Journal of Difference Equations and Applications
Vol. 00, No. 00, Month 20xx, 1–35

Local stability implies global stability for the 2-dimensional Ricker map

Ferenc A. Bartha[ab], Ábel Garab[b] and Tibor Krisztin[bc*]

[a]*CAPA group, Department of Mathematics, University of Bergen, Bergen, Norway*; [b]*Bolyai Institute, University of Szeged, Szeged, Aradi v. tere 1, 6720, Hungary*; [c]*Analysis and Stochastics Research Group of the Hungarian Academy of Sciences, Bolyai Institute, University of Szeged*

(*Received 00 Month 20xx; in final form 00 Month 20xx*)

Consider the difference equation $x_{k+1} = x_k e^{\alpha - x_{k-d}}$ where α is a positive parameter and d is a nonnegative integer. The case $d = 0$ was introduced by W.E. Ricker in 1954. For the delayed version $d \geq 1$ of the equation S. Levin and R. May conjectured in 1976 that local stability of the nontrivial equilibrium implies its global stability. Based on rigorous, computer-aided calculations and analytical tools, we prove the conjecture for $d = 1$.

Keywords: global stability; rigorous numerics; Neimark–Sacker bifurcation; graph representations; interval analysis; discrete-time single species model

AMS Subject Classification: 39A30; 39A28; 65Q10; 65G40; 92D25

1. Introduction

In 1954, Ricker [28] introduced the difference equation

$$x_{k+1} = x_k e^{\alpha - x_k} \tag{1.1}$$

with a positive parameter α to model the population density of a single species with non-overlapping generations. The function $R_1 : \mathrm{R} \ni x \mapsto xe^{\alpha - x} \in \mathrm{R}$ is called the 1-dimensional Ricker map. It has two fixed points: 0 and α. It is not difficult to show that $x = \alpha$ is stable if and only if $0 < \alpha \leq 2$, and, for $0 < \alpha \leq 2$, $x = \alpha$ attracts all points from $(0, \infty)$; or equivalently, the equilibrium $x = \alpha$ of equation (1.1) is globally stable provided it is locally stable.

In 1976, Levin and May [16] considered the case when there are explicit time lags in the density dependent regulatory mechanisms. This leads to the difference-delay equation of order $d + 1$:

$$x_{k+1} = x_k e^{\alpha - x_{k-d}}, \tag{1.2}$$

where d is a positive integer.

The map

$$R_{d+1}(\alpha) : \mathrm{R}^{d+1} \ni (x_0, \ldots, x_d) \mapsto (x_1, \ldots, x_d, x_d e^{\alpha - x_0}) \in \mathrm{R}^{d+1}$$

is called the $(d + 1)$-dimensional Ricker map, and the dynamical system generated by $R_{d+1}(\alpha)$ is equivalent to difference equation (1.2). The map $R_{d+1}(\alpha)$ has 2 fixed

*Corresponding author. Email: krisztin@math.u-szeged.hu

ISSN: 1023-6198 print/ISSN 1563-5120 online

DOI: 10.1080/1023619YYxxxxxxxx
http://www.informaworld.com

points in R^{d+1}: $(0, \dots, 0)$ and $(\alpha, \dots, \alpha)$. Levin and May [16] conjectured in 1976 that local stability of the fixed point $(\alpha, \dots, \alpha) \in \mathrm{R}^{d+1}$ implies its global stability in the sense that all points from $\mathrm{R}_+^{d+1} := (0, \infty)^{d+1}$ are attracted by $(\alpha, \dots, \alpha)$. As far as we know, the conjecture is still open for all integers $d \geq 1$.

Levin and May's conjecture and many other numerical and analytical studies suggested the folk theorem that 'The local stability of the unique positive equilibrium of a single species model implies its global stability'. A result in the opposite direction was recently obtained in a counterexample of Jiménez López [12, 13] on global attractivity for Clark's equation [6] when the delay is at least 3. Ladas also formulated some conjectures on global stability for similar delay difference equations in [15]. The proof of one of those conjectures has been recently completed by Merino [23].

Liz, Tkachenko and Trofimchuk [21] proved that if

$$0 < \alpha < \frac{3}{2(d+1)} \tag{1.3}$$

then the fixed point $(\alpha, \dots, \alpha) \in \mathrm{R}^{d+1}$ of $R_{d+1}(\alpha)$ is globally asymptotically stable, where globally means that the region of attraction of $(\alpha, \dots, \alpha)$ is R_+^{d+1}. They also suggested that condition (1.3) can be replaced by

$$0 < \alpha < \frac{3}{2(d+1)} + \frac{1}{2(d+1)^2}, \tag{1.4}$$

which was proven by Tkachenko and Trofimchuk in [31]. This result is a strong support of the conjecture of Levin and May, and it is proven for a class of maps, not only for $R_{d+1}(\alpha)$. See also [17] and [18] in the topic.

Linearising $R_2(\alpha)$ at the fixed point (α, α) shows that local exponential stability of (α, α) holds for $0 < \alpha < 1$, and (α, α) is unstable for $\alpha > 1$. Thus, the conjecture of Levin and May for the case $d = 1$ is that for $0 < \alpha < 1$ the fixed point (α, α) of the map $R_2(\alpha)$ attracts all points of $\mathrm{R}_+^2 = (0, \infty) \times (0, \infty)$. Condition (1.4) of Tkachenko and Trofimchuk [31] verifies the conjecture for all $\alpha \in (0, 0.875)$, and up to now this seemed to be the best known result concerning the conjecture.

The aim of this paper is to study the 2-dimensional Ricker map $R_2(\alpha)$. As $d = 1$ will be fixed in the remaining part of the paper, we shall use the notation F_α instead of $R_2(\alpha)$.

Our main result is the following

THEOREM 1.1. *If $0 < \alpha \leq 1$, then (α, α) is locally stable, and $F_\alpha^n(x, y) \to (\alpha, \alpha)$ as $n \to \infty$ for all $(x, y) \in \mathrm{R}_+^2$.*

Here F_α^n denotes the n-fold composition of F_α. The result of Theorem 1.1 is optimal in the sense that for $\alpha > 1$ the fixed point (α, α) is unstable. We emphasise that our result implies global stability at the critical parameter value $\alpha = 1$ as well. Theorem 1.1 is new only for parameter values α in $[0.875, 1]$. Nevertheless, for the sake of completeness we give a proof for all parameters $\alpha \in (0, 1]$.

As our approach to the problem is new, we give a short description of the main steps. We combine two main technical tools: analytical techniques and rigorous numerics.

The first step towards the proof of Theorem 1.1 is the construction of a sequence of compact neighbourhoods $(S_n^{(\alpha)})_{n=0}^\infty$ of the fixed point (α, α) so that $F_\alpha(S_n^{(\alpha)}) \subset S_n^{(\alpha)}$, and $S_n^{(\alpha)}$ attracts all points of R_+^2. In case $\alpha \in (0, 0.5]$, the sequence $(S_n^{(\alpha)})_{n=0}^\infty$

approaches the fixed point (α, α) as $n \to \infty$ yielding global stability of (α, α). For $\alpha \in (0.5, 1]$ global stability is not concluded, but we get a compact, positively invariant neighbourhood $S^{(\alpha)}$ of (α, α) in R^2_+ where each trajectory of F_α starting from R^2_+ enters eventually.

The next step is to construct a neighbourhood $N^{(\alpha)}$ of (α, α) so that $N^{(\alpha)}$ belongs to the domain of attraction of (α, α). Such a neighbourhood can be obtained by using the linear approximation of F_α at the fixed point (α, α). Naturally, the size of this neighbourhood tends to 0 as the parameter α tends to the critical value 1. So this approach is applied only in the parameter region $\alpha \in [0.5, 0.999]$. As α passes the value 1, a Neimark–Sacker bifurcation takes place at the fixed point (α, α). So, for the parameter values α near 1, we transform F_α to its normal form used in the study of the Neimark–Sacker bifurcation. We analyse this normal form and obtain a neighbourhood $N^{(\alpha)}$ of (α, α) for each parameter value $\alpha \in [0.999, 1]$ such that $N^{(\alpha)}$ belongs to the basin of attraction of the fixed point (α, α), and the size of $N^{(\alpha)}$ is far away from 0 uniformly in $\alpha \in [0.999, 1]$. We emphasize that for our purpose it is not sufficient to use only the standard techniques applied in the Neimark–Sacker bifurcation and leading to the conlusion that, for α close enough to 1 and $\alpha \leq 1$, the fixed point (α, α) attracts a small neighbourhood of (α, α), and for $\alpha > 1$ there is an invariant curve of F_α. We need quantitative results, that is, explicit estimations on the closeness of α to 1 where a neighbourhood attracted by (α, α) can be given, and we also have to estimate the size of the attracted neighbourhood. This is achieved by considering the sizes of the higher order (error) terms in the Taylor expansion of the normal form of F_α. These two approaches combined give an $\varepsilon > 0$ independently of α so that $N^{(\alpha)}$ contains the ε-neighbourhood of (α, α) for all $\alpha \in [0.5, 1]$.

The final step toward the proof of Theorem 1.1 for $\alpha \in [0.5, 1]$ is to show that the trajectories of F_α starting from $S^{(\alpha)}$ enter eventually into $N^{(\alpha)}$. This step requires only a finite number of elementary calculations. However, in order to carry out them by hand it would take too long time. Therefore, it is done by a computer using rigorous computations and validated numerics, that is, this step is a computer-aided proof. Instead of using numbers, the computations are based on intervals. The application of interval analysis makes it possible to consider this step also a rigorous result. See Section 2 for further details. In recent years the dramatically increased computational power made computers a vital tool in the analysis of dynamical systems. We mention three pioneering works in this field. The proof of the existence of the Lorenz attractor by Tucker [32], the solution of the double bubble conjecture by Hass, Hutchings and Schlafly in [11] and the proof of chaos in Lorenz equations by Mischaikow and Mrozek [24] used validated numerics. These are accepted as mathematical proofs.

The structure of the paper is as follows. We give definitions, notations, a short description of interval analysis and an overview on graph representations of discrete dynamical systems in Section 2. Section 3 contains a construction of a compact neighbourhood $S^{(\alpha)}$ of (α, α) having the property that $F_\alpha(S^{(\alpha)}) \subseteq S^{(\alpha)}$ and every trajectory enters it eventually. In particular we prove Theorem 1.1. for $\alpha \in (0, 0.5]$. Sections 4 and 5 are the most important parts of the paper. In Section 4 we construct a neighbourhood $N^{(\alpha)}$ of the fixed point (α, α) so that $N^{(\alpha)}$ belongs to the domain of attraction of (α, α), and the size of $N^{(\alpha)}$ does not approach 0 as α goes to 1. In Section 5 we demonstrate how graph representations can be used to study qualitative properties of dynamical systems. For possible future applications we formulate two approaches for general continuous maps in Euclidean spaces. In particular, the correctness of an algorithm is verified in order to enclose non-wandering points. In Section 6 we combine the computational techniques of Section 5, and rigorously show that every trajectory of F_α starting from $S^{(\alpha)}$ enters the neighbourhood $N^{(\alpha)}$ constructed in Section 4. There is an appendix containing some large formulae used in Section 4. The program codes and results of our rigorous computer-aided computations can be found on link [1].

Consideration of a local bifurcation of a given dynamical system is usually used to show that some phenomenon appears locally in the global dynamics of the system as a parameter passes a critical value. The innovation in our method is that we use the normal form of a bifurcation in combination with the tools of graph representations of dynamical systems and interval arithmetics to prove the absence of a phenomenon for certain parameter values near the critical one. As we want to construct explicitly given and computationally useful regions, the key technical difficulty is the estimation of the sizes of the higher order (error) terms in the normal forms. We hope that our proof shows that these ideas are applicable for a wide range of discrete or continuous dynamical systems, as well. For instance, the global stability in the Maynard Smith model

$$x_{n+1} = ax_n(1 - x_{n-1})$$

for $1 < a \leq 2$ has not been established yet, and it seems the same approach should work. This equation has been considered, for example, in [3].

Running the program of Dénes and Makay [9], which is developed to (nonrigorously) find and visualise attractors and basins of attraction of discrete dynamical systems, suggests that the conjecture of Levin and May stands for the 3-dimensional Ricker map, as well. Proving the conjecture for $d \geq 2$ may constitute a direction for future research. In this case an additional technical difficulty arises. Namely, first a center manifold reduction is necessary, and the construction of an attracted neighbourhood should be done on the center manifold. Among others, an explicit estimation of the size of the center manifold will play a crucial role as well.

2. Notation, definitions, interval analysis and graph representations

Throughout the paper some further notations and definitions will be used. Let N, N_0, R and C stand for the set of positive integers, non-negative integers, real numbers, and complex numbers, respectively. We use superscript $^{\mathrm{T}}$ to denote the transpose of a vector or a matrix, but where it is not confusing we sometimes omit it. The open ball in the Euclidean-norm $\|.\|$ and in the maximum norm with radius $\delta \geq 0$ around $q \in \mathrm{R}^n$ are denoted by $\mathrm{B}(q;\delta)$ and $\mathrm{K}(q;\delta)$, respectively. In Section 4 we shall often use the notation $\mathrm{B}_\delta = \{z \in \mathrm{C} : |z| < \delta\}$ for $\delta > 0$, where $|z|$ denotes the absolute value of $z \in \mathrm{C}$. For $\xi = (\xi_1, \xi_2) \in \mathrm{C}^2$ and $\zeta = (\zeta_1, \zeta_2) \in \mathrm{C}^2$ let $\langle \xi, \zeta \rangle$ denote the scalar product of them defined by $\langle \xi, \zeta \rangle = \overline{\xi_1}\zeta_1 + \overline{\xi_2}\zeta_2$. Let also $\alpha^* = (\alpha, \alpha)$. Consider the continuous map

$$f : \mathcal{D}_f \subseteq \mathrm{R}^n \to \mathrm{R}^n. \tag{2.1}$$

Let $f^{-1}(x) = \{y \in \mathcal{D}_f : f(y) = x\}$, for $x \in \mathrm{R}^n$. For $k \in \mathrm{N}_0$, f^k denotes the k-fold composition of f, i.e., $f^{k+1}(x) = f(f^k(x))$, and $f^0(x) = x$.

Definition 2.1. The point $x^* \in \mathcal{D}_f$ is called a *fixed point* of f if $f(x^*) = x^*$. The point $q \in \mathcal{D}_f$ is a *periodic point of f with minimal period m* if $f^m(q) = q$ and for all $0 < k < m : f^k(q) \neq q$; $q \in \mathcal{D}_f$ is *eventually periodic* if it is not periodic, but there is a k_0 such that $f^{k_0}(q)$ is periodic. The point $q \in \mathcal{D}_f$ is a *non-wandering point* of f if for every neighbourhood U of q and for any $M \geq 0$, there exists an integer $m \geq M$ such that $f^m(U \cap \mathcal{D}_f) \cap U \cap \mathcal{D}_f \neq \emptyset$.

Let $K \subseteq \mathcal{D}_f$ be a compact set. We denote the set of periodic points of f in K by $\mathrm{Per}\,(f; K)$, and the set of non-wandering points of f in K by $\mathrm{NonW}\,(f; K)$.

A fixed point $x^* \in \mathcal{D}_f$ of f is called *locally stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - x^*\| < \delta$ implies $\|f^k(x) - x^*\| < \varepsilon$ for all $k \in \mathrm{N}$. We say that the fixed point x^* *attracts* the region $U \subseteq \mathcal{D}_f$ if for all points $u \in U$, $\|f^k(u) - x^*\| \to 0$ as $k \to \infty$. The fixed point x^* is *globally attracting* if it attracts all of $\mathcal{D}_f$, and it is *globally stable* if it is locally stable and globally attracting.

Definition 2.2. A set $\mathcal{O} \subseteq \mathcal{D}_f$ is called *invariant* if $f(\mathcal{O}) = \mathcal{O}$. An invariant set $\mathcal{O}$ is called an *attracting set* if there exists an open neighbourhood $\mathcal{U} \subseteq \mathcal{D}_f$ of $\mathcal{O}$ such that

$$(\forall \text{ open neighbourhood } V \supseteq \mathcal{O})\ (\exists\, L(V) \geq 0) \text{ such that } \forall k \geq L(V) : f^k(\mathcal{U}) \subseteq V \tag{2.2}$$

This neighbourhood $\mathcal{U}$ is called a *fundamental neighbourhood* of $\mathcal{O}$. The *basin of attraction* of $\mathcal{O}$ is $\cup_{k \in \mathrm{N}_0} f^{-k}(\mathcal{U})$.

Interval analysis

The closed and bounded intervals of the real line are denoted by

$$\mathrm{IR} = \{[x] = [x^-, x^+] : -\infty < x^- \leq x^+ < \infty\} \cup \{\emptyset\}. \tag{2.3}$$

x^- (x^+) is the *lower* (*upper*) endpoint of the interval $[x]$. If $x^- = x^+$, then we call it a *thin* interval. The natural embedding of R into IR are the thin intervals and is given by

$$\iota : \mathrm{R} \to \mathrm{IR},\ r \mapsto [r] = [r, r]. \tag{2.4}$$

Having a set S, we denote its interval enclosure by $[S]$. The extension of the real arithmetic over the intervals results in the so-called interval arithmetic, and we refer to the extension of real valued functions over the intervals as interval analysis. We say that the function $F : \mathcal{D}_F \subseteq \mathrm{IR} \to \mathrm{IR}$ is an *interval extension* of the real function $f : \mathrm{R} \to \mathrm{R}$, if it satisfies for every $[x] \in \mathcal{D}_F$ that

$$\begin{gathered} \{f(x) : x \in [x]\} \subseteq F([x]) \text{ (range inclusion)}, \\ [y] \subseteq [z] \subseteq [x] \Rightarrow F([y]) \subseteq F([z]) \text{ (inclusion isotonicity)}. \end{gathered} \tag{2.5}$$

Throughout the calculation, the computer is forced to use directed rounding modes. Consider an arithmetic operation. Using the round up (round down) mode of the computer, the result of the operation is a number that is not smaller (larger) than the true result. This gives us rigorous endpoints, thus every numerical error that might be introduced by the computer is already taken into account. After finishing a computation, the information we obtain is that the true result is contained in the final interval.

In order to give a simple example consider the sum of the first one million terms of the harmonic series. We shall use *single* precision in this example, that means roughly eight significant digits. We have evaluated the sum in increasing order and got the following

$$\frac{1}{1000000} + ... + \frac{1}{1} = 14.392652. \tag{2.6}$$

This is a nonrigorous result, one might say, a good guess. Repeating the same with intervals having *single* precision endpoints results in

$$\frac{1}{1000000} + ... + \frac{1}{1} = [14.350339, 14.436089]. \tag{2.7}$$

We would like to point out that equation (2.7) gives us rigorous bounds, that is the true result of the sum is inside the interval. Observe that the nonrigorous result from equation (2.6) lies in the interior of the interval result. Equation (2.7) provides a computer-aided proof for the statement $\sum_{k=1}^{10^6} \frac{1}{k} \in [14.350339, 14.436089]$ or equivalently $14.350339 \leq \sum_{k=1}^{10^6} \frac{1}{k} \leq 14.436089$. Naturally it is possible to achieve better results by using higher precision, this may result in tighter intervals.

In our proof concerning the Ricker map we check for intersections of sets, that essentially means checking inequalities, which is achieved using validated numerics.

The reader is referred to Moore [25], Alefeld [2], Tucker [32, 33], Nedialkov, Jackson and Corliss [26] for a detailed introduction to rigorous computations and computer-aided proofs.

Graph representations

Different directed graphs can be associated with a given map. The vertices of these graphs are sets and the edges correspond to transitions between them. These graphs reflect the behaviour of the map, if for every point x_0 and its image x_1, it is satisfied that there is an edge going from any vertex containing x_0 to any vertex containing x_1. We give the definitions of covers and graph representations that will be used in Section 5 and 6.

Definition 2.3. $\mathcal{S}$ is called a *cover* of $\mathcal{D} \subseteq \mathrm{R}^n$ if it is a collection of subsets of R^n such that $\cup_{s\in\mathcal{S}} s \supseteq \mathcal{D}$. We denote their union $\cup_{s\in\mathcal{S}} s$ by $|\mathcal{S}|$ in the following. We define the *diameter* or *outer resolution* of the cover $\mathcal{S}$ by

$$\mathcal{R}^+(\mathcal{S}) = \mathrm{diam}(\mathcal{S}) = \sup_{s\in\mathcal{S}} \mathrm{diam}(s).$$

with

$$\mathrm{diam}(s) = \sup_{x,y\in s} \|x - y\|.$$

A cover $\mathcal{S}_2$ is said to be *finer* than the cover $\mathcal{S}_1$ if

$$(\forall s_1 \in \mathcal{S}_1)\ (\exists \{s_{2,i}, i \in \mathcal{I}\} \subseteq \mathcal{S}_2)\ \text{ such that } \bigcup_{i\in\mathcal{I}} s_{2,i} = s_1.$$

We denote this relation by $\mathcal{S}_2 \preccurlyeq \mathcal{S}_1$. The *inner resolution* of a cover $\mathcal{S}$ is the following:

$$\mathcal{R}^-(\mathcal{S}) = \sup\{r \geq 0 : \forall x \in \mathcal{D}, \exists s \in \mathcal{S} : \mathrm{B}(x; r) \subseteq s\}.$$

A cover $\mathcal{S}$ is *essential* if $\mathcal{S} \setminus s$ is not a cover anymore for any $s \in \mathcal{S}$. The cover $\mathcal{P}$ is called a *partition* if it consists of closed sets such that $|\mathcal{P}| = \mathcal{D}$ and $\forall p_1, p_2 \in \mathcal{P} : p_1 \cap p_2 \subseteq \mathrm{bd}(p_1) \cup \mathrm{bd}(p_2)$, where $\mathrm{bd}(p)$ is the boundary of the set p. Consequently, for any partition $\mathcal{P}$ the inner resolution $\mathcal{R}^-(\mathcal{P})$ is zero.

In the following we will always work with essential and finite partitions, as a consequence the supremum in the definition of the diameter $\mathcal{R}^+$ of the partition becomes a maximum.

Definition 2.4. A *directed graph* $\mathcal{G} = \mathcal{G}(\mathcal{V}, \mathcal{E})$ is a pair of sets representing the vertices $\mathcal{V}$ and the edges $\mathcal{E}$, that is: $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and $(u, v) \in \mathcal{E}$ means that $\mathcal{G}$ has a directed edge going from u to v. We say that $v_1 \to v_2 \to \cdots \to v_k$ is a *directed path* if $(v_i, v_{i+1}) \in \mathcal{E}$ for all $i = 1, \ldots, k-1$. If $v_k = v_1$, then it is a *directed cycle*.

A directed graph $\mathcal{G}$ is called *strongly connected* if for any $u, v \in \mathcal{V}$, $v \neq u$ there is a directed path from u to v and from v to u as well. The *strongly connected components* (SCC) of a directed graph $\mathcal{G}$ are its maximal strongly connected subgraphs. It is easy to see that u and v are in the same SCC if and only if there is a directed cycle going through both u and v. Every directed graph $\mathcal{G}$, can be decomposed into the union of strongly connected components and directed paths between them. If we contract each SCC to a new vertex, we obtain a directed acyclic graph, that is called the *condensation* of $\mathcal{G}$.

We say that the directed paths p_1, p_2 are from the same *family of directed paths*, if they visit the same vertices in $\mathcal{V}$ (multiple visits are possible). If the set of the visited vertices is $V \subseteq \mathcal{V}$, then we denote the family by $\Upsilon_{path}(V)$, and we say that V is the *vertex set* of the family. In a similar manner we can define the *family of directed cycles*, and denote it by $\Upsilon_{cycle}(V)$, and say that V is the vertex set of the family.

Definition 2.5. Let $f : \mathcal{D}_f \subseteq \mathrm{R}^n \to \mathrm{R}^n$, $\mathcal{D} \subseteq \mathcal{D}_f$, and $\mathcal{S}$ be a cover of $\mathcal{D}$. We say that the directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is a *graph representation of f on $\mathcal{D}$ with respect to $\mathcal{S}$*, if there is a $\iota : \mathcal{V} \to \mathcal{S}$ bijection such that the following implication is true for all $u, v \in \mathcal{V}$:

$$f(\iota(u) \cap \mathcal{D}) \cap \iota(v) \cap \mathcal{D} \neq \emptyset \Rightarrow (u, v) \in \mathcal{E},$$

and we denote it by $\mathcal{G} \propto (f, \mathcal{D}, \mathcal{S})$.

Having a graph representation $\mathcal{G}$ of f on $\mathcal{D}$ with respect to $\mathcal{S}$, we take the liberty to handle the elements of the cover as vertices and vice versa, omitting the usage of ι. It is important to emphasise that in general $(u, v) \in \mathcal{E}$ does not imply that $f(u \cap \mathcal{D}) \cap v \cap \mathcal{D} \neq \emptyset$. If we have $(u, v) \in \mathcal{E} \Leftrightarrow f(u \cap \mathcal{D}) \cap v \cap \mathcal{D} \neq \emptyset$, then we call $\mathcal{G}$ an *exact* graph representation.

3. A neighbourhood of α^* where all points enter

In this section we construct compact neighbourhoods $S_i^{(\alpha)} \subset \mathrm{R}_+^2$ of α^*, $i \in \mathrm{N}_0$, so that $F_\alpha(S_i^{(\alpha)}) \subseteq S_i^{(\alpha)}$, and $S_i^{(\alpha)}$ attracts all points of R_+^2 for all $i \in \mathrm{N}_0$ and $\alpha \in (0, 1]$. Hence an elementary proof of Theorem 1.1 is obtained for $0 < \alpha \leq 0.5$. Recall $F_\alpha(\mathrm{R}_+^2) \subseteq \mathrm{R}_+^2$. We can illustrate the image $(y, ye^{\alpha - x})$ of (x, y) under F_α as first going horizontally from (x, y) to the diagonal, proceeding upwards if $0 < x < \alpha$, otherwise downwards vertically until we reach the value $ye^{\alpha-x}$. This is shown on Figure 1.

We need the function

$$\tau_\alpha : \mathrm{R} \ni t \mapsto \alpha e^{2(\alpha-t)} \in \mathrm{R}$$

depending on a parameter $\alpha > 0$. Define a sequence $(s_n^{(\alpha)})_{n=0}^\infty$ by

$$s_0^{(\alpha)} = 0, \quad s_{n+1}^{(\alpha)} = \tau_\alpha(s_n^{(\alpha)}) \quad \text{for } n \in \mathrm{N}_0.$$

The following is satisfied for $(s_n^{(\alpha)})$.

Proposition 3.1. *For $\alpha > 0$ the relation*

$$\begin{aligned} 0 = s_0^{(\alpha)} < s_2^{(\alpha)} < \cdots < s_{2n-2}^{(\alpha)} < s_{2n}^{(\alpha)} \\ < \alpha < s_{2n+1}^{(\alpha)} < s_{2n-1}^{(\alpha)} < \cdots < s_3^{(\alpha)} < s_1^{(\alpha)} = \alpha e^{2\alpha} \end{aligned} \tag{3.1}$$

holds for all $n \in \mathrm{N}$, and $0 < l^{(\alpha)} \leq \alpha \leq L^{(\alpha)} < \alpha e^{2\alpha}$ with $l^{(\alpha)} = \lim_{n\to\infty} s_{2n}^{(\alpha)}$, $L^{(\alpha)} = \lim_{n\to\infty} s_{2n+1}^{(\alpha)}$.

If $\alpha \in (0, 0.5]$ then $l^{(\alpha)} = L^{(\alpha)} = \alpha$, that is, $\lim_{n\to\infty} s_n^{(\alpha)} = \alpha$.

Proof. As $s_0^{(\alpha)} = 0$, $s_1^{(\alpha)} = \alpha e^{2\alpha} > \alpha$, $s_2^{(\alpha)} = \alpha e^{2(\alpha - s_1^{(\alpha)})} \in (0, \alpha)$, $s_3^{(\alpha)} = \alpha e^{2(\alpha - s_2^{(\alpha)})} \in$

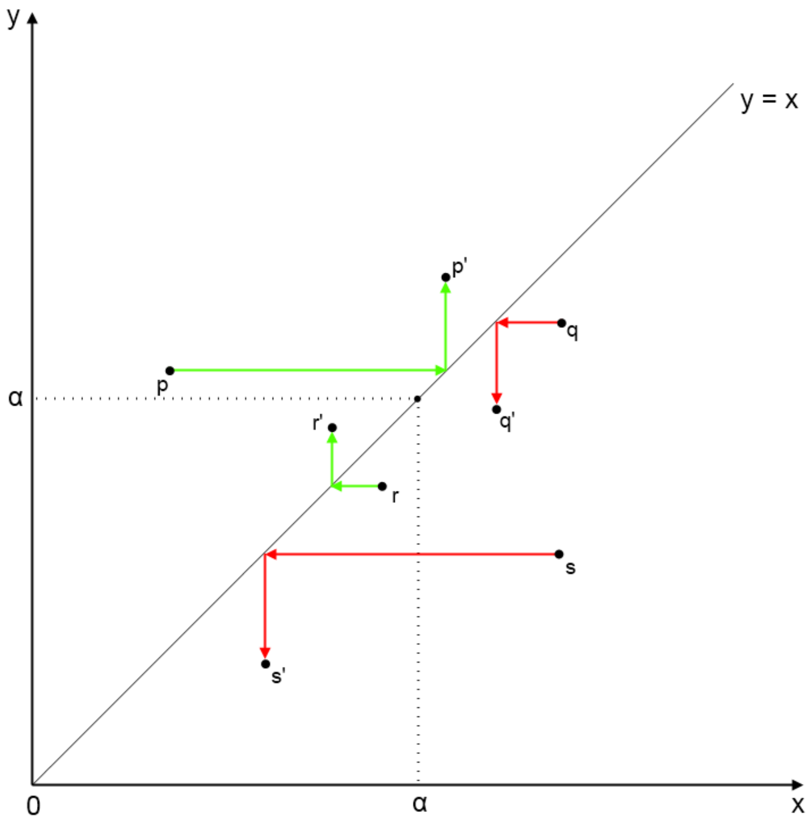

Figure 1. Graphical analysis of obtaining the first iterate $(x_1, y_1) = F_\alpha(x_0, y_0)$ for $\alpha \in (0, 1]$. *Four different initial conditions are shown on the picture, $(x_0, y_0) \in \{p, q, r, s\}$*

$(\alpha, s_1^{(\alpha)})$, the statement (3.1) is true with $n = 1$. Suppose that (3.1) holds for an integer $n \geq 1$. Then, by using the monotone property of τ_α in t, the equality $\tau_\alpha(\alpha) = \alpha$, the inequalities $s_{2n}^{(\alpha)} < \alpha$, $s_{2n}^{(\alpha)} = \tau_\alpha(s_{2n-1}^{(\alpha)})$ and $s_{2n+1}^{(\alpha)} < s_{2n-1}^{(\alpha)}$ imply $s_{2n}^{(\alpha)} < s_{2n+2}^{(\alpha)} = \tau_\alpha(s_{2n+1}^{(\alpha)}) < \alpha$, and similarly $\alpha < s_{2n+3}^{(\alpha)} < s_{2n+1}^{(\alpha)}$. By induction, the proof of (3.1) is complete. The claim for l^α and L^α easily follows.

Note that assertion $l^{(\alpha)} = L^{(\alpha)} = \alpha$ for $\alpha \in (0, 0.5)$ follows from Proposition 3.3 in [20]. However, for the sake of completeness, we give here our elementary proof of the claim. Let $\alpha \in (0, 0.5]$ be fixed. Define the map

$$\kappa_\alpha : [\alpha, \infty) \ni t \mapsto \tau_\alpha(\tau_\alpha(t)) \in \mathrm{R}.$$

From $\tau'_\alpha(t) = -2\tau_\alpha(t)$ it can be obtained that

$$\kappa'_\alpha(t) = 4\kappa_\alpha(t)\tau_\alpha(t), \tag{3.2}$$

$$\kappa''_\alpha(t) = 8\kappa_\alpha(t)\tau_\alpha(t)(2\tau_\alpha(t) - 1) \tag{3.3}$$

for all $t \geq \alpha$. Clearly, $\kappa_\alpha(\alpha) = \alpha$, and $\kappa'_\alpha(\alpha) = 4\alpha^2 \leq 1$ by (3.2). If $t > \alpha$ then from (3.3) we find $\kappa''_\alpha(t) < 0$. Therefore, $\kappa_\alpha([\alpha, \infty)) \subset [\alpha, \infty)$, κ_α is strictly increasing and strictly concave on $[\alpha, \infty)$. It is elementary to show that the fixed point α of κ_α attracts all points of $[\alpha, \infty)$. In particular,

$$s_{2n+1}^{(\alpha)} = \kappa_\alpha^n(\alpha e^{2\alpha}) \to \alpha \text{ as } n \to \infty.$$

Moreover, $s_{2n+2}^{(\alpha)} = \tau_\alpha(s_{2n+1}^{(\alpha)}) \to \tau_\alpha(\alpha) = \alpha$ as $n \to \infty$. □

Define the subsets $H_1, \ldots, H_6$ of R_+^2 by

$$\begin{aligned} H_1 &= \{(x, y) : 0 < y < x \leq \alpha\}, & H_2 &= \{(x, y) : 0 < x \leq y < \alpha\}, \\ H_3 &= \{(x, y) : 0 < x < \alpha \leq y\}, & H_4 &= \{(x, y) : \alpha \leq x < y\}, \\ H_5 &= \{(x, y) : \alpha < y \leq x\}, & H_6 &= \{(x, y) : y \leq \alpha < x\}. \end{aligned}$$

Clearly, sets $H_1, \ldots, H_6$ are pairwise disjoint, and $\cup_{i=1}^6 H_i = \mathrm{R}_+^2 \setminus \{\alpha^*\}$. See Figure 2.

For given real constants a, b with $0 \leq a < \alpha < b < \infty$, we need truncated versions of sets $H_1, \ldots, H_6$ defined by

$$\begin{aligned} &G_1(a) = H_1 \cap \{y > a\}, G_2(a) = H_2 \cap \{x > a\}, G_3(a, b) = H_3 \cap \{x > a\} \cap \{y < b\}, \\ &G_4(b) = H_4 \cap \{y < b\}, \ G_5(b) = H_5 \cap \{x < b\}, \ G_6(b, a) = H_6 \cap \{x < b\} \cap \{y > a\}. \end{aligned}$$

Proposition 3.2. *For all a, b with $0 \leq a < \alpha < b < \infty$ the following statements hold.*

(i) $F_\alpha(G_1(a) \cup G_2(a)) \subset G_2(a) \cup G_3(a, \alpha e^{\alpha - a})$,
(ii) $F_\alpha(G_3(a, b)) \subset G_4(b e^{\alpha - a})$,
(iii) $F_\alpha(G_4(b) \cup G_5(b)) \subset G_5(b) \cup G_6(b, \alpha e^{\alpha - b})$,
(iv) $F_\alpha(G_6(b, a)) \subset G_1(a e^{\alpha - b})$.

Proof. Fix a, b so that $0 \leq a < \alpha < b < \infty$.

1. Let $(x, y) \in G_1(a) \cup G_2(a)$, i.e., $a < y < x \leq \alpha$ or $a < x \leq y < \alpha$. Then we have $a < y < \alpha$ and $y \leq y e^{\alpha - x} < \alpha e^{\alpha - a}$, that is, $F_\alpha(x, y) = (y, y e^{\alpha - x}) \in G_2(a) \cup G_2(a, \alpha e^{\alpha - a})$.

2. Let (x, y) be given with $a < x < \alpha \leq y < b$. Then one clearly gets $\alpha \leq y < y e^{\alpha - x} < b e^{\alpha - a}$, that is, by $F_\alpha(x, y) = (y, y e^{\alpha - x})$, (ii) holds.

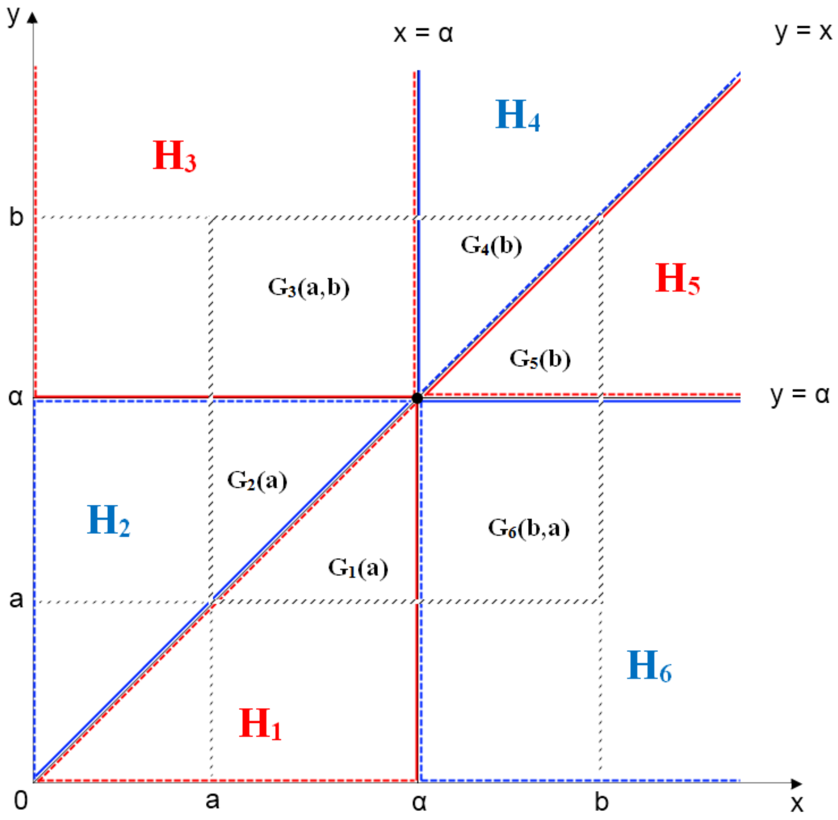

Figure 2. The sets $H_1, \dots H_6$. *See Figure 3 for the possible transitions of* F_α *between these sets.*

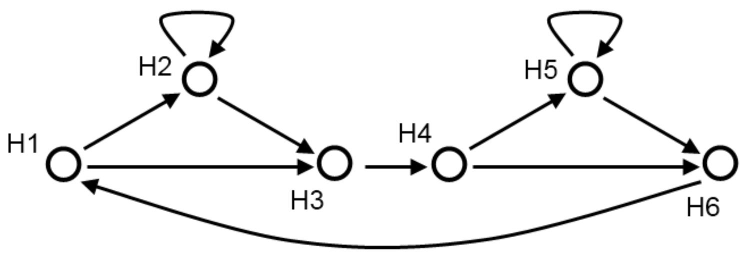

Figure 3. Transitions between the sets H_i, $i = 1, \dots, 6$

3. If $(x, y) \in G_4(b) \cup G_5(b)$, i.e., $\alpha \le x < y < b$ or $\alpha < y \le x < b$, then it is obvious that $\alpha < y < b$ and $y \ge ye^{\alpha - x} > \alpha e^{\alpha - b}$. This means that $F_\alpha(x, y) \in G_5(b) \cup G_6((b, \alpha e^{\alpha - b})$. So, (iii) is satisfied.
4. Let $(x, y) \in G_6(b, a)$, i.e., $a < y \le \alpha < x < b$. Then, trivially, $ae^{\alpha - b} < ye^{\alpha - x} < y \le \alpha$ holds, that is, $F_\alpha(x, y) = (y, ye^{\alpha - x}) \in G_1(ae^{\alpha - b})$. □

As any $(x, y) \in \mathbb{R}^2_+ \setminus \{(\alpha, \alpha)\}$ is in one of the sets $G_1(a)$, $G_2(a)$, $G_3(a, b)$, $G_4(b)$, $G_5(b)$, $G_6(b, a)$ with some $a \in (0, \alpha)$ and $b > \alpha$, the map F_α restricted to $\mathbb{R}^2_+ \setminus \{(\alpha, \alpha)\}$ can be represented graphically as follows.

For $a \in [0, \alpha)$ define

$$S^{(\alpha)}(a) = \Big([a,\alpha] \times [a, \alpha e^{\alpha-a}]\Big) \cup \Big([\alpha, \tau_\alpha(a)] \times [\alpha e^{\alpha-\tau_\alpha(a)}, \tau_\alpha(a)]\Big),$$

and, for each $n \in \mathrm{N}_0$, let $S_n^{(\alpha)} = S^{(\alpha)}(s_{2n}^{(\alpha)})$. The next result shows that sets $S_n^{(\alpha)}$ are positively invariant, and attract all points of R_+^2.

Proposition 3.3.

(a) $F_\alpha(S_k^{(\alpha)}) \subset S_k^{(\alpha)}$ *holds for all* $k \in \mathrm{N}_0$.
(b) For any $(x,y) \in \mathrm{R}_+^2$ *and for any* $k \in \mathrm{N}_0$ *there exists* $n = n(k,x,y)$ *such that* $F_\alpha^n(x,y) \in S_k^{(\alpha)}$.

Proof. 1. The proof of (a). Set $c = s_{2n}^{(\alpha)}$. Then

$$\begin{aligned} S_n^{(\alpha)} &= S^{(\alpha)}(c) \\ &= Cl\Big\{G_1(c) \cup G_2(c) \cup G_3(c, \alpha e^{\alpha-c}) \\ &\qquad \cup G_4(\tau_\alpha(c)) \cup G_5(\tau_\alpha(c)) \cup G_6\Big(\tau_\alpha(c), \alpha e^{\alpha-\tau_\alpha(c)}\Big)\Big\}, \end{aligned}$$

where Cl denotes the closure. By continuity of F_α, it is enough to show that the open set

$$G_1(c) \cup G_2(c) \cup G_3(c, \alpha e^{\alpha-c}) \cup G_4(\tau_\alpha(c)) \cup G_5(\tau_\alpha(c)) \cup G_6\Big(\tau_\alpha(c), \alpha e^{\alpha-\tau_\alpha(c)}\Big)$$

is invariant under F_α. This is a straightforward consequence of Proposition 3.2. Namely, we apply statement (i) with $a = c$, (ii) with $b = \alpha e^{\alpha-c}$, (iii) with $b = \tau_\alpha(c)$, and (iv) with $b = \tau_\alpha(c)$, $a = \alpha e^{\alpha-\tau_\alpha(c)}$. Observing $\alpha e^{2(\alpha-\tau_\alpha(c))} = s_{2n+2}^{(\alpha)}$, and using $s_{2n}^{(\alpha)} < s_{2n+2}^{(\alpha)} < \alpha$ from Proposition 3.1, we conclude

$$\begin{aligned} F_\alpha\Big(G_6\Big(\tau_\alpha(c), \alpha e^{\alpha-\tau_\alpha(c)}\Big)\Big) &\subset G_1\Big(\alpha e^{2(\alpha-\tau_\alpha(c))}\Big) \\ &= G_1\Big(s_{2n+2}^{(\alpha)}\Big) \subset G_1\Big(s_{2n}^{(\alpha)}\Big) = G_1(c). \end{aligned}$$

2. The proof of (b). Fix $(x,y) \in \mathrm{R}_+^2$ and $k \in \mathrm{N}$. The case $(x,y) = (\alpha,\alpha)$ is obvious. So, assume $(x,y) \neq (\alpha,\alpha)$. By the graph representation of F_α on Figure 3 we distinguish 3 cases:

- Case 1: There exists $n_0 \in \mathrm{N}$ such that $F_\alpha^n(x,y) \in H_2$ for all $n \geq n_0$.
- Case 2: There exists $n_0 \in \mathrm{N}$ such that $F_\alpha^n(x,y) \in H_5$ for all $n \geq n_0$.
- Case 3: There is a subsequence $(n_l)_{l=0}^\infty$ in N such that $F_\alpha^n(x,y) \in H_1$ for all $l \in \mathrm{N}_0$.

In case 1 let $(u_n, v_n) = F_\alpha^n(x,y)$, $n \geq n_0$. Then by $(u_n, v_n) \in H_2$ and $u_{n+1} = v_n$, it follows that $(u_n)_{n_0}^\infty$ and $(v_n)_{n_0}^\infty$ are bounded, monotone increasing sequences with limits u^*, v^*, respectively. Then (u^*, v^*) is a fixed point of F_α in R_+^2. Consequently $(u^*, v^*) = (\alpha, \alpha)$, and $F_\alpha^n(x,y) \to (\alpha,\alpha)$. As $S_k^{(\alpha)}$ is a neighbourhood of (α,α), there exists $n(k,x,y)$ with the desired property. Case 2 is analogous.

In case 3 we shall show by induction on k that for any $k \in \mathrm{N}_0$, there exists $l \in \mathrm{N}_0$ such that $F_\alpha^n(x,y) \in G_1(s_{2k}^{(\alpha)}) \subset S_k^{(\alpha)}$. Case $k = 0$ is trivial, since $H_1 = G_1(0) = G_1(s_0^{(\alpha)})$. Now fix $k \in \mathrm{N}_0$ and assume that $F_\alpha^l(x,y) \in G_1(s_{2k}^{(\alpha)})$ for some $l \in \mathrm{N}_0$. Let $(u,v) = F_\alpha^l(x,y)$. Now, we can apply Proposition 3.2 in the same way as in part 1 of

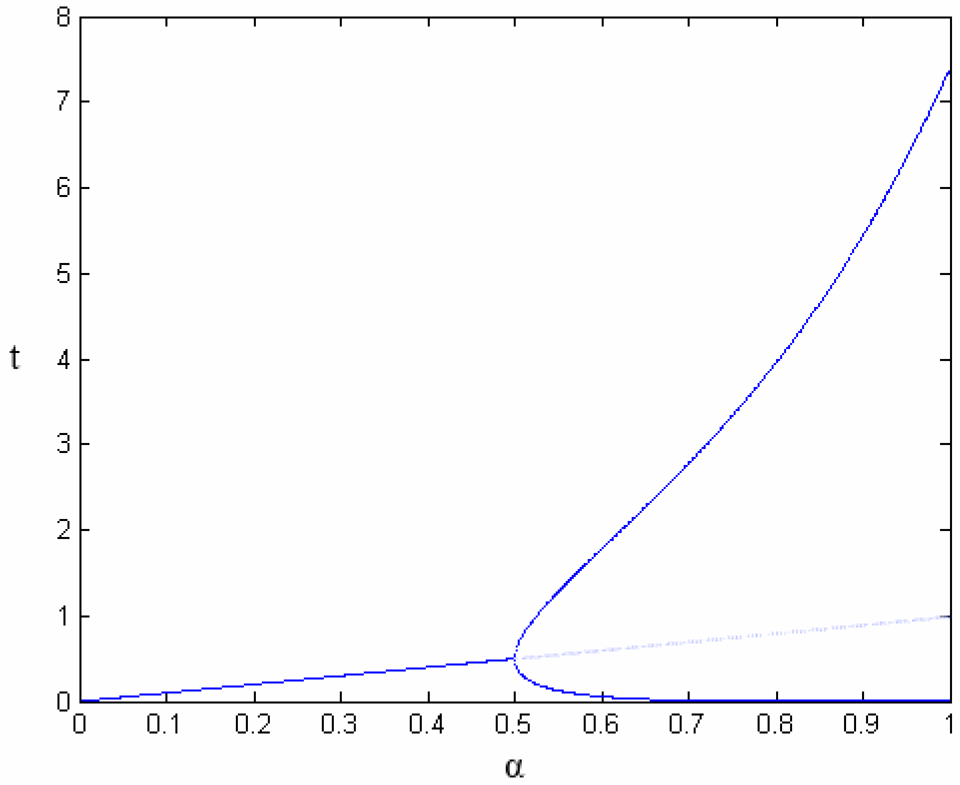

Figure 4. Bifurcation diagram of τ_α, where $\tau_\alpha(t) = \alpha e^{2(\alpha - t)}$

this proof. There will be a smallest integer $m > 0$ such that

$$F_\alpha^m(u,v) \in G_6\left(\tau_\alpha(s_{2k}^{(\alpha)}), \alpha e^{\alpha - \tau_\alpha(s_{2k}^{(\alpha)})}\right).$$

By statement (iv) of Proposition 3.2, it follows that

$$F_\alpha^{l+m+1}(x,y) = F_\alpha^{m+1}(u,v) \in G_1\left(\alpha e^{2(\alpha - \tau_\alpha(s_{2k}^{(\alpha)})}\right) = G_1\left(s_{2k+2}^{(\alpha)}\right) \subset S_{k+1}^{(\alpha)}.$$

This proves statement (b). □

An immediate corollary of the above propositions is a global attractivity result for $\alpha \in (0, 0.5]$. This is not new. More general results were shown by Liz, Tkachenko and Trofimchuk [21] based on techniques developed for delay differential equations. For the sake of completeness we include our elementary proof.

Corollary 3.4. *If $0 < \alpha \leq 0.5$ then for every $(x,y) \in \mathrm{R}_+^2$ we have $F_\alpha^n(x,y) \to (\alpha,\alpha)$ as $n \to \infty$.*

Proof. By Proposition 3.1 we have $S_n^{(\alpha)} = S^{(\alpha)}(s_{2n}^{(\alpha)}) \to S^{(\alpha)}(\alpha) = \{(\alpha,\alpha)\}$ as $n \to \infty$. Apply Proposition 3.3. □

We remark that global attractivity of (α,α) cannot be expected in the above way when $\alpha > 0.5$. In fact, the bifurcation diagram for τ_α on Figure 4 shows an attracting 2-cycle for $\alpha > 0.5$. As seen in [19], this diagram is rigorous.

Define

$$S^{(\alpha)} = \cap_{n=0}^{\infty} S_n^{(\alpha)}.$$

Clearly,

$$S^{(\alpha)} = \left[l^{(\alpha)}, \alpha\right] \times \left[l^{(\alpha)}, \alpha e^{\alpha - l^{(\alpha)}}\right] \cup \left[\alpha, L^{(\alpha)}\right] \times \left[\alpha e^{\alpha - L^{(\alpha)}}, L^{(\alpha)}\right], \tag{3.4}$$

and $S^{(\alpha)}$ is compact, positively invariant under F_α, and attracts all points of R^2_+.

4. A neighbourhood attracted by the fixed point α^*

Let us consider map F_α. In this section we are going to give $\varepsilon(\alpha) > 0$ such that $\inf_{\alpha\in[0.5,1]} \varepsilon(\alpha) > 0$ and $\mathrm{K}(\alpha^*; \varepsilon(\alpha))$ belongs to the basin of attraction of α^* for $\alpha \in [0.5, 1]$, that is, $F^n_\alpha(x_0, y_0) \to \alpha^*$ as $n \to \infty$ for all $(x_0, y_0) \in \mathrm{K}(\alpha^*; \varepsilon(\alpha))$.

Introducing the new variables $u = x - \alpha,\ v = y - \alpha$, map F_α can be written in the form

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto A(\alpha) \begin{pmatrix} u \\ v \end{pmatrix} + f_\alpha(u, v), \tag{4.1}$$

where the linear part is

$$A(\alpha) = \begin{pmatrix} 0 & 1 \\ -\alpha & 1 \end{pmatrix}$$

and the remainder is

$$f_\alpha(u, v) = \begin{pmatrix} 0 \\ v(e^{-u} - 1) + \alpha(e^{-u} - 1 + u) \end{pmatrix}.$$

The eigenvalues of $A(\alpha)$ are $\mu_{1,2}(\alpha) = \frac{1\pm i\sqrt{4\alpha-1}}{2} \in \mathrm{C}$, and the corresponding complex eigenvectors are $q_{1,2}(\alpha) = \left(\frac{1\mp\sqrt{1-4\alpha}}{2\alpha}, 1\right)^{\mathrm{T}} = \left(\frac{1\mp i\sqrt{4\alpha-1}}{2\alpha}, 1\right)^{\mathrm{T}} \in \mathrm{C}^2$, respectively for $\alpha > \frac{1}{4}$. Let $q(\alpha) = q_1(\alpha)$ and $\mu(\alpha) = \mu_1(\alpha)$. Let $p(\alpha) \in \mathrm{C}^2$ denote the eigenvector of $A(\alpha)^{\mathrm{T}}$ corresponding to $\overline{\mu(\alpha)}$ such that $\langle p(\alpha), q(\alpha)\rangle = 1$. This leads to

$$p(\alpha) = \left(-\frac{i\alpha}{\sqrt{4\alpha-1}}, \frac{\sqrt{4\alpha-1}+i}{2\sqrt{4\alpha-1}}\right)^{\mathrm{T}}.$$

We introduce a complex variable

$$z = z(u, v, \alpha) = \langle p(\alpha), (u, v)^{\mathrm{T}}\rangle = \frac{1}{2}\left(v - \frac{i(v - 2u\alpha)}{\sqrt{-1+4\alpha}}\right). \tag{4.2}$$

There is an explicit formula for $(u, v)^{\mathrm{T}}$ in terms of z, which reads as

$$(u, v)^{\mathrm{T}} = zq(\alpha) + \overline{zq(\alpha)} = \left(\frac{\mathrm{Re}z + \sqrt{4\alpha-1}\mathrm{Im}z}{\alpha}, 2\mathrm{Re}z\right)^{\mathrm{T}}. \tag{4.3}$$

Our original system (4.1) is now transformed into the complex system

$$\begin{aligned} z \mapsto G(z, \overline{z}, \alpha) &= \left\langle p(\alpha), A(\alpha)(zq(\alpha) + \overline{zq(\alpha)}) + f_\alpha(zq(\alpha) + \overline{zq(\alpha)})\right\rangle \\ &= \mu(\alpha)z + g(z, \overline{z}, \alpha), \end{aligned} \tag{4.4}$$

where g is a complex valued smooth function of z, $\overline{z}$ and α, defined by (A.1) in the Appendix. It can also be seen that for fixed α, g is an analytic function of z and $\overline{z}$ and the Taylor expansion of g with respect to z and $\overline{z}$ has only quadratic and higher order terms. That is,

$$g(z, \overline{z}, \alpha) = \sum_{2\le k+l} \frac{1}{k!\,l!} g_{kl}(\alpha) z^k \overline{z}^l, \quad \text{with } k, l = 0, 1, \dots,$$

where $g_{kl}(\alpha) = \frac{\partial^{k+l}}{\partial z^k \partial \overline{z}^l} g(z, \overline{z}, \alpha)\Big|_{z=0}$ for $k + l \geq 2,\ k, l \in \{0, 1, \dots\}$.

PROPOSITION 4.1. *Let $\alpha \in [0.5, 1)$ be fixed and*

$$\varepsilon(\alpha) = \min\left\{\frac{1}{20}\sqrt{\frac{4\alpha-1}{2+\alpha}}, \frac{9(4\alpha-1)(1-\sqrt{\alpha})}{20(1+2\sqrt{\alpha})\sqrt{2+\alpha}}\right\}.$$

Then $\{(x, y) \in \mathrm{R}^2 : |x - \alpha| < \varepsilon(\alpha), |y - \alpha| < \varepsilon(\alpha)\}$ belongs to the basin of attraction of the fixed point α^* of F_α.

Proof. Let us study the map in the form (4.4). First note that (4.3) easily implies the inequalities

$$|u| \leq \frac{2}{\sqrt{\alpha}}|z| \text{ and } |v| \leq 2|z|. \tag{4.5}$$

Assuming $|u| < 1/10$ and $|v| < 1/10$ and applying the inequalities $\left|e^{-u} - 1\right| \leq e^{1/10}|u| \leq \frac{10}{9}|u|$ and $\left|e^{-u} - 1 + u\right| \leq e^{1/10}\frac{u^2}{2} \leq \frac{5}{9}u^2$, we obtain the following estimations:

$$\begin{aligned}
|g(z, \overline{z}, \alpha)| &= \left|\left\langle p(\alpha), f_\alpha\left(zq(\alpha) + \overline{zq(\alpha)}\right)\right\rangle\right| \\
&= \left|\frac{\sqrt{4\alpha-1}+i}{2\sqrt{4\alpha-1}}\right| \left|v(e^{-u}-1) + \alpha(e^{-u}-1+u)\right| \\
&\leq \sqrt{\frac{\alpha}{4\alpha-1}}\left(|v||e^{-u}-1| + \alpha|e^{-u}-1+u|\right) \\
&\leq \sqrt{\frac{\alpha}{4\alpha-1}}\left(|uv|e^{1/10} + \alpha\frac{e^{1/10}}{2}u^2\right) \\
&\leq \frac{5}{9}\sqrt{\frac{\alpha}{4\alpha-1}}\left(u^2 + 2|uv|\right) \leq \frac{5}{9} \cdot \frac{4(1+2\sqrt{\alpha})}{\sqrt{\alpha(4\alpha-1)}}|z|^2.
\end{aligned}$$

Now, since $|\mu(\alpha)| = \sqrt{\alpha}$, hence

$$|G(z, \overline{z}, \alpha)| \leq \left(\sqrt{\alpha} + \frac{5}{9} \cdot \frac{4(1+2\sqrt{\alpha})}{\sqrt{\alpha(4\alpha-1)}}|z|\right)|z| < |z|$$

provided that $|z| \neq 0$ is so small that $|u| < \frac{1}{10}$ and $|v| < \frac{1}{10}$ and $\sqrt{\alpha} + \frac{5}{9} \cdot \frac{4(1+2\sqrt{\alpha})}{\sqrt{\alpha(4\alpha-1)}}|z| < 1$. Inequalities (4.5) imply that $|z| < \frac{\sqrt{\alpha}}{20}$ guarantees $|u| < \frac{1}{10}$ and $|v| < \frac{1}{10}$. Therefore

$$0 < |z| < \varepsilon_G(\alpha) = \min\left\{\frac{\sqrt{\alpha}}{20}, \frac{9(1-\sqrt{\alpha})\sqrt{\alpha(\alpha-1)}}{20(1+2\sqrt{\alpha})}\right\} \tag{4.6}$$

implies $|G(z, \overline{z}, \alpha)| < |z|$, which means that $|G^n(z_0, \overline{z_0}, \alpha)| \to 0$ as $n \to \infty$ if $|z_0| < \varepsilon_G(\alpha)$ is satisfied. We show this by way of contradiction. Assume that $|z_0| < \varepsilon_G(\alpha)$, $z_n = G^n(z_0, \overline{z_0}, \alpha)$ and $|z_0| > |z_1| > \cdots > |z_n| > \cdots \geq 0$ with $|z_n| \to c > 0$ as $n \to \infty$. Since G is continuous we have that $\max_{|z|=c}|G(z, \overline{z}, \alpha)| < c$ and consequently $|z_k| < c$ also holds if k is large enough, which is a contradiction.

From equation (4.2) one obtains that if $|u| < \delta,\ |v| < \delta$, then

$$|z| < \delta\sqrt{\frac{\alpha(2+\alpha)}{4\alpha-1}}. \tag{4.7}$$

From (4.7) we infer that if $|u| < \varepsilon(\alpha)$ and $|v| < \varepsilon(\alpha)$ then $|z(u,v,\alpha)| < \varepsilon_G(\alpha)$ which completes our proof. □

Note that $\varepsilon(\alpha)$ goes to 0 as α goes to 1, that is, the constructed region $\mathrm{K}(\alpha^*;\varepsilon(\alpha))$ becomes very small. Thus, it is impossible to show by interval arithmetic tools that every trajectory enters into it eventually. Nevertheless $\varepsilon(\alpha)$ might be used in the case $\alpha \in [0.5, 0.999]$. However, our following method is not only capable to give an attracting neighbourhood for all $\alpha \in [0.999, 1]$, whose size is independent of the concrete value of the parameter, but it also serves a better estimation of the attracting region even if we assume only $\alpha \in [0.875, 1]$. The main results of the section are the following two propositions. Here, we only present the proof of Proposition 4.3. The whole argument can be repeated to get a universal attracting neighbourhood when only $[0.875, 1]$ is assumed. The differences only appear in concrete values in the given estimations.

Proposition 4.2. *For all fixed $\alpha \in [0.875, 1]$, the set*

$$\left\{(x,y) \in \mathrm{R}^2 : |x-\alpha| < 1/37, |y-\alpha| < 1/37\right\}$$

belongs to the basin of attraction of the fixed point α^* of F_α.

Proposition 4.3. *For all fixed $\alpha \in [0.999, 1]$, the set*

$$\left\{(x,y) \in \mathrm{R}^2 : |x-\alpha| < 1/22, |y-\alpha| < 1/22\right\}$$

belongs to the basin of attraction of the fixed point α^* of F_α.

Proof. We follow the steps of finding the normal form of the Neimark–Sacker bifurcation, according to Kuznetsov [14]. In our calculations and estimations, although everything could be obtained by hand, we use symbolic calculations and built in symbolic interval arithmetic tools of Wolfram Mathematica v. 7 or 8.

According to Kuznetsov [14], we are aiming to transform system (4.4) to a system which takes the following form.

$$w \mapsto \mu(\alpha)w + c_1(\alpha)w^2\overline{w} + R_2(w,\overline{w},\alpha), \tag{4.8}$$

where c_1 and R_2 are smooth, real functions such that for fixed α, $R_2(w,\overline{w},\alpha) = O(|w|^4)$. We are going to show that there exists $\varepsilon_0 > 0$ such that for all $|w| < \varepsilon_0$ and $\alpha \in [0.999, 1]$,

$$\left|\mu(\alpha)w + c_1(\alpha)w^2\overline{w} + R_2(w,\overline{w},\alpha)\right| < |w|$$

holds, which implies that $\mathrm{B}_{\varepsilon_0}$ belongs to the basin of attraction of the fixed point 0 of the discrete dynamical system generated by (4.8). From this, we shall be able to show that the fixed point α^* of F_α attracts all points of $\mathrm{K}(\alpha^*; \frac{1}{22})$.

Step 1: A transformation to simplify (4.4)

For a fixed α, we are looking for a function $h_\alpha : \mathrm{C} \to \mathrm{C}$, which is invertible in a neighbourhood of $0 \in \mathrm{C}$ and which is such that in the new coordinate $w = h_\alpha^{-1}(z)$, our map (4.4) takes the form

$$w \mapsto h_\alpha^{-1}(G(h_\alpha(w), \overline{h_\alpha(w)}, \alpha)) = \mu(\alpha)w + c_1(\alpha)w^2\overline{w} + R_2(w,\overline{w},\alpha), \tag{4.9}$$

where $R_2(w,\overline{w},\alpha) = O(|w|^4)$ for fixed α. According to [14], such a function can be found in the form

$$\begin{aligned} h_\alpha(w) = w &+ \frac{h_{20}(\alpha)}{2}w^2 + h_{11}(\alpha)w\overline{w} + \frac{h_{02}(\alpha)}{2}\overline{w}^2 \\ &+ \frac{h_{30}(\alpha)}{6}w^3 + \frac{h_{12}(\alpha)}{2}w\overline{w}^2 + \frac{h_{03}(\alpha)}{6}\overline{w}^3. \end{aligned} \tag{4.10}$$

Clearly, h_α has an inverse in a small neighbourhood of $0 \in \mathbb{C}$. A formula for h_α^{-1} can be given in the form

$$h_\alpha^{-1}(z) = h_{inv,\alpha}(z) + R_3(z,\alpha),$$

where

$$h_{inv,\alpha}(z) = z + \sum_{2\leq k+l\leq 4} a_{kl}(\alpha)z^k\overline{z}^l$$

and $R_3(z,\alpha) = O(|z|^5)$. The coefficients can be obtained by substituting $w = h^{-1}(z)$ into $z = h_\alpha(w)$ and equating the coefficients of the same type of terms up to the fourth order. The result for $h_{inv,\alpha}$ in terms of $h_{20}(\alpha),\dots,h_{03}(\alpha)$ is given in (A.14). The coefficients $h_{20}(\alpha),\dots,h_{03}(\alpha)$ are determined such that

$$h_\alpha^{-1}(G(h_\alpha(w),\overline{h_\alpha(w)},\alpha))$$

has the form $\mu(\alpha) + c_1(\alpha)w^2\overline{w}$ plus at least fourth order terms in w, that is, the transformation kills all second and third order terms with one exception. This requires the condition

$$\left(\frac{\mu(1)}{|\mu(1)|}\right)^k \neq 1 \quad \text{for all } k \in \{1,2,3,4\}.$$

As $\mu(1) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, this is clearly satisfied. Formulae (A.15)–(A.20) contain the obtained results.

Step 2: A region where transformation h_α is valid, and estimation for h_α

We are going to show that h_α is injective on $\overline{\mathrm{B}_{1/3}} \subset \mathbb{C}$ and that h_α^{-1} is defined on $\overline{\mathrm{B}_{1/5}}$. Let us suppose that $z \in \mathbb{C}$ is fixed and $h_{20}(\alpha),\dots,h_{03}(\alpha)$ are given for a fixed $\alpha \in [0.999, 1]$. Let

$$H_{\alpha,z} : \mathbb{C} \ni w \mapsto w + z - h_\alpha(w) \in \mathbb{C}.$$

By this notation, $H_{\alpha,z}(w) = w$ holds if and only if $h_\alpha(w) = z$. Now, we have the following

$$\begin{aligned} |H_{\alpha,z}(w_1) - H_{\alpha,z}(w_2)| &= |w_1 - h_\alpha(w_1) - w_2 + h_\alpha(w_2)| \\ &\leq |w_1 - w_2| \cdot \Bigg(\left(\frac{|h_{20}(\alpha)|}{2} + |h_{11}(\alpha)| + \frac{|h_{02}(\alpha)|}{2}\right)(|w_1| + |w_2|) + \\ &\left(\frac{|h_{30}(\alpha)|}{6} + \frac{|h_{12}(\alpha)|}{2} + \frac{|h_{03}(\alpha)|}{6}\right)\left(|w_1|^2 + |w_1||w_2| + |w_2|^2\right)\Bigg). \end{aligned}$$

If $|h_{20}(\alpha)|/2 + |h_{11}(\alpha)| + |h_{02}(\alpha)|/2 < \delta_1$, $|h_{30}(\alpha)|/6 + |h_{12}(\alpha)|/2 + |h_{03}(\alpha)|/6 < \delta_2$, $|w| \leq \delta_3$ and $|z| \leq \delta_4$ hold, then

$$|H_{\alpha,z}(w_1) - H_{\alpha,z}(w_2)| \leq |w_1 - w_2|(2\delta_1\delta_3 + 3\delta_2\delta_3^2)$$

and

$$|H_{\alpha,z}(w)| \leq \delta_4 + \delta_1\delta_3^2 + \delta_2\delta_3^3.$$

By interval arithmetics we obtain that the first two inequalities are fulfilled if $\delta_1 = 0.76$ and $\delta_2 = 0.52$. Now, if we choose $\delta_3 = \frac{1}{3}$ and $\delta_4 = \frac{1}{5}$ we obtain that $H_{\alpha,z} : \overline{\mathrm{B}_{1/3}} \to \overline{\mathrm{B}_{1/3}}$ is a contraction. Hence for all fixed $z \in \overline{\mathrm{B}_{1/5}}$ there exists exactly one $w = w(z) \in \overline{\mathrm{B}_{1/3}}$ such that $H_{\alpha,z}(w(z)) = w(z)$, that is $h_\alpha(w(z)) = z$. This means that h_α^{-1} can be defined on $\overline{\mathrm{B}_{1/5}}$.

The obtained estimations on h_α are going to be used in the sequel. These were

$$|h_{20}(\alpha)|/2 + |h_{11}(\alpha)| + |h_{02}(\alpha)|/2 < 0.76,$$

$$|h_{30}(\alpha)|/6 + |h_{12}(\alpha)|/2 + |h_{03}(\alpha)|/6 < 0.52$$

and in particular

$$|w| - 0.76|w|^2 - 0.52|w|^3 < |h_\alpha(w)| < |w| + 0.76|w|^2 + 0.52|w|^3 \tag{4.11}$$

for all $\alpha \in [0.999, 1]$.

Let

$$H = \{(\alpha, w) \in \mathrm{R} \times \mathrm{C} : \ \alpha \in [0.999, 1], \ w \neq 0 \text{ and } |w| < 1/20\}.$$

In the sequel, we shall always assume that $(\alpha, w) \in H$. From this assumption and inequality (4.11) we readily get that $|w| < 1.05|h_\alpha(w)|$ from which we get in particular that $|z| = |h_\alpha(w)| < 1/19$.

Step 3: A region of attraction for the fixed point 0 of system (4.9)

Our goal now is to give $\varepsilon_0 \in (0, 1/20]$, independent of α such that for every $\alpha \in [0.999, 1]$, if $0 < |w| < \varepsilon_0$, then $\left|h_\alpha^{-1}\left(G\left(h_\alpha(w), \overline{h_\alpha(w)}, \alpha\right)\right)\right| < |w|$ holds which guarantees that $\mathrm{B}_{\varepsilon_0}$ belongs to the basin of attraction of the fixed point 0 of the discrete dynamical system generated by (4.9). For this, we turn our attention to the estimation of function R_2 in (4.9).

Step 3.1: Estimation of g

First, we go back to (4.4). Let us consider

$$g(z, \overline{z}, \alpha) = \sum_{k+l=2,3} \frac{g_{kl}(\alpha)}{k!\, l!} z^k \overline{z}^l + R_1(z, \overline{z}, \alpha).$$

The explicit formulae for $g_{20}(\alpha), \ldots, g_{03}(\alpha)$ can be found in equations (A.2)–(A.8). By interval arithmetics, one may obtain that for all $\alpha \in [0.999, 1]$,

$$\frac{|g_{20}(\alpha)|}{2} + |g_{11}(\alpha)| + \frac{|g_{02}(\alpha)|}{2} = \frac{1 - \alpha + \sqrt{\alpha(2+\alpha)}}{\sqrt{\alpha(-1+4\alpha)}} < 1.01, \tag{4.12}$$

$$\frac{|g_{30}(\alpha)|}{6} + \frac{|g_{21}(\alpha)|}{2} + \frac{|g_{12}(\alpha)|}{2} + \frac{|g_{03}(\alpha)|}{6} = \sqrt{\frac{(6+\alpha)}{9\alpha(4\alpha-1)}} + \sqrt{\frac{2+(\alpha-2)\alpha}{\alpha^2(4\alpha-1)}} < 1.09. \tag{4.13}$$

We also have that $R_1(z, \overline{z}, \alpha) = \sum_{k+l=4} \frac{g_{kl}(\alpha)}{k!l!} z^k \overline{z}^l + \tilde{R}_1(z, \overline{z}, \alpha)$, where $\tilde{R}_1(z, \overline{z}, \alpha) = O(|z|^5)$ for fixed α. For explicit formulae of the fourth order coefficients see equations (A.9)-(A.13) in the appendix. It is clear from equations (4.1), (4.3) and (4.4) that

$$\tilde{R}_1(z, \overline{z}, \alpha) = \frac{\sqrt{4\alpha-1} - i}{2\sqrt{4\alpha-1}} \left(v \sum_{k=4}^{\infty} \frac{(-u)^k}{k!} + \alpha \sum_{k=5}^{\infty} \frac{(-u)^k}{k!} \right),$$

where u and v are defined by equation (4.3). Using $0 < |z| < 1/19$ and (4.5) we have that $|u| < 1/8$, $|v| < 1/8$ and obtain

$$\begin{aligned}\left|\tilde{R}_1(z,\overline{z},\alpha)\right| &\leq \sqrt{\frac{\alpha}{4\alpha-1}}\left(|v|\frac{e^{1/8}}{4!}|u|^4+\alpha\frac{e^{1/8}}{5!}|u|^5\right)\\ &< \sqrt{\frac{\alpha}{4\alpha-1}}\frac{8}{7}\left(2|z|\frac{16}{24\alpha^2}+\frac{32}{\alpha^{3/2}120}|z|\right)|z|^4\\ &\leq \sqrt{\frac{\alpha}{4\alpha-1}}\frac{8}{7}\left(\frac{4}{57\alpha^2}+\frac{4}{285\alpha^2}\right)|z|^4=\frac{64}{665}\sqrt{\frac{1}{(4\alpha-1)\alpha^3}}|z|^4.\end{aligned}$$

Now for all $(\alpha,w)\in H$ with $z=h_\alpha(w)$ we get that

$$\begin{aligned}|R_1(z,\overline{z},\alpha)| &\leq \sum_{k+l=4}\left|\frac{g_{kl}(\alpha)}{k!\,l!}z^k\overline{z}^l\right|+\left|\tilde{R}_1(z,\overline{z},\alpha)\right|\\ &= \frac{6-3\alpha+\sqrt{\alpha(12+\alpha)}+4\sqrt{3+\alpha^2}}{12\sqrt{\alpha^3(-1+4\alpha)}}+\left|\tilde{R}_1(z,\overline{z},\alpha)\right|\\ &< \frac{4758-1995\alpha+665\sqrt{\alpha(12+\alpha)}+2660\sqrt{3+\alpha^2}}{7980\sqrt{\alpha^3(-1+4\alpha)}}.\end{aligned}$$

By interval arithmetics we obtain for all $\alpha\in[0.999,1]$ that

$$|R_1(z,\overline{z},\alpha)| < 0.76|z|^4. \tag{4.14}$$

Step 3.2: Estimation of $h_\alpha^{-1}(z)$

Let us recall that $z=h_\alpha(w)$ and

$$w=h_\alpha^{-1}(z)=h_{inv,\alpha}(z)+R_3(z,\alpha).$$

As $h_{inv,\alpha}(z)$ is a polynomial of z and $\overline{z}$ of degree four (see formulae (A.14)–(A.20)), we denote the coefficient corresponding to $z^k\overline{z}^l$ by $h_{inv}^{kl}(\alpha)$. Calculating these coefficients and using interval arithmetics, we obtain that for all $\alpha\in[0.999,1]$,

$$\sum_{k+l=2}\left|h_{inv}^{kl}\right| < 0.76, \qquad \sum_{k+l=3}\left|h_{inv}^{kl}\right| < 1.06 \quad \text{and} \quad \sum_{k+l=4}\left|h_{inv}^{kl}\right| < 1.39. \tag{4.15}$$

See formulae (A.21) – (A.27). Let us recall that for all $(\alpha,w)\in H$, $|w| < 1.05|h_\alpha(w)|$ holds. Now, for the fifth and higher order terms in R_3, first we give an estimation of type $|R_3(h_\alpha(w),\alpha)| < K_3|w|^4$ and then we get that $|R_3(z,\alpha)| < K_3 1.05^4|z|^4$, with $z=h_\alpha(w)$.

From the definition of $h_{inv,\alpha}$, it follows that $R_3(h_\alpha(w),\alpha)=w-h_{inv,\alpha}(h_\alpha(w))$ is a polynomial of w and $\overline{w}$ and it has only fifth and higher order terms. Let $r_3^{kl}(\alpha)$ denote the coefficient of $R_3(h_\alpha(w),\alpha)$ corresponding to $w^k\overline{w}^l$. We use a bit rougher estimation for $\left|\sum_{5\leq k+l} r_3^{k+l}(\alpha)w^k\overline{w}^l\right|$. Namely, first we give the estimations

$$\begin{aligned}&|h_{20}(\alpha)| < 1.01; \quad |h_{11}(\alpha)| < 0.001; \quad |h_{02}(\alpha)| < 0.51\\ &|h_{30}(\alpha)| < 0.89; \quad |h_{12}(\alpha)| < 0.45; \quad |h_{03}(\alpha)| < 0.89\end{aligned} \tag{4.16}$$

for all $\alpha\in[0.999,1]$. Now, in $R_3(h_\alpha(w),\alpha)$ we replace w and $\overline{w}$ by $|w|$, $h_{nm}(\alpha)$ by the estimates given in (4.16) (for $2\leq n+m\leq 3$, $(n,m)\neq(2,1)$), and then we turn every $-$ sign into $+$ to get a real polynomial $\hat{R}_3(|w|)$, with nonnegative coefficients

$\hat{r}_3^k$ (independent of α) corresponding to $|w|^k$. If we use $0 < |w| < 1/20$, then we get that

$$\sum_{5\leq k+l\leq 12} |r_3^{kl}(\alpha)||w|^{k+l} < \sum_{5\leq k\leq 12} \hat{r}_3^k|w|^k < \sum_{5\leq k\leq 12} \hat{r}_3^k|w|^4\left(\frac{1}{20}\right)^{k-4} < 1.02|w|^4. \quad (4.17)$$

This implies that

$$|R_3(z,\alpha)| < 1.05^4 \cdot 1.02|z|^4 < 1.24|z|^4 \quad (4.18)$$

for all $(\alpha, w) \in H$, where $z = h_\alpha(w)$. It is now clear that

$$h_{inv}(z) < |z| + 0.76|z|^2 + 1.06|z|^3 + 2.63|z|^4 \quad (4.19)$$

holds for all $(\alpha, w) \in H$ with $z = h_\alpha(w)$.

Step 3.3: Estimating R_2

Now, we are able to give an estimation of R_2 in (4.9). First, according to our previous estimations, let us define the following real polynomials.

$$h^{max}(s) = s + 0.76s^2 + 0.52s^3,$$

$$G^{max}(s) = s + 1.01s^2 + 1.09s^3 + 0.76s^4,$$

$$h^{max}_{inv}(s) = s + 0.76s^2 + 1.06s^3 + 2.63s^4$$

and

$$Q(s) = \sum_{k=1}^{48} q_k s^k = h^{max}_{inv} \circ G^{max} \circ h^{max}(s).$$

By now, it is obvious that for $(\alpha, w) \in H$, $|R_2(w, \overline{w}, \alpha)| < \sum_{k=4}^{48} q_k|w|^4\left(\frac{1}{20}\right)^{k-4}$ holds, which leads to $|R_2(w, \overline{w}, \alpha)| < 23.9|w|^4$. For our purposes this approach is too rough – the obtained neighbourhood would be too small ($\approx$ K(α^*; 1/80)) and we could not prove that every trajectory enters it eventually. Hence we have to be as sharp as we can in our estimations to obtain as large neighbourhood as possible. So, instead of only estimating these functions separately, let us consider the composite function $h_\alpha^{-1} \circ G_\alpha \circ h_\alpha$, where $G_\alpha(z)$ denotes $G(z, \overline{z}, \alpha)$. Now, we are only interested in the fourth and higher order terms. Since h_α is a known function and we also know functions G_α and h_α^{-1} up to fourth order terms, hence we are able to compute the fourth order coefficients of $h_\alpha^{-1} \circ G_\alpha \circ h_\alpha$, denoted by $r_2^{kl}(\alpha)$, where $k + l = 4$. By interval arithmetics we show that

$$\sum_{k+l=4} |r_2^{kl}(\alpha)| < 1.02. \quad (4.20)$$

See equations (A.28) – (A.32) for the formulae. Using this we infer that

$$|R_2(w, \overline{w}, \alpha)| < 1.02|w|^4 + \sum_{k=5}^{48} q_k|w|^4\left(\frac{1}{20}\right)^{k-4} < 4.6|w|^4. \quad (4.21)$$

for all $(\alpha, w) \in H$.

Now, we turn our attention to $c_1(\alpha)$ in (4.9). The formula for $c_1(\alpha)$ can be found in (A.33).

Step 3.4: Set $\mathrm{B}_{1/20}$ belongs to the basin of attraction of the fixed point 0 of (4.9)

According to [14] and using inequality (4.21) we get the following

$$\begin{aligned}&\left|\mu(\alpha)w + c_1(\alpha)w^2\overline{w} + R_2(w,\overline{w},\alpha)\right| \le |w|\left|\mu(\alpha) + c_1(\alpha)|w|^2\right| + |R_2(w,\overline{w},\alpha)| \\ &= |w|\left||\mu(\alpha)|+d(\alpha)|w|^2\right| + |R_2(w,\overline{w},\alpha)| < |w|\left|\sqrt{\alpha} + d(\alpha)|w|^2\right| + 4.6|w|^4,\end{aligned} \tag{4.22}$$

for all $(\alpha, w) \in H$, where $d(\alpha) = \frac{|\mu(\alpha)|}{\mu(\alpha)} c_1(\alpha)$. Let

$$R_4(w,\alpha) = \left|\sqrt{\alpha} + d(\alpha)|w|^2\right| - (\sqrt{\alpha} + a(\alpha)|w|^2),$$

where $a(\alpha)$ denotes the real part of $d(\alpha)$.

In the following we are going to prove that $|R_4(w,\alpha)| < 0.1|w|^3$ holds for all $(\alpha, w) \in H$. First of all, the formula for function a is the following

$$a(\alpha) = \frac{4 + \alpha\left(-10 + \alpha + \alpha^2\right)}{4\alpha^{3/2}(-1 + \alpha(4+\alpha))}. \tag{4.23}$$

It can be readily shown that

$$-1 < a(\alpha) \le -\frac{1}{4} \tag{4.24}$$

holds for all $\alpha \in [0.999, 1]$. Using the definition of d and a, the estimation above and the assumption $(\alpha, w) \in H$ we get the following.

$$\begin{aligned}&\left|\left|\sqrt{\alpha} + d(\alpha)|w|^2\right| - (\sqrt{\alpha} + a(\alpha)|w|^2)\right| \\ &= \left|\sqrt{\alpha + 2\sqrt{\alpha}a(\alpha)|w|^2 + |d(\alpha)|^2|w|^4} - (\sqrt{\alpha} + a(\alpha)|w|^2)\right| \\ &= \left|\frac{(|d(\alpha)|^2 - (a(\alpha))^2)|w|^4}{\sqrt{\alpha + 2\sqrt{\alpha}a(\alpha)|w|^2 + |d(\alpha)|^2|w|^4} + \sqrt{\alpha} + a(\alpha)|w|^2}\right| \\ &\le \frac{(|d(\alpha)|^2 - (a(\alpha))^2)|w|^4}{\sqrt{400\alpha|w|^2 + 2\sqrt{\alpha}a(\alpha)|w|^2} + 20\sqrt{\alpha}|w| + a(\alpha)|w|} \le \frac{(|d(\alpha)|^2 - (a(\alpha))^2)}{\sqrt{400\alpha - 2} + 20\alpha - 1}|w|^3 \\ &\le \frac{(|d(\alpha)|^2 - (a(\alpha))^2)}{19\sqrt{\alpha} + 18\alpha} \cdot |w|^3 \le \frac{(|d(\alpha)|^2 - (a(\alpha))^2)}{37\alpha} \cdot |w|^3 \\ &= \frac{\left(\alpha^4 + 3\alpha^3 - 12\alpha^2 + 20\alpha - 4\right)^2}{16 \cdot 37\alpha^6(4\alpha - 1)(\alpha^2 + 4\alpha - 1)^2} \cdot |w|^3.\end{aligned}$$

From this last formula it can be proven that $|R_4(w,\alpha)| < 0.1|w|^3$. From inequalities (4.22), (4.24) and from the above estimate we obtain that for all $(\alpha, w) \in H$ the inequalities

$$\begin{aligned}&\left|\mu(\alpha)w + c_1(\alpha)w^2\overline{w} + R_2(w,\overline{w},\alpha)\right| < |w|\left(\sqrt{\alpha} + a(\alpha)|w|^2 + R_4(w,\alpha)\right) + 4.6|w|^4 \\ &< \sqrt{\alpha}|w| - \frac{1}{4}|w|^3 + 4.7|w|^4 < |w|\left(1 - \frac{1}{4}|w|^2(1 - 4 \cdot 4.7|w|)\right) < |w|\end{aligned} \tag{4.25}$$

hold provided that $|w| < \frac{1}{20} = \min\left\{\frac{1}{20}, \frac{1}{4 \cdot 4.7}\right\}$. This means that for all $\alpha \in [0.999, 1]$ the set $\mathrm{B}_{1/20}$ belongs to the basin of attraction of fixed point 0 of the map given by (4.9).

Step 4: A region of attraction of the fixed point α^* of F_α

Now, we only have to show that for any $(x, y) \in K(\alpha^*; 1/22)$, after our transformations, $|w|< \varepsilon_0 = \frac{1}{20}$ holds. First, we need ε_G such that for $|z|< \varepsilon_G$, $|w|= |h_\alpha^{-1}(z)|< \frac{1}{20}$ holds for all $\alpha \in [0.999, 1]$. From (4.11) we obtain that $\varepsilon_G = \frac{1}{21}$ is an appropriate choice. Now, if $|u|< \varepsilon$, $|v|< \varepsilon$, then from (4.7) we get that

$$|z(u, v, \alpha)|\leq \varepsilon\sqrt{\frac{\alpha(2+\alpha)}{4\alpha-1}}.$$

Thus for

$$\varepsilon < \min_{\alpha\in[0.999,1]} \sqrt{\frac{4\alpha-1}{\alpha(2+\alpha)}} \cdot \frac{1}{21} \tag{4.26}$$

we get that if $|u|< \varepsilon$ and $|v|< \varepsilon$ then $|z(u, v, \varepsilon)|< \frac{1}{21}$. It is easily shown that $\varepsilon = \frac{1}{22}$ fulfils inequality (4.26) which proves Proposition 4.3. □

PROPOSITION 4.4. *Fixed point α^* of map F_α is locally asymptotically stable if and only if $\alpha \in (0, 1]$.*

Proof. By linearisation one readily gets that α^* is locally asymptotically stable if $\alpha \in (0, 1)$, and unstable if $\alpha \in (1, \infty)$. Transforming map F_α to the form (4.9) and using inequality (4.25) yields that $(1, 1)$ is a locally asymptotically stable fixed point of F_1. □

5. Graph representations

Covers and graph representations

From a graph representation of the given map it is posible to derive properties of the generated dynamical system. These techniques appeared in many articles, in both rigorous and non-rigorous computations for maps by Hohmann, Dellnitz, Junge, Rumpf [7, 8], Galias [10], Luzzatto and Pilarczyk [22], and computations for the time evolution of a continuous system with a given timestep by Wilczak [34]. In this section we introduce two applications. One to enclose the non-wandering points and the other one to estimate the basin of attraction.

Both methods (Algorithms 1 and 2) combined with local estimations of the type of Section 4 at the critical points, can be theoretically applied to prove different dynamical properties. On the one hand these algorithms are included for possible future references, on the other hand certain elements of these algorithms proved to be useful for the map F_α in Section 6. In particular, the correctness of Algorithm 1 is crucial in Section 6.

Enclosure of the non-wandering points

Instead of directly studying the map (2.1), we will analyse different graph representations of f. Let $K \subseteq \mathcal{D}_f$ be a compact set satisfying $f(K) \subseteq K$, $\mathcal{P}$ a partition of K, and $\mathcal{G} \propto (f, K, \mathcal{P})$. We shall use the algorithm from [10] to enclose the non-wandering points in K.

Algorithm 1 Enclosure of non-wandering points

function EnclosureNonW(f, K, δ_0) $\triangleright$ f is the function, K is the starting region.
  $k \leftarrow 0$
  $\mathcal{V}_0 \leftarrow \text{Partition}(K, \delta_0)$ $\triangleright$ $\mathcal{V}_0$ is a partition of K, $\text{diam}(\mathcal{V}_0) \leq \delta_0$.
  loop
    $\mathcal{E}_k \leftarrow \text{Transitions}(\mathcal{V}_k, f)$ $\triangleright$ Extra edges may occur.
    $\mathcal{G}_k \leftarrow \text{GRAPH}(\mathcal{V}_k, \mathcal{E}_k)$ $\triangleright$ $\mathcal{G}_k \propto (f, |\mathcal{V}_k|, \mathcal{V}_k)$
    for all $v \in \mathcal{V}_k$ **do**
      if v is **not** in a directed cycle **then**
        remove v from $\mathcal{G}_k$
      end if
    end for
    if $\mathcal{G}_k$ is empty **then**
      return $\emptyset$ $\triangleright$ $\text{NonW}(f; K) = \emptyset$.
    else
      $\delta_{k+1} \leftarrow \delta_k / 2$
      $\mathcal{V}_{k+1} \leftarrow \text{Partition}(|\mathcal{V}_k|, \delta_{k+1})$ $\triangleright$ $\text{diam}(\mathcal{V}_{k+1}) \leq \delta_{k+1}$
      $k \leftarrow k + 1$
    end if
  end loop
end function

This algorithm for enclosing non-wandering points appeared in [10] without a full proof. We will give a proof here not just for the sake of completeness, but because some steps are non-trivial. We need to take special care when a non-wandering point is on the boundary of a partition element.

For any $x \in K$, define

$$\tilde{\mathcal{P}}_x := \{u \in \mathcal{P} : x \in u\}.$$

Since we are working with finite covers, both $\mathcal{P}$ and $\tilde{\mathcal{P}}_x$ are finite.

Lemma 5.1. *For every $q \in K$, there is an $\eta_q > 0$ such that for any $u \in \mathcal{P}$, if $u \cap B(q; \eta_q) \neq \emptyset$, then $q \in u$.*

Proof. Since we are working with finite partitions, the proof is straightforward. If $\tilde{\mathcal{P}}_q = \mathcal{P}$ then any positive number satisfies the condition. Otherwise define

$$\eta := \min_{u \in \mathcal{P} \setminus \tilde{\mathcal{P}}_q} d(q, u).$$

This is positive, since $\mathcal{P} \setminus \tilde{\mathcal{P}}_q$ is a finite set and for $u \in \mathcal{P} \setminus \tilde{\mathcal{P}}_q$, u is compact and $q \notin u$. Now we can take any number for η_q from $(0, \eta)$. □

Lemma 5.2. *For every $q \in \text{NonW}(f; K)$, there are $u, v \in \mathcal{P}$ such that*

(1) $q \in u \cap v$
(2) for every $\varepsilon > 0$, there are $x = x(\varepsilon), N = N(x(\varepsilon), \varepsilon) \in \mathrm{N}$ such that $\lim_{\varepsilon \to 0} N(x(\varepsilon), \varepsilon) = \infty$, $x \in B(q; \varepsilon)$, $f^N(x) \in B(q; \varepsilon)$, $x \in u$ and $f^N(x) \in v$.

Proof. Consider a decreasing sequence of positive numbers $\{\varepsilon_k\}_{k=0}^{\infty}$ with $\lim_{k\to\infty} \varepsilon_k = 0$ and $\varepsilon_0 < \eta_q$. Since q is non-wandering,

$$\exists N_k \geq k : f^{N_k}(\mathrm{B}(q;\varepsilon_k) \cap \mathcal{D}_f) \cap \mathrm{B}(q;\varepsilon_k) \cap \mathcal{D}_f \neq \emptyset.$$

Therefore

$$\exists x_k \in \mathrm{B}(q;\varepsilon_k) : f^{N_k}(x_k) \in \mathrm{B}(q;\varepsilon_k).$$

Since $\tilde{\mathcal{P}}_q$ is finite, we can pick $u \in \tilde{\mathcal{P}}_q$ such that it contains infinitely many x_k-s. We may assume, by switching to an appropriate subsequence and reindexing, that $\forall k : x_k \in u$. It is now possible to choose – because of finiteness again – an element of the partition $v \in \tilde{\mathcal{P}}_q$ such that it contains infinitely many of $f^{N_k}(x_k)$. Switching again to the subsequence, the required conditions are now satisfied. □

Remark 1. If there exists $u \in \mathcal{P}$ such that $q \in \mathrm{int}(u)$ then it follows from the definition, that u and $v = u$ is a good choice. By $\mathrm{int}(u)$ we mean the interior of the set u.

Lemma 5.3. *For every $q \in \mathrm{NonW}\,(f;K)\backslash\mathrm{Per}\,(f;K)$ there is an element $u \in \tilde{\mathcal{P}}_q$ and a family of directed cycles $\Upsilon_{cycle}(V)$ in $\mathcal{G}$ such that $u \in V$, and the family encloses infinitely many trajectories in K of the form $\left(x_k, f(x_k), f^2(x_k), \ldots, f^{N_k}(x_k)\right)$ with*

$$\lim_{k\to\infty} N_k = \infty \quad \textit{and} \quad \lim_{k\to\infty} x_k = \lim_{k\to\infty} f^{N_k}(x_k) = q.$$

Proof. If there is a $u \in \mathcal{P}$ such that $q \in \mathrm{int}(u)$, then since $q \in \mathrm{NonW}\,(f;K)$, there are infinitely many such trajectories, and each one of them is enclosed by a directed cycle that passes through u. Since there is only finite number of families of directed cycles, therefore we can pick one family $\Upsilon_{cycle}(V)$ that encloses infinitely many trajectories, and $u \in V$.

If such u cannot be found, then we will do the following: for a series of positive numbers $\lim_{k\to\infty} \varepsilon_k = 0$, with the use of Lemma 5.2, we obtain the sets $u, v \in \tilde{P}_q$, the points $x_k \in u$, $f^{N_k}(x_k) \in v$ with $N_k \to \infty$, and that $\lim_{k\to\infty} x_k = \lim_{k\to\infty} f^{N_k}(x_k) = q$.

Since $\{f^{N_k-1}(x_k)\} \subseteq K$, and K is compact, we may assume that this sequence converges to a point $q' \in K$, that is $f^{N_k-1}(x_k) \to q'$. From the continuity of f it follows that

$$f(q') = \lim_{k\to\infty} f^{N_k}(x_k) = q.$$

Since $\tilde{\mathcal{P}}_{q'}$ is finite, and because of Lemma 5.1, infinitely many of the points $f^{N_k-1}(x_k)$ are inside one of its elements, without loss of generality we may assume, that

$$\forall k \in \mathrm{N} : f^{N_k-1}(x_k) \in v' \in \tilde{\mathcal{P}}_{q'}.$$

With similar argument we may assume that

$$\forall k \in \mathrm{N} : f(x_k) \in u' \in \tilde{\mathcal{P}}_{f(q)}.$$

We are thus considering directed paths in the graph of the following form:

$$u \to u' \to \cdots \to v' \to v.$$

Since there is only finite number of families of directed paths in $\mathcal{G}$ and infinitely many trajectories of the desired property are enclosed by them, there must be at least one such family $\Upsilon_{path}(V)$ that encloses infinitely many trajectories itself. The directed

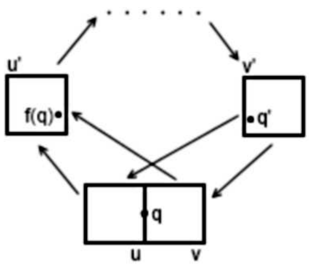

Figure 5. The directed paths close to q

paths $v' \to v \to u'$ and $v' \to u \to u'$ are present in $\mathcal{G}$, since $f(q') = q$. The situation is depicted in Figure 5.

We can conclude our argument that the set V is a vertex set for a family of directed cycles as well, since

$$u \to u' \to \cdots \text{ vertices from } V \cdots \to v' \to v \to u' \to \cdots \text{ vertices from } V \cdots \to v' \to u,$$

is a directed cycle. Thus $\Upsilon_{cycle}(V)$ is well defined and encloses infinitely many trajectories of the desired type. □

Now we are ready to prove the correctness of Algorithm 1.

THEOREM 5.4. *If* $\mathrm{NonW}(f;K) \neq \emptyset$*, then Algorithm 1 will never stop and* $\mathrm{NonW}(f;K) \subseteq \mathcal{V}_k$ *is satisfied for every* k*.*

Proof. Assume that $q \in \mathrm{NonW}(f;K)$. If q is a periodic point with period $N+1$, then its orbit $\{q, f(q), \dots, f^N(q)\}$ is enclosed in a directed cycle and none of these vertices is removed in the first step. Therefore q is a periodic point of f restricted to $|\mathcal{V}_1|$ as well. Repeating the same argument gives that q is always enclosed.

If $q \in \mathrm{NonW}(f;K)$, but it is not periodic, then we obtain a family of directed cycles $\Upsilon_{cycle}(V)$ from Lemma 5.3, that encloses infinitely many trajectories of the type mentioned before. The vertices in V are not removed, since they are in a directed cycle, thus the enclosures for all of these trajectories are preserved. Therefore q is a non-wandering point of f restricted to $|\mathcal{V}_1|$ as well. Since $|\mathcal{V}_k|$ is compact, we can repeat the same argument and obtain families of directed cycles $\Upsilon_{cycle}(V_k)$, that enclose infinitely many of these trajectories, that ensure that q stays a non-wandering point when we restrict f to $|\mathcal{V}_k|$ and $q \in |\mathcal{V}_k|$ is satisfied. □

Remark 2. The theorem does not imply that at one step each vertex containing q is kept.

Remark 3. We decide whether a vertex is in a directed cycle by decomposing the graph into strongly connected components. The vertices that form a component by themselves and have no self edges are the ones that are not in directed cycles. To find this decomposition, we will use the algorithm of Tarjan [30], that runs in linear time.

Inner enclosure of the basin of attraction

Assume now that $K \subseteq \mathcal{D}_f$ is a compact set such that $f(K) \subseteq K$. When analysing the forward orbits starting from K we can work with $f|_K$. Assume that there is an attracting set $\mathcal{O} \subseteq K$ for $f|_K$ and a neighbourhood U such that $\mathcal{O} \subseteq U \subseteq K$ and U is inside the basin of attraction of $\mathcal{O}$. We want to find a set B in the basin of attraction of $\mathcal{O}$ so that $U \subset B$. We will use the algorithm from [10]:

Algorithm 2 Inner enclosure of the basin of attraction

```
function BASIN_OF_ATTRACTION(f, K, δ_0; U)          ▷ U is attracted by O.
    k ← 0
    W ← ∅                 ▷ We collect the vertices in the basin of attraction into W.
    V_0 ← Partition(K, δ_0)              ▷ V_0 is a partition of K, diam(V_0) ≤ δ_0.
    loop
        E_k ← Transitions(V_k ∪ W, f)                     ▷ Extra edges may occur.
        G_k ← GRAPH(V_k ∪ W, E_k)                ▷ G_k ∝ (f, |V_k ∪ W|, V_k ∪ W)
        repeat
            ready ← TRUE
            for all v ∈ V_k do
                if v ⊆ U ∪ |W| or f(v) ⊆ U ∪ |W| then
                    move v from V_k to W                  ▷ v is attracted by O.
                    ready ← FALSE
                end if
            end for
        until ready  ▷ The remaining vertices are not attracted at this resolution
        if STOP(k, V_k, W, δ_k) then                  ▷ Some stopping condition
            return W
        end if
        δ_{k+1} ← δ_k/2
        V_{k+1} ← Partition(|V_k|, δ_{k+1})          ▷ V_{k+1} diam(V_{k+1}) ≤ δ_{k+1}
        k ← k + 1
    end loop
end function
```

In Algorithm 2, we move those partition elements into W, that lie inside or are mapped into the initial attracted neighbourhood, or the other elements in W. Then we refine our remaining partition and continue our procedure with a new one, that has a diameter half as big as before. Since at the beginning W was empty, it will only contain sets that are contained in the basin of attraction of $\mathcal{O}$. Thus after each cycle, $|W|$ is an inner enclosure of the basin of attraction of $\mathcal{O}$. We stop our iteration when a stopping condition is satisfied, for example $\delta_k < \Delta$, where Δ is a small positive number given in advance.

6. The completion of the proof of Theorem 1.1

Consider now a parameter value α for F_α such that $0.5 \leq \alpha \leq 1$. In order to prove that the fixed point α^* is globally attracting, we need the following observation: given any starting point (x_0, y_0), the accumulation points of the sequence $\left(F_\alpha^k(x_0, y_0)\right)_{k=0}^{\infty}$ are non-wandering points of F_α. We want to show that the only non-wandering point of F_α in R_+^2 is the fixed point α^*. We know from Proposition 3.2 that it is enough to look for points in $S_i^{(\alpha)}$, $i \in \mathrm{N}_0$. Thus our goal is to prove that

(1) $S_i^{(\alpha)}$ is entirely in the basin of attraction of α^*,
(2) or equivalently, $S_i^{(\alpha)}$ contains exactly one non-wandering point, and that is α^*.

Our strategy is to divide the parameter range $[0.5, 1] = [0.5, 0.875] \cup [0.875, 0.999] \cup [0.999, 1]$ into small subintervals $[\alpha]$. The diameter of these subintervals will vary between $10^{-3}, 10^{-4}$ and 10^{-5} in practice. For one small parameter interval $[\alpha]$ we shall follow these steps:

(1) Let $i_0 \geq 1$ be the smallest integer such that $|s_{2i_0}^{([\alpha])} - s_{2i_0+2}^{([\alpha])}| + |s_{2i_0+1}^{([\alpha])} - s_{2i_0+3}^{([\alpha])}| < 10^{-9}$.
(2) The function `ConstructRegion`($[\alpha]$) returns, using Proposition 3.3 and equation (3.4), a rigorous enclosure $[S]$ such that
$$\bigcup_{\alpha \in [\alpha]} S_{i_0}^{(\alpha)} \subseteq [S].$$
(3) Using Propositions 4.1, 4.2 and 4.3, the function `FindAttractionDomain`($[\alpha]$) returns an $\varepsilon_0 > 0$ such that $\mathrm{K}(\alpha; \varepsilon_0)$ is contained in the basin of attraction of α^* for every $\alpha \in [\alpha]$.
(4) Enclose rigorously $\cup_{\alpha \in [\alpha]} \mathrm{NonW}\,(F_\alpha; [S]) \setminus \{\alpha^*\})$ by removing parts of $[S]$ that do not contain non-wandering points of F_α or are in the basin of attraction of the fixed point α^* for every $\alpha \in [\alpha]$. We do this by simultaneously checking the criteria from line 8 of Algorithm 1 and line 11 of Algorithm 2. If we obtain an empty enclosure at some step, then we have proved that the fixed point in the given parameter region is globally attracting.

We sum it in the following algorithm:

Algorithm 3 Proving the global stability of α^* for the Ricker-map

procedure RICKER($[\alpha], \delta$)
  $[S] \leftarrow$ `ConstructRegion`($[\alpha]$) ▷ from Proposition 3.3 and (3.4)
  $\varepsilon_0 \leftarrow$ `FindAttractionDomain`($[\alpha]$) ▷ from Propositions 4.1, 4.2 and 4.3
  $[U] \leftarrow \mathrm{K}([\alpha^*]; \varepsilon_0 - (\alpha^+ - \alpha^-))$
  $\mathcal{V} \leftarrow \mathrm{Partition}([S], \delta)$ ▷ $\mathcal{V}$ is a partition of $[S]$, $\mathrm{diam}(\mathcal{V}) \leq \delta$
  repeat
    $\mathcal{E} \leftarrow \mathrm{Transitions}(\mathcal{V}, F_{[\alpha]})$ ▷ Extra edges may occur.
    $\mathcal{G} \leftarrow \mathrm{GRAPH}(\mathcal{V}, \mathcal{E})$ ▷ $\mathcal{G} \propto (F_{[\alpha]}, |\mathcal{V}|, \mathcal{V})$
    $T \leftarrow \{v : v \text{ is in a directed cycle }\}$ ▷ with the use of Tarjan's algorithm
    for all $v \in \mathcal{V}$ **do**
      if $v \notin T$ **or** $v \subseteq [U]$ **or** $F_{[\alpha]}(v) \subseteq [U]$ **then**
        remove v from $\mathcal{G}$
      end if
    end for

```
15:         δ ← δ/2
16:         𝒱 ← Partition(|𝒱|, δ)
17:     until |𝒱|= ∅
18: end procedure
```

We know that `FindAttractionDomain`($[\alpha]$) returns an $\varepsilon_0 > 0$ such that for every $\alpha \in [\alpha] = [\alpha^-, \alpha^+]$, the set $\mathrm{K}(\alpha^*; \varepsilon_0)$ is in the basin of attraction of α^*. Assume that $\alpha^+ - \alpha^- < \varepsilon_0$ and let

$$\varepsilon = \varepsilon_0 - (\alpha^+ - \alpha^-).$$

Now $\varepsilon > 0$ and the set $\mathrm{K}([\alpha^*]; \varepsilon)$ is in the basin of attraction of α^* for every $\alpha \in [\alpha]$. Observe that, using subintervals with $\alpha^+ - \alpha^- \leq 10^{-3}$ and the ε_0 obtained from Propositions 4.1, 4.2 and 4.3, $\alpha^+ - \alpha^- < \varepsilon_0$ is satisfied.

After each step in the main cycle in Algorithm 3, $|\mathcal{V}|$ is a rigorous enclosure of all the non-wandering points of F_α in $[S] \setminus \{\alpha^*\}$ for every $\alpha \in [\alpha]$. This is easy to see since vertices are removed for two possible reasons which are both checked in line 11 of Algorithm 3. First if $v \notin T$, then v does not contain non-wandering points for any F_α, $\alpha \in [\alpha]$ as we have seen in the proof of the correctness of Algorithm 1. Second if $v \subseteq [U]$ or $F_{[\alpha]}(v) \subseteq [U]$ that is v is inside or mapped into the small neighbourhood attracted by every fixed point. Note that if a vertex is inside the basin of attraction of a fixed point α^*, then it cannot contain any other non-wandering point of F_α, not even on the boundary. This is a similar criterion to what we have used in line 11 of Algorithm 2. The difference is that now we remove these vertices, consequently we do not have to collect them into a list. If the procedure ends in finite time, then we have established, that there are no other non-wandering points in $[S]$, thus the fixed point is globally attracting for all $\alpha \in [\alpha]$. We state this as

Proposition 6.1. *If Algorithm 3 ends in finite time with input parameters* $([\alpha], 10^{-1})$*, that is, after finite number of steps,* $|\mathcal{V}| = \emptyset$ *is satisfied, then* α^* *is a globally attracting fixed point of the two dimensional Ricker-map* F_α *for every* $\alpha \in [\alpha]$.

We implemented our program in C++, using the CAPD Library [5] for rigorous computations, and the Boost Graph Library [29] for handling the directed graphs. The recursion number in Tarjan's algorithm was very high, therefore we converted it into a sequential program, using virtual stack structures from the Standard Library in order to avoid overflows. Instead of simulating the Ricker-map itself, we used its third iterate, the formula is still compact enough not to cause big overestimation in interval arithmetics and it considerably speeds up the calculations.

As an example, we ran our program for the parameter slice $[0.9, 0.90001]$, with $\delta = 10^{-1}$ as the initial diameter for the partition. We show the evolution of the enclosure during the first 8 iterations on Figure 6. The small rectangle is the attracted neighbourhood $[U]$. After 6 iterations the diameter of the partition is sufficiently small in order to have some boxes removed from the inside even though they are in directed cycles. This happens because they are contained in, or get mapped into the basin of attraction of the fixed point.

We used different sizes for the parameter intervals and ran the computations on a cluster of the NIIF HPC centre at the University of Szeged (48 cores, 128 GB memory per cluster) parallelising it with OpenMP. We summarise some technical details in Table 1.

Table 1. Resources used by the program

parameter	size of slices	# of CPU	max. memory	wall clock time	total time
$[0.5, 0.875]$	10^{-3}	48	2.30 GB	1.2s	38.5s
$[0.875, 0.95]$	10^{-3}	48	3.05 GB	2.4s	52.3s
$[0.95, 0.99]$	10^{-4}	48	3.07 GB	33.7s	22m 5.9s
$[0.99, 1]$	10^{-5}	20	65.30 GB	204m 42.6s	1800m 37.1s

Remark 4. Here *wall clock time (real)* refers to the actual running time of the process, whilst the *total time (user + sys)* is the sum of the time spent on individual CPUs.

The complexity of the computations for some parameter slices is shown in Table 2.

Table 2. Complexity of the computations

parameter slice	$[S]$	# iteration	max # of vertices	max # of edges
$[0.875, 0.876]$	$[2.072e-04, 5.049031]^2$	13	242	1,676
$[0.999, 0.99901]$	$[2.928e-06, 7.369087]^2$	27	729,528	4,193,329
$[0.99999, 1]$	$[2.822e-06, 7.389015]^2$	33	3,105,304	118,751,916

The program ran successfully for every parameter slice, thus Proposition 6.1 implies that α^* is globally attracting for $\alpha \in [0.5, 1]$. The output of these computations can be found at [1]. Combining this result with Corollary 3.4 and Proposition 4.4, the proof

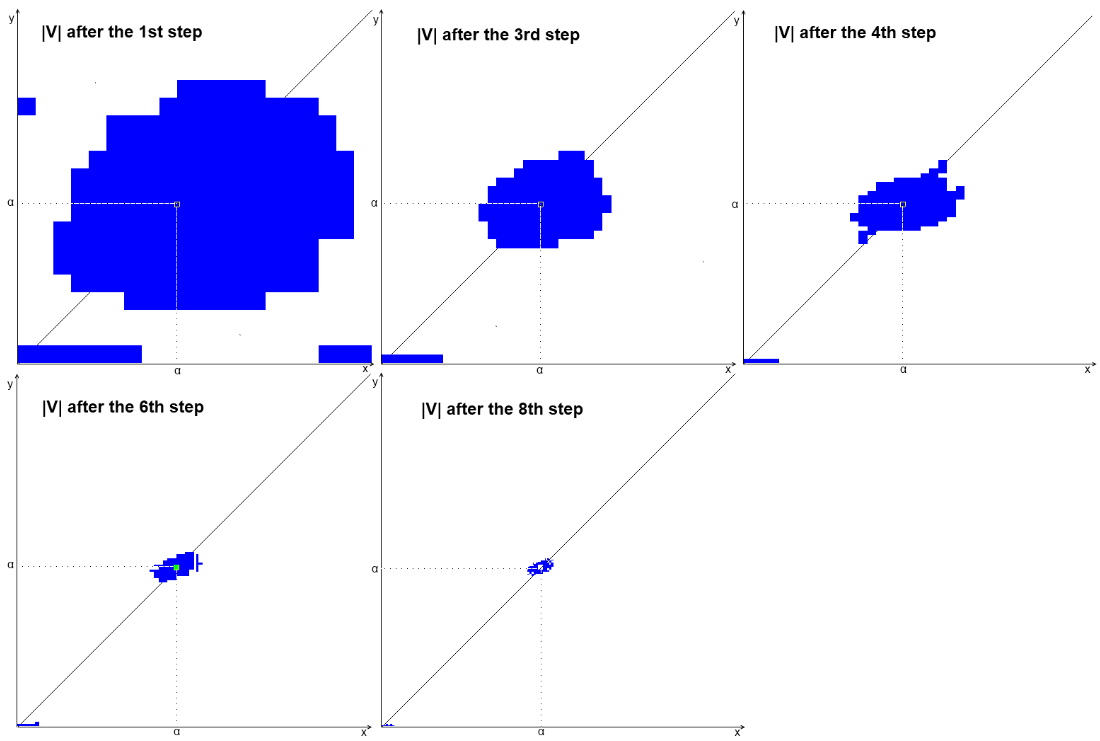

Figure 6. $|\mathcal{V}|$ after 1, 3, 4, 6, 8 steps having $[\alpha] = [0.9, 0.90001]$

of Theorem 1.1 is completed. Thus we established using both rigorous, computer-aided calculations and analytical tools that the fixed point α^* is globally attracting for $\alpha \in (0, 1]$.

Acknowledgements

The authors thank Warwick Tucker from the CAPA group [4], Daniel Wilczak and Tomasz Kapela, members of the CAPD group [5] for useful suggestions and their help.

We also acknowledge the useful comments of the anonymous referees, which helped us to improve our paper.

The computations were performed on the HPC centre provided by the Hungarian National Information Infrastructure Development Institute [27] at the University of Szeged.

The first author was supported by Bergens forskningsstiftelse - The Bergen Research Foundation. Project title: "Computer-Aided Proofs in Mathematical Analysis." Project number: 801458. The second and third authors were supported by the Hungarian Scientific Research Fund, Grant No. K 75517. This work was partially supported by the European Union and co-funded by the European Social Fund under the project "Telemedicine-focused research activities on the field of Matematics, Informatics and Medical sciences" of project number "TÁMOP-4.2.2.A-11/1/KONV-2012-0073". Ábel Garab was also supported by the European Union and co-funded by the European Social Fund. Project title: "Broadening the knowledge base and supporting the long term professional sustainability of the Research University Centre of Excellence at the University of Szeged by ensuring the rising generation of excellent scientists." Project number: TÁMOP-4.2.2/B-10/1-2010-0012.

References

[1] `http://www.math.u-szeged.hu/~krisztin/ricker`.

[2] G. Alefeld, *Introduction to interval analysis*, SIAM Rev. 53 (2011), pp. 380–381.

[3] D.G. Aronson, M.A. Chory, G.R. Hall, and R.P. McGehee, *Bifurcations from an invariant circle for two-parameter families of maps of the plane: a computer-assisted study*, Comm. Math. Phys. 83 (1982), pp. 303–354, URL `http://projecteuclid.org/getRecord?id=euclid.cmp/1103920867`.

[4] CAPA, *Computer-aided proofs in analysis*, `http://www2.math.uu.se/~warwick/CAPA/`, at the University of Uppsala & University of Bergen.

[5] CAPD, *Capd library*, software available at `http://capd.ii.uj.edu.pl`, the Computer Assisted Proofs in Dynamics group at the Jagiellonian University in Krakow.

[6] C.W. Clark, *A delayed-recruitment model of population dynamics, with an application to baleen whale populations*, Journal of Mathematical Biology 3 (1976), pp. 381–391, URL `http://dx.doi.org/10.1007/BF00275067`.

[7] M. Dellnitz and A. Hohmann, *A subdivision algorithm for the computation of unstable manifolds and global attractors*, Numer. Math. 75 (1997), pp. 293–317, URL `http://dx.doi.org/10.1007/s002110050240`.

[8] M. Dellnitz, A. Hohmann, O. Junge, and M. Rumpf, *Exploring invariant sets and invariant measures*, Chaos 7 (1997), pp. 221–228, URL `http://dx.doi.org/10.1063/1.166223`.

[9] A. Dénes and G. Makay, *Attractors and basins of dynamical systems*, Electron. J. Qual. Theory Differ. Equ. (2011), pp. No. 20, 11.

[10] Z. Galias, *Rigorous investigation of the Ikeda map by means of interval arithmetic*, Nonlinearity 15 (2002), pp. 1759–1779, URL `http://dx.doi.org/10.1088/0951-7715/15/6/304`.

[11] J. Hass, M. Hutchings, and R. Schlafly, *The double bubble conjecture*, Electron. Res. Announc. Amer. Math. Soc. 1 (1995), pp. 98–102, URL `http://dx.doi.org/10.1090/S1079-6762-95-03001-0`.

[12] V. Jiménez López, *A counterexample on global attractivity for Clark's equation*, in *Proceedings of the Workshop Future Directions in Difference Equations*, Colecc. Congr., Vol. 69, Univ. Vigo, Serv. Publ., Vigo, 2011, pp. 97–105.

[13] V. Jiménez López and E. Parreño, *L.A.S. and negative Schwarzian derivative do not imply G.A.S. in Clark's equation* .
[14] Y.A. Kuznetsov, Applied Mathematical Sciences, Vol. 112, 2nd ed., Springer-Verlag, New York 1998.
[15] G. Ladas, *On the recursive sequence* $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-1})/(A + Bx_n + Cx_{n-1})$, J. Differ. Equations Appl. 1 (1995), pp. 317–321, URL http://dx.doi.org/10.1080/10236199508808030.
[16] S.A. Levin and R.M. May, *A note on difference-delay equations*, Theoret. Population Biology 9 (1976), pp. 178–187.
[17] E. Liz, *Local stability implies global stability in some one-dimensional discrete single-species models*, Discrete Contin. Dyn. Syst. Ser. B 7 (2007), pp. 191–199 (electronic), URL http://dx.doi.org/10.3934/dcdsb.2007.7.191.
[18] E. Liz, *Stability of non-autonomous difference equations: simple ideas leading to useful results*, J. Difference Equ. Appl. 17 (2011), pp. 203–220, URL http://dx.doi.org/10.1080/10236198.2010.549007.
[19] E. Liz and G. Röst, *Dichotomy results for delay differential equations with negative schwarzian derivative*, Nonlinear Analysis: Real World Applications 11 (2010), pp. 1422 – 1430, URL http://www.sciencedirect.com/science/article/pii/S1468121809001424.
[20] E. Liz, V. Tkachenko, and S. Trofimchuk, *A global stability criterion for scalar functional differential equations*, SIAM J. Math. Anal. 35 (2003), pp. 596–622, URL http://dx.doi.org/10.1137/S0036141001399222.
[21] E. Liz, V. Tkachenko, and S. Trofımchuk, *Global stability in discrete population models with delayed-density dependence*, Math. Biosci. 199 (2006), pp. 26–37, URL http://dx.doi.org/10.1016/j.mbs.2005.03.016.
[22] S. Luzzatto and P. Pilarczyk, *Finite resolution dynamics*, Found. Comput. Math. 11 (2011), pp. 211–239, URL http://dx.doi.org/10.1007/s10208-010-9083-z.
[23] O. Merino, *Global attractivity of the equilibrium of a difference equation: an elementary proof assisted by computer algebra system*, J. Difference Equ. Appl. 17 (2011), pp. 33–41, URL http://dx.doi.org/10.1080/10236190902932718.
[24] K. Mischaikow and M. Mrozek, *Chaos in the Lorenz equations: a computer-assisted proof*, Bull. Amer. Math. Soc. (N.S.) 32 (1995), pp. 66–72, URL http://dx.doi.org/10.1090/S0273-0979-1995-00558-6.
[25] R.E. Moore, SIAM Studies in Applied Mathematics, Vol. 2, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa. 1979.
[26] N.S. Nedialkov, K.R. Jackson, and G.F. Corliss, *Validated solutions of initial value problems for ordinary differential equations*, Appl. Math. Comput. 105 (1999), pp. 21–68, URL http://dx.doi.org/10.1016/S0096-3003(98)10083-8.
[27] NIIF, *National information infrastructure development institute*, http://www.niif.hu.
[28] W.E. Ricker, *Stock and recruitment*, Journal of the Fisheries Research Board of Canada 11 (1954), pp. 559–623, URL http://www.nrcresearchpress.com/doi/abs/10.1139/f54-039.
[29] J.G. Siek, L.Q. Lee, and A. Lumsdaine 2002.
[30] R. Tarjan, *Depth-first search and linear graph algorithms*, SIAM Journal on Computing 1 (1972), pp. 146–160, URL http://link.aip.org/link/?SMJ/1/146/1.
[31] V. Tkachenko and S. Trofimchuk, *A global attractivity criterion for nonlinear non-autonomous difference equations*, J. Math. Anal. Appl. 322 (2006), pp. 901–912, URL http://dx.doi.org/10.1016/j.jmaa.2005.09.052.
[32] W. Tucker, *A rigorous ODE solver and Smale's 14th problem*, Found. Comput. Math. 2 (2002), pp. 53–117.
[33] W. Tucker, Princeton University Press, Princeton, NJ 2011, a short introduction to rigorous computations.
[34] D. Wilczak, *Uniformly hyperbolic attractor of the Smale-Williams type for a Poincaré map in the Kuznetsov system*, SIAM J. Appl. Dyn. Syst. 9 (2010), pp. 1263–1283, URL http://dx.doi.org/10.1137/100795176, with online multimedia enhancements.

Appendix

$$g(z,\bar{z},\alpha) = \frac{-i+\sqrt{-1+4\alpha}}{2\sqrt{-1+4\alpha}}\Bigg(\left(-1+e^{-\frac{z-iz\sqrt{-1+4\alpha}+\bar{z}+i\sqrt{-1+4\alpha}\bar{z}}{2\alpha}}\right)(z+\bar{z}) + \alpha\left(-1+e^{-\frac{z-iz\sqrt{-1+4\alpha}+\bar{z}+i\sqrt{-1+4\alpha}\bar{z}}{2\alpha}} + \frac{2z}{1+i\sqrt{-1+4\alpha}} + \frac{2i\bar{z}}{i+\sqrt{-1+4\alpha}}\right)\Bigg) \quad (A.1)$$

$$g_{20}(\alpha) = -\frac{1}{2} + \frac{3i}{2\sqrt{-1+4\alpha}} \quad (A.2)$$

$$g_{11}(\alpha) = \frac{2(-1+\alpha)}{-1+4\alpha+i\sqrt{-1+4\alpha}} \quad (A.3)$$

$$g_{02}(\alpha) = \frac{i\alpha\left(3i+\sqrt{-1+4\alpha}\right)}{\sqrt{-1+4\alpha}\left(i-2i\alpha+\sqrt{-1+4\alpha}\right)} \quad (A.4)$$

$$g_{30}(\alpha) = -\frac{i\left(1+\alpha-i\sqrt{-1+4\alpha}\right)}{\alpha\sqrt{-1+4\alpha}} \quad (A.5)$$

$$g_{21}(\alpha) = \frac{-3i+2i\alpha+\sqrt{-1+4\alpha}}{2\alpha\sqrt{-1+4\alpha}} \quad (A.6)$$

$$g_{12}(\alpha) = \frac{2\left(3i-2i\alpha+\sqrt{-1+4\alpha}\right)}{\sqrt{-1+4\alpha}\left(i+\sqrt{-1+4\alpha}\right)^2} \quad (A.7)$$

$$g_{03}(\alpha) = \frac{4i\alpha\left(5i+\sqrt{-1+4\alpha}\right)}{\sqrt{-1+4\alpha}\left(i+\sqrt{-1+4\alpha}\right)^3} \quad (A.8)$$

$$g_{40}(\alpha) = \frac{8\left(-2+\alpha-2i\sqrt{-1+4\alpha}\right)}{\sqrt{-1+4\alpha}\left(-i+\sqrt{-1+4\alpha}\right)^3} \quad (A.9)$$

$$g_{31}(\alpha) = -\frac{2\left(-2i+i\alpha+\sqrt{-1+4\alpha}\right)}{\alpha\left(-i+4i\alpha+\sqrt{-1+4\alpha}\right)} \quad (A.10)$$

$$g_{22}(\alpha) = \frac{2(-2+\alpha)}{\alpha\left(-1+4\alpha+i\sqrt{-1+4\alpha}\right)} \quad (A.11)$$

$$g_{13}(\alpha) = \frac{8\left(2-\alpha-i\sqrt{-1+4\alpha}\right)}{\sqrt{-1+4\alpha}\left(i+\sqrt{-1+4\alpha}\right)^3} \quad (A.12)$$

$$g_{04}(\alpha) = \frac{8\alpha\left(7i+\sqrt{-1+4\alpha}\right)}{\sqrt{-1+4\alpha}\left(i+\sqrt{-1+4\alpha}\right)^4} \quad (A.13)$$

$$\begin{aligned}
h_{inv,\alpha}(z) = {} & z - \frac{h_{20}(\alpha)z^2}{2} + \frac{1}{6}z^3\left(3h_{20}(\alpha)^2 - h_{30}(\alpha) + 3h_{11}(\alpha)\overline{h_{02}(\alpha)}\right) \\
& + \frac{1}{24}z^4\left(-15h_{20}(\alpha)^3 + 10h_{20}(\alpha)h_{30}(\alpha) - 30h_{11}(\alpha)h_{20}(\alpha)\overline{h_{02}(\alpha)}\right. \\
& \left. -3h_{02}(\alpha)\overline{h_{02}(\alpha)}^2 + 4h_{11}(\alpha)\overline{h_{03}(\alpha)} - 12h_{11}(\alpha)\overline{h_{02}(\alpha)h_{11}(\alpha)}\right) \\
& - h_{11}(\alpha)z\bar{z} + \frac{1}{2}z^2\left(3h_{11}(\alpha)h_{20}(\alpha) + h_{02}(\alpha)\overline{h_{02}(\alpha)} + 2h_{11}(\alpha)\overline{h_{11}(\alpha)}\right)\bar{z} \\
& + \frac{1}{6}z^3\left(-15h_{11}(\alpha)h_{20}(\alpha)^2 + 4h_{11}(\alpha)h_{30}(\alpha) - 12h_{11}(\alpha)^2\overline{h_{02}(\alpha)}\right. \\
& + 3h_{12}(\alpha)\overline{h_{02}(\alpha)} - 6h_{02}(\alpha)h_{20}(\alpha)\overline{h_{02}(\alpha)} + h_{02}(\alpha)\overline{h_{03}(\alpha)} \\
& - 12h_{11}(\alpha)h_{20}(\alpha)\overline{h_{11}(\alpha)} - 6h_{02}(\alpha)\overline{h_{02}(\alpha)h_{11}(\alpha)} - 6h_{11}(\alpha)\overline{h_{11}(\alpha)}^2 \\
& \left. +3h_{11}(\alpha)\overline{h_{12}(\alpha)} - 3h_{11}(\alpha)\overline{h_{02}(\alpha)h_{20}(\alpha)}\right)\bar{z} - \frac{h_{02}(\alpha)\bar{z}^2}{2} + \frac{1}{2}z\left(2h_{11}(\alpha)^2\right. \\
& \left. -h_{12}(\alpha) + h_{02}(\alpha)h_{20}(\alpha) + 2h_{02}(\alpha)\overline{h_{11}(\alpha)} + h_{11}(\alpha)\overline{h_{20}(\alpha)}\right)\bar{z}^2 \\
& + \frac{1}{4}z^2\left(-12h_{11}(\alpha)^2h_{20}(\alpha) + 3h_{12}(\alpha)h_{20}(\alpha) - 3h_{02}(\alpha)h_{20}(\alpha)^2\right. \\
& + h_{02}(\alpha)h_{30}(\alpha) + h_{03}(\alpha)\overline{h_{02}(\alpha)} - 9h_{02}(\alpha)h_{11}(\alpha)\overline{h_{02}(\alpha)} - 12h_{11}(\alpha)^2\overline{h_{11}(\alpha)} \\
& + 4h_{12}(\alpha)\overline{h_{11}(\alpha)} - 6h_{02}(\alpha)h_{20}(\alpha)\overline{h_{11}(\alpha)} - 6h_{02}(\alpha)\overline{h_{11}(\alpha)}^2 + 2h_{02}(\alpha)\overline{h_{12}(\alpha)} \\
& \left. -3h_{11}(\alpha)h_{20}(\alpha)\overline{h_{20}(\alpha)} - 3h_{02}(\alpha)\overline{h_{02}(\alpha)h_{20}(\alpha)} - 6h_{11}(\alpha)\overline{h_{11}(\alpha)h_{20}(\alpha)}\right)\bar{z}^2 \\
& + \frac{1}{6}\left(-h_{03}(\alpha) + 3h_{02}(\alpha)h_{11}(\alpha) + 3h_{02}(\alpha)\overline{h_{20}(\alpha)}\right)\bar{z}^3 \\
& + \frac{1}{6}z\left(-6h_{11}(\alpha)^3 + 6h_{11}(\alpha)h_{12}(\alpha) + h_{03}(\alpha)h_{20}(\alpha) - 9h_{02}(\alpha)h_{11}(\alpha)h_{20}(\alpha)\right. \\
& - 3h_{02}(\alpha)^2\overline{h_{02}(\alpha)} + 3h_{03}(\alpha)\overline{h_{11}(\alpha)} - 18h_{02}(\alpha)h_{11}(\alpha)\overline{h_{11}(\alpha)} \\
& - 6h_{11}(\alpha)^2\overline{h_{20}(\alpha)} + 3h_{12}(\alpha)\overline{h_{20}(\alpha)} - 3h_{02}(\alpha)h_{20}(\alpha)\overline{h_{20}(\alpha)} \\
& \left. -12h_{02}(\alpha)\overline{h_{11}(\alpha)h_{20}(\alpha)} - 3h_{11}(\alpha)\overline{h_{20}(\alpha)}^2 + h_{11}(\alpha)\overline{h_{30}(\alpha)}\right)\bar{z}^3 \\
& + \frac{1}{24}\left(4h_{03}(\alpha)h_{11}(\alpha) - 12h_{02}(\alpha)h_{11}(\alpha)^2 + 6h_{02}(\alpha)h_{12}(\alpha)\right. \\
& - 3h_{02}(\alpha)^2h_{20}(\alpha) - 12h_{02}(\alpha)^2\overline{h_{11}(\alpha)} + 6h_{03}(\alpha)\overline{h_{20}(\alpha)} \\
& \left. -18h_{02}(\alpha)h_{11}(\alpha)\overline{h_{20}(\alpha)} - 15h_{02}(\alpha)\overline{h_{20}(\alpha)}^2 + 4h_{02}(\alpha)\overline{h_{30}(\alpha)}\right)\bar{z}^4
\end{aligned} \tag{A.14}$$

$$h_{20}(\alpha) = \frac{2\left(1 + \sqrt{1-4\alpha} - \alpha\right)}{\alpha\left(1 - 4\alpha + i\sqrt{-1+4\alpha}\right)} \tag{A.15}$$

$$h_{11}(\alpha) = \frac{2(-1+\alpha)}{\alpha\left(-1+4\alpha-i\sqrt{-1+4\alpha}\right)} \tag{A.16}$$

$$h_{02}(\alpha) = \frac{2\alpha\left(3i+\sqrt{-1+4\alpha}\right)}{\left(i+\sqrt{-1+4\alpha}\right)^2\left(-i+4i\alpha+\alpha\sqrt{-1+4\alpha}\right)} \tag{A.17}$$

$$\begin{aligned} h_{30}(\alpha) = \Big(16\Big(&-12-12i\sqrt{-1+4\alpha} \\ &+\alpha\left(45+21i\sqrt{-1+4\alpha}+\alpha\left(-27+2\alpha-5i\sqrt{-1+4\alpha}\right)\right)\Big)\Big) \\ \Big(\left(1+\sqrt{1-4\alpha}\right)^4&\sqrt{-1+4\alpha}\Big(i+\sqrt{-1+4\alpha} \\ &+\alpha\left(-2i+\alpha\left(-5i+\sqrt{-1+4\alpha}\right)\right)\Big)\Big)^{-1} \end{aligned} \tag{A.18}$$

$$\begin{aligned} h_{12}(\alpha) = \Big(\Big(&4i+\alpha\Big(-4\left(5i+3\sqrt{-1+4\alpha}\right)+\alpha\Big(28i+18\sqrt{-1+4\alpha} \\ &+\alpha\left(-31i-9\sqrt{-1+4\alpha}+\alpha\left(9i+\sqrt{-1+4\alpha}\right)\right)\Big)\Big)\Big)\Big) \\ &\Big(\Big(\alpha^3\Big(-i+\sqrt{-1+4\alpha}+\alpha\Big(6i+\alpha\Big(-13i-15\sqrt{-1+4\alpha}+ \\ &2\alpha\left(10i+\sqrt{-1+4\alpha}\right)\Big)\Big)\Big)\Big)\Big)^{-1} \end{aligned} \tag{A.19}$$

$$\begin{aligned} h_{03}(\alpha) = \Big(\Big(2\Big(&5i+7\sqrt{-1+4\alpha}+\alpha\Big(-29i-14\sqrt{-1+4\alpha} \\ &+\alpha\left(17i-i\alpha+3\sqrt{-1+4\alpha}\right)\Big)\Big)\Big)\Big) \\ &\Big(\Big(i+\sqrt{-1+4\alpha}+\alpha\Big(-2\left(7i+6\sqrt{-1+4\alpha}\right) \\ &+\alpha\Big(70i+46\sqrt{-1+4\alpha}+\alpha\Big(-6\left(23i+10\sqrt{-1+4\alpha}\right) \\ &+\alpha\left(73i-4i\alpha+13\sqrt{-1+4\alpha}\right)\Big)\Big)\Big)\Big)\Big)^{-1} \end{aligned} \tag{A.20}$$

$$\begin{aligned} &\left|h^{20}_{inv}(\alpha)\right|+\left|h^{11}_{inv}(\alpha)\right|+\left|h^{02}_{inv}(\alpha)\right| = \\ &\left(-2(-1+\alpha)\alpha+\sqrt{\alpha^3(2+\alpha)}+\sqrt{\frac{\alpha^5(2+\alpha)}{-1+\alpha(4+\alpha)}}\right)\left(2\sqrt{\alpha^5(-1+4\alpha)}\right)^{-1} \end{aligned} \tag{A.21}$$

$$\left|h^{30}_{inv}(\alpha)\right|+\left|h^{21}_{inv}(\alpha)\right|+\left|h^{12}_{inv}(\alpha)\right|+\left|h^{03}_{inv}(\alpha)\right| = \tag{A.22}$$

$$\begin{aligned} &\frac{1}{6}\sqrt{\frac{36+15\alpha-12\alpha^2+4\alpha^3}{\alpha^5\left(-2+7\alpha+4\alpha^2\right)}}+\sqrt{\frac{6-40\alpha+75\alpha^2-21\alpha^4+4\alpha^5+4\alpha^6}{(2-15\alpha+22\alpha^2+23\alpha^3+4\alpha^4)4\alpha^6}} \\ &\quad+\frac{1}{6}\sqrt{\frac{12-54\alpha+66\alpha^2-13\alpha^3+4\alpha^4+4\alpha^5}{\alpha^3\left(-1+13\alpha-57\alpha^2+88\alpha^3-18\alpha^4+7\alpha^5+4\alpha^6\right)}} \\ &\quad+\frac{1}{2}\sqrt{\frac{2-16\alpha+13\alpha^2+124\alpha^3-198\alpha^4+15\alpha^5+54\alpha^6+9\alpha^7}{\alpha^6(-1+4\alpha)\left(-1+4\alpha+\alpha^2\right)^2}} \end{aligned}$$

$$\begin{aligned} &|h^{40}_{inv}(\alpha)| = \\ &\frac{1}{24}\Big(-1140+10874\alpha-12660\alpha^2-143073\alpha^3+423177\alpha^4 \\ &-211261\alpha^5+1356\alpha^6+30869\alpha^7-13766\alpha^8-2238\alpha^9+1489\alpha^{10} \\ &+256\alpha^{11}\Big)^{\frac{1}{2}}\left((1-4\alpha)^2\alpha^6\left(-1+4\alpha+\alpha^2\right)^3\left(-1+5\alpha-2\alpha^2+\alpha^3\right)\right)^{-\frac{1}{2}} \end{aligned} \tag{A.23}$$

$$\begin{aligned}
&|h_{inv}^{31}(\alpha)|= \\
&\big(-36+912\alpha-9354\alpha^2+49026\alpha^3-133548\alpha^4+155248\alpha^5 \\
&\quad+24851\alpha^6-182342\alpha^7+127014\alpha^8-47122\alpha^9-12543\alpha^{10} \\
&\quad+34528\alpha^{11}-329\alpha^{12}+1925\alpha^{13}+2452\alpha^{14}+361\alpha^{15}\big)^{\frac{1}{2}}\big(-1 \\
&\quad+5\alpha-2\alpha^2+\alpha^3\big)^{-\frac{1}{2}}\left(6\alpha^5\left(2-15\alpha+22\alpha^2+23\alpha^3+4\alpha^4\right)\right)^{-1}
\end{aligned} \tag{A.24}$$

$$\begin{aligned}
&|h_{inv}^{22}(\alpha)|= \\
&\big(-16+4\alpha+2240\alpha^2-14868\alpha^3+19782\alpha^4+91297\alpha^5 \\
&\quad-309731\alpha^6+259610\alpha^7+81080\alpha^8-147591\alpha^9+12815\alpha^{10} \\
&\quad+19871\alpha^{11}-8236\alpha^{12}+241\alpha^{13}+2132\alpha^{14}+361\alpha^{15}\big)\big(-1 \\
&\quad+5\alpha-2\alpha^2+\alpha^3\big)^{-\frac{1}{2}}\left(4\alpha^5\left(2-15\alpha+22\alpha^2+23\alpha^3+4\alpha^4\right)\right)^{-1}
\end{aligned} \tag{A.25}$$

$$\begin{aligned}
&|h_{inv}^{13}(\alpha)|= \\
&\frac{1}{6}\big(\big(36-840\alpha+7728\alpha^2-35454\alpha^3+84157\alpha^4-96139\alpha^5+39017\alpha^6 \\
&\quad+15361\alpha^7-22836\alpha^8+10489\alpha^9+5142\alpha^{10}-397\alpha^{11}+922\alpha^{12}+529\alpha^{13} \\
&\quad+64\alpha^{14}\big)\big)^{\frac{1}{2}}\left(\alpha^9\left(-1+4\alpha+\alpha^2\right)^3\left(2-17\alpha+35\alpha^2+4\alpha^3-\alpha^4+4\alpha^5\right)\right)^{-\frac{1}{2}}
\end{aligned} \tag{A.26}$$

$$\begin{aligned}
&|h_{inv}^{04}(\alpha)|= \\
&\frac{1}{24}\big(36-396\alpha+1350\alpha^2-1422\alpha^3+318\alpha^4+441\alpha^5-145\alpha^6+100\alpha^7 \\
&\quad+25\alpha^8\big)^{\frac{1}{2}}\left(\alpha^6\left(-1+4\alpha+\alpha^2\right)^2\left(1-9\alpha+22\alpha^2-9\alpha^3+4\alpha^4\right)\right)^{-\frac{1}{2}}
\end{aligned} \tag{A.27}$$

$$\begin{aligned}
&|r_2^{40}(\alpha)|= \\
&\frac{1}{24}\big(-432+2808\alpha-5016\alpha^2+4692\alpha^3-2584\alpha^4 \\
&\quad+930\alpha^5-210\alpha^6+7\alpha^7+27\alpha^8-6\alpha^9+\alpha^{10}\big)^{\frac{1}{2}} \\
&\left(\alpha^4\left(-1+4\alpha+\alpha^2\right)^2\left(1-9\alpha+22\alpha^2-9\alpha^3+4\alpha^4\right)\right)^{-\frac{1}{2}}
\end{aligned} \tag{A.28}$$

$$\begin{aligned}
&|r_2^{31}(\alpha)|= \\
&\frac{1}{6}\big(90-810\alpha+1635\alpha^2+1485\alpha^3-3462\alpha^4+4056\alpha^5-659\alpha^6 \\
&\quad-108\alpha^7-320\alpha^8+24\alpha^9+87\alpha^{10}-102\alpha^{11}+36\alpha^{12}+\alpha^{13}\big)^{\frac{1}{2}} \\
&\left(\alpha^7\left(2-17\alpha+33\alpha^2+13\alpha^3+4\alpha^5+\alpha^6\right)(-1+4\alpha)^2\right)^{-\frac{1}{2}}
\end{aligned} \tag{A.29}$$

$$\begin{aligned}
&|r_2^{22}(\alpha)|= \\
&\frac{1}{4}\big(64-608\alpha+1348\alpha^2+728\alpha^3-1692\alpha^4+1948\alpha^5-688\alpha^6-2008\alpha^7 \\
&\quad+1184\alpha^8+742\alpha^9-355\alpha^{10}-136\alpha^{11}+34\alpha^{12}+14\alpha^{13}+\alpha^{14}\big)^{\frac{1}{2}} \\
&\left(\alpha^9(-1+4\alpha)\left(-2+7\alpha+6\alpha^2+\alpha^3\right)^2\right)^{-\frac{1}{2}}
\end{aligned} \tag{A.30}$$

$$|r_2^{13}(\alpha)|= \frac{1}{6}\left(-54+918\alpha-5973\alpha^2+18921\alpha^3-32250\alpha^4 + 32742\alpha^5-15643\alpha^6+2506\alpha^7+3246\alpha^8-3551\alpha^9 + 1327\alpha^{10}-156\alpha^{11}-48\alpha^{12}+59\alpha^{13}+16\alpha^{14}+\alpha^{15}\right)^{\frac{1}{2}} \left(\alpha^7\left(-2+9\alpha+\alpha^2+\alpha^4\right)(6-48\alpha+90\alpha^2+4\alpha^3)^2\right)^{-\frac{1}{2}} \tag{A.31}$$

$$|r_2^{40}(\alpha)|= \frac{1}{24}\left(-432+2808\alpha-5016\alpha^2+4692\alpha^3-2584\alpha^4 + 930\alpha^5-210\alpha^6+7\alpha^7+27\alpha^8-6\alpha^9+\alpha^{10}\right)^{\frac{1}{2}} \left(\alpha^4\left(-1+4\alpha+\alpha^2\right)^2\left(1-9\alpha+22\alpha^2-9\alpha^3+4\alpha^4\right)\right)^{-\frac{1}{2}} \tag{A.32}$$

$$c_1(\alpha) = \left(2i-2\sqrt{-1+4\alpha}+\alpha\left(2\left(-7i+5\sqrt{-1+4\alpha}\right) + \alpha\left(25i-13\sqrt{-1+4\alpha}+\alpha\left(-25i+7\sqrt{-1+4\alpha}-\alpha\left(-7i+\sqrt{-1+4\alpha}\right)\right)\right)\right)\right) \left(2\alpha^3\sqrt{-1+4\alpha}\left(-i+\sqrt{-1+4\alpha}\right)\left(i\alpha+\sqrt{-1+4\alpha}\right)\right)^{-1} \tag{A.33}$$