Global asymptotic stability of a generalization of the Pielou difference equation

Ábel Garab, Víctor Jiménez López and Eduardo Liz

Abstract. The Pielou equation is a well-known discrete-time population model in which the per capita growth rate depends on the population size, but the density dependence operates with a delay of d generations. Thus, the between-year dynamics are governed by a difference equation of order $d+1$. The main result in this paper establishes the global stability of the unique positive equilibrium for a generalization of the 2-dimensional Pielou equation. Our proof is based on a rather natural combination of two techniques which could be, in principle, applicable to obtain global asymptotic stability in other problems: some dominance conditions and the determination of a first integral for a related equation, which turns out to be a quasi-Lyanupov function for the generalized Pielou equation.

We provide additional results on the global dynamics of the generalized Pielou equation for dimensions higher than two, and discuss its relationship with other families of difference equations traditionally employed for modelling population dynamics.

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1. Introduction

Population growth of many species is modelled by difference equations, in such a way that the population size x_n after n generations is described by a production function f depending on the previous generations:

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-d}), \qquad n = 0, 1, 2, \ldots, \tag{1.1}$$

where the initial conditions $x_{-d}, \ldots, x_{-1}, x_0$ are nonnegative real numbers with $x_0 > 0$.

A well-known example is given by the Levin–May model [16]:

$$x_{n+1} = x_n F(x_{n-d}), \qquad n = 0, 1, 2, \dots, \tag{1.2}$$

where $F : [0, \infty) \to (0, \infty)$ is continuous and decreasing. For some specific forms of F, see Table I in [16]. In particular, for $F(x) = \beta/(1 + \delta x)$, $\beta > 0$, $\delta > 0$, (1.2) gives the Pielou equation

$$x_{n+1} = \frac{\beta x_n}{1 + \delta x_{n-d}}, \qquad n = 0, 1, 2, \dots, \tag{1.3}$$

introduced by Pielou in [23].

Another example of (1.1) is the Clark model [6], given by

$$x_{n+1} = \alpha x_n + F(x_{n-d}), \qquad n = 0, 1, 2, \dots, \tag{1.4}$$

where $0 < \alpha < 1$, and $F : [0, \infty) \to [0, \infty)$ is continuous.

Both equations (1.2) and (1.4) are simple models to take into account some age structure in a population, and x_n denotes the adult population. Equation (1.2) assumes that the per capita growth rate depends on the population size, but the density dependence operates with a delay of d generations; equation (1.4) assumes that a proportion αx_n of the adult population survives to the next generation, and the new borns need a maturation delay d to become adults.

This paper is focused on the following generalization of the Pielou equation (1.3):

$$x_{n+1} = \frac{\beta x_n^\gamma}{1 + \delta x_{n-d}}, \qquad n = 0, 1, 2, \dots, \tag{1.5}$$

where β, γ, δ are positive real numbers and d is a nonnegative integer. In the one-dimensional case $d = 0$, equation (1.5) has been recently studied in [18], and it belongs to a family of first-order difference equations given by

$$x_{n+1} = x_n^\gamma F(x_n), \qquad n = 0, 1, 2, \dots, \tag{1.6}$$

which includes other models with applications in population dynamics, economics, and social science (see [17, 19] and their references).

In this way, our study of equation (1.5) can be considered as a step to understand some qualitative properties of the family of delayed difference equations

$$x_{n+1} = x_n^\gamma F(x_{n-d}), \qquad n = 0, 1, 2, \dots, \tag{1.7}$$

where $\gamma > 0$ and d is a positive integer. In particular, equation (1.7) has been recently considered in [3] with $\gamma \geq 1$ and $F(x) = e^{r(1-x)}$, $r > 0$, to investigate Allee effects in a Ricker-type population model with delay.

Note that (1.7) with $\gamma = 1$ gives the Levin–May model (1.2), and a change of variables transforms (1.7) into the Clark model (1.4). We provide more details in Section 2. These two facts make the study of equation (1.7) of interest from a theoretical point of view. From the perspective of its applications, assuming that F is decreasing, equation (1.7) can be viewed as a population model where the per capita growth rate x_{n+1}/x_n is the product of two density-dependent factors: the delayed factor of intraspecific competition $F(x)$ already present in the Levin–May equation, and a new factor $x^{\gamma-1}$, which can be a cooperation factor ($\gamma > 1$) or an interference factor ($0 < \gamma < 1$). We emphasize that for $\gamma > 1$, the cooperation factor induces a strong Allee effect, in such a way that populations below a threshold are doomed to extinction [3, 17]. For $0 < \gamma \leq 1$, (1.7) has a unique positive equilibrium (the additional assumption $F(0) > 1$ is necessary when $\gamma = 1$), and the parameter γ is used to gain flexibility to fit population data; for details and more references, see [17, 18, 24].

Our main results concern local and global stability properties for the unique positive equilibrium of (1.5) in the case $0 < \gamma < 1$. In particular, we prove that this equilibrium is globally asymptotically stable in the 2-dimensional case ($d = 1$), thus extending the result for $\gamma = 1$ (Pielou equation), proved by Kuruklis and Ladas [15, Theorem 4]. Our proof is based on a rather natural combination of two techniques which could be, in principle, applicable to obtain global asymptotic stability in other problems: the dominance conditions introduced in [8, 11], and the determination of a first integral for a related equation, which turns out to be a quasi-Lyanupov function for (1.5) in a certain parameter regime, where we can show global stability with the aid of a Lyapunov-type result (Lemma 3.11).

For other values of d, we get some global stability results that work for the more general equation (1.7) with $\gamma \in (0, 1)$. In this case, we apply some previous results of Tkachenko and Trofimchuk [25], using the relationship between (1.7) and the Clark equation (1.4).

The paper is organized as follows: in Section 2, we provide some results on the global stability of the positive equilibrium of (1.7). Section 3 is devoted to equation (1.5): in the subsequent subsections, we address the local stability of the equilibrium (Subsection 3.1), the global stability in the 2-dimensional case (Subsection 3.2), some partial global stability results for dimensions higher than two (Subsections 3.3 and 3.4). Finally, Section 4 contains some remarks and open problems.

2. Preliminary results

As we have stated before, a change of variables transforms (1.7) into the Clark model (1.4). Since this fact will be useful in some of the subsequent results, in this section we give the explicit form of this transformation and we use it to derive some results.

We consider (1.7) with $0 < \gamma < 1$ and $F\colon [0, \infty) \to (0, \infty)$ continuous and decreasing. If p is the unique equilibrium of (1.7), then the change of variables $y_n = -\log(x_n/p)$ transforms (1.7) into

$$y_{n+1} = \gamma y_n + g(y_{n-d}), \qquad n = 0, 1, 2, \dots, \tag{2.1}$$

where $g(y) = -\log\left(p^{\gamma-1}F(pe^{-y})\right)$.

The following global stability result is a straightforward consequence of Theorem 7 in the paper [25] by Tkachenko and Trofimchuk:

Theorem 2.1. *Assume that $0 < \gamma < 1$, and let $A = -p^{-\gamma}(1-\gamma)/F'(p)$. The positive equilibrium p of (1.7) is globally asymptotically stable if the following conditions hold:*

(H1) *Either $A \geq 1$, or $A < 1$ and*

$$\gamma^{d+1} > \frac{1}{A} \log\left(\frac{1+A}{1+A^2}\right). \tag{2.2}$$

(H2) *F is $\mathcal{C}^3$-differentiable, $F'(x) < 0$ for all $x > 0$, and*

$$x^2\left(2\,\frac{F'''(x)}{F'(x)} - 3\left(\frac{F''(x)}{F'(x)}\right)^2 + \left(\frac{F'(x)}{F(x)}\right)^2\right) < 1, \qquad \forall\, x > 0. \tag{2.3}$$

For $\gamma = 1$, equation (1.2) has also been considered by Tkachenko and Trofimchuk, and the following global stability result can be derived from [26, Theorem 1.3]:

Proposition 2.2. *If* **(H2)** *holds, then the positive equilibrium p of (1.2) is globally asymptotically stable if the following inequality is satisfied:*

$$-\,pF'(p) < \frac{3}{2(d+1)} + \frac{1}{2(d+1)^2}. \tag{2.4}$$

Remark 2.3. The technical condition (2.3) is equivalent to say that the Schwarzian derivative of the transformed map $g(y) = -\log\left(p^{\gamma-1}F(pe^{-y})\right)$ is negative (see [19, Proposition 6]).

3. Results for the gamma-Pielou equation

In this section we consider equation (1.5), which we refer to as the gamma-Pielou equation, in analogy with the one-dimensional gamma-model (1.6) introduced in [19]. Since we are mainly interested in global stability results, we only consider the case $0 < \gamma \leq 1$. As we mentioned in the Introduction, in this case (1.5) has a unique positive equilibrium p. Our first remark is that we can normalize the equilibrium to 1, thus dropping one of the parameters in (1.5); indeed, since $\beta p^{\gamma-1} = 1 + \delta p$, the change of variables $x_n \mapsto x_n/p$ transforms (1.5) into

$$x_{n+1} = \frac{(\alpha+1)x_n^{\gamma}}{\alpha + x_{n-d}}, \qquad n = 0, 1, 2, \ldots, \tag{3.1}$$

with $\alpha = 1/(\delta p) > 0$.

Note also that $x_n > 0$ holds for all $n \geq 1$ under any admissible initial conditions, i.e. provided $x_{-d}, \ldots, x_0$ are all nonnegative, and $x_0 > 0$, so it is not restrictive to assume that our state-space is the open set $\mathbb{R}_+^{d+1}$.

3.1. Local stability

We begin with the study of the local asymptotic stability of the equilibrium 1 in (3.1). For completeness, we consider the general case $\gamma > 0$.

The linearized equation about the positive equilibrium 1 of (3.1) is

$$x_{n+1} = \gamma x_n - q x_{n-d}, \qquad q = \frac{1}{\alpha+1} < 1. \tag{3.2}$$

Applying the Schur–Cohn criterion (see, e.g., [13, Theorem 1.3.4]), we get the following result for $d = 1$:

Proposition 3.1. *The equilibrium $p = 1$ of*

$$x_{n+1} = \frac{(\alpha+1)x_n^\gamma}{\alpha + x_{n-1}}, \qquad n = 0, 1, 2, \ldots,$$

is locally asymptotically stable if and only if $0 < \gamma < (\alpha+2)/(\alpha+1)$.

Proof. The Schur–Cohn conditions are equivalent to

$$\gamma < 1 + q < 2 \Longleftrightarrow \gamma < \frac{\alpha+2}{\alpha+1} < 2.$$

The second inequality trivially holds for all $\alpha > 0$. □

Corollary 3.2. *If $\gamma \in (0, 1]$, then the equilibrium $p = 1$ of*

$$x_{n+1} = \frac{(\alpha+1)x_n^\gamma}{\alpha + x_{n-1}}, \qquad n = 0, 1, 2, \ldots,$$

is locally asymptotically stable for all $\alpha > 0$.

For $d > 1$, we use Corollary 4.2 in [5], which provides the following result:

Theorem 3.3. *The equilibrium $p = 1$ of (3.1) is locally asymptotically stable if and only if either $\gamma \le \alpha/(1+\alpha)$, or*

$$\frac{\alpha}{\alpha+1} < \gamma < \frac{\alpha+2}{\alpha+1} \tag{3.3}$$

and

$$d < \frac{\arccos\left(\frac{(\gamma^2-1)(1+\alpha)^2+1}{2\gamma(1+\alpha)}\right)}{\arccos\left(\frac{(\gamma^2+1)(1+\alpha)^2-1}{2\gamma(1+\alpha)^2}\right)}. \tag{3.4}$$

Corollary 3.4. *If $\gamma \ge 2$, then the equilibrium $p = 1$ of (3.1) is unstable for all positive integer d and all real number $\alpha > 0$.*

If we denote by $G(\gamma, \alpha)$ the function in the right-hand side of (3.4), it is clear that $\lim_{\alpha \to 0^+} G(\gamma, \alpha) = 1$ for every fixed $\gamma > 0$. Thus, if $d > 1$, for every $\gamma > 0$, there is a value of $\alpha > 0$ such that the equilibrium $p = 1$ of (3.1) is unstable. This means that we cannot extend the result of Corollary 3.2 to larger values of the delay.

Analogously, Theorem 3.3 establishes a sufficient condition $\gamma \le \alpha/(1+\alpha)$ for the asymptotic stability of the equilibrium independent of the delay d (absolute local stability). This condition is also necessary because if $\gamma > \alpha/(1+\alpha)$, then there is a value of d for which (3.4) does not hold. The region R_d of absolute local stability is represented in Figure 1 in the parameter plane (α, γ).

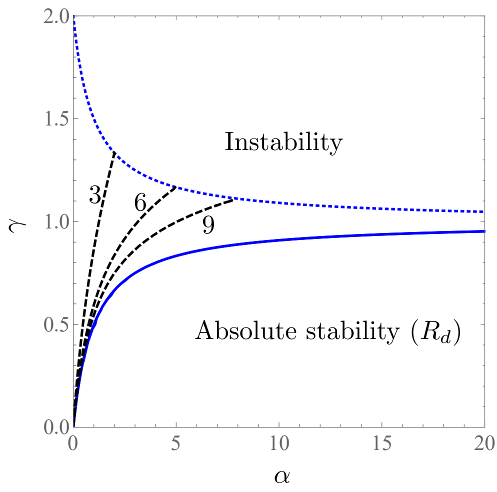

FIGURE 1. Stability diagram of the positive equilibrium $p = 1$ of (3.1) in the parameter plane (α, γ). The equilibrium is asymptotically stable for every d in the region R_d below the blue solid line, and unstable above the dotted blue line. For particular values of d, the stability region is the one below the dotted blue line and the corresponding black dashed line (examples for $d = 3, 6, 9$).

3.2. Main results: global stability in the 2-dimensional case

This subsection is devoted to show that the conclusion of Corollary 3.2 has a global character, that is, the positive equilibrium $p = 1$ of (3.1) is globally asymptotically stable if $d = 1$ and $0 < \gamma \leq 1$.

In the case $\gamma = 1$ (Pielou equation), this result was proved by Kuruklis and Ladas [15, Theorem 4]. We notice that in the one-dimensional case $d = 0$, the equilibrium is also globally asymptotically stable if $0 < \gamma \leq 1$. See [15, Theorem 3] for $\gamma = 1$ and [18, Theorem 3.1] for $0 < \gamma < 1$.

For convenience, we state the form of (3.1) for $d = 1$:

$$x_{n+1} = \frac{(\alpha+1)x_n^{\gamma}}{\alpha + x_{n-1}}, \qquad n = 0, 1, 2, \ldots, \tag{3.5}$$

with positive initial conditions x_{-1}, x_0.

Theorem 3.5. *The equilibrium $p = 1$ is globally asymptotically stable for (3.5) if $\alpha > 0$ and $0 < \gamma \leq 1$.*

Firstly, we list some notation. If $x, y > 0$, we introduce

$$z = \frac{(\alpha+1)y^{\gamma}}{\alpha+x}, \qquad a = \frac{(\alpha+1)z^{\gamma}}{\alpha+y} = y^{\gamma^2}\frac{(\alpha+1)^{\gamma+1}}{(\alpha+x)^{\gamma}(\alpha+y)},$$

and we denote

$$\rho_1 = \gamma(1-\gamma), \qquad \rho_2 = 1+\gamma(1-\gamma),$$
$$\rho_3 = \gamma(1+\gamma)-1, \qquad \rho_4 = \frac{(1-\gamma)(2+\gamma)}{\gamma(1+\gamma)-1}.$$

Also, we consider the maps

$$h(t) = \left(\frac{\alpha+1}{\alpha+t}\right)^{\mu+1} \quad (\mu>0), \qquad f(t) = t^{\gamma^2}\left(\frac{\alpha+1}{\alpha+t}\right)^{\gamma-\mu} \quad (\mu>0),$$
$$u(t) = \frac{\alpha+t}{\alpha+1}, \qquad v(t) = t^{\rho_3},$$

for $t > 0$, and the one-dimensional equation

$$t_{n+1} = h(t_n), \qquad n = 0, 1, 2, \ldots, \tag{3.6}$$

with positive initial condition t_0. Finally, we consider the scalar map

$$V(x,y) = \log^2 x - \gamma \log x \log y + \log^2 y,$$

for $x, y > 0$. Note that $V(1,1) = 0$ and $V(x,y) > 0$ otherwise. In fact, this map is a first integral (that is, it is invariant for the orbits) of (3.5) in the limit case $\alpha = 0$; one can easily find V by taking logarithms in this limit equation and integrating the ensuing linear equation.

Lemma 3.6. *If $\alpha \geq \mu$, then 1 is globally asymptotically stable for* (3.6)*. If $\alpha < \mu$, then there is a 2-periodic orbit attracting all orbits of* (3.6) *(except the equilibrium 1).*

Proof. This is an easy consequence of three facts: h is decreasing, the Schwarzian derivative of h is negative, and $h'(-1) = -(1+\mu)/(1+\alpha)$. (See, e.g., [20, Theorem 1]). □

The following notion of dominance is necessary for stating our next lemma, and it will also play an essential role in our arguments.

Definition 3.7 **([11, p.106]).** Let $I \subseteq \mathbb{R}$ be an open interval, and let $G\colon I^{k+1} \to I$ and $H\colon I \to I$ be continuous maps, and consider the difference equation

$$x_{n+1} = G(x_n, \ldots, x_{n-k}), \qquad n \geq 0, \quad (x_0, \ldots, x_{-k}) \in I^{k+1}. \tag{3.7}$$

Then G is said to be dominated by H if the following condition is fulfilled:

(DC) There exists $l \geq 0$ such that for every semi-orbit $(x_n)_{n=-k}^{l+1}$ of (3.7), the following holds: if $x_{l+1} \geq \max\{x_l, \ldots, x_{-k}\}$ (respectively $x_{l+1} \leq \min\{x_l, \ldots, x_{-k}\}$), then there exists some $x \in \operatorname{conv}\{x_l, \ldots, x_{-k}\}$, for which inequality $h(x) \geq x_{l+1}$ (respectively $h(x) \leq x_{l+1}$) holds, where $\operatorname{conv} A$ denotes the smallest connected set containing the set A.

Lemma 3.8. *Assume $\mu \geq \rho_1$. Then h dominates (3.5) in the sense of Definition 3.7, with $l = 1$.*

Proof. Note first that, due to the condition on μ, the map f is increasing.

We must show that if $a \geq \max\{x, y, z\}$ (respectively, $a \leq \min\{x, y, z\}$), then there is $t \in \operatorname{conv}\{x, y, z\}$ such that $h(t) \geq a$ (respectively, $h(t) \leq a$). We just consider the cases i) $1 \leq x \leq y$, ii) $1 \leq y \leq x$ and iii) $y < 1 < x$, because the argument in the other cases is analogous. Of course, we can discard the trivial possibility $x = y = 1$; hence, in cases i) and ii), we also assume, respectively, $y > 1$ and $x > 1$.

Case i): $1 \leq x \leq y$. If $z > 1$, then $a < z$, and if $z \leq 1$, then $a < y$. Thus, in either case, $a < \max\{x, y, z\}$, so our statement follows after proving $h(y) \leq a$, that is,

$$\left(\frac{\alpha+1}{\alpha+y}\right)^{\mu+1} \leq y^{\gamma^2} \frac{(\alpha+1)^{\gamma+1}}{(\alpha+x)^{\gamma}(\alpha+y)}.$$

If fact, since $x \leq y$, it suffices to show

$$\left(\frac{\alpha+1}{\alpha+y}\right)^{\mu+1} \leq y^{\gamma^2} \left(\frac{\alpha+1}{\alpha+y}\right)^{\gamma+1},$$

that is, $f(y) \geq 1$, which is true because f is increasing and $y > 1$.

Case ii): $1 \leq y \leq x$. As in Case i), if $z > 1$, then $a < z$, and if $z \leq 1$, then $a < x$. Thus it suffices to show now that $h(x) \leq a$, that is,

$$\left(\frac{\alpha+1}{\alpha+x}\right)^{\mu+1} \leq y^{\gamma^2} \frac{(\alpha+1)^{\gamma+1}}{(\alpha+x)^{\gamma}(\alpha+y)},$$

or, equivalently,

$$\frac{(\alpha+1)^{\mu+1}}{(\alpha+x)^{\mu+1-\gamma}} \leq y^{\gamma^2} \frac{(\alpha+1)^{\gamma+1}}{\alpha+y},$$

Since $y \leq x$, it suffices to show

$$\frac{(\alpha+1)^{\mu+1}}{(\alpha+y)^{\mu+1-\gamma}} \leq y^{\gamma^2} \frac{(\alpha+1)^{\gamma+1}}{\alpha+y},$$

which, again, amounts to $f(y) \geq 1$.

Case iii): $y < 1 < x$. Now $z < 1$, so $a > z^{\gamma} > z$. Then it suffices to show that $h(y) \geq a$, that is,

$$\left(\frac{\alpha+1}{\alpha+y}\right)^{\mu+1} \geq y^{\gamma^2} \frac{(\alpha+1)^{\gamma+1}}{(\alpha+x)^{\gamma}(\alpha+y)},$$

Since $x > y$, we just need to show

$$\left(\frac{\alpha+1}{\alpha+y}\right)^{\mu+1} \geq y^{\gamma^2} \left(\frac{\alpha+1}{\alpha+y}\right)^{\gamma+1},$$

that is, $f(y) \leq 1$, which follows, once again, because f is increasing and now $y < 1$. $\square$

Lemma 3.9. *If $\alpha \geq \rho_1$, then* 1 *is globally asymptotically stable for* (3.5)*. If $\alpha < \rho_1$, then* (3.5) *is permanent and the ω-limit set of every orbit of* (3.5) *is included in the interval $[r(\alpha), s(\alpha)]$ whose endpoints are the points of the* 2*-periodic orbit of* (3.6) *for $\mu = \rho_1$.*

Proof. This follows from Lemmas 3.6 and 3.8 after applying [11, Theorem B] to h with $\mu = \rho_1$. (We emphasize that, although strictly speaking [11, Theorem B] just states permanence in the case $\mu = \rho_1$, its proof guarantees that the ω-limit sets of the orbits of (3.5) are included in any compact invariant (for h) interval M whose interior contains all ω-limit sets of the orbits of (3.6). In our present situation, if $x = r(\alpha) - \epsilon$ for an arbitrarily small number $\epsilon > 0$, then the interval $M = [x, h(x)]$ has this property. As a conclusion, all ω-limit sets of (3.5) are in fact included in $[r(\alpha), s(\alpha)]$.) □

Remark 3.10. Observe that, in particular, we have just shown global asymptotical stability for the classical Pielou equation ($\gamma = 1$), firstly proved, as previously mentioned, by Kuruklis and Ladas in [15]. While elementary as well, their proof relies on very specific properties of the equation (to begin with, the equilibrium is relocated in the origin after a linear change of variables, which would be rather unpractical in the generalized case $0 < \gamma < 1$), allowing them to estimate the relative distance to the origin of consecutive blocks of positive and negative points of any given orbit. In contrast to this, both our choice of the dominance map and the checking of the dominance condition (**DC**) in Lemma 3.8 are completely natural, and the argument works (partially) as well when $\gamma < 1$.

The rest of the proof basis on the following Lyapunov-type result.

Lemma 3.11. *Consider the discrete-time semi-dynamical system $S\colon \mathbb{N}_0 \times X \to X$, $(n, \xi) \mapsto S(n, \xi)$ on an arbitrary metric space X, which possesses a global attractor. Assume that $\xi^* \in X$ is a locally attractive fixed point and that $V\colon X \to [0, \infty)$ is a quasi-Lyapunov function in the following sense: V is continuous on X, $V(\xi^*) = 0$, whereas for all $\xi^* \neq \xi \in X$, $V(\xi) > 0$ holds, and there exists $n = n(\xi) \geq 1$ such that $V(S(n, \xi)) < V(\xi)$.*

Then ξ^* is the global attractor of the semi-dynamical system.

The main idea of quasi-Lyapunov technique has been already used as an effective tool to prove global stability in rational difference equations; this technique appeared in the paper of Merino [22] to prove global stability in another generalization of the Pielou equation (solving the so-called Y2K problem suggested by G. Ladas). Motivated by the approach of Merino, Bastien and Rogalski [4] introduced the concept of quasi-Lyapunov functions for discrete dynamical systems of arbitrary order k, and used them in the context of Lyness type difference equations. A similar idea already appeared in a paper by Kruse and Nesemann [14], which was later successfully applied in by Hogan and Zeilberger [10] to obtain an algorithmic approach for proving global asymptotic stability for rational difference equations.

In spite of the similarities in the definition, in the statement and in the technique of the proof, there are also differences in each part, so we give our own proof of the lemma below.

Proof. Since ξ^* is locally attractive, it is sufficient to show that for all $\xi \in X$, $\xi^* \in \omega(\xi)$ holds. Note that the existence of the global attractor implies that $\omega(\xi)$ is nonempty, compact and invariant for all $\xi \in X$ (see e.g. [9, Corollary 2.2.4]).

Now, seeking for contradiction, suppose that there exists $\xi \in X$ such that $\xi^* \notin \omega(\xi)$. Let $m := \min_{\eta\in\omega(\xi)} V(\eta)$ and $\eta_0 \in \omega(\xi)$ such that $V(\eta_0) = m$. Our assumptions on V yield on the one hand that $m > 0$, and on the other hand, there exists $n = n(\eta_0)$ such that $V(S(n, \eta_0)) < V(\eta_0) = m$. This, together with invariance of $\omega(\xi)$, contradicts the minimality property of η_0. □

In the following lemma we show that $V(x, y)$ is a quasi-Lyapunov function for (3.5) if γ or α are not too large. In the following lemmas we assume $(x, y) \neq (1, 1)$.

Lemma 3.12. *Assume $\rho_3 \leq 0$. Then either $V(y, z) < V(x, y)$ or $V(z, a) < V(x, y)$. As a consequence, 1 is globally asymptotically stable for (3.5).*

Proof. Concerning the first statement, observe that

$$\Delta V(x, y) := V(x, y) - V(y, z) = \log\left(\frac{x(\alpha + x)}{(\alpha + 1)y^\gamma}\right)\log\left(\frac{x(\alpha + 1)}{\alpha + x}\right).$$

Therefore, $\Delta V(x, y) > 0$ trivially holds if $x \leq 1 \leq y$ or $y \leq 1 \leq x$, so we assume in what follows that either $x, y \leq 1$ or $x, y \geq 1$. In fact, $\Delta V(x, y) \leq 0$ if and only if

$$(x, y) \in \left\{(x, y) \in \mathbb{R}^2_+ : x \geq 1 \text{ and } \left(\frac{x(\alpha + x)}{\alpha + 1}\right)^{1/\gamma} \leq y\right\}$$

or

$$(x, y) \in \left\{(x, y) \in \mathbb{R}^2_+ : x \leq 1 \text{ and } \left(\frac{x(\alpha + x)}{\alpha + 1}\right)^{1/\gamma} \geq y\right\},$$

so we assume that we are in this case. This, in particular, implies $y < x$ (in the case $x, y \leq 1$) and $y > x$ (in the case $x, y \geq 1$). Proving $V(z, a) < V(x, y)$ is equivalent to showing $-\Delta V(x, y) < \Delta V(y, z)$, that is

$$\log\left(\frac{(\alpha + 1)y^\gamma}{x(\alpha + x)}\right)\log\left(\frac{x(\alpha + 1)}{\alpha + x}\right) < \log\left(\frac{y(\alpha + y)}{(\alpha + 1)z^\gamma}\right)\log\left(\frac{y(\alpha + 1)}{\alpha + y}\right).$$

In fact, it is easy to see – by eliminating z from the right-hand side, and comparing terms on both sides – that it suffices to show

$$\log(y^\gamma)\log\left(\frac{x(\alpha + 1)}{\alpha + x}\right) < \log\left(\frac{y^{1-\gamma^2}(\alpha + y)}{\alpha + 1}\right)\log\left(\frac{y(\alpha + 1)}{\alpha + y}\right),$$

or just

$$|\log(y^\gamma)| < \left|\log\left(\frac{y^{1-\gamma^2}(\alpha+y)}{\alpha+1}\right)\right|,$$

which in turn amounts to check that $u(y) < v(y)$ (respectively, $u(y) > v(y)$) whenever $y < 1$ (respectively, $y > 1$). This is obvious because u is increasing and, due to the hypothesis on γ, v is decreasing, and $v(1) = u(1)$.

In order to prove the last statement we simply need to show that Lemma 3.11 can be applied to the semi-dynamical system corresponding to (3.5) and generated by the two-dimensional mapping

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ \frac{(\alpha+1)y^\gamma}{\alpha+x} \end{pmatrix}. \tag{3.8}$$

with the state-space $X := \mathbb{R}^2_+$.

In the first statement of the lemma we showed that V has the quasi-Lyapunov property (with $\xi^* = (1,1)$). Furthermore, Corollary 3.2 implies that $(1,1)$ is locally asymptotically stable, hence it is only left to verify that there exists a global attractor, which is ensured by permanence (see Lemma 3.9).

Therefore, application of Lemma 3.11 yields global asymptotic stability of the equilibrium 1 of (3.5). □

Lemma 3.13. *Assume $\rho_3 > 0$ and $\alpha < \rho_1$. Then there is exactly one point $0 < q < 1$ satisfying $u(q) = v(q)$, and $u(t) < v(t)$ (respectively, $u(t) > v(t)$) whenever $q < t < 1$ (respectively, $t > 1$). Moreover, $q < r(\alpha)$.*

Proof. The first statement follows easily from $u(1) = v(1)$, $v(0) < u(0)$, the concavity of v, and from $\alpha < \rho_1 < \rho_4$, which implies

$$v'(1) = \rho_3 < \frac{1}{\alpha+1} = u'(1).$$

Proving $q < r(\alpha)$ amounts to show $h^2(q) > q$ (when $\mu = \rho_1$), that is,

$$\left(\frac{\alpha+1}{\alpha+q}\right)^{\rho_2} q^{1/\rho_2} < 1 + \alpha(1 - q^{1/\rho_2}).$$

In fact, the stronger statement

$$\left(\frac{\alpha+1}{\alpha+q}\right)^{\rho_2} q^{1/\rho_2} < 1,$$

that is,

$$q^{1/\rho_2 - \rho_2\rho_3} < 1,$$

holds because

$$1 - \rho_2^2\rho_3 = (1-\gamma)((3\gamma+1)(1-\gamma^2) + 1 - \gamma^3 + \gamma^5) > 0. \qquad \square$$

The following lemma is probably true as well without any restrictions on α or y, but this is enough for our purposes.

Lemma 3.14. *Assume $\rho_3 > 0$ and $\alpha < \rho_1$ and set q as defined in Lemma 3.13. If $y > q$, then either $V(y,z) < V(x,y)$ or $V(z,a) < V(x,y)$. As a consequence, if $\rho_3 > 0$ and $\alpha < \rho_1$, then 1 is globally asymptotically stable for (3.5).*

Proof. The statement concerning V can be proved, due to the properties of u and v to the right of q (Lemma 3.13), with the same argument as that of Lemma 3.12. To deduce asymptotic stability recall first that, by Lemma 3.13, $q < r(\alpha) < s(\alpha) < h(q)$ and $h^2(q) > q$. This last inequality implies, because h is decreasing, that h maps $(q, h(q))$ into itself. Then, as a consequence of the dominance condition, the two-dimensional mapping (3.8) maps the square $X = (q, h(q))^2$ into itself, hence the corresponding semi-dynamical system $S : \mathbb{N} \cup \{0\} \times X \to X$ is well-defined. Now $q < r(\alpha) < s(\alpha) < h(q)$ and Lemma 3.9 guarantee the existence of a global attractor for S, and we can apply Lemma 3.11 to get asymptotic stability for this "restricted" system. But, again by Lemma 3.9, every orbit of (3.5) eventually lies between q and $h(q)$. This finishes the proof. □

Proof of Theorem 3.5. The proof follows from Lemmas 3.9, 3.12 and 3.14. □

3.3. Global stability for $d > 1$

In this subsection we address the global stability of the equilibrium $p = 1$ of (3.1) for $\gamma \in (0, 1]$ and $d > 1$.

From Theorem 2.1, we get the following result:

Corollary 3.15. *Assume that $\alpha > 0$, $0 < \gamma < 1$, and let $A = (\alpha+1)(1-\gamma)$. The equilibrium $p = 1$ of (3.1) is globally asymptotically stable if either $A \geq 1$, or $A < 1$ and*

$$\gamma^{d+1} > \frac{1}{A} \log\left(\frac{1+A}{1+A^2}\right). \tag{3.9}$$

Proof. In this case, $F(x) = (\alpha+1)/(\alpha+x)$, and $F'(1) = -1/(\alpha+1)$. Hence, the value of A in the statement of Theorem 2.1 is

$$A = \frac{-p^{-\gamma}(1-\gamma)}{F'(p)} = \frac{-(1-\gamma)}{F'(1)} = (\alpha+1)(1-\gamma).$$

On the other hand, inequality (2.3) reads

$$\frac{x^2}{\alpha + x^2} < 1, \qquad \forall\, x > 0,$$

which obviously holds for all $\alpha > 0$. □

It is worth clarifying that in the special case $d = 1$ Theorem 3.5 is more general than Corollary 3.15.

For the Pielou equation ($\gamma = 1$), we can apply Proposition 2.2 to get the following result:

Corollary 3.16. *Assume that $\alpha > 0$ and $\gamma = 1$. The equilibrium $p = 1$ of (3.1) is globally asymptotically stable if the following inequality holds:*

$$\alpha > \frac{2d^2 + d - 2}{3d + 4}. \tag{3.10}$$

Since condition $A \geq 1$ is equivalent to the absolute stability condition $\gamma \leq \alpha/(1 + \alpha)$, we have the following consequence of Corollary 3.15:

Corollary 3.17. *The equilibrium $p = 1$ of (3.1) is globally asymptotically stable for all $d \geq 0$ if and only if $\gamma \leq \alpha/(1 + \alpha)$.*

In Figure 2, we plot the stability diagram of the positive equilibrium $p = 1$ of (3.1) in the parameter plane (γ, d) for $\alpha = 1$.

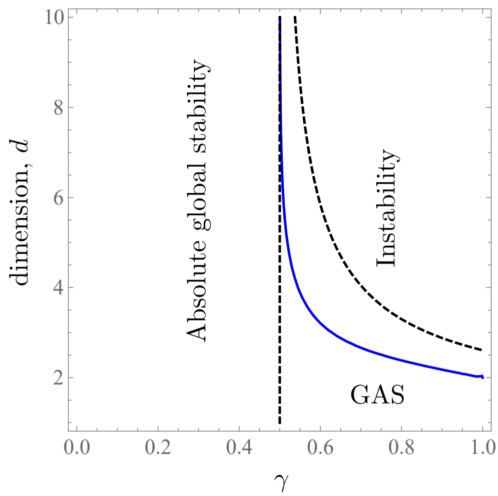

FIGURE 2. Stability diagram of the positive equilibrium $p = 1$ of (3.1) in the parameter plane (γ, d) for $\alpha = 1$. By Corollary 3.17, the equilibrium is asymptotically stable for every d if $\gamma \leq 1/2$. The blue solid line corresponds to the global stability condition (3.9), and the black dashed line is the boundary of local asymptotic stability given by (3.4). Notice that the relevant values on the d axes are the positive integers.

Remark 3.18. Dominance allows to prove Corollary 3.17 in a very simple, direct way. Actually we prove that condition

$$\alpha \geq \gamma + \cdots + \gamma^d = \frac{\gamma - \gamma^{d+1}}{1 - \gamma} \tag{3.11}$$

is sufficient for the global stability in (3.1). Thus for every d, 1 is GAS if $\alpha \geq \gamma/(1 - \gamma)$, which is equivalent to $\gamma \leq \alpha/(1 + \alpha)$.

Indeed, in view of Lemma 3.6, the statement follows after proving that $h(t)$ dominates (3.1) when $\mu = \gamma + \cdots + \gamma^d$ and $l = d$.

We must show that if $(x_n)_{n=-d}^{d+1}$ is a semi-orbit of (3.1) and $x_{d+1} \geq \max\{x_d, \ldots, x_{-d}\}$ (respectively, $x_{d+1} \leq \min\{x_d, \ldots, x_{-d}\}$), then there is $t \in \operatorname{conv}\{x_d, \ldots, x_{-d}\}$ such that $h(t) \geq x_{d+1}$ (respectively, $h(t) \leq x_{d+1}$). Say for instance $x_{d+1} \geq \max\{x_d, \ldots, x_{-d}\}$ (the other case is similar). Then either $x_d \leq 1$ or $x_0 \leq 1$, and also $t := \min\{x_d, \ldots, x_{-d}\} \leq 1$. If $x_d \leq 1$, then

$$x_{d+1} = \frac{(\alpha+1)x_d^{\gamma}}{\alpha + x_0} \leq \frac{\alpha+1}{\alpha+t} \leq h(t).$$

If $x_0 \leq 1$, then

$$x_{d+1} = \frac{(\alpha+1)^{1+\gamma+\cdots+\gamma^d} x_0^{\gamma^{d+1}}}{(\alpha+x_0)(\alpha+x_{-1})^{\gamma}\cdots(\alpha+x_{-d})^{\gamma^d}} \leq \frac{(\alpha+1)^{1+\gamma+\cdots+\gamma^d}}{(\alpha+t)^{1+\gamma+\cdots+\gamma^d}} = h(t).$$

This finishes the proof.

For parameters $\gamma \in (0,1)$ for which global stability is not granted we can show permanence:

Proposition 3.19. *Equation (3.1) is permanent: if $\alpha < \gamma + \cdots + \gamma^d$, then the ω-limit set of every orbit of (3.1) is included in the interval $[C, D]$ whose endpoints are the points of the 2-periodic orbit of (3.6) for $\mu = \gamma + \cdots + \gamma^d$.*

Proof. For $\alpha \geq \gamma + \cdots + \gamma^d$, the 1 equilibrium of equation (3.1) is GAS (see Remark 3.18), therefore (3.1) is also permanent. Thus we can focus on parameters with $\alpha < \gamma + \cdots + \gamma^d$. First we note that permanence was already proved in [1, Theorem 1].

We have seen in Remark 3.18 that the map h with $\mu = \gamma + \cdots + \gamma^d$ dominates g in the sense of Definition 3.7. The rest of the proof coincides with that of Lemma 3.9 (with (3.5) replaced by (3.1)). □

Note that monotonicity of h immediately implies the uniform ultimate upper bound

$$D \leq D_{\alpha,\gamma} := \left(\frac{\alpha+1}{\alpha}\right)^{(1-\gamma^{d+1})/(1-\gamma)}. \tag{3.12}$$

3.4. A complementary result for global stability

Making use of the uniform ultimate upper bound (3.12) and the main idea of Remark 3.18 we can give another criteria for global stability of (3.1). This extends somewhat the GAS regime (in the (α, γ) parameter-plane) in the $d = 2, 3$ cases (see Figure 3 for $d = 2$).

For convenience, let us use the notation $c\colon [0,\infty) \to [0,\infty)$,

$$c(x) = \frac{x^{\gamma^{d+1}}}{\alpha + x}.$$

Theorem 3.20. *Assume that $\alpha > 0$ and $\gamma \in (0,1)$. Then the equilibrium 1 of (3.1) is GAS provided the following three assumptions are fulfilled.*

(i) $\gamma(1-\gamma^d) > 1-\gamma$,

(ii) $c(D_{\alpha,\gamma}) \geq \frac{1}{\alpha+1}$,
(iii) $(1+\alpha)(1-\gamma) \geq \gamma(1-\gamma^d)$.

Proof. Proposition 3.19 allows us to concentrate only on solutions for which $x_n \leq D_{\alpha,\gamma}$ holds for all $n \geq 0$.

We show that h dominates (3.1) under the above assumptions with $l = d$ and $\mu = \frac{\gamma(1-\gamma^d)}{1-\gamma} - 1 = \gamma + \cdots + \gamma^d - 1$. Then (i) guarantees that $\mu > 0$, while (iii) ensures that $\alpha \geq \mu$ holds, and therefore, by virtue of Lemma 3.6, (3.1) is GAS.

To show dominance of h, we need to consider two cases.

Case 1: $x_{d+1} \geq \max\{x_d, \ldots, x_{-d}\}$. Then $x_d \leq 1$ or $x_0 \leq 1$ must hold. Let $t = \min\{x_d, \ldots, x_{-d}\} \leq 1$.

If $x_d \leq 1$, then

$$x_{d+1} = \frac{(\alpha+1)x_d^\gamma}{\alpha+x_0} \leq \frac{\alpha+1}{\alpha+t} \leq \left(\frac{\alpha+1}{\alpha+t}\right)^{\gamma+\cdots+\gamma^d} = h(t)$$

holds, where the last inequality follows from assumption (i) and $t \leq 1$.

If $x_0 \leq 1$, then consider

$$\begin{aligned} x_{d+1} &= \frac{(\alpha+1)^{1+\gamma+\cdots+\gamma^d} x_0^{\gamma^{d+1}}}{(\alpha+x_0)(\alpha+x_{-1})^\gamma \cdots (\alpha+x_{-d})^{\gamma^d}} \\ &= \frac{(\alpha+1)^{1+\gamma+\cdots+\gamma^d} c(x_0)}{(\alpha+x_{-1})^\gamma \cdots (\alpha+x_{-d})^{\gamma^d}}. \end{aligned} \tag{3.13}$$

Note that $c(0) = 0$, $c(1) = \frac{1}{\alpha+1}$, and moreover, c is strictly increasing on $(0, \xi)$ and strictly decreasing on (ξ, ∞), where $\xi = \frac{\alpha\gamma^{d+1}}{1-\gamma^{d+1}}$. These together with $x_0 \leq 1 < D_{\alpha,\gamma}$ and assumption (ii) infer that $c(x_0) \leq \frac{1}{\alpha+1}$, and hence

$$x_{d+1} \leq \frac{(\alpha+1)^{\gamma+\cdots+\gamma^d}}{(\alpha+x_{-1})^\gamma \cdots (\alpha+x_{-d})^{\gamma^d}} \leq \frac{(\alpha+1)^{\gamma+\cdots+\gamma^d}}{(\alpha+t)^{\gamma+\cdots+\gamma^d}} = h(t)$$

Case 2: $x_{d+1} \leq \min\{x_d, \ldots, x_{-d}\}$. Then, similarly to Case 1, $x_d \geq 1$ or $x_0 \geq 1$ must hold. Let $t = \max\{x_d, \ldots, x_{-d}\} \geq 1$.

The case $x_d \geq 1$ is completely analogous to the $x_d \leq 1$ part of Case 1, and therefore omitted.

It remains the case if $x_0 \geq 1$. Then using the properties of c and (ii) ensures that $c(x_0) \geq \frac{1}{\alpha+1}$ and thus from formula (3.13) we obtain that

$$x_{d+1} \geq \frac{(\alpha+1)^{\gamma+\cdots+\gamma^d}}{(\alpha+t)^{\gamma+\cdots+\gamma^d}} = h(t).$$

This completes the proof. ∎

4. Final remarks

In this section, we provide some remarks about some related papers and suggest some open problems.

- Agarwal *et al.* [1] considered a more general equation

$$x_{n+1} = x_n^{\gamma} f(x_{n-k_1}, x_{n-k_2}, \dots, x_{n-k_r}), \qquad n = 0, 1, \dots, \tag{4.1}$$

  where $\gamma > 0$, $k_1 < k_2 < \cdots < k_r$ are nonnegative integers, and the continuous map $f\colon [0,\infty)^r \to (0,\infty)$ is nonincreasing in each of its arguments. For $\gamma < 1$, their Theorem 2 establishes that the unique positive equilibrium p of (4.1) is a global attractor of all positive solutions if p is a global attractor for the related one-dimensional map

$$h(x) = p^{\gamma^d} (f(x, x, \dots x))^{(1-\gamma^d)/(1-\gamma)},$$

  where $d = k_r$. However, there is a mistake in the proof of their theorem, and the map h should be defined as

$$h(x) = p^{\gamma^{d+1}} (f(x, x, \dots x))^{(1-\gamma^{d+1})/(1-\gamma)}.$$

  We can apply the corrected version of this result to equation (3.1), using Lemma 3.6. The sufficient condition for the global attractivity of the equilibrium 1 provided by that result is exactly the dominance condition (3.11).
- The global stability results given in this paper are particular examples of global stability results for the Clark equation (1.4). Actually, Theorem 3.5 provides an additional example for which the conjecture 'LAS implies GAS' in Clark's equation holds for $d = 1$. For related results in this direction, see [2, 12]. Our conjecture is that, at least for $d = 2$, the equilibrium 1 of (3.1) is a global attractor when it is locally asymptotically stable, that is, in the conditions of Theorem 3.3 (with $\gamma < 1$).
- The paper [25] by Tkachenko and Trofimchuk gives additional results that allow to sharpen the global stability region in some cases. For example, for $d = 2$, Corollary 4 in [25] establishes the global stability of 1 in (3.1) if $\gamma \leq 0.796$ and

$$\gamma^3 \geq \frac{1-A}{1+A},$$

  where $A = (\alpha+1)(1-\gamma)$ was defined in Corollary 3.15 (See Figure 3).
- In case $\gamma < 1$, one may get some information about the global dynamics even if the positive equilibrium of (1.5) is not globally attractive: the interested reader may easily check that the results of [7] can be applied for equation (1.5) to obtain a Morse decomposition of the global attractor, which is based on a discrete valued Lyapunov functional.

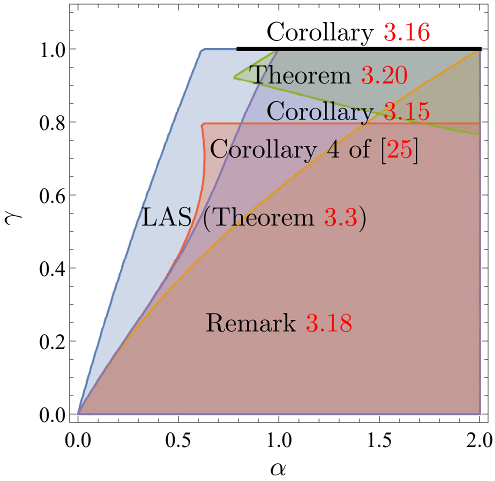

FIGURE 3. Comparison of the regions of LAS and GAS provided by various criteria for $d = 2$.

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Ábel Garab
Institut für Mathematik
Universität Klagenfurt
Universitätsstr. 65–67
9020 Klagenfurt
Austria
e-mail: `abel.garab@aau.at`

Víctor Jiménez López
Departamento de Matemáticas
Universidad de Murcia
Campus de Espinardo
30100 Murcia
Spain
e-mail: `vjimenez@um.es`

Eduardo Liz
Departamento de Matemática Aplicada II
Universidad de Vigo
36310 Vigo
Spain
e-mail: `eliz@uvigo.es`