{1}{2\pi e} + \delta\big(\cos t + \cos\sqrt{t}\big)\right) x(t-2\pi) = 0, \qquad t \ge 0, \tag{3.1}$$

where $\delta \in (0, \frac{1}{4\pi e})$. Eq. (3.1) is a special case of (1.1) when $\tau = 2\pi$, $t_0 = 0$, and

$$p(t) = \frac{1}{2\pi e} + \delta\big(\cos t + \cos\sqrt{t}\big).$$

Note that the function $\cos\sqrt{t}$ is slowly varying at infinity, since its derivative vanishes there. Hence the decomposition (2.20) holds with $q(t) = \frac{1}{2\pi e} + \delta\cos t$ and $r(t) = \delta\cos\sqrt{t}$. The conditions in (2.21) reduce to $1/e - 2\pi\delta > 0$ and $1/e + 2\pi\delta > 1/e$, respectively. By the choice of δ both inequalities are satisfied. Therefore Corollary 7 applies and all solutions of (3.1) are oscillatory.

Now we show that in this example neither Theorems 1 nor Theorem 3 applies. In view of (2.22) and (2.23), we have

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds = \frac{1}{e} - 2\pi\delta$$

and

$$B = B(\delta) = \limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds = \frac{1}{e} + 2\pi\delta.$$

Thus (1.3) does not hold and Theorem 1 cannot be applied.

As $\cos\sqrt{t}$ is slowly varying at infinity, we have $\cos\sqrt{t+\pi} - \cos\sqrt{t} \to 0$ as $t\to\infty$, and hence

$$p(t+\pi) - p(t) = -2\delta\cos t + \delta\big(\cos\sqrt{t+\pi} - \cos\sqrt{t}\big) \not\to 0 \quad \text{as } t\to\infty.$$

This shows that p is not slowly varying at infinity, and hence Theorem 3 cannot be applied.

Note also, that as $B(\delta)\to 1/e$ as $\delta\to 0+$, choosing $\delta > 0$ small enough we can rule out the application of various other sufficient conditions for the oscillation of all solutions of (3.1) (see e.g. conditions (C_3)–(C_{12}) from [8]).

Example 9. Consider the equation

$$x'(t) + \left(\frac{1}{\pi} + \delta \cos t\right) e^{-1-2\delta \sin t} x(t-\pi) = 0, \qquad t \geq 0, \tag{3.2}$$

where $\delta \in (0, 1/\pi)$. Eq. (3.2) is a special case of (1.1), where

$$p(t) = \left(\frac{1}{\pi} + \delta \cos t\right) e^{-1-2\delta \sin t},$$

$\tau = \pi$ and $t_0 = 0$. It is easy to see that Eq. (3.2) has a positive solution given by

$$x(t) = e^{-\frac{t}{\pi} - \delta \sin t}.$$

We are going to verify that all the assumptions of Theorem 4 are satisfied except that A is not slowly varying at infinity.

Clearly, p is nonnegative and bounded. Further, p is 2π-periodic and continuous and therefore it is uniformly continuous on $[t_0, \infty)$.

Since $p(t) \geq (1/\pi - \delta) e^{-1-2\delta}$ for $t \in \mathbb{R}$, we have that

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)\, ds \geq (1 - \delta\pi) e^{-1-2\delta} > 0.$$

Next we show that condition (2.2) also holds. Since p is 2π-periodic, we have

$$A((2n+2)\pi) = \int_{(2n+1)\pi}^{(2n+2)\pi} p(s)\, ds = \int_{\pi}^{2\pi} p(s)\, ds = A(2\pi)$$

for all $n \in \mathbb{N}$. Taking into account that $\sin t < 0$ for $t \in (\pi, 2\pi)$, we find that

$$A(2\pi) = \int_{\pi}^{2\pi} p(s)\, ds = \frac{1}{\pi e} \int_{\pi}^{2\pi} e^{-2\delta \sin s}(1 + \delta\pi \cos s)\, ds > \frac{1}{\pi e} \int_{\pi}^{2\pi} (1 + \delta\pi \cos s)\, ds = \frac{1}{e}.$$

Thus, $A((2n+2)\pi) = A(2\pi) > 1/e$ for all $n \in \mathbb{N}$ and hence

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)\, ds = \limsup_{t\to\infty} A(t) \geq A(2\pi) > \frac{1}{e}.$$

Finally, we show that A is not slowly varying at infinity, i.e. there exists $s_0 \in \mathbb{R}$ and a sequence $t_n \to \infty$ such that $A(t_n + s_0) - A(t_n)$ does not converge to 0 as $n \to \infty$. Thereto, note that function p is 2π-periodic, and so is A. Thus, it is sufficient to show that A is nonconstant. Indeed, if there exist $t_0 \leq s_1 < s_2 < \infty$ such that $A(s_1) \neq A(s_2)$, then letting $s_0 = s_2 - s_1$ and $t_n = s_1 + 2n\pi$, we get $A(t_n + s_0) - A(t_n) = A(s_2) - A(s_1) \neq 0$ for all $n \in \mathbb{N}$. Since A is continuously differentiable and periodic, in order to see that A is nonconstant, it suffices to find a $t \in \mathbb{R}$ such that $A'(t) \neq 0$. Choosing $t = \pi/2$ one obtains

$$A'\left(\frac{\pi}{2}\right) = p\left(\frac{\pi}{2}\right) - p\left(-\frac{\pi}{2}\right) = \frac{e^{1-2\delta} - e^{-1+2\delta}}{\pi} \neq 0.$$

This proves that A is not slowly varying at infinity.

Example 9 shows that in Theorem 4 the assumption that A is slowly varying at infinity cannot be omitted.

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